Single-particle steering and nonlocality: The consecutive Stern-Gerlach experiments

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Quantum nonlocality and quantum steering are fundamental correlations of quantum systems which cannot be created using classical resources only. Nonlocality describes the ability to influence the possible results of measurements carried out in distant systems, in quantum steering where Alice remotely steers Bob's state. Research in nonlocality and steering is of fundamental interest for the development of quantum information and in many applications requiring nonlocal resources like quantum key distribution. On the other hand, the Stern-Gerlach experiment holds an important place in the history, development, and teaching of quantum mechanics and quantum information. In particular, the thought experiment of consecutive Stern-Gerlach experiments is commonly used to exemplify the concept of noncommutativity between quantum operators. However, to the best of our knowledge, the consecutive Stern-Gerlach experiments have not been treated in a fully fledged quantum manner yet, and it is a widely accepted idea that atoms crossing consecutive Stern-Gerlach experiments follow classical paths. Here we demonstrate that two consecutive Stern-Gerlach experiments generate nonlocality and steering, and these nonlocal effects strongly modify our usual understanding of this experiment. Also, we discuss the implications of this result and its relation with entanglement. This suggests the use of quantum correlations, of particles possessing mass, to generate nonlocal tasks using this experiment.

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I. INTRODUCTION

Nonlocality, one of the fundamental features of quantum mechanics, refers to the fact that for entangled states the result of a measurement in one observable depends on the choice of measurement in the other observable; this result manifests itself at spacelike separation [1]. Steering refers to the ability to steer a state of one subsystem by measurements made in another subsystem [2–6]; it is a nonlocal property that differs from nonlocality and nonseparability [2,3]. Additionally, nonlocality rests in the uncertainty principle and in the steering of physical states [7,8]. The nonlocality of the collapse of the wave function of a single particle was experimentally proved for photons in Ref. [9], for atoms in Ref. [10], and for an experimental proof of nonclassical collapses using box games [11]. Steering was experimentally proven in a single-photon experiment by Guerreiro *et al.* [12]. Additionally, there is a strong relation between steering and joint measurability [13,14]. Nowadays, it is understood that quantum nonlocality is a fundamental resource for quantum information tasks which cannot be generated by using random data only [15]. Nevertheless, see the interesting discussion over quantum nonlocality given by Khrennikov [16–18].

Moreover, although in some works it has been conjectured that single-particle entanglement does not possess nonlocal correlations [19,20], the fascinating fact is that these nonlocal effects could be generated using the Stern-Gerlach experiment (SGE) by measuring the internal degree of freedom of the particle that traverses it, i.e., by measuring $\hat{\sigma}_z$ or $\hat{\sigma}_x$, the position of the particle could manifest itself at either of two different places at (possible) spacelike distance [21]. These nonlocal properties of the SGE were associated with the spreading of the wave function and named single-particle steering [21]. Hence, depending on the kind of states, the spreading of the wave function plays also a crucial role in the nonlocality of quantum mechanics [21].

Furthermore, single-particle entanglement [20] has many applications in quantum information like quantum key distribution [22,23], bidirectional quantum teleportation [24], swapping states [25], and entanglement concentration [26]. Recently, the single-particle entanglement between the spin and orbital angular momentum (OAM) of photons was reported in metamaterials [27] and the entanglement between polarization and OAM was studied in Ref. [28]. As mentioned earlier, steering was experimentally proven using the single-photon entanglement [12]; this effect was called *single-photon steering* by Brunner [29].

On the other hand, the SGE is a pillar in the historical development of quantum mechanics and nowadays is an active field of research, both of its experimental capabilities and theoretical studies [30–49] (in other words, the fundamental understanding of the SGE is evolving), and it also is an important tool in the teaching of both quantum mechanics and quantum information.

The quantum phenomena that is usually introduced in quantum mechanics courses using the Stern-Gerlach experiment include the concept of spin and the noncommutativity of quantum operators [50]. For the latter, the of most importance is the thought experiment of consecutive Stern-Gerlach experiments (CSGE), formally introduced by Feynman in his famous lectures [50], although its first conception comes from Heisenberg in 1927 [51]. In quantum information and computation theory the CSGE is used to exemplify the structure of the qubit and the collapse of the wave function.

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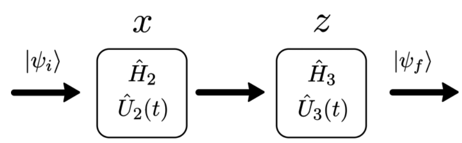

FIG. 1. Scheme of the consecutive Stern-Gerlach experiments. The inhomogeneity of the magnetic field in the first experiment is in direction x, which is associated with a Hamiltonian, $\hat{H}_2$, and an evolution operator, $\hat{U}_2$; the inhomogeneity of the magnetic field of the second is in direction z, with its respective Hamiltonian, $\hat{H}_3$, and evolution operator, $\hat{U}_3$.

Despite its importance, there has been a lack of quantum analyses of the CSGE. This is surprising given the amount of research dedicated to describing and studying the usual SGE; in recent years, there have been important advances describing the SGE in a completely quantum manner [30–37,43–47]. In particular, it was shown that the SGE behaves as an entangled device and not as a measurement device, as it had been regarded for decades [30,31]. Additionally, the violation of Bell inequalities in the SGE was shown in Ref. [52]; this violation of the Bell inequalities was associated with quantum nonlocality, instead of quantum contextuality, and the difference is explained in Ref. [21].

In this article, we analyze CSGE using the tools of quantum mechanics; in order to do that, we calculate the complete quantum evolution of a spin-1/2 particle in such consecutive configuration by means of the evolution operator method [53,54]. These results also could serve to define a basis of comparison for a possible experimental realization of this experiment. The quantum treatment of the CSGE gives us the opportunity to study the quantum characteristics of the system, such as the quantum correlations, entanglement [32,52], and steering [21].

The paper is organized as follows. In Sec. II, starting from the time-dependent Schrödinger equation for the particle, we found its time evolution by applying a factorized evolution operator to the initial state of the particle, similar to that done in Refs. [30,31]. We calculate in this way the evolution in each of the Stern-Gerlach apparatuses that are arranged consecutively and arrive at the result for the total evolution by applying the sequential evolutions. In Sec. III, we analyze the steering produced by changing the measurement basis. In Sec. IV, we describe the quantum correlations of the CSGE and we test three different Bell-type inequalities, with which we find that this system is nonlocal. In Sec. V, we visualize the relation between the quantum correlations and the entanglement in the CSGE by means of the works of Piceno *et al.* [32] and Roston *et al.* [36]. Finally, the main results obtained are summarized in Sec. VI, and we close the paper with some concluding remarks.

II. DYNAMIC EVOLUTION OF THE CSGE

The array of consecutive experiments that we take into account appears in Fig. 1; this configuration has two Stern-Gerlach apparatuses (each apparatus acts as an entangling device). The initial state entering the SGE in x direction is a state previously prepared by another SGE not shown in the figure; that is, the initial state is given by the product of a spin-up state (internal degree of freedom) with a wave packet (external degree of freedom), which we can write as follows:

$$|\psi_i\rangle = \frac{1}{(2\pi\sigma_0^2)^{\frac{3}{4}}} \exp\left(-\frac{(x^2+y^2+z^2)}{4\sigma_0^2} + ik_y y\right)|\uparrow_z\rangle, \quad (1)$$

with σ_0 being the initial width of the wave packet and k_y being the component y of the wave vector.

