Topological Uhlmann phase transitions for a spin-*j* particle in a magnetic field

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The generalization of the geometric phase to the realm of mixed states is known as the Uhlmann phase. Recently, applications of this concept to the field of topological insulators have been made and an experimental observation of a characteristic critical temperature at which the topological Uhlmann phase disappears has also been reported. Here we study the case of the Uhlmann phase of a paradigmatic system such as the spin-j particle in the presence of a slowly rotating magnetic field at finite temperature in an exact analytical form. We find that the Uhlmann phase is given by the argument of a complex-valued second-kind Chebyshev polynomial of order $2j$. Correspondingly, the Uhlmann phase displays $2j$ singularities, occurring at the roots of such polynomials which define the critical temperatures at which the system undergoes topological order transitions. Appealing to the argument principle of complex analysis, each topological order is characterized by a winding number, which happens to be $2j$ for the ground state and decreases by unity each time the increasing temperature passes through a critical value. We hope this study encourages experimental verification of this phenomenon of thermal control of topological properties, as has already been done for the spin-$1/2$ particle.

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I. INTRODUCTION

The emergence of the geometric phase in quantum physics has been a groundbreaking event [1]. It has served as a tool to comprehend its fundamentals, such as the spin statistics theorem [2] or the Aharanov-Bohm effect [1]. In the field of condensed-matter physics, the geometric phase is a quantity that comes out in the theoretical description of the quantum Hall effect [3,4], the correct expression for the velocity of Bloch electrons [5], ferromagnetism [6,7], and topological insulators [8], among other phenomena. It is also a central concept in holonomic quantum computation [9].

First proposed for adiabatic cyclic evolution [1], the geometric phase concept has already been broadened to the arbitrary evolution of a quantum state [10–12] and is thus ubiquitous in quantum systems. Nevertheless, the Berry approach [1] to the geometric phase considers pure states only, while a more realistic description of quantum phenomena requires making use of the density matrix formalism.

An extension of the geometric phase to mixed states was developed by Uhlmann [13,14] and is thus called the *Uhlmann phase*. It has been theoretically studied in the context of one-dimensional (1D) and 2D topological insulators [15–18], for example the 1D Su-Schrieffer-Heeger model [19] and the Qi-Wu-Zhang 2D Chern insulator [20]. A key feature of these systems is the appearance of a critical temperature above which the Uhlmann phase vanishes, regardless of the topological character of the system in the ground state. This temperature sets a regime of stability of the topological properties in such systems, and has already been experimentally observed in a superconducting qubit [21], which gives the Uhlmann phase a higher ontological status.

Not only has this phase emerged as an interesting quantity in physics, but also the related one called the *Uhlmann mean curvature* [22], which might be regarded as an analog of the Berry curvature for mixed states. Notably, this quantity can formally be linked to the dissipative part of the dynamical susceptibility within the linear response theory [23,24], which allows one to connect measurable physical quantities to topological invariants of the system. It has also been useful for characterization of finite-temperature topological properties in the Kitaev chain model [25].

For pure quantum systems, a paradigmatic example to illustrate the abstract notions of quantum holonomies is the spin-j particle interacting with a slowly rotating magnetic field. In its original paper [1], Berry obtained the beautiful solid angle formula for the eigenstates of this system. More recently, a generalization of the solid angle formula for arbitrary spin-j states has been found in terms of the Majorana constellation [26,27], which also gives insight into their entanglement properties [28]. From a more practical standpoint, the study of the geometric phase for a spin-$1/2$ particle is the workhorse to build one-qubit holonomic quantum gates [9,29], with a possible extension to SU(2) qudit gates for higher spins [30]. This makes the study of the geometric phase for spin-j particles of vital importance.

