Switchability of multimodal optical phases in a leaky and nonlinear quantum cavity

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(Received 5 October 2020; accepted 3 March 2021; published 26 March 2021)

In the present report, the question of U(1) symmetry breaking in a system of atoms and electromagnetic fields, interacting inside a leaky cavity, filled with a nonlinear medium, is addressed. In particular, the Z_2 discrete symmetry of the system and the emergence of optical phases are fully discussed. For the nonlinearity of a general order, with an electromagnetic field of any number of modes, conditions under which the resulting field-field interactions destroy the U(1) (and Z_2) symmetry are determined. We then apply the theory to the case of a collection of two-level atoms and three electromagnetic modes. Taking one of the fields as a classically adjustable pumping one, it is demonstrated that the quantized field-field coupling profoundly depends upon the pump field strength. The presence of two competing phenomena, namely, the cavity photonic dissipation and nonlinearity, is shown to lead the system towards steady behavior. The steady-state solutions to the atomic population and field quadratures exhibit the normal and superradiant phases, depending on the strengths of pumping field and atom-field couplings. The conditions for the stability of such steady-state solutions are also discussed in detail. A notable result of the present article is that by adjusting the parameters involved in the system one can switch from the normal to electric and/or magnetic superradiant phases.

DOI: 10.1103/PhysRevA.103.033712

I. INTRODUCTION

Since the early seventies, when the notion of quantum optical phases (QOP) was first introduced [1–3], it has gradually attracted much attention, both theoretically [4–10] and experimentally [11–16]. In its primitive description, QOP refers to the *trivial* (TP) (normal) and *superradiant* (SP) phases that exist in a system of many atoms (Dicke states) and quantized electromagnetic (QEM) fields (photons). To be more specific, the trivial phase is defined as a steady state of the system in which all atoms occupy the ground state, while the fields are in the vacuum state [17–19]. On the other hand, in the superradiant phase the atoms raise to exited states along with the QEM fields being in some combination of nonvanishing occupational states [18–20]. It is, moreover, well understood that a transition from the trivial phase to superradiant phase in such systems is possible by adjusting the parameters describing the specifications of the particular system. That is to say, one can in principle break the U(1), as well as discrete Z_2, symmetry of the noninteracting subsystems (atoms and fields), causing a transition from the trivial phase to the superradiant one. It is emphasized that the emergence of QOPs is due to the destruction of discrete Z_2 symmetry, which may remain intact in spite of the breakage of U(1) symmetry. The resulting separation of the corresponding two-fold degenerate states then gives rise to QOPs. The breakage of U(1), as is well known [21–24], strongly depends upon the form of interaction that couples the subsystems. In what follows, we investigate the U(1) in general and, particularly, the Z_2 symmetry in the system of atoms/photons, placed in a cavity of nonlinear susceptibility of arbitrary orders.

In the simplest form, the atom-photon interaction is assumed to take place in vacuum, via the field-dipole (atomic) interaction. To this end, under the customarily used rotating-wave-approximation, the total Hamiltonian is known [25,26] to possess such a symmetry. The inclusion of the counter-rotating terms has been demonstrated to destroy the U(1) symmetry [20,27], while retaining the Z_2 [a subgroup of U(1)] symmetry [28–30]. These conclusions, however, strongly depends on the atom-field coupling strength. This point comes about from the fact that, in general, the system's quantum states (and eigenvalues) make abrupt changes from one to another, at some *critical* value of the atom-field coupling [31–34]. It is worth mentioning that the occurrence of the two optical phases and, thereby, transitions between them, closely resembles the topological phases in solid state physics [35–39]. Although the above description nicely explains the occurrence of optical phases in the atom-field systems, for more generality and applicability, two more points are taken into account in the present paper. Firstly, the quantum cavity is unavoidably leaky: as the cavity walls are not perfectly reflective, the photons dissipate into the surroundings. Accordingly, appropriate operators, which are responsible for such dissipations [40–43], should be included in the dynamical equations [44–47]. Moreover, when the atom-field interactions take place in a susceptible surrounding, electrical properties of the latter may be used as a controlling mechanism. Therefore the second point of importance in our treatment of atom-field interaction is the inclusion of nonlinear properties of the medium. Although the connection between the U(1) symmetry of the system of atom-field and dissipation is by now well established [17,22,48], to the best of our knowledge,

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consideration of the effects of the lowest order of nonlinearity has been reported [24,49–51]. To this end, a part of the present paper is devoted to a thorough investigation of how an arbitrary degree of the nonlinearity affects the OPT. Moreover, if some of the interacting EM fields are supplied from out side of the cavity, then the pumping fields and the cavity leakage form two competing mechanisms which can, in principle, lead to steady solutions [52] for the system. This would occur when the two mechanisms balance each other [53]. It is then feasible to unambiguously observe the phenomenon of OPT in such an atom-field system. To this end, we may mention that the steady solutions for the two optical phases have been employed to write (read) quantum data onto (from) superradiant phase, while the trivial phase has been used for data storage, with more efficiency relative to conventional means [54]. Moreover, the existence of such steady solutions, trivial or superradiant states, have paved the road to the development of ultra-narrow-linewidth superradiant lasers [55,56]. Other possible applications of optical phases and transition between them can be found in Refs. [57–59].

The remaining part of the present section serves us to describe the problem under investigation and the organization of the paper. To this end, Sec. II is devoted to the description of the system, along with the definition of U(1) generator and symmetry. We cover the latter point when the atom-field (multimodal) system is described by the linear Dicke model (LDM) [60], as well as the linear Tavis-Cummings model (LTCM), which is the LDM within the rotating-wave-approximation [61,62]. The U(1) symmetry is then addressed when the models include field-field interactions, arising from the nonlinearity of the medium, NLDM and NLTCM, respectively. To pursue this aim, we develop the field-field interaction to any order of nonlinearity in this section also. As an important result, it is demonstrated that one can adjust the nature of interacting fields to admit any desired field-field interactions into the formulation. The fact that the whole process occurs in a leaky cavity, however, is ignored in Sec. II. We then apply the formulation developed in Sec. II to the case of a system consisting of two-level atoms interacting with *three-mode* fields. This forms a part of Sec. III. In this section, the dynamical equations for the seven operators of the system (four field quadratures and three atomic operators) are calculated. The steady-state solutions of the dynamical equations, along with their stability is covered in Sec. IV. To this end, we supply contour diagrams for S_z which distinctly show the occurrence of trivial and superradiant optical phases. Graphs of the field quadratures, as functions of the couplings involved, presented in Sec. IV then reveals how one can select the couplings to make a transition between the two optical phases. Needless to say that the physical reasons behind the results of each section are also discussed in detail. Finally, we conclude the paper by highlighting our results in Sec. V.

