Noncommutative graphs based on finite-infinite system couplings: Quantum error correction for a qubit coupled to a coherent field

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Quantum error correction plays a key role for quantum information transmission and quantum computing. In this work, we develop and apply the theory of noncommutative operator graphs to study error correction in the case of a finite-dimensional quantum system coupled to an infinite-dimensional system. We consider as an explicit example a qubit coupled via the Jaynes-Cummings (JC) Hamiltonian with a bosonic coherent field. We extend the theory of noncommutative graphs to this situation and construct, using Gazeau-Klauder coherent states, the corresponding noncommutative graph. As the result, we find the quantum anticlique, which is the projector on the error-correcting subspace, and analyze it as a function of the frequencies of the qubit and the bosonic field. The general treatment is also applied to the analysis of the error-correcting subspace for certain experimental values of the parameters of the Jaynes-Cummings Hamiltonian. The proposed scheme can be applied to any system that possess the same decomposition of spectrum of the Hamiltonian into a direct sum as in JC model, where eigenenergies in the two direct summands form strictly increasing sequences.

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I. INTRODUCTION

Quantum error-correcting codes (or, in other terminology, quantum anticliques), introduced theoretically in the pioneering papers [1–3] and implemented experimentally, e.g., in Ref. [4], play an important role in quantum information theory [5]. Analog error correction for continuous variables, such as position and momentum, was considered [6], symmetry breaking in open quantum systems for photonic cat qubits [7], etc. In particular, bosonic codes which use encoding of information in states of bosonic field are of high interest [8–12].

Error-correction theory studies the possibility of encoding information in quantum states in a way to allow zero-error decoding in the presence of a given fixed acceptable set of errors. Mathematically, each error is described by some completely positive map acting on the set of states of the quantum system. In the general setting [13], for any given set of errors it is possible to define a unique noncommutative operator graph [14] such that the knowledge of this graph allows us to define all error-correcting codes for this set of errors. The correspondence between sets of errors and noncommutative operator graphs is not one to one, but in the case of a separable Hilbert space each noncommutative operator graph describes codes for some set of errors [15–17]. As was established in the finite-dimensional [15,16] and infinite-dimensional [17] cases, the noncommutative graph describing errors in information transmission is always generated by some positive operator-valued measure.

Various models of error correction were analyzed by using the approach based on the use of noncommutative graphs. It was applied for quantum error correction for the models of coupled finite-dimensional systems [18–20] and for coupled infinite-dimensional systems [21]. In Ref. [21], the noncommutative graph generated by the dynamics of a bipartite bosonic quantum system in an infinite-dimensional Hilbert space was defined. The graph consists of orbits driven by the unitary group which is the solution of the Schrödinger equation for a two interacting bosonic oscillators. In this framework possible error-correcting codes are given by coherent states in the bosonic Fock space. In all these cases, it was possible to find quantum anticliques which are projectors onto error-correcting subspaces.

In this work, we extend the theory of operator graphs to the case when one system is finite dimensional while another is infinite dimensional. Currently, several quantum error-correction [22,23] and entanglement protection [24] techniques were introduced for systems of such structure. Explicitly, we consider the situation where the information is encoded in a joint state of a qubit (a two-level quantum system) coupled via the Jaynes-Cummings (JC) Hamiltonian [25] with a bosonic oscillator or bosonic coherent field. The Jaynes-Cummings Hamiltonian is the key model for the various theoretical and experimental works in quantum optics and for studying the interactions between light and matter, see, e.g., Refs. [26–39], including in the strong [34], ultrastrong [27,28,31], and deep strong-coupling regimes [27–29,31–33].

We develop for this model the theory of noncommutative operator graphs and apply it to find the corresponding quantum anticlique. The construction is based upon

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Gazeau–Klauder coherent states [40]. With this setting, we explicitly find the error-correcting subspace for any values of the parameters of the Jaynes-Cummings model.

The structure of the paper is the following: In Sec. II, we discuss the problem of finding the existence of an error-correcting procedure for a quantum channel, and also describe quantum channels corresponding to operator graphs of the class which includes the noncommutative graphs later constructed in Sec. V. In Sec. III, the Jaynes-Cummings model is discussed. Section IV describes the construction of Gazeau-Klauder coherent states. In Sec. V, using the Gazeau-Klauder coherent states for the Jaynes-Cummings model, we construct the noncommutative operator graphs that have quantum error-correcting codes, and find quantum anticlique and error-correction subspaces.