The quantum evolution of the Stern-Gerlach experiment has already been described in Ref. [30]; therefore, from this we know that if the initial state is a product between a Gaussian wave packet and a spin degree of freedom (DoF) the resulting evolved wave packet widens with time and gets translated in z depending on the spin component, thanks to the interaction with the inhomogeneous magnetic field when we select the spin component. With these considerations, the general packet of Eq. (1) can be seen as that resulting from such preparation with a final width σ_0 and taking the z position of such state as our origin [30,31].

To obtain the evolution as the state goes through the first experiment, we have the associated evolution operator from Refs. [30,31],

$$\hat{U}_2(t) = \exp\left(-\frac{1}{6}\kappa\right)\exp\left[-\frac{it}{2m\hbar}(p_y^2+p_z^2)\right] \times \exp\left[-\frac{it\mu_c}{\hbar}(B_2+b_2x)\sigma_x\right]\exp\left(\frac{it^2\mu_c b_2}{2m\hbar}p_x\sigma_x\right) \times \exp\left(-\frac{it}{2m\hbar}p_x^2\right), \quad (2)$$

with $\kappa = (it^2\mu_c^2 b_2^2)/(m\hbar)$, B_2 the homogeneity parameter of the magnetic field, b_2 the inhomogeneity parameter, σ_x the Pauli operator, $\mu_c = ge\hbar/(4m)$, g the gyromagnetic ratio, m the mass of the particle, and e the unit charge.

Applying $\hat{U}_2(t)$ during certain time t_2 to $|\psi_i\rangle$, we have obtained the middle state $|\psi_m(t_2)\rangle$. See Appendix A for this calculation. Next, following our description, we apply the evolution operator associated with the last SGE for a fixed time t_3, that is, the operator in z direction to the state $|\psi_m(t_2)\rangle$. This operator is given by

$$\hat{U}_3(t) = \exp\left(-\frac{1}{6}\kappa\right)\exp\left[-\frac{it}{2m\hbar}(p_x^2+p_y^2)\right]\exp\left[-\frac{it\mu_c}{\hbar}(B_3+b_3z)\sigma_z\right]\exp\left(\frac{it^2\mu_c b}{2m\hbar}p_z\sigma_z\right)\exp\left(-\frac{it}{2m\hbar}p_z^2\right). \quad (3)$$

Then we obtain the final state as

$$|\psi_f\rangle = |\psi_+\rangle|\uparrow_z\rangle + |\psi_-\rangle|\downarrow_z\rangle, \quad (4)$$

in which we define

$$|\psi_\pm\rangle = M \exp\left(\mp i\sqrt{2}k_3^3\tau_3 z_0\right) \exp\left\{-\frac{\left(Z \pm \left\{\sqrt{2}k_3^3\tau_3^2 + i2\sqrt{2}k_3^3\tau_3[1+i(\tau_2+\tau_3)]\right\}\right)^2}{4[1+i(\tau_2+\tau_3)]}\right\}$$
$$\times \left(\exp\left(-i\sqrt{2}k_2^3\tau_2 x_0\right) \exp\left\{-\frac{\left[X + \sqrt{2}k_2^3\tau_2^2 + i2\sqrt{2}k_2^3\tau_2(1+i\tau_2)\right]^2}{4[1+i(\tau_2+\tau_3)]}\right\}\right.$$
$$\left.\pm \exp\left(i\sqrt{2}k_2^3\tau_2 x_0\right) \exp\left\{-\frac{\left[X - \sqrt{2}k_2^3\tau_2^2 - i2\sqrt{2}k_2^3\tau_2(1+i\tau_2)\right]^2}{4[1+i(\tau_2+\tau_3)]}\right\}\right), \quad (5)$$

with M being a normalization factor. Equation (5) is dimensionless thanks to the following definitions:

$$\tau_{2,3} = \frac{\hbar t_{2,3}}{2m\sigma_0^2}, \quad k_{2,3} = \sqrt{2}\sigma_0\left(\frac{m\mu_c b_{2,3}}{2\hbar^2}\right)^{1/3},$$
$$x_0 = \frac{B_2}{\sigma_0 b_2}, \quad z_0 = \frac{B_3}{\sigma_0 b_3},$$
$$Z = \frac{z}{\sigma_0}, \quad X = \frac{x}{\sigma_0}. \quad (6)$$

The definitions in Eq. (6) make all the variables dimensionless and our results completely comparable with those of other works that study the entanglement in the SGE [32,36]. We found that the final state is a superposition state of the spin eigenstates—entangled with the position DoF—in the same way as the state coming out from a single SGE [30,31], and therefore it does not represent a state following a definite trajectory to the screen. This state presents hybrid entanglement between the position DoFs and the spin DoF, congruently with the known effect present in the SGE, the separation of spins. The presence of the Z variable in this entangled state attests to the noncommutativity between the different spin operators for the different spatial orientations; this is in good accordance with the observations of the semiclassical argument for the thought experiment. We also found an interesting effect for the configuration proposed here in the presence of entanglement also with the position DoF in the X coordinate.

III. STEERING

As was mentioned earlier, steering is a manifestation of quantum correlations. It was defined by Schrödinger (as a property of entangled systems) as the possibility that by suitable measurements taken on one subsystem only, the state of the other subsystem can be determined by the choice of the measurement and without interacting with it. That is to say, it is possible to steer the state of a subsystem by choosing what kind of measurement to implement on the other subsystem.

The actual understanding of steering comes from the operational definition given by Wiseman *et al.* [2], who define steering as a task; i.e., the task of Alice is to convince Bob that she can prepare a bipartite entangled state. To do it, Alice prepares a bipartite quantum state and sends one of them to Bob; then, they measure their respective subsystem. If the correlations between their measurement can be explained by a local hidden state model for Bob, then Alice could have taken a pure state at random and sent it to Bob. But if the measurements cannot be explained by a local hidden state model, then Alice steers Bob's state. Wiseman *et al.* [2] demonstrated that steering is stronger than nonseparability and weaker than nonlocality. The existence of nonlocality rules out local hidden variables and, in essence, steering rules out the existence of local hidden state models [2]. Steering was experimentally proven in a single-photon experiment by Guerreiro *et al.* [12]. In the case of pure tripartite states, He and Reid have demonstrated that it is enough that each party can be steered by one or both of the other two to certify steering [55].

In Ref. [21], it was demonstrated that in the case of a single SGE Alice can steer Bob's state depending on the measurement she chooses. In that paper [21], a thought experiment where Alice is located in Tokyo and Bob in Paris was posed. Hence, by choosing which one of the possible observables to measure, Alice can steer Bob's state. This confirms the nonlocality of the entangled wave function of the SGE and Ref. [21] associates this nonlocality with the spreading of the entangled wave function.