Here, we calculated the Uhlmann phase of a spin-j particle subjected to a slowly rotating magnetic field in a circular simple closed loop. We derived a compact analytical expression in terms of the argument of the complex-valued second-kind Chebyshev polynomials [31,32] $U_{2j}(z)$ multiplied by the Pauli sign $(-1)^{2j}$. When the field describes a circle instead of a cone, the Uhlmann phase becomes topological with respect

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to temperature: the change from zero to π (or vice versa) at certain critical temperatures related to the roots of $U_{2j}(z)$. Based on a theorem of complex analysis [33,34], we define the Chern-like Uhlmann numbers [35] as winding numbers. For arbitrary direction of the external field, and as a function of temperature, the colored plot of the phase allows one to visually identify the number of critical temperatures and the Uhlmann topological numbers for a given spin number j.

The Uhlmann holonomy of density matrices describing mixed states of high-spin systems has been previously addressed. For example, in hydrogenlike atoms with spin-orbit coupling, this phase was studied [36] in relation to the entanglement properties of the internal degrees of freedom. Some time ago, Uhlmann [13,37] treated the case of the spin-j particle in a Gibbsian ensemble under the action of time-independent SU(2) Hamiltonians, where the holonomy of the density operator was reported. Later, the holonomy of this system was used to perform a numerical comparison between the Uhlmann geometric phase and the Sjöqvist interferometric phase [38–40]. We note that the problem considered in those references can be seen to be equivalent to the one in our paper, since the evolution of the density matrix in both cases is along closed O(3) orbits in the projective space. However, an exact closed form of the Uhlmann phase was not given, nor was mention of the temperature-dependent topological phase transitions made.

More recently, the Uhlmann phase for the evolution of thermal density matrices when the magnetic field winds around the equator n times was derived [41]. They found an expression of the Uhlmann phase in terms of the diagonal entries of the Wigner $d^{(j)}_{m,m'}$ matrices [42], which allowed them to show the emergence of n and $2n$ topological phase transitions, when j equals 1/2 and 1, respectively. They explained how to experimentally observe these topological transitions, which we did not address here. Nevertheless, an analysis of critical temperatures at $j > 1$ is absent, as well as the definition of the appropriate Chern-like Uhlmann numbers that characterize different topological orders at different temperatures. Even though we restricted our calculation to single closed loops ($n = 1$), for this particular case the Uhlmann phase derived here is more general since it contains evolution of the magnetic field with arbitrary values of θ. For $n > 1$, our analysis, as we will argue in the next section, still yields the Uhlmann phase in terms of Chebyshev polynomials for any θ.

The paper is organized as follows. In Sec. II, we derive the Uhlmann phase for the spin-j particle. In Sec. III, we show the emergence of multiple thermal topological transitions and obtain their corresponding Chern-like numbers. In Sec. IV, we analyze the Uhlmann phase with respect to temperature and the magnetic field's polar angle. Section V contains the conclusions.

II. UHLMANN PHASE FOR AN ARBITRARY SPIN *J* IN AN EXTERNAL MAGNETIC FIELD

The Uhlmann phase of a mixed quantum state is given by the expression [17,18]

$$\Phi_U = \arg(\mathrm{Tr}[\hat{\rho}\,\mathcal{P}e^{\oint \hat{A}_U}]), \tag{1}$$

where $\hat{\rho}$ is the system's density matrix with a spectral decomposition $\sum_k p_k |k\rangle\langle k|$ and $\mathcal{P}$ is the path-ordering operator [5]. Expression (1) assumes $\hat{\rho}$ to be isospectral along the cyclic evolution, which turns out to be the case for this problem. The Uhlmann connection $\hat{A}_U$ is given by [18]

$$\hat{A}_U = \sum_{l,k} \frac{(\sqrt{p_l} - \sqrt{p_k})^2}{p_l + p_k} \langle l|(d|k\rangle)|l\rangle\langle k|, \tag{2}$$

where d is the exterior derivative operator [5]. This equation is written in the density matrix eigenbasis, and thus a parameter dependence on the eigenkets is to be understood.