II. QUANTUM CHANNELS AND NONCOMMUTATIVE GRAPHS

Consider encoding information in states of a quantum system with Hilbert space $\mathcal{H}$. The (convex) set $\mathfrak{S}(\mathcal{H})$ of quantum states is the set of positive unit trace operators in $\mathcal{H}$. Errors which can occur under information transmission can be described by a quantum channel $\Phi : \mathfrak{S}(\mathcal{H}) \to \mathfrak{S}(\mathcal{H})$ which is a completely positive trace preserving (CPTP) map. As any CPTP map, it possesses the Kraus operator-sum representation (Kraus OSR) [41]

$$\Phi(\rho) = \sum_{k\in K} V_k \rho V_k^*, \quad \rho \in \mathfrak{S}(\mathcal{H}). \tag{1}$$

The Kraus operators $\{V_k,\ k \in K\}$ are parametrized by some set K. They should satisfy the property

$$\sum_{k\in K} V_k^* V_k = \mathbb{I} \tag{2}$$

to preserve trace of the density matrix. In infinite-dimensional spaces, K is not necessarily countable and the sum in (1) and (2) can be replaced by an integral (see, e.g., Ref. [42]). Nevertheless, for any channel Φ there exists a countable set K parametrizing Kraus operators such that (1) holds true. Note that the Kraus OSR of a quantum channel is nonunique. The same quantum channel also has a Kraus OSR with operators $\tilde{V}_j = \sum_i U_{ji} V_i$, $j = 1, \dots, m$ $(m \geqslant |K|)$, where $U^+U = \mathbb{I}$ (unitary matrix).

The linear space $\mathcal{V}$ plays an important role in the theory of optimal coding:

$$\mathcal{V} = \overline{span}\{V_k^* V_j,\ k, j \in K\}. \tag{3}$$

The linear space $\mathcal{V}$ does not depend on the choice of the operators $\{V_i\}$ used for Kraus OSR of a given quantum channel; despite the nonuniqueness of the Kraus OSR, it is unique for a given quantum channel. Notice that in the finite-dimensional case there is not need to take the closure in (3). For the infinite-dimensional case, see Refs. [17,21].

The linear space $\mathcal{V}$ has the properties of a noncommutative graph. Such objects were introduced in Ref. [43] as operator systems and recently redefined as noncommutative graphs in quantum information theory [14]. A *noncommutative graph* is a linear subspace $\mathcal{V}$ of bounded operators in a Hilbert space $\mathcal{H}$ possessing the properties

(i) $\mathbf{V} \in \mathcal{V}$ implies that $\mathbf{V}^* \in \mathcal{V}$;

(ii) $\mathbf{I} \in \mathcal{V}$.

The famous Knill–Laflamme condition [3,13] claims that a zero-error transmission via some channel Φ is possible if and only if for some orthogonal projector P for all $A \in \mathcal{V}$ holds $PAP = \alpha(A)P$, where $\alpha(A) \in \mathbb{C}$. Here P is the projector on the subspace generated by error-correction code [3]. The optimal code belongs to the subspace $\mathcal{H}_P = P\mathcal{H}$. The dimension of the subspace $\mathcal{H}_P$ is the maximal amount of quantum information that could be transmitted via Φ with zero error. An orthogonal projection P such that $\dim(P\mathcal{H}) \geqslant 2$ is a *quantum anticlique* for a noncommutative graph $\mathcal{V}$ if it satisfies:

$$\dim P\mathcal{V}P = 1. \tag{4}$$

The most natural quantum channel is given by the projection measurement

$$\Phi_{\mathcal{P}}(\rho) = \sum_{k\in K} P_k \rho P_k, \quad \rho \in \mathfrak{S}(\mathcal{H}), \tag{5}$$

where $\mathcal{P} = (P_k)$ is the orthogonal resolution of identity

$$\sum_{k\in K} P_k = \mathbb{I}.$$

For the channel (5), the noncommutative graph (3) is

$$\mathcal{V} = \overline{span}\{P_k, k \in K\}.$$

In the finite-dimensional case, it is enough to consider only discrete sets K, while for the infinite-dimensional case $P_k = E(B_k)$ are generated by some projection valued measure on the real line, where $B_k \subset \mathbb{R}$ are some Borel sets possessing the property $\cup_{k\in K} B_k = \mathbb{R}$. In this case, different choices of B_k produce different projectional measurements and a reachable set of admissible errors.