II. HAMILTONIAN, SYMMETRY CONSIDERATION OF NONLINEAR TAVIS-CUMMINGS MODEL

In this section, we first present a brief discussion on the symmetry properties of the LDM, as well as the LTCM, followed by a full scrutiny of a multimode field-field interaction caused by the nonlinearity of the medium. The latter then leads to the nonlinear version of TCM, NLTCM. These two parts shall appear in the following subsections. Before attending to the main object, however, we recall that for a system of A number of atoms interacting with an N-mode electromagnetic field, the U(1) symmetry is generated by the operator, $C = \sum_{\alpha=1}^{N} a_\alpha^\dagger a_\alpha + S_z + A/2$. Here, a_α ($a_\alpha^\dagger$) denotes the field annihilation (creation) operator for the αth mode and S_z ($= \sum_{j=1}^{A} S_z^j$) is the z component of the collective atomic pseudospin operator. When the unitary operator, $U = \exp(-iC\phi)$ is applied, we easily find [63]

$$U^\dagger f(a,\, a^\dagger,\, S_-, S_+)U = e^{iC\phi} f(a,\, a^\dagger,\, S_-, S_+)e^{-iC\phi} = f(e^{-i\phi}a,\, e^{i\phi}a^\dagger,\, e^{-i\phi}S_-,\, e^{i\phi}S_+), \tag{1}$$

where $S_\pm$ denotes collective atomic ladder operators. The notations we employ in this paper shall be further discussed in the first subsection. In the following two sections, we repeatedly use Eq. (1) to investigate symmetry properties of linear, as well as nonlinear models of atom-field interactions. At this point, it is reminded that when the total system, described by the Hamiltonian $\mathcal{H}$, is invariant under U(1) symmetry, i.e., if $U^\dagger \mathcal{H} U = \mathcal{H}$ holds, then the eigenvalues of C becomes a constant of motion, with an infinite-fold degeneracy.

A. Symmetry properties of LDM and LTCM

The behavior of a system consisting of two-level atoms, immersed in a linear isotropic medium, and a multimodal electromagnetic field is governed by the LDM Hamiltonian,

$$\mathcal{H} = \mathcal{H}_a + \mathcal{H}_f + \mathcal{H}_{af}, \tag{2}$$

where the collective free atomic Hamiltonian is ($\hbar = 1$), $\mathcal{H}_a = \Omega \sum_{j=1}^{A} (1/2)\sigma_z^{(j)}$, with Ω denoting the atomic transition frequency and $\sigma_\beta^{(j)}$ ($\beta = x, y, z$) stands for the βth component of Pauli matrices for jth atom. The linear part of the field Hamiltonian, $\mathcal{H}_f$, reads $\mathcal{H}_f = 1/2 \int (\epsilon E^2 + \mu_0 H^2) dv = \sum_{\alpha=1}^{N} \omega_\alpha a_\alpha^\dagger a_\alpha$, where the multimodality of the electromagnetic field has been taken into account (N is the number of modes). At this point, it is reminded that the system's Hilbert space is spanned by $|k, A> \otimes |n_1> \otimes |n_2> \cdots \otimes |n_N> \equiv |D, f>$, where $\mathcal{H}_a |k, A> = (\Omega/2)(k - A/2)|k, A>$ and $a_\alpha^\dagger a_\alpha |n_\alpha> = n_\alpha |n_\alpha>$. The states $|k, A>$ (Dicke states) describe an atomic state in which k number (out of A atoms) of atoms is excited. As a straight forward application of Eq. (1) shows, $\mathcal{H}_a + \mathcal{H}_f$ inhabits the U(1) symmetry for each ϕ. Assuming that the fields are all polarized along the x axis, the atom-field interaction, in the electric-dipole approximation, reads

$$\mathcal{H}_{af} = \frac{S_x}{\sqrt{A}} \sum_{\alpha=1}^{N} \lambda_\alpha (a_\alpha + a_\alpha^\dagger), \tag{3}$$

where $S_\beta = (1/2)\sum_{j=1}^{A} \sigma_\beta^{(j)}$ ($\beta = x, y, z$) indicates an atomic collective operator and λ_α, that depends upon atomic dipole moments, denotes the atom-field coupling strengths. As it is well established [20] and can be readily verified through an application of Eq. (1), the inclusion of $\mathcal{H}_{af}$ in the linear part of the Hamiltonian breaks the U(1) symmetry while the inhabitant Z_2 symmetry of the system remains intact. On the other hand, the Hamiltonian of Eq. (3), in the rotating wave approximation (LTCM), $\sum_{\alpha=1}^{N}\frac{\lambda_\alpha}{\sqrt{A}}(S_+a_\alpha+S_-a_\alpha^\dagger)$, with the atomic ladder operators $S_\pm=S_x\pm iS_y$, preserves the U(1) symmetry [64]. The linear Hamiltonian of Eq. (2), Therefore it is invariant under a U(1) transformation, in this approximation. Since the eigenstates of the LTCM Hamiltonian, $|E>$, is a linear combinations of $|D,f>$ and $|D-1,f+1>$, in the degenerate subspace of the operator C, then a particular combination, $|E>$, and $e^{-iC\phi}|E>$ are degenerate for all values of the continuous variable ϕ, an *infinite-fold* degeneracy.