The Hamiltonian of a spin-j particle interacting with a magnetic field is expressed as

$$\hat{H} = B\hat{\mathbf{n}}\cdot\hat{\boldsymbol{J}}, \tag{3}$$

where all physical constants which give rise to the interaction are taken into account in B. The vector operator $\hat{\boldsymbol{J}}$ has the components $(\hat{J}_x, \hat{J}_y, \hat{J}_z)$, where $\hat{J}_i$ are the usual angular momentum matrices of spin j [42]. We will consider the familiar fixed magnitude magnetic field that rotates along the $\hat{\mathbf{z}}$ axis at constant frequency. The unit vector $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is taken with fixed θ, while ϕ changes during the evolution from 0 to 2π. For a thermal ensemble, the corresponding unnormalized density matrix is written as

$$\hat{\rho} = e^{-\beta B\hat{\mathbf{n}}\hat{\boldsymbol{J}}}, \tag{4}$$

where $\beta = 1/k_BT$. The partition function Z can be ignored in the calculation of the Uhlmann phase (1), since it is real and represents just a scaling factor of the complex number $\mathrm{Tr}[\hat{M}]$, where $\hat{M} = \hat{\rho}\,\mathcal{P}e^{\oint \hat{A}_U}$. The thermal basis is, in this case, a rotated $|j, m\rangle$ basis, which, in Euler angle representation, is given by

$$|j, m; \hat{\mathbf{n}}\rangle = e^{-i\phi\hat{J}_z}e^{-i\theta\hat{J}_y}e^{i\phi\hat{J}_z}|j, m\rangle. \tag{5}$$

The thermal occupation probabilities p_m are readily seen to be $e^{-\beta Bm}/Z$.

We now proceed to calculate the the Uhlmann connection for the problem at hand. The factor involving the thermal occupation probabilities in (2) can be expressed as

$$\frac{(\sqrt{p_m} - \sqrt{p_{m'}})^2}{p_m + p_{m'}} = 1 - \mathrm{sech}[\beta B(m - m')/2], \tag{6}$$

while the factor involving the exterior derivative d becomes

$$\langle m'|d|m\rangle = ie^{-i(m'-m)\phi}\sin\theta\,\langle m'|\hat{J}_x|m\rangle, \tag{7}$$

plus a negligible diagonal term. Inserting Eqs. (6) and (7) into (2) straightforwardly yields

$$\hat{A}_U = -i\eta\big(\hat{J}_z\sin\theta - e^{-i\phi\hat{J}_z}\hat{J}_x e^{i\phi\hat{J}_z}\cos\theta\big)d\phi, \tag{8}$$

where $\eta = \sin\theta[1 - \mathrm{sech}(\beta B/2)]$. Calculation of the time-ordered integral $\mathcal{P}e^{\oint \hat{A}_U}$ is equivalent to solving a Schrödinger equation,

$$i\frac{d}{d\phi}\hat{U} = \hat{V}(\phi)\hat{U}, \tag{9}$$

where the solution $\hat{U}$ is just the time-ordered exponential and $\hat{V}(\phi)$ is $i\hat{A}_U$. Solving Eq. (9) for a closed loop followed by the direction of the external field results in [5]

$$\mathcal{P}e^{\oint \hat{A}_U} = (-1)^{2j} e^{-i2\pi[(\eta \sin\theta - 1)\hat{J}_z - \eta\cos\theta \hat{J}_x]}. \quad (10)$$

In order to obtain the Uhlmann phase, we need to take the trace of $\hat{M}$. Except for the lowest total angular momentum representations, its exact closed form is very cumbersome. Also we would need to find the specific matrix $\hat{M}$ for every j, which is not very practical for our purposes. A way out of this pothole is noticing that the object we need to trace out belongs to the Lie group $SL(2,\mathbb{C})$ in the $(j,0)$ representation. Basic representation theory of this group [43] tells us that once the eigenvalues (λ_+,λ_-) of the $(1/2,0)$ representation are known, the eigenvalues for higher j are given by

$$\lambda^{2j}, \lambda^{2j-2}, \dots, \lambda^{-2j+2}, \lambda^{-2j}, \quad (11)$$

with $\lambda = \lambda_+ = \lambda_-^{-1}$, where $\lambda_\pm$ are the eigenvalues obtained from $j = 1/2$. The equality above follows from the property that the group has unit determinant. Having noted this, the trace of $\hat{M}$ is readily seen to be

$$\mathrm{Tr}[\hat{\rho}\,\mathcal{P}e^{\oint \hat{A}_U}] = \frac{\lambda^{2j+1} - \lambda^{-2j-1}}{\lambda - \lambda^{-1}}. \quad (12)$$