Suppose that some unitary group $\mathcal{U} = \{U_t = e^{-itG},\ t \geqslant 0\}$ acts in the Hilbert space H. Then the possible set of errors can be extended to all possible projection measurements

$$\Phi^t(\rho) = \sum_{k\in K} U_t P_k U_t^* \rho U_t P_k U_t^*, \quad t \in \mathbb{R}. \tag{6}$$

For the goal of constructing a quantum anticlique allowing to correct errors of the form (6) for any fixed t, it is natural to define the noncommutative graph corresponding to all these errors as follows:

$$\mathcal{V} = \overline{span}\{U_t P_k U_t^*,\ t \in \mathbb{R},\ k \in K\}.$$

Our interpretation of such a graph is that the quantum system distorts the transmitted information by a set of time-dependent errors.

Based upon this interpretation, suppose that there is a set of orthogonal projections $\{P_\alpha,\ \alpha \in \mathfrak{A}\}$ parametrized by some set $\mathfrak{A}$, and the operator space is generated by orbits of some unitary group $\mathcal{U}$ as follows:

$$\mathcal{V} = \overline{span}\{U_t P_\alpha U_t^*,\ t \in \mathbb{R},\ \alpha \in \mathfrak{A}\}. \tag{7}$$

It is known [15,17] that (7) is a noncommutative operator graph corresponding to some channel if and only if $\mathbb{I} \in \mathcal{V}$. We shall construct the explicit example of graph for the Jaynes-Cummings model. Moreover, we shall show that there exists an anticlique for this graph.

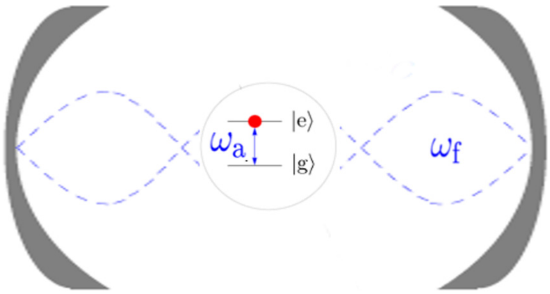

FIG. 1. Jaynes-Cummings model of a qubit interacting with bosonic reservoir.

III. JAYNES-CUMMINGS MODEL

We consider a two-level quantum system (qubit) coupled to a coherent field. The Hilbert space of the qubit is $\mathcal{H}_s = \mathbb{C}^2$. Ground and excited basis states of the qubit are denoted $\{|g\rangle, |e\rangle\}$. The Hilbert space of the coherent field is $\mathcal{H}_f = L^2(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \int_{\mathbb{R}} |f(x)|^2 dx < \infty\}$. Fock states of the field are denoted as $\{|k\rangle, k \in \mathbb{N}_0\}$. We use the set of natural numbers including zero $\mathbb{N}_0 = 0 \cup \mathbb{N}$ to enumerate the states. The qubit and the field are assumed to be coupled via the Jaynes-Cummings Hamiltonian acting in $\mathcal{H} = \mathcal{H}_s \otimes \mathcal{H}_f$:

$$H = \omega_f a^+ a^- + \frac{\omega_s}{2}\sigma_z + \frac{\kappa}{2}(\sigma^- a^+ + \sigma^+ a^-). \quad (8)$$

Here ω_s, $\omega_f \in \mathbb{R}_+$ are the frequencies of the qubit and the field, respectively, $\kappa \geqslant 0$ is the coupling constant, σ_z is the Pauli matrix, σ^+, σ^- are the rising and lowering operators of the qubit and a^+, a^- are the creation and annihilation operators of the field. The detuning parameter is $\Delta = \omega_f - \omega_s$. We use the normalization of the physical units such that $\hbar = 1$. Denote the basis in $\mathcal{H}$ as $|q\rangle \otimes |p\rangle = |q, p\rangle$, where the first number $q \in \mathbb{N}_0$ denotes the coherent state and the second number $p \in \{e, g\}$ denotes the qubit state. The schematic picture of the Jaynes-Cumming model states interaction is provided in Fig. 1.