B. Symmetry properties of nonlinear field-field interaction

The field-field Hamiltonian, arising from nonlinear processes in the medium may readily be calculated from $H_{ff}=\sum_s\mathcal{H}^{(s)}$, where $\mathcal{H}^{(s)}=\int\overrightarrow{E}\cdot\overrightarrow{P_s}^{NL}dv$ denotes the contribution of the sth order nonlinearity. It is emphasized that in the foregoing expression, $\overrightarrow{P_s}^{NL}$ is the sth order nonlinear polarization of the medium. Assuming that the nonlinear medium is isotropic, the sth-order field-field interaction Hamiltonian becomes

$$\mathcal{H}^{(s)}=\frac{1}{2}\int_V\chi^{(s)}E^{s+1}dv, \tag{4}$$

where $E=\sum_{\alpha=1}^{N}E_\alpha$, is the total field. The fact that the components in the total field are commutative, along with a multinomial expansion, may be employed to arrive at

$$\left(\sum_{\alpha=1}^{N}E_\alpha\right)^{s+1}=\sum_{\{p_\alpha\}}\frac{(s+1)!}{\prod_{\alpha=1}^{N}p_\alpha!}\prod_{\alpha=1}^{N}(E_\alpha)^{p_\alpha}. \tag{5}$$

The set of integers in Eq. (5), $\{p_\alpha\}$ is such that $\sum_{\alpha=1}^{N}p_\alpha=s+1$ for each mode. When the quantized form of the electric field,

$$E_\alpha=\sqrt{\frac{\omega_\alpha}{2\epsilon V}}\left(u_\alpha a_\alpha+u_\alpha^*a_\alpha^\dagger\right)\quad\alpha=1,2,\ldots,N, \tag{6}$$

with u_αs representing the orthonormal solutions of the Helmholtz equation, is explicitly used in Eq. (5) and the result is substituted in Eq. (4) one finds

$$\mathcal{H}^{(s)}=(s+1)!\sum_{\{p_\alpha\}}\sum_{\{m_\alpha=0\}}^{\{[\frac{p_\alpha}{2}]\}}\sum_{\{l_\alpha=0\}}^{\{p_\alpha-2m_\alpha\}}\lambda_{p_\alpha m_\alpha l_\alpha}\times\prod_{\alpha=1}^{N}C_{p_\alpha m_\alpha l_\alpha}a_\alpha^{\dagger l_\alpha}a_\alpha^{p_\alpha-2m_\alpha-l_\alpha}, \tag{7}$$

where $[p_\alpha/2]$ denotes the floor integer of $p_\alpha/2$,

$$\lambda_{p_\alpha m_\alpha l_\alpha}=\int dv\frac{\chi^{(s)}}{2}\prod_{\alpha=1}^{N}u_\alpha^{p_\alpha-l_\alpha-m_\alpha}u_\alpha^{*l_\alpha+m_\alpha} \tag{8}$$

and

$$C_{p_\alpha m_\alpha l_\alpha}=\frac{\left(\frac{\omega_\alpha}{2\epsilon V}\right)^{p_\alpha/2}}{2^{m_\alpha}m_\alpha!l_\alpha!(p_\alpha-2m_\alpha-l_\alpha)!}. \tag{9}$$

Needless to say that in deriving Eq. (7), the commutation relation between the operators belonging to each mode has been employed repeatedly. In spite of the look of Eq. (7), moreover, $\mathcal{H}^{(s)}$ is indeed Hermitian. The reason for this conclusion of course lies in the summations over the sets, $\{p_\alpha\}$, $\{m_\alpha\}$, and $\{l_\alpha\}$. It is also worth mentioning that analytical evaluation of Eq. (8) is formidably nontrivial, simply because each integral involves multiplications of the u_αs. In order to shed light on the physical significance of $\lambda_{p_\alpha m_\alpha l_\alpha}$, we assume that each mode of the total field is a plane wave. In that case, the integrals become the definition of delta functions:

$$\lambda_{p_\alpha m_\alpha l_\alpha}=\frac{\chi^{(s)}}{2}\int e^{i\overrightarrow{K}\cdot\overrightarrow{r}}dv=\frac{\chi^{(s)}}{2}\delta(\overrightarrow{K})V, \tag{10}$$

where $\overrightarrow{K}=\sum_{\alpha=1}^{N}(p_\alpha-2m_\alpha-2l_\alpha)\overrightarrow{k_\alpha}$, with $\overrightarrow{k_\alpha}$ specifying each mode, defines the field total momentum. We observe that by appropriately selecting the orientation of the field components one can control the combination of field operators appearing in Eq. (7). As a concrete example, we set $p_\alpha=s+1$, $m_\alpha=0$ and $l_\alpha=0$ or $s+1$, for any mode, the contribution to $\mathcal{H}^{(s)}$ reads $\sum_{\alpha=1}^{N}\delta((s+1)k_\alpha)(a^{s+1}+a^{\dagger s+1})$ which vanishes identically. Consequently, in the field-field Hamiltonian, Eq. (7), operators of each free mode (those which do not mix different modes) are absent. Such points become more clear where we consider an specific example involving a three-mode field. Moreover, when the transformation of Eq. (1) is applied to $\mathcal{H}^{(s)}$, its form is preserved except for the $\lambda_{p_\alpha m_\alpha l_\alpha}$s of Eq. (10), which now reads

$$\tilde{\lambda}^{(s)}_{p_\alpha m_\alpha l_\alpha}=\lambda^{(s)}_{p_\alpha m_\alpha l_\alpha}e^{i\phi\sum_{\alpha=1}^{N}(p_\alpha-2m_\alpha-2l_\alpha)}. \tag{11}$$

It is evident that the exponent in Eq. (11) determines the symmetry of the participating field-field interactions. To be more specific, the condition $\sum_{\alpha=1}^{N}(p_\alpha-2m_\alpha-2l_\alpha)\overrightarrow{k_\alpha}=0$ generates the particular interacting fields, while the condition $\sum_{\alpha=1}^{N}(p_\alpha-2m_\alpha-2l_\alpha)=0$ imposes the invariance of field-field interaction. As we mentioned earlier, the set of p_αs must satisfy $\sum_{\alpha=1}^{N}p_\alpha=s+1$, from which the invariance condition becomes $\sum_{\alpha=1}^{N}(m_\alpha+l_\alpha)=(s+1)/2$. Since m_αs and l_αs are integers, we deduce that the U(1), as well as Z_2, symmetry always breaks down for even order of nonlinearity. For odd order of nonlinearity, however, we may or may not have the U(1) symmetry, depending upon the orientation of the interacting quantized fields. The general points about the symmetry properties of a system of atom-field as presented here becomes more tangible in the next section where we consider second order of nonlinearity and a three-mode electromagnetic field.