In the case of the CSGE, a similar situation—in fact more rich—can be conceived. This is depicted in Fig. 2: Alice in Tokyo is in full control of a fully automatized CSGE located

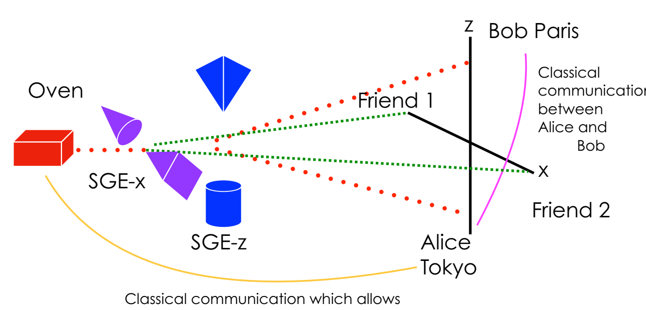

FIG. 2. CSGE featuring the Einstein's boxes; see Ref. [21]. The red box is the oven, in violet a SGE in the x direction is depicted, in blue the SGE in z direction is shown, and the red and green dots represent the fact that there are not classical trajectories; see Refs. [30,31]. Alice could communicate with Bob by using the classical channel in magenta. Moreover, Alice is in full control of the SGE by using the classical channel in yellow, and she possess the ability to turn it on and to choose between a single or N atoms. The red dots end in the z axis in Tokyo with Alice and in Paris with Bob. The green dots in the x axis arrive at suitable places where there are two friends of Alice.

at a suitable place, can turn it on and off, and can send a single or many atoms one by one as she wishes.

Therefore, if Alice makes a measurement of $|\downarrow_z\rangle$ and she finds $-\hbar/2$, then the entangled wave function given by Eq. (4), i.e., $|\psi_f\rangle = |\psi_+\rangle|\uparrow_z\rangle + |\psi_-\rangle|\downarrow_z\rangle$, would collapse to $|\psi_-\rangle|\downarrow_z\rangle$ in Tokyo. Notice that $|\psi_-\rangle$ comprises a superposition state in X. On the other hand, if she detected nothing, then the wave function would collapse to $|\psi_+\rangle|\uparrow_z\rangle$ in Paris (see Ref. [21] for an explanation of this effect in terms of the Einstein's boxes); similarly, $|\psi_+\rangle$ comprises a superposition state in X.

However, if Alice decides to measure in a different basis, for example, $\hat{\sigma_x}$, to ascertain the possibilities we must rewrite Eq. (4) in the $\hat{\sigma_x}$ basis, getting

$$|\psi_f\rangle = \{|\psi_+\rangle + |\psi_-\rangle\}|\uparrow_x\rangle + \{|\psi_+\rangle - |\psi_-\rangle\}|\downarrow_x\rangle. \quad (7)$$

Then, the following situations can arise:

(i) If she measures spin down in the basis of $\hat{\sigma}_x$ and obtains $-\hbar/2$, then the wave function would collapse toward $\{|\psi_+\rangle - |\psi_-\rangle\}|\downarrow_x\rangle$. Notice that $\{|\psi_+\rangle - |\psi_-\rangle\}$ comprises a superposition state in Z.

(ii) If she checks spin up in the basis of $\hat{\sigma}_x$ and obtains $\hbar/2$, then the wave function would collapse toward $\{|\psi_+\rangle + |\psi_-\rangle\}|\uparrow_x\rangle$.

Therefore, we must conclude that Bob's state is steered depending on the kind of observable Alice decides to measure, confirming the nonlocality of the wave function generated by the CSGE.

On the other hand, a similar procedure could be used by Bob in Paris to steer Alice's state, or by one of the friends of Alice to steer states to Alice or Bob. Therefore, the He and Reid criterion cited above is fulfilled: The CSGE possesses the nonlocal property of steering. As can be seen in Fig. 2, a richer situation than the one given in Ref. [21] arises, because in the CSGE case there could be participation by four people to address all the possibilities. For example, if Alice's friend 2 in a different location decides to check for $\hbar/2$ in the $\hat{\sigma_x}$ basis and obtains nothing in their measurement, then the wave function would collapse toward the state $\{|\psi_+\rangle - |\psi_-\rangle\}|\downarrow_x\rangle$. This is equivalent to Alice measuring $\hat{\sigma_x}$ and obtaining $-\hbar/2$ and is also equivalent to friend 1 measuring $\hat{\sigma_x}$ and obtaining $-\hbar/2$; both of them will collapse the wave function (by effects of their measurements) toward $\{|\psi_+\rangle - |\psi_-\rangle\}|\downarrow_x\rangle$, i.e., the same function as that obtained by friend 2 when testing $\hbar/2$ in the basis $\hat{\sigma_x}$ and obtaining nothing.

IV. NONLOCALITY OF THE CONSECUTIVE STERN GERLACH EXPERIMENTS

The study of quantum correlations of quantum systems is strongly related to nonclassical tasks [15,30–32,36,43,46,52,56–64] that open the way to important applications. We quantify the quantum correlations of the CSGE with a correlation function for hybrid spin systems that has already been tested for bipartite states [52,61]

$$\boldsymbol{C}(x, p_x, z, p_z, \theta) = \langle\psi_f|\hat{W}(x, p_x, z, p_z)\hat{\sigma}(\theta)|\psi_f\rangle, \quad (8)$$

that is, the generalized Banaszek-Wódkiewicz (BW) correlation function [62], with

$$\hat{W}(x, p_x, z, p_z) = \int_{-\infty}^{\infty} dq_x dq_z \left|x - \frac{1}{2}q_x, z - \frac{1}{2}q_z\right\rangle \times \exp[-i(p_z q_z + p_x q_x)/\hbar] \times \left\langle x + \frac{1}{2}q_x, z + \frac{1}{2}q_z\right|, \quad (9)$$

and the generalized Wigner operator, originally defined by Ben-Benjamin *et al.* [65], for coordinates x, z with their respective moments p_x, p_z; $\hat{\sigma}(\theta)$ is the usual Pauli operator for an arbitrary θ direction in the plane. It has been demonstrated that in one dimension the Wigner operator for the translational degree of freedom is equivalent to the parity operator [65,66], and thereby dichotomized results (for the measurement of a particle that arrives in the positive or negative subplane of the CSGE screen) are produced. Accordingly, the Wigner operator given by Eq. (9) is the displaced party operator $\hat{\Pi}(x, z)$ and its mean value can be seen as a correlation in a Bell-type experiment [56,61,67] (please see Appendix A).

Hence, in this way we can calculate the correlation function for the final state of the CSGE that is given by Eq. (4). The relation between the correlations and entanglement in the CSGE will be discussed later in this paper.

Results

Our quantum description of the experiments allows us to delve into the quantum characteristics of the system; in this manner, we study the quantum correlations that are present by means of the CHSH, Bell-Klyshko-Mermin, and Svetlichny inequalities.

1. CHSH inequality

From Eq. (8), the correlation function of the CSGE has the following form:

$$\begin{aligned}\boldsymbol{C}(X, P_x, Z, P_z, \theta) &= \exp[\omega_z'(Z, P_z) + \omega_x'(X, P_x)][4\cos(\theta)(\exp[\omega_z]\{\exp[\omega_x] \\ &\times \cosh[d_x(X, P_x)]\sinh[d_z(Z, P_z)] + \exp[-\omega_x] \\ &\times \cos[\delta_x(X, P_x)]\cosh[d_z(Z, P_z)]\}) + 4\sin(\theta)(\exp[-\omega_z] \\ &\times \{\exp[\omega_x]\sinh[d_x(X, P_x)]\cos[\delta_z(Z, P_z)] \\ &+ \exp[-\omega_x]\sin[\delta_x(X, P_x)]\sin[\delta_z(Z, P_z)]\})] \quad (10)\end{aligned}$$

up to a normalization factor. The functions $\omega_z'(Z, P_z)$, $\omega_x'(X, P_x)$, ω_z, ω_x, $d_z(Z, P_z)$, $d_x(X, P_x)$, $\delta_z(Z, P_z)$, and $\delta_x(X, P_x)$ are real valued functions. See Appendix B for the complete definitions.