The Schrödinger equation with the Jaynes-Cummings Hamiltonian has an exact solution. The Hamiltonian has the following eigenstates:

$$|0, g\rangle,$$
$$|n, +\rangle = \cos\left(\frac{\theta_n}{2}\right)|n-1, e\rangle + \sin\left(\frac{\theta_n}{2}\right)|n, g\rangle,$$
$$|n, -\rangle = \sin\left(\frac{\theta_n}{2}\right)|n-1, e\rangle - \cos\left(\frac{\theta_n}{2}\right)|n, g\rangle,$$

where $\theta_n = \tan^{-1}(\kappa\sqrt{n}/\Delta)$ and $n \in \mathbb{N}$, for the nonresonant case $\Delta \neq 0$. For the resonant case $\Delta = 0$ the eigenstates are

$$|0, g\rangle,$$
$$|n, +\rangle = |n-1, e\rangle + |n, g\rangle,$$
$$|n, -\rangle = |n, g\rangle - |n-1, e\rangle.$$

In both cases the corresponding eigenenergies are

$$E_{0,g} = \frac{\omega_f + \Delta}{2},$$
$$E_{n,\pm} = \omega_f\left(n - \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{\Delta^2 + \kappa^2 n}, \quad n \in \mathbb{N}.$$

Below we follow closely to the method provided in Ref. [44], where a new class of coherent states was constructed for the Jaynes-Cummings model with strictly increasing sequences of the eigenenergies $E_{n,+}$ and $E_{n,-}$. Our goal is to divide the space $\mathcal{H}$ into three direct summands, two of which are generated by eigenstates corresponding to strictly increasing sequences of eigenenergies and one is finite dimensional. The sequence

$$J_k = E_{k+1,+}, \quad k \in \mathbb{N}_0 \quad (9)$$

is known to be strictly increasing. On the other hand, the sequence $S_0 = E_{0,g}$, $S_k = E_{k,-}$, $k \in \mathbb{N}$ may have degenerate levels $S_{k_1} = S_{k_2}$, $k_1 \neq k_2$. We want to keep only a strictly increasing tail of the sequence S_k. Let us show that there exists $M_0 \in \mathbb{N}$ such that, for all $l_2 > l_1 \geqslant M_0$, one gets $S_{l_2} > S_{l_1}$. It is equivalent to

$$S_{n+1} - S_n = \omega_f - \tfrac{1}{2}(\sqrt{\Delta^2 + \kappa^2(n+1)} - \sqrt{\Delta^2 + \kappa^2 n}) > 0 \; \forall \, n \geqslant M_0. \quad (10)$$

From this one gets that M_0 is the minimal integer solution of the inequality

$$(\sqrt{\Delta^2 + \kappa^2(M_0+1)} + \sqrt{\Delta^2 + \kappa^2 M_0})^{-1} < \frac{2\omega_f}{\kappa^2}. \quad (11)$$

Thus, the sequence

$$S_k = E_{k,-}, \quad k \geqslant M_0, \quad (12)$$

becomes strictly increasing.

Let us fix any number $K_0 \in \mathbb{N}$, $K_0 \geqslant M_0$. Then the sequence $S_k = E_{k,-}$, $k \geqslant K_0$ will also be strictly increasing and we can separate the pieces where we assured to have strictly increasing eigenenergies. This allows us to represent the Hilbert space $\mathcal{H}$ as the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where

$$\mathcal{H}_1 = span\{|n, +\rangle, \; n \in \mathbb{N}\}, \quad (13)$$

$$\mathcal{H}_2 = span\{|n, -\rangle, \; n \geqslant K_0\}, \quad (14)$$

$$\mathcal{H}_3 = span\{|g, 0\rangle\} \cup \{|n, -\rangle, \; 1 \leqslant n < K_0\}. \quad (15)$$

The subspace $\mathcal{H}_3$ later will be shown to be the error-correcting subspace for this system. The error dynamics will be shown to interchange states in $\mathcal{H}_1$ and $\mathcal{H}_2$, while keeping states in $\mathcal{H}_3$ unchanged.

In Sec. 5, following the ideas of Ref. [44] we will define the Gazeau-Klauder coherent states in $\mathcal{H}_1$ and $\mathcal{H}_2$.

IV. GAZEAU-KLAUDER COHERENT STATES

Here we introduce the construction of Gazeau-Klauder coherent states [40]. Let us consider an infinite-dimensional Hilbert space $\mathcal{H}$ with the basis $|k\rangle$, $k \in \mathbb{N}_0$ and a self-adjoint operator G which is diagonal in this basis. In Ref. [40] Gazeau and Klauder defined the generalized coherent states corresponding to the operator G as a two-parameter system of vectors $\{|x, y\rangle, x \in \mathbb{R}_+,\ y \in \mathbb{R}\} \subset \mathcal{H}$ with the following properties:

(1) *Continuity:* $(x, y) \to (x_0, y_0) \Rightarrow |x, y\rangle \to |x_0, y_0\rangle$;
(2) *Resolution of identity:* $\int |x, y\rangle \langle x, y| \, d\nu(x, y) = \mathbb{I}_{\mathcal{H}}$;
(3) *Temporal stability:* $e^{-itG} |x, y\rangle = |x, y + \omega t\rangle$;
(4) *Action identity:* $\langle x, y| G |x, y\rangle = \omega x$;

for some real constant ω and some measure ν.