III. DYNAMICAL DESCRIPTION OF THREE-MODE FIELDS AND TWO-LEVEL ATOMS IN A MEDIUM OF SECOND-ORDER SUSCEPTIBILITY

In the following, we consider the dynamical behavior of three interacting modes ($\alpha=1,2,3$) and Dicke atoms, in a medium with second order nonlinear susceptibility $\chi^{(2)}$. Moreover, we assume that one of the modes is a controllable classical field (k_3, for instance), while one other (k_2, for instance) plays the role of a signal. The classical field is also assumed to be way far from resonance with the atoms, so that its interaction with atoms may be neglected. If we again suppose that the fields propagate along the z axis and set $k_1+k_2=k_3$, the contribution to the field-field interaction in the Hamiltonian of Eq. (7) consists of only two terms corresponding to the sets, $p_\alpha = 1, m_\alpha = 0, (\alpha = 1, 2, 3)$, along with $l_1 = l_2 = 0, l_3 = 1$ or $l_1 = l_2 = 1, l_3 = 0$. Hence, $\mathcal{H}_{ff}$ reduces to

$$\mathcal{H}_{ff} = G(a_1 a_2 a_3^\dagger + a_1^\dagger a_2^\dagger a_3) \tag{12}$$

in which $G = 6(\frac{\omega_1\omega_2\omega_3}{8\epsilon_1\epsilon_2\epsilon_3 V})^{1/2}$. As the third mode is a classical field pump, one has to substitute $E_p e^{-i(2\omega_p t)}$ $(E_p e^{i(2\omega_p t)})$ for a_3 $(a_3^\dagger)$ in Eq. (12). Here, E_p and ω_p denote the classical (*pump*) field amplitude and frequency, respectively. We then find

$$\mathcal{H}_{ff} = GE_p\big(a_1 a_2 e^{i(2\omega_p t)} + a_1^\dagger a_2^\dagger e^{-i(2\omega_p t)}\big). \tag{13}$$

To remove the time-dependent exponentials, it is customary to apply the unitary transformation, $e^{-iC\omega_p t}$, to the total Hamiltonian, Eq. (2) along with the nonlinear field-field interaction of Eq. (12). The resulting time-independent total Hamiltonian then reads (here on we denote the field coupling as, $g = GE_p$),

$$\mathcal{H} = \sum_{\alpha=1}^{2} \Delta_\alpha a_\alpha^\dagger a_\alpha + \Delta S_z + \sum_{\alpha=1}^{2} \frac{\lambda_\alpha}{\sqrt{A}}(S_+ a_\alpha + S_- a_\alpha^\dagger) + g\big(a_1 a_2 + a_1^\dagger a_2^\dagger\big), \tag{14}$$

where the rotating-wave approximation has been explicitly used. It is evident that an application of Eq. (1) to the above Hamiltonian breaks the U(1) symmetry while preserving the Z_2 one. In the present work, we assume that the ensemble formed by atoms and photons dissipate through a cavity damping only. We thus represent the dissipation Liouvillian operator (to be presented soon) which acts solely on photonic states. In the system of bimodal photons and two-level atoms, the corresponding operator space is spanned by four photonic and three atomic operators, along with the relevant identity. We take the photonic operators to be the field quadratures, $Q_\alpha = (1/\sqrt{2})(a_\alpha^\dagger + a_\alpha)$ and $P_\alpha = (i/\sqrt{2})(a_\alpha^\dagger - a_\alpha)$ $(\alpha = 1, 2)$, while for the atomic subspace S_β $(\beta = x, y, z)$ are chosen. As can be straightforwardly shown, however, $S^2 = S_x^2 + S_y^2 + S_z^2$ (the total pseudospin) commutes with the total Hamiltonian of Eq. (14) so that it is a constant of motion. The atomic unknown operators then reduce to two.

When in the cavity there is a damping, the dissipation may be expressed as an operator, $\mathcal{D}(y_i) = \sum_j \kappa_j(2L_j y_i L_j^\dagger - L_j L_j^\dagger y_i - y_i L_j L_j^\dagger)$ so that the quantum mechanical Langvin equation of motion reads, $\dot{y_j} = i[\mathcal{H}, y_j] + \mathcal{D}(y_j)$. Here, y_i represents a system's dynamical variable which is under a damping generated by the operators L_i. Choosing $y_i = \rho$ and $L_j = a_\alpha$, the density operator evolves in time as

$$\dot{\rho} = i[\rho, \mathcal{H}] + \sum_{\alpha=1,2} \kappa_\alpha(2a_\alpha\rho a_\alpha^\dagger - a_\alpha a_\alpha^\dagger\rho - a_\alpha^\dagger a_\alpha\rho), \tag{15}$$

In the derivation of Eq. (15) (and hereafter), we neglect the atomic dampings. Moreover, letting $y_i = Q_\alpha$ or $y_i = P_\alpha$ along with $L_j = a_\alpha^\dagger$, the equations of motion for the field quadratures are obtained as

$$\dot{Q}_\alpha = \Delta_\alpha P_\alpha - \sqrt{2}\lambda_\alpha S_y - \kappa_\alpha Q_\alpha - gP_{\beta\neq\alpha}, \tag{16a}$$

$$\dot{P}_\alpha = -\Delta_\alpha Q_\alpha - \sqrt{2}\lambda_\alpha S_x - \kappa_\alpha P_\alpha - gQ_{\beta\neq\alpha}. \tag{16b}$$

Although we may include atomic damping by the selection, $y_i = S_{x,y,z}$ and $L_i = S_+$ in the following two equations this effect is neglected. The undamped, but coupled, equations of motion for the atomic operators then become,

$$\dot{S}_x = -\Delta S_y - \sqrt{2}(\lambda_1 P_1 + \lambda_2 P_2)S_z, \tag{17a}$$

$$\dot{S}_y = \Delta S_x - \sqrt{2}(\lambda_1 Q_1 + \lambda_2 Q_2)S_z. \tag{17b}$$

Needless to say that in deriving Eqs. (16) and (17) the total Hamiltonian of Eq. (14) has been employed. For future use, where the stability of steady solutions is discussed, we also state that

$$\dot{S}_z = (\lambda_1 P_1 + \lambda_2 P_2)S_x + (\lambda_1 Q_1 + \lambda_2 Q_2)S_y. \tag{18}$$

It is worth noting that dynamics of field operators [Eqs. (16)] are indeed *linear*, while those for atomic operators [Eqs. (17)] are *nonlinear*. In the next section we first solve the set of Eqs. (15)–(17), in the steady state, then discuss the stability of the resulting solutions.