We have the extra dimensionless definitions given by

$$P_x = \frac{p_x\sigma_0}{\hbar}, \quad P_z = \frac{p_z\sigma_0}{\hbar}. \quad (11)$$

The CHSH inequality [57,61,64,68] that we use to test the existence of nonlocality between the Z-θ pair is as

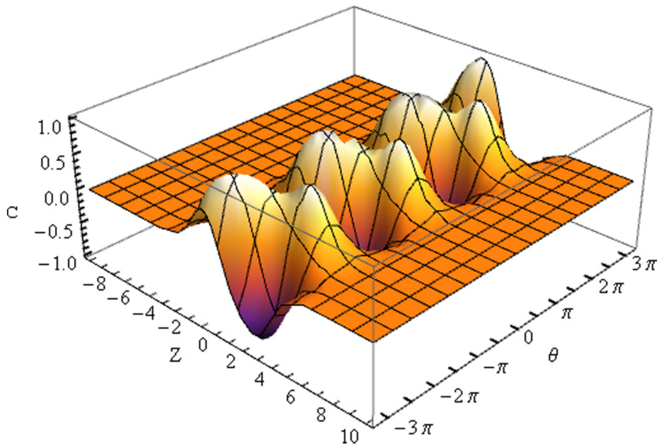

FIG. 3. Correlation function for the Z-θ pair taking the fixed dimensionless quantities as $X = 0.1$, $P_x = 0.129$, $P_z = 0.049$, $\tau_2 = 6.8$, $\tau_3 = 2.6$, $k_2 = 0.3$, $k_3 = 0.3$, $x_0 = 4$, and $z_0 = 4$.

follows:

$$\begin{aligned}-2 \leqslant B_{\text{CHSH}} = &\, \boldsymbol{C}(X, P_x, Z, P_z, \theta) + \boldsymbol{C}(X, P_x, Z, P_z, \theta') \\ &+ \boldsymbol{C}(X, P_x, Z', P_z, \theta) - \boldsymbol{C}(X, P_x, Z', P_z, \theta') \\ &\leqslant 2. \end{aligned} \tag{12}$$

To study the nonlocality between z and θ, we need to fix the other variables X, P_x, P_z, Z', and θ'.

The plot of the correlation function for this case appears in Fig. 3 for the values of the fixed parameters $X = 0.1$, $P_x = 0.129$, $P_z = 0.049$, $\tau_2 = 6.8$, $\tau_3 = 2.6$, $k_2 = 0.3$, $k_3 = 0.3$, $x_0 = 4$, and $z_0 = 4$. The Bell function for this case, B_{CHSH}, is shown in Fig. 4 for the primed quantities $Z' = 2.4$ and $\theta' = \frac{\pi}{5}$. We report the violation of the CHSH inequality for the Z, θ pair for a minimum value of B_{CHSH} of (negative) -2.62405.

2. Bell-Klyshko-Mermin inequality

Our treatment for the study of correlations and the number of DOFs in the system allows us to evaluate tripartite nonlocality; the case of major interest is the X-Z-θ triad. The Bell-Klyshko-Mermin (BKM) inequality that we use for this purpose is the following [57,64]:

$$\begin{aligned}-2 \leqslant B_{\text{BKM}} = &\, \boldsymbol{C}(X, P_x, Z, P_z, \theta') + \boldsymbol{C}(X, P_x, Z', P_z, \theta) \\ &+ \boldsymbol{C}(X', P_x, Z, P_z, \theta) - \boldsymbol{C}(X', P_x, Z', P_z, \theta') \\ &\leqslant 2. \end{aligned} \tag{13}$$

In this case, we need to fix the P_X, P_Z, X', Z', and θ' variables. The verification of Eq. (13) can only be handled numerically and it is not possible to visualize a plot of B_{BKM} due to the number of DoFs in the Bell-Klyshko-Mermin inequality. Our study of the inequality of Eq. (13) gives us as maximum value for B_{BKM} of 2.43258, demonstrating the violation of the inequality (13) for the fixed quantities $P_x = 0.051$, $P_z = 0.089$, $X' = 0.83$, and $Z' = 3.3$ and $\theta' = \frac{\pi}{2}$, $\tau_2 = 2.7$, $\tau_3 = 4.7$, $k_2 = 0.3$, $k_3 = 0.3$, $x_0 = 4$, and $z_0 = 4$.

3. Svetlichny inequality

Our third test of nonlocality is given by the Svetlichny inequality [60,69,70] for X, Z, and θ. The Svetlichny inequality is given as

$$\begin{aligned}-4 \leqslant B_S = &\, \boldsymbol{C}(X, P_x, Z, P_z, \theta) + \boldsymbol{C}(X, P_x, Z, P_z, \theta') \\ &+ \boldsymbol{C}(X, P_x, Z', P_z, \theta) - \boldsymbol{C}(X, P_x, Z', P_z, \theta') \\ &+ \boldsymbol{C}(X', P_x, Z, P_z, \theta) \\ &- \boldsymbol{C}(X', P_x, Z, P_z, \theta') - \boldsymbol{C}(X', P_x, Z', P_z, \theta) \\ &- \boldsymbol{C}(X', P_x, Z', P_z, \theta') \leqslant 4, \end{aligned} \tag{14}$$

fixing P_x, P_z, X', Z', and θ'. Our exhaustive study of the inequality of Eq. (14) permits us to report a possible nonviolation of the Svetlichny inequality for our system. We obtain the maximum of $B_S \lessapprox 4$ for the fixed quantities $P_x = 0.043$, $P_z = 0.066$, $X' = 0.52$, and $Z' = 1.49$ and $\theta' = \frac{\pi}{5}$, $\tau_2 = 2.3$, $\tau_3 = 3.5$, $k_2 = 0.3$, $k_3 = 0.3$, $x_0 = 4.5$, and $z_0 = 5.2$.

Nonetheless, we have tested our system with two weaker versions of the Svetlichny inequality given by Ref. [60],

$$\begin{aligned}B_{\text{SV1}} = &\, -\boldsymbol{C}(X, P_x, Z, P_z, \theta) + \boldsymbol{C}(X, P_x, Z', P_z, \theta) \\ &+ \boldsymbol{C}(X', P_x, Z, P_z, \theta) + \boldsymbol{C}(X', P_x, Z', P_z, \theta) \\ &+ \boldsymbol{C}(X, P_x, Z, P_z, \theta') \\ &+ \boldsymbol{C}(X, P_x, Z', P_z, \theta') + \boldsymbol{C}(X', P_x, Z, P_z, \theta') \\ &- \boldsymbol{C}(X', P_x, Z', P_z, \theta') \leqslant 4, \end{aligned} \tag{15}$$

and by inequality (185) in Ref. [15],

$$\begin{aligned}B_{\text{SV2}} = &\, -\boldsymbol{C}(X, P_x, Z, P_z, \theta) - \boldsymbol{C}(X, P_x, Z', P_z, \theta) \\ &+ \boldsymbol{C}(X', P_x, Z, P_z, \theta) - \boldsymbol{C}(X', P_x, Z', P_z, \theta) \\ &- \boldsymbol{C}(X, P_x, Z, P_z, \theta') \\ &+ \boldsymbol{C}(X, P_x, Z', P_z, \theta') - \boldsymbol{C}(X', P_x, Z, P_z, \theta') \\ &- \boldsymbol{C}(X', P_x, Z', P_z, \theta') \leqslant 4. \end{aligned} \tag{16}$$