Consider the set of eigenvalues $h_k = \langle k| G |k\rangle$ for the operator G. In the case $h_0 = 0$ and strictly increasing h_k, Gazeau and Klauder gave the explicit construction for the system of coherent states. If $h_0 > 0$ and the sequence h_k is strictly increasing, their construction describes the set of vectors that satisfy the first two properties and the following version of the time stability condition:

$$e^{-itG} |x, y\rangle \langle x, y| e^{itG} = |x, y + \omega t\rangle \langle x, y + \omega t| \,.$$

The main property of generalized coherent states is that they form the resolution of identity. It was shown that the measure ν has the form

$$d\nu(x, y) = \tau(x) dx dy,$$

where $\tau(x)$ is some probability distribution density on the half axis.

Below we give an explicit description of this construction. Consider a sequences of weights

$$c_k > 0, \quad k \in \mathbb{N}_0,$$

with the convergence condition

$$\limsup_{k\to\infty} \sqrt[k]{c_k} = R > 0. \tag{16}$$

Suppose that these weights are the moments of probability distributions with density $\rho(x) > 0$ on the interval $[0, R)$,

$$c_k = \int_0^R \rho(x) x^k dx < +\infty, \quad k \in \mathbb{N}_0.$$

We also need the normalization factor and the density defined by the formulas

$$N^2(x) = \sum_{k=0}^{\infty} \frac{x^k}{c_k}, \quad 0 \leqslant x < R,$$
$$\tau(x) = N^2(x)\rho(x). \tag{17}$$

The radius of convergence in (17) is equal to R by the property (16). Now the Gazeau–Klauder coherent states are defined as follows:

$$|x, y\rangle = \frac{1}{N(x)} \sum_{k=0}^{\infty} \frac{x^{k/2} e^{-ih_k y}}{\sqrt{c_k}} |k\rangle \,. \tag{18}$$

We suppose the constant ω is equal to one. Following the definition in Ref. [40], for $f : \mathbb{R} \to B(\mathcal{H})$, where $B(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$, we introduce its integration as the weak-limit of averages of weak integrals,

$$I(f) = \int_{-\infty}^{+\infty} f(y) d\mu(y) = \lim_{R\to+\infty} \frac{1}{2R} \int_{-R}^{R} f(y) dy.$$

Note that if such integral converges for some f with the image lying in the weakly closed subspace $\mathrm{Im}(f) \subset \mathcal{W} \subset B(H)$, then the integral also lies in this subspace, $I(f) \in \mathcal{W}$. The resolution of identity property for coherent states results in

$$\int_0^R \int_{-\infty}^{+\infty} |x, y\rangle \langle x, y| \, \tau(x) dx d\mu(y) = \mathbb{I}_{\mathcal{H}}. \tag{19}$$

V. GRAPHS GENERATED BY GAZEAU-KLAUDER COHERENT STATES

Now we are able to define systems of Gazeau–Klauder coherent states [40] in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Take two sequences of weights

$$c_k^{(j)} > 0, \quad d \in \mathbb{N}_0, \quad j = 1, 2$$

with the same convergence condition

$$\limsup_{k\to\infty} \sqrt[k]{c_k^{(j)}} = R > 0.$$

Suppose that the weights have the corresponding probability densities $\rho_1(x),\ \rho_2(x) > 0$ on the interval $[0, R]$ such that

$$c_k^{(j)} = \int_0^R \rho_j(x) x^k dx < +\infty, \quad k \in \mathbb{N}_0, \quad j = 1, 2.$$

Then, the normalization factors and the densities for measures defining resolutions of identity are given by the formulas

$$N_j^2(x) = \sum_{k=0}^{\infty} \frac{x^k}{c_k^{(j)}}, \quad 0 \leqslant x < R,$$
$$\tau_j(x) = N_j^2(x)\rho_j(x), \quad j = 1, 2.$$

Consider the Gazeau–Klauder coherent states

$$|J, x, y\rangle = \frac{1}{N_1(x)} \sum_{k=0}^{\infty} \frac{x^{k/2} e^{-iJ_k y}}{\sqrt{c_k^{(1)}}} |k + 1, +\rangle \,,$$

$$|S, x, y\rangle = \frac{1}{N_2(x)} \sum_{k=0}^{\infty} \frac{x^{k/2} e^{-iS_{k+K_0} y}}{\sqrt{c_k^{(2)}}} |k + K_0, -\rangle \,,$$

where the strictly increasing sequences of eigenenergies are given by (9) and (12), respectively.