It remains to obtain the exact form of the eigenvalue λ. By diagonalizing $\hat{M}$ in the $(1/2,0)$ representation, we find

$$\lambda = z + \sqrt{z^2 - 1}, \quad (13)$$

with $z(\theta)$ being the complex variable,

$$z = \cosh(\beta B/2)\cos(\pi C) - i\sinh(\beta B/2)\sin(\pi C)\frac{\cos\theta}{C}, \quad (14)$$

where $C(\theta) = \sqrt{1 - \sin^2\theta \tanh^2(\beta B/2)}$. The function $z(\theta)$ defines a simple closed curve in the complex plane, with this property being of fundamental importance in what follows. With all these ingredients, the Uhlmann phase is readily obtained,

$$\Phi_U^{(j)} = \arg\left[(-1)^{2j}\frac{(z+\sqrt{z^2-1})^{2j+1} - (z-\sqrt{z^2-1})^{2j+1}}{2\sqrt{z^2-1}}\right]$$
$$= \arg[(-1)^{2j} U_{2j}(z)], \quad (15)$$

where $U_{2j}(z)$ are the second-kind Chebyshev polynomials [32]; we will refer to them just as Chebyshev polynomials to simplify matters. Equation (15) is valid under cyclic adiabatic evolution and is exact in this regime. The upper index (j) tells us that the Uhlmann phase is that of a spin-j particle. Also, in the low-temperature limit, this equation reduces to the corresponding Berry phase [1,5,17]. We note that the nice compact form of result (15) is not simply a phase portrait of the Chebyshev polynomials in the whole complex plane because the point $z(\theta)$ lies on a curve, with its shape depending on the parameter βB. However, the relation between the phase of polynomials $U_{2j}(z)$ and an observable phase of a quantum system is interesting. The appearance of Chebyshev polynomials in the Uhlmann phase can be traced back to $\hat{H}$ pertaining to $\mathfrak{su}(2)$ algebra and the particle being in thermal equilibrium. This generates an element of $SL(2,\mathbb{C})$ via $\hat{\rho}\mathcal{P}e^{\oint \hat{A}_U}$, whose eigenvalues can always be written as $v \pm \sqrt{v^2-1}$, $v \in \mathbb{C}$. This is also true when the magnetic field rotates n times around the z axis since the path-ordered integral, in this case, is just $\hat{U}^n$, where $\hat{U}$ is given by Eq. (10), and thus the product $\hat{\rho}\hat{U}^n$ also pertains to $SL(2,\mathbb{C})$. We will not explore the $n > 1$ cases here. In Ref. [41], the case with the magnetic field rotating on the equatorial plane ($\theta = \pi/2$) is treated and the results for the cases with $n = 1$ and 2 are shown. It would be interesting to explore what kind of mathematical object the Uhlmann phase would be when considering a Hamiltonian that belongs to $\mathfrak{su}(n)$ for $n > 2$.

III. TOPOLOGICAL UHLMANN PHASE TRANSITIONS

A. Critical temperatures

The Uhlmann phase just obtained is determined by the argument of the complex Chebyshev polynomials $U_{2j}(z)$. The function U_{2j} has real roots only, $2j$ in number, lying in the open interval $(-1, 1)$ [32]. The zeros of any polynomial $P_n(z)$ define points in the complex plane where its magnitude becomes zero, implying that its argument becomes undefined [33]. These points are referred to as phase singularities and are a general phenomenon of wave physics. In the field of optics, for example, they allow one to define optical vortices, which have found a number of applications [44].

The Uhlmann phase displays $2j$ phase singularities. The restriction on the variable $z(\theta)$ to acquire real values implies that $\theta = \pi/2$, that is, the magnetic field should lie on the equator of the sphere of directions. As a consequence, there are as many as $2j$ *critical* temperatures $T_{c,k}^{(j)}$ $(k = 1,\dots,2j)$ determined by

$$\cosh(\beta_k B/2)\cos[\pi\,\mathrm{sech}(\beta_k B/2)] = \cos\left(\frac{k\pi}{2j+1}\right), \quad (16)$$

where the kth root of the Chebyshev polynomial $U_{2j}(x)$, with $x = z(\theta = \pi/2)$, is on the right-hand side.