IV. STEADY-STATE SOLUTIONS OF THE DYNAMICAL EQUATIONS

When we substitute the steady values for the dynamical variables (denoted by an over-head $\sim$) in the right hand side of Eqs. (16) and (17), the resulting equations should vanish identically. Using the steady equations (algebraic) for $\tilde{S}_{x(y)}$,

$$\tilde{S}_x = 2\frac{\lambda_1\tilde{Q}_1 + \lambda_2\tilde{Q}_2}{\Delta}\tilde{S}_z, \tag{19a}$$

$$\tilde{S}_y = -2\frac{\lambda_1\tilde{P}_1 + \lambda_2\tilde{P}_2}{\Delta}\tilde{S}_z, \tag{19b}$$

and conservation of $S^2 = \sum_{i=1}^3 S_i^2$, one can eliminate $\tilde{S}_{x(y)}$ from the field equations and obtain

$$\eta_\alpha\tilde{P}_\alpha - \kappa_\alpha\tilde{Q}_\alpha - \xi^-\tilde{P}_{\beta\neq\alpha} = 0, \tag{20a}$$

$$-\eta_\alpha\tilde{Q}_\alpha - \kappa_\alpha\tilde{P}_\alpha - \xi^+\tilde{Q}_{\beta\neq\alpha} = 0, \tag{20b}$$

where $\xi^\pm = g \pm 2\lambda_1\lambda_2\tilde{S}_z/\Delta$, $\eta_\alpha = \Delta_\alpha + 2\lambda_\alpha^2\tilde{S}_z/\Delta$. From Eqs. (20), it is obvious that the trivial steady solutions to the equations for the field quadratures are, $\tilde{P}_\alpha = \tilde{Q}_\alpha = 0$. Equations (19) then show that the trivial state of the system is described by no photons (of any kind) and either upper or lower atomic state ($\tilde{S}_x = \tilde{S}_y = 0$). The trivial values, $\tilde{S}_z = \pm 1/2$ arise from the initial condition, which we take as the fields being in their corresponding vacuum states and the atoms fill the ground state. As it is shown in a moment, these trivial solutions describe the optical normal phase of the system. Arranging the field quadratures in a column matrix, $[\tilde{Q}_1, \tilde{P}_1, \tilde{Q}_2, \tilde{P}_2]^T$, Eqs. (20) can be cast into a matrix equation. For non trivial solutions to the field quadratures, Therefore the determinant of the coefficient matrix must vanish,

$$\begin{vmatrix} \kappa_1 & -\eta_1 & 0 & \xi^- \\ \eta_1 & \kappa_1 & \xi^+ & 0 \\ 0 & \xi^- & \kappa_2 & -\eta_2 \\ \xi^+ & 0 & \eta_2 & \kappa_2 \end{vmatrix} = 0. \tag{21}$$

Through the definitions of η_α and ξ_α, Eq. (21) provides an equation for the only unknown $\tilde{S}_z$. Solving this equation, one finds

$$\tilde{S}_z = \frac{\Delta}{2A}(-B \pm \sqrt{B^2 - AC}), \tag{22}$$

where the real parameters A, B, and C, are defined as

$$A = (\kappa_1\lambda_2^2 + \kappa_2\lambda_1^2)^2 + (\Delta_1\lambda_2^2 + \Delta_2\lambda_1^2)^2 - 4g^2\lambda_1^2\lambda_2^2, \tag{23a}$$

$$B = \kappa_2^2\Delta_1\lambda_1^2 + \kappa_1^2\Delta_2\lambda_2^2 + (\Delta_1\lambda_2^2 + \Delta_2\lambda_1^2)(\Delta_1\Delta_2 - g^2), \tag{23b}$$

$$C = (\Delta_1\Delta_2 + \kappa_1\kappa_2 - g^2)^2 + (\Delta_1\kappa_2 - \Delta_2\kappa_1)^2. \tag{23c}$$

In arriving to Eqs. (23), the explicit forms of η_α and ξ_α have been used. Since the four equations given in Eq. (20) are not independent [because of Eq. (21)], we first solve for $\tilde{P}_1$, $\tilde{Q}_2$, and $\tilde{P}_2$ in terms of $\tilde{Q}_1$, followed by casting Eqs. (19) in terms of the latter, and substituting the results into $\tilde{S}_x^{\,2} + \tilde{S}_y^{\,2} + \tilde{S}_z^{\,2} = 1/4$, one finds

$$\tilde{Q}_1 = \pm\frac{\Delta}{2\tilde{S}_z}\sqrt{\frac{(1/4) - \tilde{S}_z^{\,2}}{(\gamma_1+\gamma_2)^2 + (\lambda_1+\gamma_3)^2}}, \tag{24}$$

where we have defined the real parameters:

$$\gamma_1 = -\lambda_1\frac{\xi^+\xi^-\kappa_2 - \eta_2^2\kappa_1 - \kappa_1\kappa_2^2}{\eta_1\eta_2^2 + \eta_1\kappa_2^2 - \eta_2\xi^{-^2}}, \tag{25a}$$

$$\gamma_2 = -\lambda_2\frac{\xi^+\eta_1\kappa_2 - \eta_2\kappa_1\xi^-}{\eta_1\eta_2^2 + \eta_1\kappa_2^2 - \eta_2\xi^{-^2}}, \tag{25b}$$

$$\gamma_3 = -\lambda_2\frac{\xi^+\eta_1\eta_2 - \xi^+\xi^{-^2} + \xi^-\kappa_1\kappa_2}{\eta_1\eta_2^2 + \eta_1\kappa_2^2 - \eta_2\xi^{-^2}}, \tag{25c}$$