Since the Gazeau–Klauder coherent states from Sec. IV form the resolution of identity, we get for the projections $P_{\mathcal{H}_1}$, $P_{\mathcal{H}_2}$ on the subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ that

$$\int_0^R \int_{-\infty}^{+\infty} |J, x, y\rangle \langle J, x, y| \, \tau_1(x) dx d\mu(y) = P_{\mathcal{H}_1}, \tag{20}$$

$$\int_0^R \int_{-\infty}^{+\infty} |S, x, y\rangle \langle S, x, y| \, \tau_2(x) dx d\mu(y) = P_{\mathcal{H}_2}. \tag{21}$$

Consider the unitary group $\mathcal{U} = \{U_t = e^{-itH}, t \in \mathbb{R}\}$, where the Hamiltonian H is determined by (8). Systems

$|J,x,y\rangle$, $|S,x,y\rangle$ satisfy the temporal stability property (IV) with respect to $\mathcal{U}$,

$$U_t\,|J,x,y\rangle\,\langle J,x,y|\,U_t^* = |J,x,y+t\rangle\,\langle J,x,y+t|\,, \tag{22}$$

$$U_t\,|S,x,y\rangle\,\langle S,x,y|\,U_t^* = |S,x,y+t\rangle\,\langle S,x,y+t|\,. \tag{23}$$

Consider the two families of orthogonal projections

$$P_x^1 = |J,x,0\rangle\,\langle J,x,0|\,,\quad P_x^2 = |S,x,0\rangle\,\langle S,x,0|\,,$$

for $x \in [0,R]$. The projections P_x^1, P_x^2 and $P_x^3 \equiv P_{\mathcal{H}_3}$ are pairwise orthogonal for any fixed value of $x \in [0,R]$.

Theorem 1. The subspace

$$\mathcal{V} = \overline{span}\{U_t P_x^j U_t^*,\ t\in\mathbb{R},\ x\in[0,R],\ j\in\{1,2,3\}\}$$

is a noncommutative operator graph with the anticlique $P_{\mathcal{H}_3}$.

Proof. Consider the operator

$$Q_x = P_x^1 + \frac{\tau_2(x)}{\tau_1(x)}P_x^2 + \frac{1}{\tau_1(x)}P_{\mathcal{H}_3}.$$

It follows from (22) and (23) that

$$U_tQ_xU_t^* = |J,x,t\rangle\,\langle J,x,t| + \frac{\tau_2(x)}{\tau_1(x)}\,|S,x,t\rangle\,\langle S,x,t| + \frac{1}{\tau_1(x)}P_{\mathcal{H}_3}.$$

Then, (20) and (21) result in

$$\int_0^R\int_{-\infty}^{+\infty}\tau_1(x)\bigg(|J,x,t\rangle\,\langle J,x,t| + \frac{1}{\tau_1(x)}\mathbb{I}_{\mathcal{H}_2} + \frac{\tau_2(x)}{\tau_1(x)}\,|S,x,t\rangle\,\langle S,x,t|\bigg)dx\,d\mu(t) = \mathbb{I}_{\mathcal{H}} \in \mathcal{V}. \tag{24}$$

Since K_0 is given by the rule (25) the dimension of $\mathcal{H}_3$ is at least two. From the equalities

$$P_{\mathcal{H}_3}\,|J,x,t\rangle\,\langle J,x,t|\,P_{\mathcal{H}_3} = 0,$$
$$P_{\mathcal{H}_3}\,|S,x,t\rangle\,\langle J,x,t|\,P_{\mathcal{H}_3} = 0,$$