Figure 1 shows the Uhlmann phase as a function of βB for $\theta = \pi/2$ and several values of j. Note that for half-integer j, there is a negative sign multiplying $U_{2j}(x)$. It can be seen that $\Phi_U^{(j)}(\beta B) = \arg[(-1)^{2j}U_{2j}(x)]$ is 0 or π and, in this manner, *topological*. The topological transitions, between trivial and nontrivial phases, occur at temperatures (or field magnitudes) such that the Chebyshev polynomials vanish, and the precise value, 0 or π, of the phase $\Phi_U^{(j)}$ is determined by the sign of the polynomials times the Pauli sign $(-1)^{2j}$. Note that for very high temperatures ($\beta B \ll 1$), the Uhlmann phase vanishes, as expected for a system under thermal noise [17]. For very low temperatures ($\beta B \gg 1$), the phase $\Phi_U^{(j)}$ is either π or zero for half-integer and integer values of j, respectively, which is expected because the Uhlmann phase of a thermal ensemble approaches the geometric phase of a pure system in its ground state as we approach zero temperature [17]. For example, a ground-state spin-j particle in a slowly rotating planar ($\theta = \pi/2$) magnetic field acquires a Berry phase [1] $\gamma_{-j} = 2\pi j$, consistent with the aforementioned description. What is remarkable about this result is the emergence of many critical temperatures, distributed in a nonuniform way as j varies. There are $2j$ critical temperatures, some of which are at higher or lower values from that of the $j = 1/2$ case. Thus, additional nontrivial topological phases appear at higher temperatures in comparison to the simplest spin-1/2 particle,

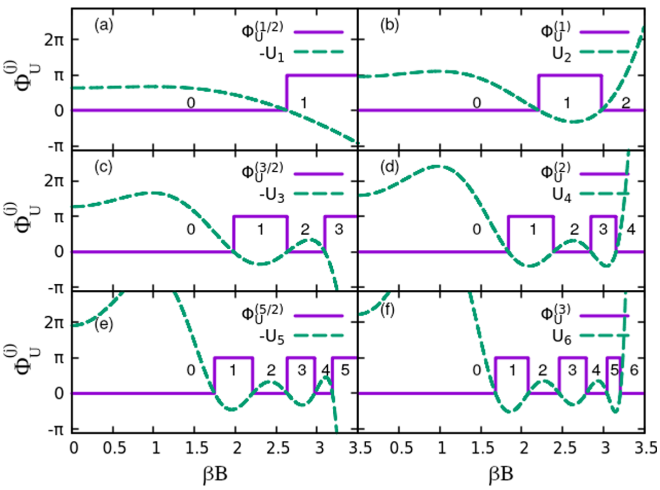

FIG. 1. Uhlmann topological phases $\Phi_U^{(j)}$ (solid) and Chebyshev polynomials $(-1)^{2j}U_{2j}$ (dashed) as functions of βB, for $\theta = \pi/2$. The left column presents the phase for the half-integer values (a) $j = 1/2$, (c) $=3/2$, and (e) $=5/2$. The right column displays the phase for the integer values (b) $j = 1$, (d) $=2$, and (f) $=3$. The integers between the zeros of the polynomials indicate associated winding numbers, corresponding to Uhlmann numbers $n_U^{(j)}$ (see text in Sec. III B).