to simplify the expression. Needless to say that one can express the γ_is in terms of A, B, C, and vice versa. Following the forgoing prescription, we also find for the field quadratures, $\tilde{P}_1 = (\gamma_1/\lambda_1)\tilde{Q}_1$, $\tilde{P}_2 = (\gamma_2/\lambda_2)\tilde{Q}_1$ and $\tilde{Q}_2 = (\gamma_3/\lambda_2)\tilde{Q}_1$ along with, $\tilde{S}_x = (2/\Delta)(\lambda_1+\gamma_3)Q_1S_z$ and $\tilde{S}_y = (-2/\Delta)(\gamma_1+\gamma_2)Q_1S_z$ for atomic operators. It is also worth noting that when there is only one quantized field (say the first field in our presentation) and a field pump, then $a_1 = a_2$, $\kappa_1 = \kappa_2$, $\lambda_1 = \lambda_2$ and $\Delta_1 = \Delta_2 = \Delta$. Under these circumstances our results, in particular Eq. (22), reduce exactly to those reported in Ref. [24]. We emphasize at this point that the emergence of two distinct solutions for $\tilde{Q}_1$ is a profound indication of Z_2 symmetry breaking. Out of the two distinct solutions for $\tilde{S}_z$, however, only the upper sign give rise to stable steady states. As it shall be shown, the upper (positive) sign in the steady solution, $\tilde{S}_z$, proves to be stable, we take this sign in Eq. (22) hereafter. A discussion of the stability of the corresponding steady-state solutions follows. To this end, we write each equation of motion for the operators ρ, Q_α, P_α ($\alpha = 1, 2$) and $S_{x,y,z}$ in the concise form,

$$\dot{y}_i = f_i(y_1, y_2, \dots, y_7) \qquad i = 1, 2, \dots, 7, \tag{26}$$

where y_is denote any of the operators, as listed above. In this manner, Eqs. (16)–(18) take the short-hand form, $\dot{y}_i = f(y_j\text{s})$, $i, j = 1, \dots, 7$. To study the stability, it is assumed that the solutions undergoes a small variation about the steady-state solutions, $\tilde{y}_i$, in the long run. Accordingly, we make a Taylor expansion of long-run quantities, $f_i = f_i(\tilde{y}_k) + \sum_{j=1}^7 J_{ij}\delta y_j$, where, $J_{ij} = \frac{\partial f_i}{\partial y_j}|_{y_k=\tilde{y}_k}$ for all ks. Writing δy_j as a superposition of time varying exponentials, $\delta y_i = \sum_{ik} C_{ik}e^{\alpha_k t}$, one finds, $\sum_{j=1}^7 (J_{ij} - \alpha_k\delta_{ij})C_{jk} = 0$, from the corresponding equations of motion (we recall that $f_i = f_i(\tilde{y}_k)$ vanishes for all i and k). It is then concluded that the nature of the eigenvalues, α_k, of J_{ij}, determines the stability of the steady-state solutions. It is obvious that these eigenvalues explicitly depend on $\tilde{y}_k$s, which are themselves functions of the parameters (couplings, frequencies, etc.) specifying the system. We thus arrive at the conclusion that for specific combinations of such parameters, the real part of α_ks assume non positive values. When this happens, the time-dependence of δy_is vanish in the long run, giving rise to stable steady-state solutions. It is straight forward to show that the trivial solution corresponding to $\tilde{S}_z = +1/2$, as anticipated, under no circumstances is stable.

Having discussed the conditions for the stability of the optical phases and established the relation between ensemble averages (steady solutions) of field quadratures, as well as those of atomic operators, and $\tilde{S}_z$, we now examine the limitations physically imposed on the results. First, it is noted that all steady values, in particular $\tilde{S}_z$, should be real, since the corresponding operators remain Hermitian at any time. For $\tilde{S}_z$ to be real, one has the condition, $B^2 \geqslant AC$. Secondly, the reality of $\tilde{Q}_1$ necessitates the condition that $-1/2 \leqslant \tilde{S}_z \leqslant 1/2$, consistent with what one expects for $\tilde{S}_z$. The first condition (simultaneously the second one) can be satisfied by adjusting the parameters appearing in the equations of motion. When for a set of adjustable parameters these two conditions are satisfied, $\tilde{S}_z$, along with all other relevant quantities, fall into the superradiant phase. A collection of such data then defines the so-called phase diagrams (= density profile of $\tilde{S}_z$) of the system. In Fig. 1, the phase diagram is plotted as a function of photon-photon and atom-photon couplings [$g(E_p)$ and $\lambda = \lambda_1 = (1/2)(or(1/3))\lambda_2$, respectively]. In this figure, all the other parameters, namely, the dissipation constants and the participating frequencies, are fixed. The colors (light to dark blue, light to dark red; if gray-scaled: light gray to dark gray), as the bars indicate, gives the values of $\tilde{S}_z$ for a set of $g(E_p)$ and λ_is. Moreover, the boundaries of stability regions are also indicated by dashed lines (curves) in Fig. 1. Panel (a) of the figure corresponds to $\lambda_1 = (1/2)\lambda_2$, while $\lambda_1 = (1/3)\lambda_2$ is the subject of part (b). Although both parts can be extended to the larger values of the two parameters, in order to avoid crowdedness, these extensions are omitted in Fig. 1. We just utter that for larger values of $g(E_p)$, more regions of superradiant phase appear. These additional regions, as can be easily shown, do not fall into the stability region. Meanwhile, for larger values of λ, the steady solutions for S_z converges to zero (note the upper bounds on the bars). From Fig. 1, it is also clear that, for a range of the two couplings, one can reach a point in a region where the superradiant optical phase occurs. If $g(E_p)$ and λ_i s are chosen outside this region, the system remains in the trivial optical phase (blue region, dark gray if gray-scaled). It is further noted that the stability region coincides with that of superradiant phase in the present system, as has been observed in similar cases [24]. For actual applications, in the following, we outline some important points which can be easily observed from Fig. 1. (i) For a specific atom-field