we obtain that $P_{\mathcal{H}_3}$ is an anticlique. ■

VI. THE ERROR-CORRECTING SUBSPACE

As Theorem 1 states, the subspace $\mathcal{H}_3$ is the error-correcting subspace. In some cases the number is $M_0 = 1$, so in this case for $K_0 = M_0$ the error-correcting subspace would be empty (since its dimension is $K_0 - 1$). However, in our construction one can take any natural $K_0 \geqslant M_0$. To satisfy this condition for our coding procedure, that is the dimension of error-correcting subspace is greater or equal to two, we should take any

$$K_0 \geqslant K_0^* = \max\{3, M_0\}. \tag{25}$$

To analyze the minimal dimension of the error-correcting subspace for various parameters of the Jaynes-Cummings Hamiltonian, consider the coupling rates $\gamma_f = \kappa/\omega_f$ and $\gamma_s = \kappa/\omega_s$. In terms of these quantities, the inequality (11) takes the following form:

$$\sqrt{\left(\gamma_f^{-1}-\gamma_s^{-1}\right)^2 + M_0 + 1} + \sqrt{\left(\gamma_f^{-1}-\gamma_s^{-1}\right)^2 + M_0} > \frac{\gamma_f}{2}. \tag{26}$$

Now it is evident that the key quantity defining the dimension for the resonant case (when $\Delta = \omega_f - \omega_s = 0$ and hence $\gamma_f = \gamma_s$) is the coupling rate γ_f. For having M_0 equal or larger than four or greater, we need the γ_f to be at least $2(2+\sqrt{3})$. For the nonresonant case for fixed γ_f, decreasing the value γ_s will decrease M_0. Figure 2 shows the behavior of the minimal possible dimension $D_{\min} = K_0^* - 1$ of the error-correcting subspace $\mathcal{H}_3$ vs coupling rates γ_s and γ_f. The figure clearly shows that the behavior is nonsymmetric with respect to γ_s and γ_f, as is also evident from inequality (26). The resonant case is the extremal case in the inequality (26), what means if M_0 satisfies it for some γ_f, then for the same parameter γ_f the number M_0 also solves the inequality in the resonant case. Figure 3 shows the dependence of $D_{\min}$ on the coupling rate γ_f for equal frequencies.

The Jaynes-Cummings Hamiltonian is used in various theoretical and experimental analysis in quantum optics, cavity QED, see, e.g., Refs. [26–39], including in the strong [34], ultrastrong [27,28,31], and deep strong-coupling regimes [27–29,31–33]. We take, for example, from the weak-coupling regime the parameters used in experiments performed by the group of Haroche [36]. In the setup of that experiment, the cavity is designed to have the frequency equal to the frequency of the atom, i.e., $\Delta = 0$. The approximate parameters for the experiment are $\kappa = 2\pi \times 47$ kHz and

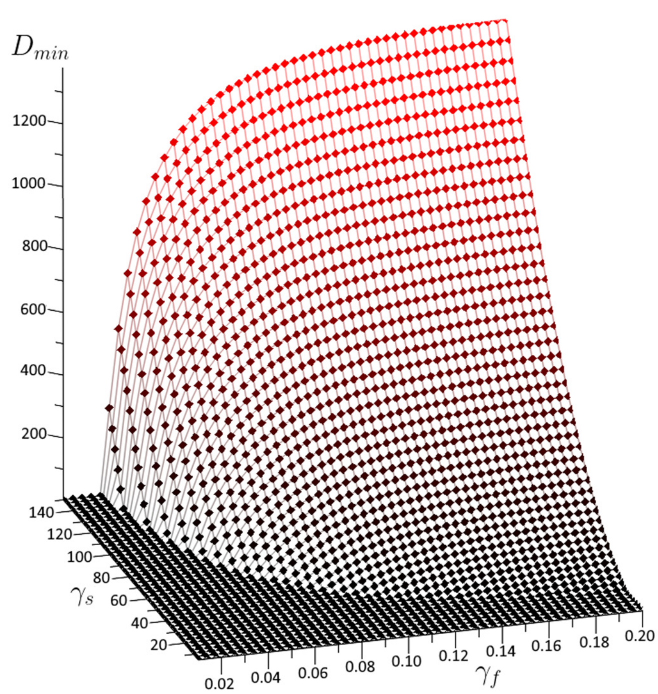

FIG. 2. Behavior of the minimal dimension of the error-correcting subspace vs coupling rates of the Jaynes-Cummings Hamiltonian.