In both cases, for B_{SV1} and B_{SV2}, we have not found any violation for a wide variety of cases with different variables; after a wide search for conditions to find a possible violation, this suggest possibly no violation of this quantities. This result is in concordance with the maximum found in the Bell-Klyshko-Mermin inequality, Eq. (13), because the maximal

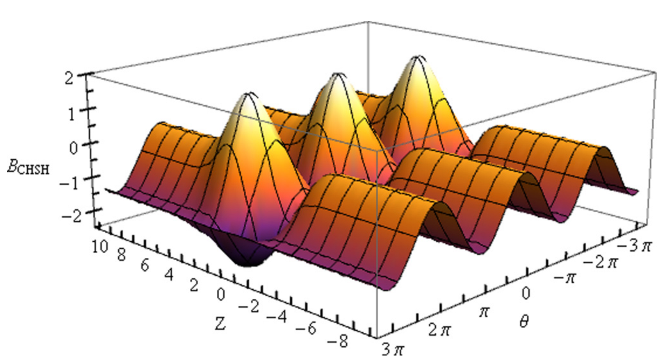

FIG. 4. Plot of B_{CHSH} for the primed quantities $Z' = 2.4$ and $\theta' = \frac{\pi}{5}$. We found a minimum of -2.62405 for this function; in this way, there exists a violation for the inequality (12) by an amount of 0.62405, i.e., around 75% of the maximal amount of violation, ≈0.8284, given by the Cirel'son's bound [57,64].

violation is not greater than $2\sqrt{2}$ for our case [64,71]. Additionally, it is convenient to recall that in certain cases stronger nonlocal effects are due to weaker entangled states [72–74], in such cases, the maximum entangled states do not produce the maximum nonlocality [74].

V. RELATION BETWEEN CORRELATIONS AND ENTANGLEMENT IN THE CSGE

The entanglement in the SGE has been widely studied in recent years, for instance, in Refs. [32,36,43]. One of the primary goals of these descriptions of entanglement is its quantification by means of entanglement measures either in discrete, continuous, or hybrid systems.

Our treatment of the quantum correlations is directly comparable with the study of the entanglement of the SGE in Refs. [32,36] due the similar quantum description of the SGE in Ref. [32] and the same temporal dependence in the correlation function, Eq. (10), and the entanglement in Refs. [32,36]. These studies of entanglement focus on the entanglement between the spatial and the spin degrees of freedom in a single SGE only. Our analysis focuses on the qualitative behavior of the entanglement, which is quantified by using the entanglement entropy, given in Ref. [32]. The description of the entanglement between the spin (discrete) and position (continuous) degrees of freedom was possible in Ref. [32] by a discretization of the SGE space, particularly by the use of the partial trace contained in the definition of the entanglement entropy. We consider that the qualitative temporal behavior of the creation of entanglement between position and spin can be implemented into each of the Stern-Gerlach apparatuses that constitute our CSGE setup, where correlations will be provided between the spin component and the spatial degree of freedom corresponding to the direction of the field inhomogeneity of the apparatus, as in the evolutions of Eqs. (2) and (3). Here the behavior of the entanglement is compared with the temporal evolution of the correlations in the CSGE. However, a deeper explanation of the temporal behavior of the entanglement in the SGE is beyond the scope of the present paper.

For the case of the CHSH inequality, the maximum violation found in the inequality of Eq. (12) is reached for $\tau_2 = 6.8$, $\tau_3 = 2.6$, $k_2 = 0.3$, and $k_3 = 0.3$. For values of the nondimensional time $\tau_2 = 6.8$ with $k_2 = 0.3$, the maximum entanglement between X and θ is almost achieved after the first SGE, while for $\tau_3 = 2.6$ with $k_3 = 0.3$ in a single SGE, the entanglement would be small; see Fig. 5. Therefore, in this case, to get the maximum quantum nonlocality the atom does not need to spend much time in the second SGE of the CSGE; the only thing that we can assume is that in the presence of maximal quantum correlations exists entanglement.

We report a nonviolation of the CHSH inequality of the CSGE for times, $\tau_{2,3}$, where the entanglement is maximum, for example, $\tau_{2,3} = 10$ for $k_{2,3} = 0.3$, a counterintuitive result at first glance. Its explanation is that the distinguishablity that the SGE state presents when time tends to infinity, a fact that is endorsed by the probability of the final state of the SGE [30,31], which also could be calculated for the CSGE final state given by Eq. (B7).

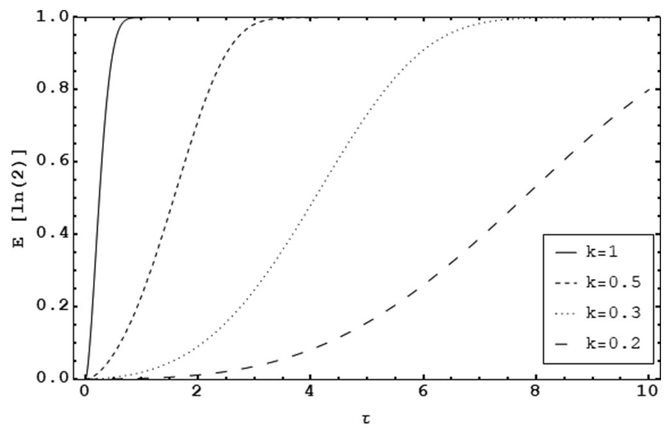

FIG. 5. Entanglement entropy (E) of the SGE for various values of k trough the time τ. In the CSGE, we have $k_{2,3}$ and $\tau_{2,3}$, which are equivalent to these nonlabeled constants. In this way, we can directly compare the temporal behavior of the quantum correlations and entanglement in the CSGE, relying on the parameter k. Plot is similar to the one given in Ref. [32].

We expect that for the case with more DoFs engaged, the entanglement in the CSGE will have a behavior similar to that of the original SGE; this is a reasonable expectation. In this manner, we also report a nonviolation of the Bell-Klyshko-Mermin and the weaker Svetlichny inequalities when $\tau_{2,3} \to \infty$, meaning that, in these cases, we anticipate the lack of quantum correlations when the multipartite entanglement on those cases will be maximum [72–74].

VI. CONCLUSIONS

One of our main conclusions is that the fundamental understanding of the SGE is evolving, as well as the basic understanding of the CSGE. In this work, we have studied the nonlocality and steering in the CSGE. We have demonstrated that the CSGE could be used to steer quantum states between two different places. Our analysis suggests that the spreading of the entangled wave function allows the particle to sense which observable is being measured by Alice, which hence suggests that the particle senses which observable is being measured, for example, $\hat{\sigma}_x$, $\hat{\sigma}_z$, or $\hat{x}$.

Additionally, we have found violation of the CHSH and Bell-Klyshko-Mermin inequalities; however, we have not found data suggesting that the evolved state of the CSGE violates the strong Svetlichny inequality or the two weaker forms of the Svetlichny inequality. We then conclude the presence of nonlocality in the CSGE in the bipartite case but we cannot confirm the existence of tripartite nonlocality or real tripartite entanglement due to the lack of violation of the Svetlichny inequalities [64,70,71]. This result does not imply that a violation of the Svetlichny inequality cannot exist for the CSGE, just that we cannot provide a proof of violation of such inequality; further research is necessary to find out whether it is violated by this state.