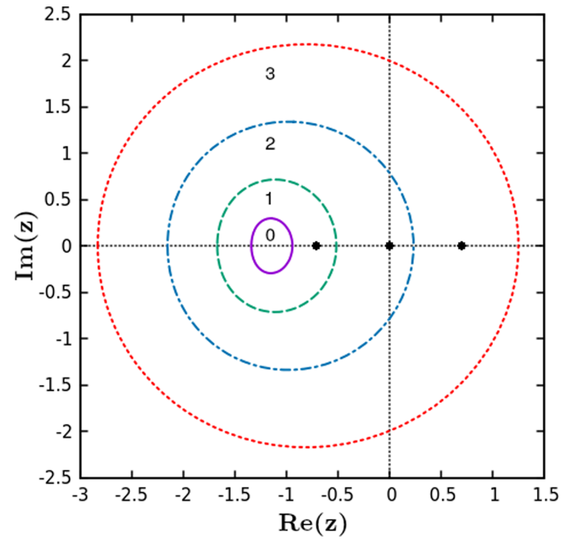

FIG. 2. Argand diagram of $z(\theta)$ for several values of βB. Considering the case $j = 3/2$, the points marked on the real axis are the roots of polynomial $U_3(z)$. At high enough temperatures, the curve $z(\theta)$ (solid purple) does not enclose any root. As the temperature reduces, the curve expands and progressively encloses the roots. The Uhlmann numbers are given by the number of roots inside $z(\theta)$.

and are in this sense more robust against thermal noise. On the other hand, the critical temperatures cannot reach arbitrary large values for higher j, given the constraint imposed by Eq. (16) or, equivalently, due to the fact that all the roots of $U_{2j}(z)$ lie in the interval $(-1, 1)$ [32].

Viyuela *et al.* predict [35] the existence of two critical temperatures in a 2D topological insulator with high Chern numbers, suggesting the possibility of purely thermal topological transitions. Furthermore, the thermal topological phase transition for $j = 1/2$ has already been confirmed experimentally [21] in a superconducting qubit. We regard this as a strong suggestion of the physical existence of multiple Uhlmann topological transitions for a spin-j particle, and thus hope our results encourage experimental verification of this phenomenon.

B. Topological Uhlmann numbers

The case $j = 1/2$ is illustrative. There is only one topological transition, occurring at $\beta B = 2\ln(2+\sqrt{3})$ [Fig. 1(a)]. Viyuela *et al.* [15] report this single critical temperature [45] for three representative 1D models of topological insulators and superconductors. At zero temperature, $T = 0$, the ground state of the system acquires a Berry phase of π and a Chern number of $+1$. At finite temperature, for $T < T_{c,1}^{(1/2)}$, the same phase is preserved, but above the critical temperature, the Uhlmann topological phase becomes trivial. This is in sharp contrast to the zero-temperature behavior. A question that naturally arises at this point is about the invariants associated with the topological phases that occur for higher j. A look at Fig. 2 will give insight about writing the proper definition of them. The figure depicts the $z(\theta)$ curve (14) for four temperatures, where the dots mark the roots of $U_3(z(\theta))$. The smallest curve (solid purple) corresponds to the highest temperature, while the largest (dotted red) is for the lowest temperature. As the temperature decreases, the curve expands and progressively encloses the roots of U_3, whenever the temperature crosses a critical value $T_{c,k}^{(j)}$. The number of enclosed roots is zero at high temperatures, and ends up being $2j$ for low enough temperatures.

According to the argument principle of complex analysis [33,34], if $z(\theta)$ encloses k roots of U_{2j}, then the curve [46] $U_{2j}(z(\theta))$ winds around the origin k times, so the number of closed roots equals the winding number. The winding number of the curve, $(-1)^{2j}U_{2j}(z(\theta))$, tells us how many times its phase changes from 0 to 2π [47], but this phase is just $\Phi_U^{(j)}(\theta)$. This suggests the definition of the Uhlmann number,

$$n_U^{(j)}(T) = \frac{1}{2\pi}\int_0^{\pi} \frac{d\Phi_U^{(j)}(z(\theta))}{d\theta} d\theta, \tag{17}$$

to be the winding numbers of the $(-1)^{2j}U_{2j}(z(\theta))$ curve, for a temperature between two successive critical values. These integer numbers are the equivalent of the Chern numbers of pure states [7]. In fact, expression (17) is consistent with that proposed as the definition of Uhlmann numbers for two-dimensional topological insulators [17]. Here, we have followed a more *ad hoc* path, motivated by the specific form of the Uhlmann phase (15), given in terms of polynomials.