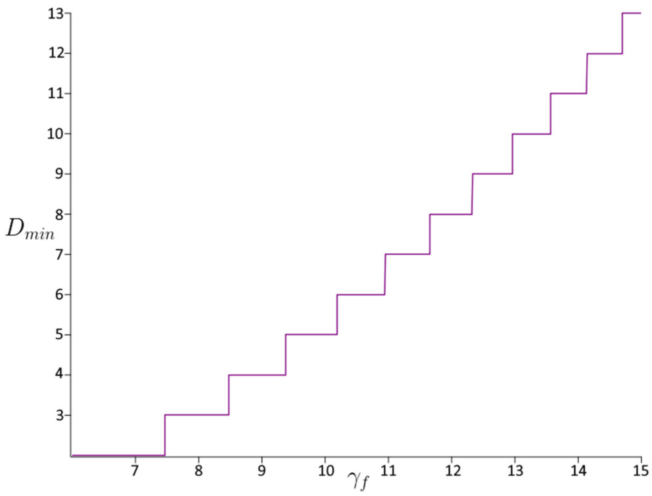

FIG. 3. Behavior of the minimal dimension of the error-correcting subspace for the resonant case $\Delta = 0$ vs the coupling rate of the field.

$\omega_f = \omega_s = 2\pi \times 51.1$ GHz. For this case the inequality (11) becomes

$$\sqrt{M_0 + 1} + \sqrt{M_0} > \frac{\kappa}{2\omega_f} = \frac{47}{2 \times 51.1 \times 10^6} \approx 0.46 \times 10^{-6}. \tag{27}$$

The minimal natural solution of this inequality is $M_0 = 1$, so $K_0 = 3$ and the minimal dimension is $D_{\min} = K_0 - 1 = 2$. This minimal two-dimensional error-correcting subspace is spanned by the two vectors $|g, 0\rangle$ and $|1, -\rangle = |1, g\rangle - |0, e\rangle$. We remark that this is the error-correcting subspace of minimal dimension. One could choose arbitrary large K_0 and the corresponding noncommutative operator graph will have the error-correcting subspace H_3 of dimension $K_0 - 1$.

The Jaynes-Cummings model is derived from Rabi model via the rotating-wave approximation (RWA). This approximation is typically valid for $\gamma_f < 0.1$ and $\omega_f \approx \omega_s$. In this case the minimal dimension is $D_{\min} = 2$. As can be seen from Figs. 2 and 3, to have $D_{\min} > 2$ one has to consider values of γ_f in the range of the deep strong-coupling regime [27–29]. This regime, as well as the less intense ultrastrong regime, is of interest now. In these regimes the Rabi model is nonintegrable, and investigating these regimes motivates describing eigenenergy approximations for this model [29,30]. One can show [31] that introducing a special type of frequency modulations applied to the field and the qubit will give the dynamics governed by Jaynes-Cummings Hamiltonian with γ_f in the range of deep strong-coupling regime. In circuit-QED simulations rates of γ_f for Rabi model up to 2.1 are achieved [32,33]. Thus value is lower, while not that much, than the minimal $\gamma_f \approx 7.5$ that is necessary to see the effect in which minimal dimension of the error-correcting code in the proposed scheme will be three or greater. Our analysis allows us to construct nontrivial quantum error-correcting codes for all possible values γ_f, γ_s.

We remark that our scheme could be applied to any system that possess the same decomposition into the direct sum, where eigenenergies in the two direct summands form strictly increasing sequences. Potentially, this property could be exploited for more complex Hamiltonian beyond the Jaynes-Cummings model, such as, for example, for the Jaynes-Cummings-Hubbard Hamiltonian, describing the interaction of several qubit-cavity systems, or perhaps directly for the Rabi Hamiltonian.

VII. CONCLUSION

In this work, the theory of noncommutative operator graphs has been developed for error correction in the case of a finite-dimensional quantum system coupled to an infinite-dimensional quantum system. We have constructed the noncommutative operator graph generated by orbits of the unitary group driven by Hamiltonian (8) of the Jaynes-Cummings model. We have shown that, for a positive integer K_0 that satisfies (25), using for encoding the eigenstates $|g, 0\rangle$ with $|n, -\rangle$ for $0 \leqslant n < K_0$ (9) allows us to transmit information with zero error via quantum channels with operator graphs belonging to the constructed graph. Thus the error-correcting subspace is explicitly computed for all values of the parameters of the Jaynes-Cummings model. Our scheme could be applied to any system that possesses the same decomposition of eigenenergies into the direct sum as for the JC Hamiltonian, where eigenenergies in the two direct summands form strictly increasing sequences.

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