The striking result of the SGE with the spatial separation of the spins is carried to the CSGE evolution, where our description of the evolved state indicates the presence of entanglement between spin and position. The quantum character of the SGE is then raised to attention when proving the correlations between position and spin to be nonlocal and therefore nonclassical in nature. The quantum correlations of the hybrid tripartite state of the CSGE were successfully characterized by a generalization of the BW correlation function. The proposed function dichotomizes the spatial DoFs using the parity operator and combines this description with the usual one for discrete variables [61]. With these results, the CSGE presents an important case of hybrid nonlocality.

Furthermore, the violation of the CHSH inequality by the correlated Z-θ pair, Z being the direction of the last separation by the SGE apparatus, was found to have a maximum value of 2.62. This is a sizable violation, appreciably close to the maximum value for the violation of the two-party CHSH inequality of $2\sqrt{2}$ and a very good violation for this type of Bell operator [58,59].

On the other hand, the maximal violation of the tripartite Bell-Klyshko-Mermin inequality found for the CSGE state is 2.43 for the X-Z-θ triad. This proves tripartite nonlocality in the CSGE, but this inequality detects as well the nonlocality of the multiple two-party correlations. The violation found is in this case is far away from the possible maximum of 4 for the tripartite inequality [64].

We cannot establish the presence of genuine tripartite (three-way) nonlocality from the lack of violation of the Svetlichny inequalities; moreover, real hybrid tripartite entanglement is also not assured in our system [15,60,71]. It is worth noting that the quantum correlations studied here are between the degrees of freedom of a single particle; nevertheless, the spreading of the entangled wave function implies a clear separation of the components of the wave functions corresponding to the spin up and down that can be seen in Eq. (5); see Ref. [21]. Consequently, nonlocality naturally arises from our study when the degrees of freedom of the SGE (in the spin degree of freedom on one side and the position on the other one) are measured at separate distant location; for example, the steered state will depend on the kind of the chosen measurement device, as shown in Sec. III and Ref. [21]. The nonlocality features of SGE have been certified in Sec. IV, so we conclude that there exists nonlocality in our system. Additionally, the nonlocality is in principle experimentally detectable in an efficient manner due to the nature of the Stern-Gerlach arrangement.

We can further correlate these results with the presence of entanglement in the SGE. In the usual SGE, hybrid entanglement is created as soon as we have interaction with the inhomogeneous field, with it increasing with the time of evolution in the field until it reaches the maximum possible entanglement [32,36]. At this point, the correlations between position and spin are perfect. We can reasonably expect these results to be carried over to the CSGE. However, in the whole configuration space two-party nonlocality has not been found for parameters of the CSGE evolution where we expect maximal entanglement. Tripartite nonlocality in the form of a violation of the Bell-Klyshko-Mermin inequality is also not found for the parameters that would give maximal entanglement. This result is in concordance with previous studies, where it is shown that systems with maximum entanglement present slight or null nonlocality.

We can ensure that in the points where nonlocality has been found there exists tripartite entanglement in the state of the CSGE, but this entanglement is not perfect.

ACKNOWLEDGMENTS

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APPENDIX A: CONSTRUCTION OF THE CHSH INEQUALITY

In this Appendix, we discuss the construction of the correlation function for the CSGE that appears in Eq. (8) and of the Bell operator corresponding to the CHSH inequality in Eq. (12).

It was demonstrated by Banazek and Wódkiewicz that the Wigner operator (function) is itself a mathematical entity appropriate to measure correlations including a continuous degree of freedom in a Bell-type experiment [67]. To quote Ref. [67], p. 4346: "As the measurement of the parity operator yields only one of two values: $+1$ or -1, there exists an apparent analogy between the measurement of the parity operator and of the spin-1/2 projectors." The Wigner function then gives a measurement of the parity operator, corresponding to the expected value of the parity operator Π_{rp} generating the reflection in phase space about r, p [61,66]

$$\begin{aligned}\Pi_{rp} &= \int dq e^{-2ipq} \left| r-q \right\rangle \left\langle r+q \right| \\ &= \int dk e^{-2ikr} \left| p+k \right\rangle \left\langle p-k \right| . \end{aligned} \tag{A1}$$

The modern definition by Ben-Benjamin *et al.* [65] of the Wigner operator endorses the conclusions of Banaszek and Wódkiewicz that the Wigner operator given in terms of the parity operator dichotomizes the phase space of the system and, for the case of the CSGE, we have that our complete observable, given by $\hat{W}\hat{\sigma}(\theta)$, describes adequately, on average, the correlations between spin and position for our final state of Eq. (4); i.e., in principle this observable can distinguish between the positive and negative parts of the position variables and relate them with the spin degree of freedom.

This is the central part of the Banaszek-Wódkiewicz correlation function [62,67], which gives the joint measurement of the spin degree of freedom and of a spatial degree of freedom, dichotomized in this way by the Wigner operator. This correlation function has been used in this way to describe a measurement of the position in only one coordinate [56,61,63], as was used for the SGE in Ref. [52]. The evolution of the CSGE, given in Eq. (5), ensures that a dichotomization in both the x and z directions can be done to obtain the measurement in the position degree of freedom. Therefore, we construct the BW correlation function using the Wigner operator of Eq. (9).

In this way, to construct the function explored by the CHSH inequality (and similarly for the other inequalities present in this article), we take the correlation function of Eq. (10) that defines by itself a correlation in a Bell-type experiment, and we compose the B_{CHSH} function with the appropriate pairings of variables needed by the CHSH inequality. We fix the rest of the variables to study the Z-θ pair, in order to make the resulting function depend only on Z and θ. As we can see from Eq. (12), we take into consideration the correlations in the primed quantities, Z' and θ', and in the not primed quantities Z and θ, that represent two measurements made by two different persons in different separate places.

APPENDIX B: DYNAMIC EVOLUTION OF THE CSGE

In this Appendix, we will find the explicit evolution of the initial wave packet as it traverses the configuration of CSGE shown in Fig. 1.

The effective Hamiltonian related with the first experiment is [30,31]

$$\hat{H}_2 = -\frac{\hbar^2}{2m}\nabla^2 + \mu_c(\boldsymbol{\sigma}\cdot\boldsymbol{B}_2), \tag{B1}$$

with the effective inhomogeneous magnetic field $\boldsymbol{B}_2 = (B_2 + b_2 x)\hat{\imath}$, $\mu_c = ge\hbar/(4m)$, where g is the gyromagnetic ratio, m is the mass of the particle, e is the unit charge, and $\boldsymbol{\sigma}$ is the Pauli matrices vector. In a similar way, we have the effective Hamiltonian associated with the second experiment,

$$\hat{H}_3 = -\frac{\hbar^2}{2m}\nabla^2 + \mu_c(\boldsymbol{\sigma}\cdot\boldsymbol{B}_3), \tag{B2}$$

with the effective field $\boldsymbol{B}_3 = (B_3 + b_3 z)\hat{k}$.