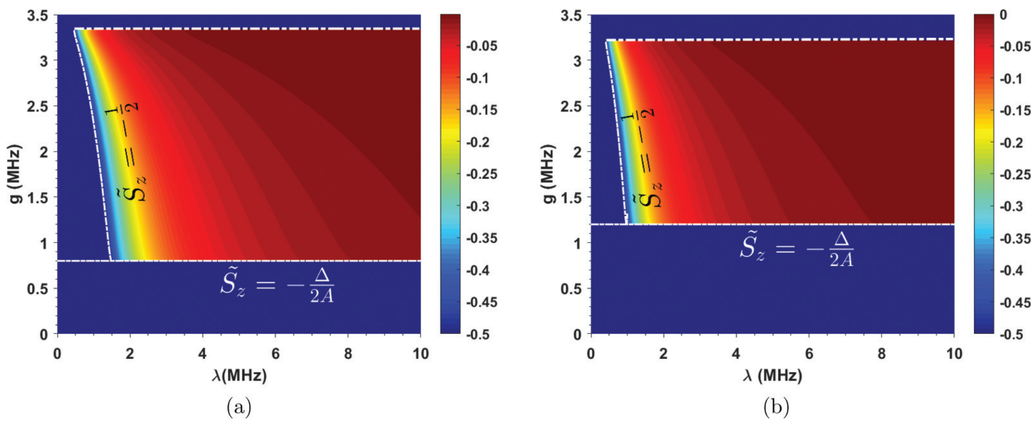

FIG. 1. The system's optical phase diagram. The blue (dark gray, if gray-scaled) region indicates the normal phase while the redish (light gray, if gray-scaled) part corresponds to the superradiant phase. The diagram is drawn for $\Delta = 5$ MHz, $\Delta_1 = 6$ MHz, $\Delta_2 = 2MHz$, $\kappa_1 = 0.5$ MHz, $\kappa_2 = 0.25$ MHz, and $\lambda = \lambda_1 = (1/2)\lambda_2$ (a) and $(1/3)\lambda_2$ (b).

system, there is boundaries within which the system exhibits superradiant phase. The boundary widens as the relative atom-field (quantized) coupling is reduced. (ii) The critical values for the pump field strength reduces when such relative couplings are small. For completeness, we also provide Fig. 2, in which the behavior of $\tilde{S}_z$, panel (a) as a function of $g(E_p)$ (λ_i fixed), and (part b) λ_i ($g(E_p)$ fixed), respectively, is depicted. Panel (a) of Fig. 2 clearly demonstrates the lower and the upper limits imposed on the photon-photon couplings for which the superradiant phase occurs. It is also evident from this part of Fig. 2 that as the atom-photon coupling increases, the lower limit on $g(E_p)$ shifts toward lower values. The upper limit on $g(E_p)$, however, remains the same for any value of λ. From part (b) of Fig. 2 it is observed, as for $\tilde{S}_z$, that as the photon-photon coupling is increased, the lower limit of λ moves toward the origin. That is to say, the superradiant phase occurs at a lower value of atom-photon coupling. Moreover, we call attention to the fact that when the non linearity of the medium is neglected (i.e., the photon-photon coupling vanishes), the corresponding graph (blue one) in part (b) of Fig. 2 coincides with the horizontal axis (purple curve), indicating that our atom-photon system remains in the trivial phase, irrespective of atom-photon coupling [22]. Since the red curve corresponds to a $g(E_p)$ less than the critical photon-photon coupling, one expects that the system would remain in the normal phase. In part (b), the combination of blue and red colors then coincide with the horizontal axis, indicated by purple color. Since the behavior of photonic quadratures are of much interest, in Fig. 3, we present such quantities as a function of photon-photon coupling, for different values of atom-field couplings.

An inspection of Fig. 3 reveals that there exists a discontinuity in all quadratures for large values of atom-field couplings ($\lambda > 1.3429$ MHz, for the data used in generating

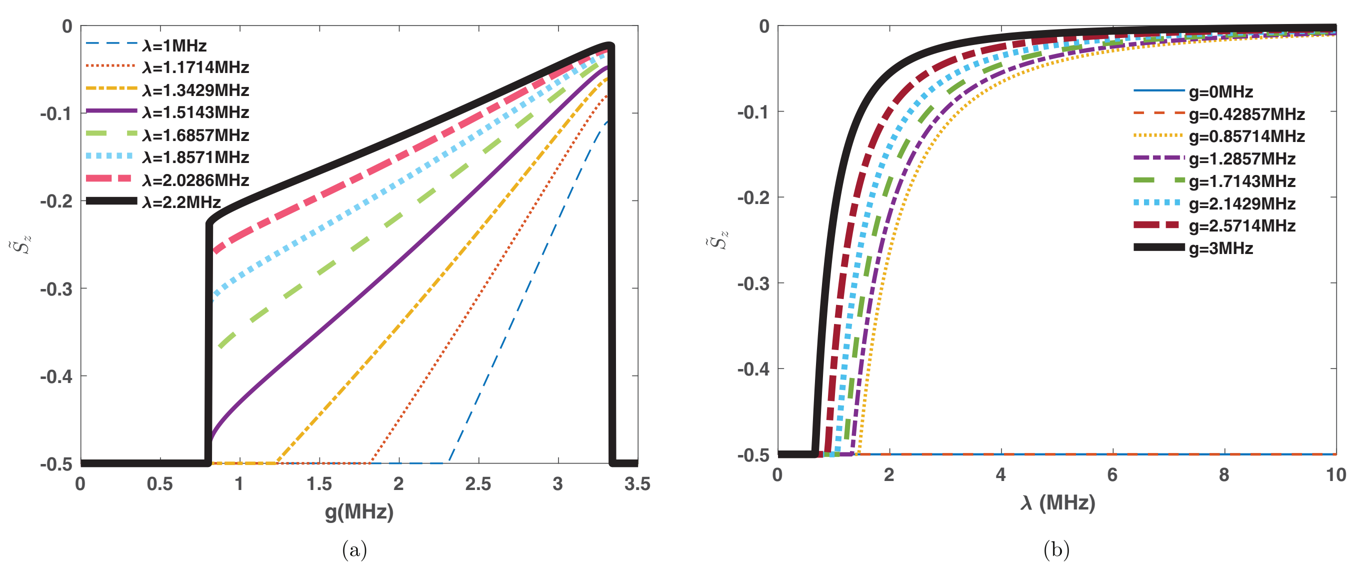

FIG. 2. (a) $\tilde{S}_z$ vs the photon-photon coupling, for different values of atom-field couplings. (b) $\tilde{S}_z$ vs the atom-field coupling, for different values of photon-photon coupling. The figure is drawn for $\Delta = 5$ MHz, $\Delta_1 = 6$ MHz, $\Delta_2 = 2$ MHz, $\kappa_1 = 0.5$ MHz, $\kappa_2 = 0.25$ MHz, and $\lambda_1 = \lambda = \frac{1}{2}\lambda_2$. The insets clearly identify, colored or otherwise, each curve.