Figure 3 shows the Uhlmann numbers for different values of j. The steps at which $n_U^{(j)}$ change by one are located at the critical temperatures $T_{c,k}^{(j)}$. Note that the maximum value that $n_U^{(j)}$ takes on is $2j$, which equals the Chern number of a spin-j particle in its ground state [5]. The figure illustrates how the appearance of multiple critical temperatures makes possible transitions between nontrivial topological orders of the type $2j \to 2j-1 \to 2j-2 \to \cdots \to 0$ for increasing temperature.

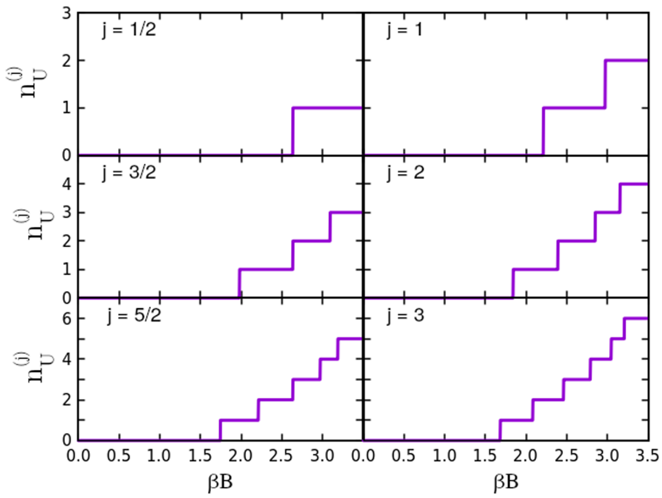

FIG. 3. Uhlmann number as a function of temperature, for some values of the spin number j. The steps are located at a set of values of $\beta_k B$, which define critical temperatures $T_{c,k}^{(j)}$, with $k = 1, \ldots, 2j$.

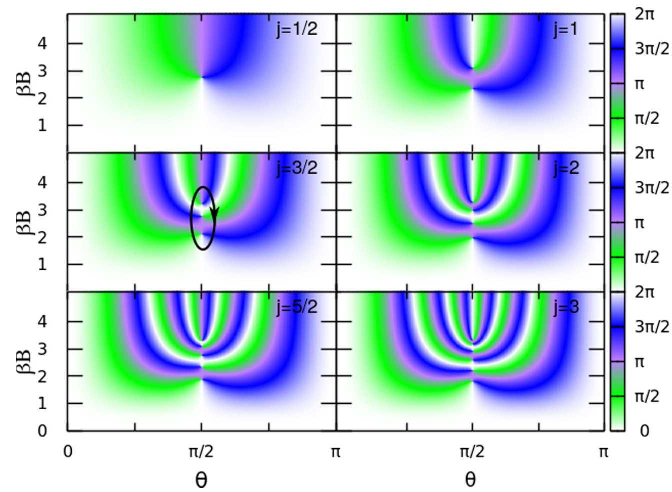

FIG. 4. Uhlmann phases $\Phi_U^{(j)}(\theta, \beta B)$ for $j = 1/2, 1, \ldots, 3$. There are $2j$ singularities along the $\theta = \pi/2$ line. The Uhlmann number characterizing a topological order can be obtained visually by encircling one or more singularities with a simple closed curve and counting the number of times an isochromatic line is repeated (see text).

In the model of a 2D topological insulator which presents two critical temperatures [35], there are three topological phases, i.e., one trivial with $n_U = 0$ and two nontrivial with $n_U = 1$ and 2, which can be accessed by varying the temperature. The appearance of $2j + 1$ distinct Uhlmann numbers in the spin-j particle is a more dramatic example of a system with more than one nontrivial order.