The initial state is given in Eq. (1) of the article as

$$|\psi_i\rangle = \frac{1}{(2\pi\sigma_0^2)^{\frac{3}{4}}}\exp\left(-\frac{(x^2+y^2+z^2)}{4\sigma_0^2} + ik_y y\right)|\uparrow_z\rangle. \tag{B3}$$

To obtain the evolution through the first experiment, we have the evolution operator associated with the Hamiltonian $\hat{H}_2$ of Eq. (B1), appearing in Eq. (2) of the article,

$$\hat{U}_2(t) = \exp\left(-\frac{1}{6}\kappa\right)\exp\left[-\frac{it}{2m\hbar}\left(p_y^2+p_z^2\right)\right]\exp\left[-\frac{it\mu_c}{\hbar}(B_2+b_2x)\sigma_x\right]\exp\left(\frac{it^2\mu_c b_2}{2m\hbar}p_x\sigma_x\right)\exp\left(-\frac{it}{2m\hbar}p_x^2\right), \tag{B4}$$

with $\kappa = (it^2\mu_c^2 b_2^2)/(m\hbar)$ where B_2 is the homogeneity parameter of the magnetic field, b_2 is the inhomogeneity parameter, and σ_x is the Pauli operator. The evolution operator of this Eq. (B4) is factorized utilizing the evolution operator factorization method of Refs. [53,54], as done in Refs. [30,31].

Applying $\hat{U}_2(t)$ for a certain time t_2 to $|\psi_i\rangle$, we have the intermediate state $|\psi_m\rangle$,

$$\begin{aligned}\hat{U}_2(t_2)|\psi_i\rangle = |\psi_m(t_2)\rangle &= \frac{1}{\sqrt{2}}\exp\left(-\frac{1}{6}\kappa_2\right)\sigma_0^{\frac{3}{2}}\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)^{-\frac{3}{4}}\left\{\left[(2\pi)^{\frac{1}{2}}\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)^{\frac{1}{2}}\right]^{-\frac{1}{2}}\right\}^3\\ &\times\exp\left(-k_y^2\sigma_0^2\right)\exp\left\{\frac{-\left[\left(y-2ik_y\sigma_0^2\right)^2+z^2\right]}{4\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)}\right\}\left\{\exp\left[-\frac{it_2\mu_c}{\hbar}(B_2+b_2x)\right]\exp\left[\frac{-\left(x+\frac{t_2^2\mu_c b_2}{2m}\right)^2}{4\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)}\right]|\uparrow_x\rangle\right.\\ &\left.+\exp\left[\frac{it_2\mu_c}{\hbar}(B_2+b_2x)\right]\exp\left[\frac{-\left(x-\frac{t_2^2\mu_c b_2}{2m}\right)^2}{4\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)}\right]|\downarrow_x\rangle\right\},\end{aligned} \tag{B5}$$

where we remember $|\uparrow_z\rangle = \frac{1}{\sqrt{2}}(|\uparrow_x\rangle + |\downarrow_x\rangle)$ and $\kappa_2 = \kappa(t_2)$.

Now, following our description of the CSGE, we apply the evolution operator associated with the last SGE for a fixed time t_3. That is, we apply the evolution operator in direction z direction, corresponding to the Hamiltonian of Eq. (B2), to the state of the Eq. (B5). This operator is given by

$$\hat{U}_3(t) = \exp\left(-\frac{1}{6}\kappa\right)\exp\left[-\frac{it}{2m\hbar}\left(p_x^2+p_y^2\right)\right]\exp\left[-\frac{it\mu_c}{\hbar}(B_3+b_3z)\sigma_z\right]\exp\left(\frac{it^2\mu_c b}{2m\hbar}p_z\sigma_z\right)\exp\left(-\frac{it}{2m\hbar}p_z^2\right). \tag{B6}$$

Then we obtain the final state of the system, $|\psi_f\rangle$,

$$\begin{aligned}\hat{U}_3(t_3)|\psi_m(t_2)\rangle = |\psi_f\rangle = A_3'\times\exp&\left\{\frac{-\left(y-2ik_y\sigma_0^2\right)^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}\left[\exp\left[-\frac{it_3\mu_c}{\hbar}(B_3+b_3z)\right]\exp\left\{\frac{-\left(z+\frac{t_3^2\mu_c b_3}{2m}\right)^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}\right.\\ &\times\left(\exp\left\{-\frac{it_2\mu_c}{\hbar}B_2\right\}\exp\left\{\frac{-\left[x+\frac{t_2^2\mu_c b_2}{2m}+2i\frac{t_2\mu_c b_2}{\hbar}\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)\right]^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}\right.\\ &\left.+\exp\left\{\frac{it_2\mu_c}{\hbar}B_2\right\}\exp\left\{\frac{-\left[x-\frac{t_2^2\mu_c b_2}{2m}-2i\frac{t_2\mu_c b_2}{\hbar}\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)\right]^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}\right)|\uparrow_z\rangle\end{aligned}$$

$$+\exp\left[\frac{it_3\mu_c}{\hbar}(B_3+b_3z)\right]\exp\left\{\frac{-\left(z-\frac{t_3^2\mu_c b_3}{2m}\right)^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}$$
$$\times\left(\exp\left\{-\frac{it_2\mu_c}{\hbar}B_2\right\}\exp\left\{\frac{-\left[x+\frac{t_2^2\mu_c b_2}{2m}+2i\frac{t_2\mu_c b_2}{\hbar}\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)\right]^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}\right.$$
$$\left.-\exp\left\{\frac{it_2\mu_c}{\hbar}B_2\right\}\exp\left\{\frac{-\left[x-\frac{t_2^2\mu_c b_2}{2m}-2i\frac{t_2\mu_c b_2}{\hbar}\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)\right]^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}\right)|\downarrow_z\rangle\Bigg] \quad \text{(B7)}$$

with $\kappa_3=\kappa(t_3)$ and

$$A_3'=\exp\left(\frac{-\kappa_2-\kappa_3}{6}\right)\frac{1}{2}\left[\frac{\sigma_0}{(2\pi)^{\frac{1}{2}}}\right]^{\frac{3}{2}}\left(\sigma_0^2+\frac{i\hbar}{2m}(t_2+t_3)\right)^{-\frac{3}{2}}\exp\left(-k_y^2\sigma_0^2\right)\exp\left(\frac{it_2}{\hbar}\mu_c b_2\cdot\frac{t_2^2\mu_c b_2}{2m}\right)$$
$$\times\exp\left[-\left(\frac{t_2\mu_c b_2}{\hbar}\right)^2\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)\right]. \quad \text{(B8)}$$

To obtain Eq. (4) in the article, we rewrite the final state of Eq. (B7) as follows:

$$|\psi_f\rangle=|\psi_+\rangle|\uparrow_z\rangle+|\psi_-\rangle|\downarrow_z\rangle, \quad \text{(B9)}$$

defining, as in Eq. (5) of the article,

$$|\psi_\pm\rangle=M\exp\left(\mp i\sqrt{2}k_3^2\tau_3 z_0\right)\exp\left\{-\frac{\left(Z\pm\left\{\sqrt{2}k_3^3\tau_3^2+i2\sqrt{2}k_3^3\tau_3[1+i(\tau_2+\tau_3)]\right\}\right)^2}{4[1+i(\tau_2+\tau_3)]}\right\}$$
$$\times\left(\exp\left(-i\sqrt{2}k_2^3\tau_2x_0\right)\exp\left\{-\frac{\left[X+\sqrt{2}k_2^3\tau_2^2+i2\sqrt{2}k_2^3\tau_2(1+i\tau_2)\right]^2}{4[1+i(\tau_2+\tau_3)]}\right\}\right.$$
$$\left.\pm\exp\left(i\sqrt{2}k_2^3\tau_2x_0\right)\exp\left\{-\frac{\left[X-\sqrt{2}k_2^3\tau_2^2-i2\sqrt{2}k_2^3\tau_2(1+i\tau_2)\right]^2}{4[1+i(\tau_2+\tau_3)]}\right\}\right), \quad \text{(B10)}$$

with M being the normalization factor given from Eqs. (B5) and (B8) by

$$M=\exp\left(\frac{-\kappa_2-\kappa_3}{6}\right)\frac{1}{2}\left[\frac{\sigma_0}{(2\pi)^{\frac{1}{2}}}\right]^{\frac{3}{2}}\left(\sigma_0^2+\frac{i\hbar}{2m}(\tau_2+\tau_3)\right)^{-\frac{3}{2}}\exp\left(-k_y^2\sigma_0^2\right)\times\exp\left[-\left(\frac{t_2\mu_c b_2}{\hbar}\right)^2\sigma_0^2\right]$$
$$\times\exp\left[-\left(\frac{t_3\mu_c b_3}{\hbar}\right)^2\left(\sigma_0^2+\frac{i\hbar t_2}{2m}\right)\right]\exp\left\{\frac{-\left(y-2ik_y\sigma_0^2\right)^2}{4\left[\sigma_0^2+\frac{i\hbar(t_2+t_3)}{2m}\right]}\right\}, \quad \text{(B11)}$$

and the dimensionless definitions of Eq. (6) in the article

$$\tau_{2,3}=\frac{\hbar t_{2,3}}{2m\sigma_0^2},\quad k_{2,3}=\sqrt{2}\sigma_0\left(\frac{m\mu_c b_{2,3}}{2\hbar^2}\right)^{1/3},$$
$$x_0=\frac{B_2}{\sigma_0 b_2},\quad z_0=\frac{B_3}{\sigma_0 b_3},$$
$$Z=\frac{z}{\sigma_0},\quad X=\frac{x}{\sigma_0}. \quad \text{(B12)}$$

APPENDIX C: THE CORRELATION FUNCTION FOR THE CSGE

In the discussion surrounding Eq. (9) of the article, we found that the correlation function for the CSGE can be written as

$$\boldsymbol{C}(X,P_x,Z,P_z,\theta)=\exp[\omega_z'(Z,P_z)+\omega_x'(X,P_x)][4\cos(\theta)(\exp[\omega_z]\{\exp[\omega_x]\cosh[d_x(X,P_x)]\sinh[d_z(Z,P_z)]$$
$$+\exp[-\omega_x]\cos[\delta_x(X,P_x)]\cosh[d_z(Z,P_z)]\})+4\sin(\theta)(\exp[-\omega_z]\{\exp[\omega_x]\sinh[d_x(X,P_x)]\cos[\delta_z(Z,P_z)]$$
$$+\exp[-\omega_x]\sin[\delta_x(X,P_x)]\sin[\delta_z(Z,P_z)]\})] \quad \text{(C1)}$$

up to a normalization factor.

We do so by defining the following functions that appear in Eq. (C1):

$$\begin{aligned}\omega'_z(Z,P_z) = {} & \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big[-(\tau_2+\tau_3)^2\big(Z^2+2k_3^6\tau_3^4\big)\big] - 2k_3^6\tau_3^2[1+(\tau_2+\tau_3)^2] \\ & + 2k_3^6\tau_3^3(\tau_2+\tau_3) - 2[1+(\tau_2+\tau_3)^2]P_z^2 - \frac{1}{4}Z^2 + 2P_zZ(\tau_2+\tau_3) \\ & - \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big(Z^2+2k_3^6\tau_3^4\big) - 4k_3^6\tau_3^2,\end{aligned} \tag{C2}$$

$$\begin{aligned}\omega'_x(X,P_x) = {} & \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big(-(\tau_2+\tau_3)^2\big(X^2+2k_2^6\tau_2^4\big) - 8k_2^6\tau_2^2\big[4\big(1+\tau_2^2\big)^2 - 2\tau_3^2\big(1-\tau_2^2\big) + 4\tau_2\tau_3\big(1+\tau_2^2\big)\big] \\ & + 8k_2^6\tau_2^3\{4(\tau_2+\tau_3) - 2\tau_2[1-(\tau_2+\tau_3)^2]\}\big) - 2[1+(\tau_2+\tau_3)^2]P_x^2 - \frac{1}{4}X^2 + 2XP_x(\tau_2+\tau_3) \\ & - \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big\{X^2 + 2k_2^6\tau_2^4 - 8k_2^6\tau_2^2\big[2\big(1+\tau_2^2\big) + 4\tau_2\tau_3\big]16k_2^3\tau_2^3\tau_3\big\},\end{aligned} \tag{C3}$$

$$\omega_z = -\frac{1}{2}k_3^6\tau_3^4 - 2k_3^6\tau_3^2[1+(\tau_2+\tau_3)^2] + 2(\tau_2+\tau_3)k_3^6\tau_3^3, \tag{C4}$$

$$\omega_x = -\frac{1}{2}k_2^6\tau_2^4 - 2k_2^6\tau_2^2, \tag{C5}$$

$$\begin{aligned}d_z(Z,P_z) = {} & \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big[-2\sqrt{2}k_3^3\tau_3^2(\tau_2+\tau_3)^2Z\big] + \sqrt{2}k_3^3\tau_3(\tau_2+\tau_3)Z \\ & - \frac{1}{\sqrt{2}}k_3^3\tau_3^2Z + \sqrt{2}k_3^3\tau_3(\tau_2+\tau_3)Z - \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big[2\sqrt{2}k_3^3\tau_3^2Z\big] \\ & + P_z\big\{-4\sqrt{2}k_3^3\tau_3 + (\tau_2+\tau_3)\big[2\sqrt{2}k_3^3\tau_3^2 - 4\sqrt{2}k_3^3\tau_3(\tau_2+\tau_3)\big]\big\},\end{aligned} \tag{C6}$$

$$\begin{aligned}d_x(X,P_x) = {} & \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big(-2\sqrt{2}k_2^3\tau_2^2(\tau_2+\tau_3)^2X + 2\sqrt{2}k_2^3\tau_2X\{4(\tau_2+\tau_3) - 2\tau_2[1-(\tau_2+\tau_3)^2]\}\big) \\ & - \frac{1}{\sqrt{2}}k_2^3\tau_2^2X + P_x\big[2\sqrt{2}k_2^3\tau_2^2(\tau_2+\tau_3) - 4\sqrt{2}k_2^3\tau_2\big(1+\tau_2^2+\tau_2\tau_3\big)\big] \\ & - \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big(2\sqrt{2}k_2^3\tau_2^2X + 8\sqrt{2}k_2^3\tau_2\tau_3X\big),\end{aligned} \tag{C7}$$

$$\delta_z(Z,P_z) = -P_z2\sqrt{2}k_3^3\tau_3^2 + 2\sqrt{2}k_3^3\tau_3Z + 2\sqrt{2}k_3^3\tau_3z_0, \tag{C8}$$

$$\begin{aligned}\delta_x(X,P_x) = {} & \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big\{2\sqrt{2}k_2^3\tau_2X[2-2(\tau_2+\tau_3)^2+4\tau_2(\tau_2+\tau_3)]\big\} + \sqrt{2}k_2^3\tau_2X - P_x\big(2\sqrt{2}k_2^3\tau_2^2 + 4\sqrt{2}k_2^3\tau_2\tau_3\big) \\ & - \frac{1}{4[1+(\tau_2+\tau_3)^2]}\big\{-4\sqrt{2}k_2^3\tau_2X[2+2\tau_2(\tau_2+\tau_3)]\big\} + 2\sqrt{2}k_2^3\tau_2x_0.\end{aligned} \tag{C9}$$

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