IV. UHLMANN GEOMETRIC PHASES FOR ARBITRARY FIELD DIRECTION

In Fig. 1, we show the temperature dependence of the Uhlmann phase for $\theta = \pi/2$. We now turn to analyze this dependence for directions in the whole interval $0 \leqslant \theta \leqslant \pi$. Figure 4 shows a color map of $\Phi_U^{(j)}(\theta, \beta B)$ for distinct spin number j. In each panel, $2j$ vortices can be distinguished along the line $\theta = \pi/2$, which correspond to the zeros of $U_{2j}(z)$ or, equivalently, the critical temperatures. Note that for all cases, the phase $\Phi_U^{(j)}(\theta, \beta B) \to 0$ for high temperatures $\beta B \ll 1$, as expected. On the other hand, for very low temperatures $\beta B \gg 1$, the Uhlmann phase must converge to the Berry phase [1,5], $\Phi_U^{(j)}(\theta, \beta B) \to 2j\pi(1 - \cos\theta)$. For example, in the $j = 1/2$ panel at low temperatures, the sequence of colors, as θ goes from zero to π , is that of the color boxes on the right: the Uhlmann phase is 0 for $\theta = 0$ and increases up to 2π for $\theta = \pi$. Let us call that sequence of colors a phase cycle. For panels with higher j at low temperatures, we see that the phase cycles appear $2j$ times. Lowering the temperature, the number of phase cycles decreases by a unit when crossing a critical temperature. A look to the $j = 1$ panel illustrates this point. For low temperatures, there are two phase cycles as θ goes from 0 to π. When increasing the temperature above the first critical value $T_{c,1}^{(1)}$, the Uhlmann phase only traverses one phase cycle, and none of them above the second critical temperature.

This behavior can also be illustrated in a similar way to that used to see the argument principle in action through the colored phase portrait of a complex function [33]. To take a concrete example, consider a simple closed path encircling the three singularities of the phase in Fig. 4 for $j = 3/2$, and follow the number of phase cycles occurring when it is crossed. It can be seen that this number is exactly $2j$, the number of zeros enclosed, which is also the number of critical temperatures. The isochromatic lines (for instance, the green ones) appear just $2j$ times. The number of cycles diminishes by one each time the path shrinks to leave out a zero, where shrinks means to increase the temperature, in line with the geometrical interpretation of the Uhlmann numbers suggested by Fig. 2. Thus, in our problem, the Uhlmann numbers can also be obtained from the number of cycles displayed by the function $\Phi_U^{(2j)}(\theta, \beta B)$ in the vicinity of singularities.

V. CONCLUSIONS

In this paper, we have studied the Uhlmann phase of a spin-j particle interacting with a slowly varying magnetic field. We obtained a simple expression for that phase given by the argument of complex-valued second-kind Chebyshev polynomials $U_{2j}(z)$ multiplied by the Pauli sign $(-1)^{2j}$, with the complex variable z being a function of the direction of the external field and temperature. As a consequence, $2j$ phase singularities appear, which implies the possibility of topological phase transitions at $2j$ distinct critical temperatures. This is remarkably in contrast to the temperature dependence of the Uhlmann phase reported for topological insulators and superconductors. Based on the principle argument of complex analysis, we derived a proper topological invariant, the Uhlmann number, as a winding number associated to a topological order of the system, existing between two successive critical values of the temperature. The Uhlmann number lies between 0 and $2j$.

Our study suggests a purely thermal manipulation of topological transitions of a spin-j particle. This nontrivial effect has already been observed for the $j = 1/2$ case and thus we hope this study encourages experimental verification of this phenomenon.

The physical implications of multiple topological phase transitions remains as an open issue. For example, in the Kitaev chain model, there are critical temperatures at which the Uhlmann phase changes from zero to π [25]. The intricate distribution of the critical temperatures is not related to the topology of the band structure and direct connection to the physical properties seems hard to grasp [25]. Budich and Diehl [48] noted that the lack of an additive group structure of the Uhlmann phase makes ambiguous the definition of topological invariants as 1D winding numbers for general 2D systems. They showed that by applying constraints on the spectrum of the density matrix, nontrivial topological invariants can be defined by means of the Uhlmann curvature. Nevertheless, these topological numbers do not have any temperature dependence, beyond vanishing at the infinite-temperature limit. Thus, interpretation of the temperature-dependent Uhlmann numbers remains elusive at the present time and requires further investigation.

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