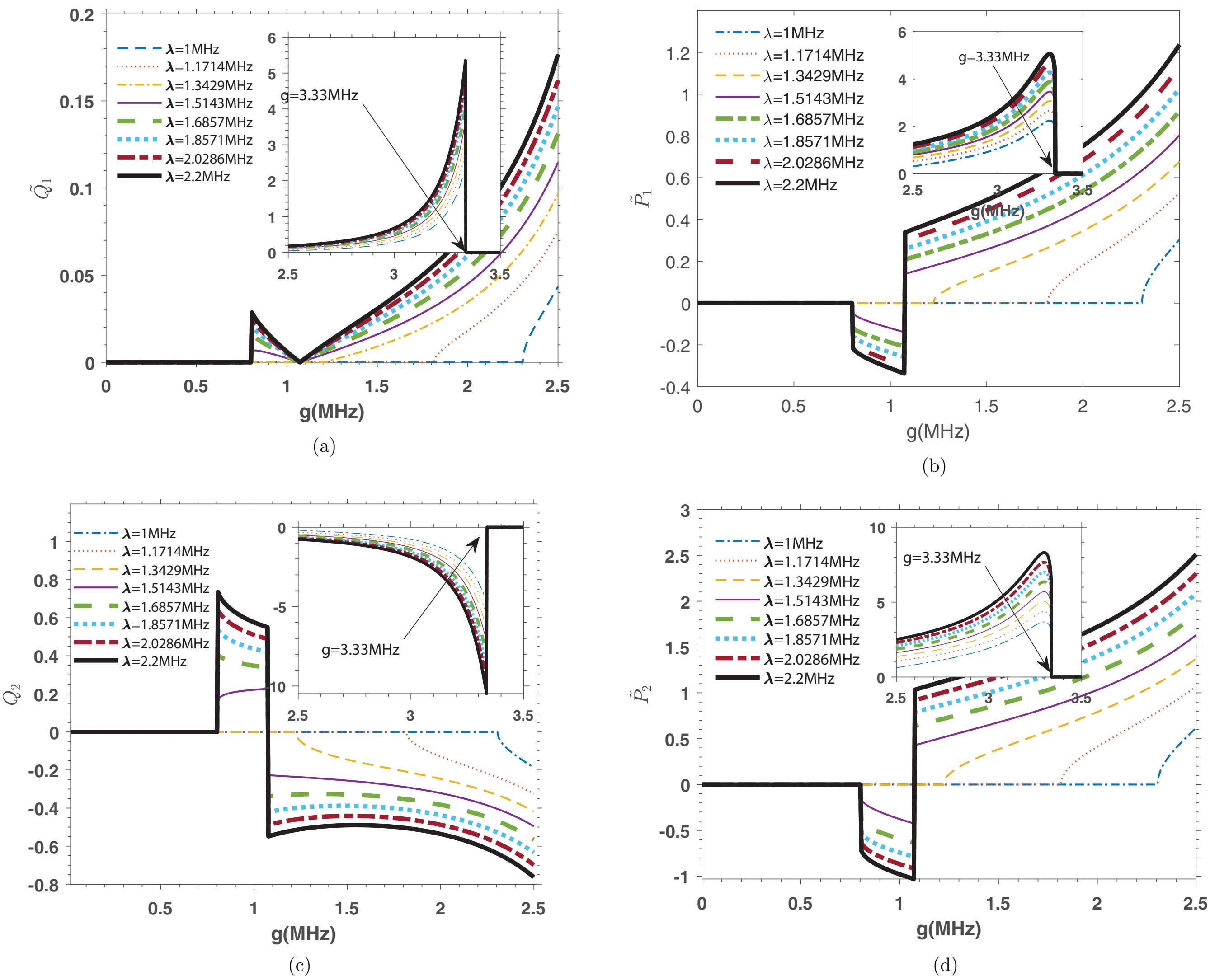

FIG. 3. Field quadratures versus $g(E_p)$, for different values of $\lambda = \lambda_1 = \frac{1}{2}\lambda_2$. All other entries are the same as in the previous figures. The insets clearly demonstrate the corresponding behavior at large values of field-field couplings. Identification of the curves is similar to Fig. 1.

the figure). It is also evident that for $g(E_p) \approx 1.072$ MHz, the first quantized field vanishes for $\lambda > 1.3429$ MHz. At these values, however, all the other quadratures assume non zero values, accompanied by a reversal of polarization directions. The vanishing of Q_1 is due to the fact that the γ s in Eq. (25), a combination of which appears in the denominator of Eq. (24), approach infinity for the data used. However, the limits of the other quadratures [see the lines immediately following Eq. (25)] do exist and are equal to $\tilde{P}_1 \approx \pm\frac{\Delta}{2\lambda\tilde{S}_z}\frac{\gamma_1}{|\gamma_1|}\sqrt{\frac{1}{4}-\tilde{S}_z}$, $\tilde{Q}_2 \approx \pm\frac{\Delta}{2\lambda\tilde{S}_z}\frac{\gamma_2}{|\gamma_2|}\sqrt{\frac{1}{4}-\tilde{S}_z}$ and $\tilde{P}_2 \approx \pm\frac{\Delta}{2\lambda\tilde{S}_z}\frac{\gamma_3}{|\gamma_3|}\sqrt{\frac{1}{4}-\tilde{S}_z}$. We, moreover, note that for $\lambda \leqslant 1.3429$ MHz, the discontinuity of the quadratures disappear. The discontinuity of the first derivatives in all the quadratures persist, indicating that the phase transformation in our case is second order [65]. It is worthwhile to note that there exists sets of parameters for which one or more quadratures vanish, thereby, suggesting means of on-off switching of the quantized fields [66]. For completion, we illustrate the behavior of field quadratures versus $\lambda = \lambda_1 = \frac{1}{2}\lambda_2$ for various values of $g(E_p)$, in Fig. 4. Although the physical points surrounding the optical phases in the present system have already stated, Fig. 4 serves to demonstrate the consistency of our treatment of the problem.

V. CONCLUSION

The present report is mainly concerned with the U(1), in particular its subgroup Z_2, symmetry of a system of a collection of two-level atoms and photons, interacting inside a cavity filled by a *nonlinear* medium. To this end, we present a thorough discussion of the nonlinear field-field interaction Hamiltonian, and its properties under the action of U(1) (or Z_2) transformation. It is emphasized that in treating this part, the number of interacting fields, as well as the order of nonlinearity, is arbitrary, so that the results are quite general. We then proceed to apply the theory to the particular case of one classical field (pump field) and two quantized fields (signal and idler fields), interacting with each other and the two-level atoms. In this application of the theory, the second order nonlinearity is assumed and field-field interaction Hamilto-

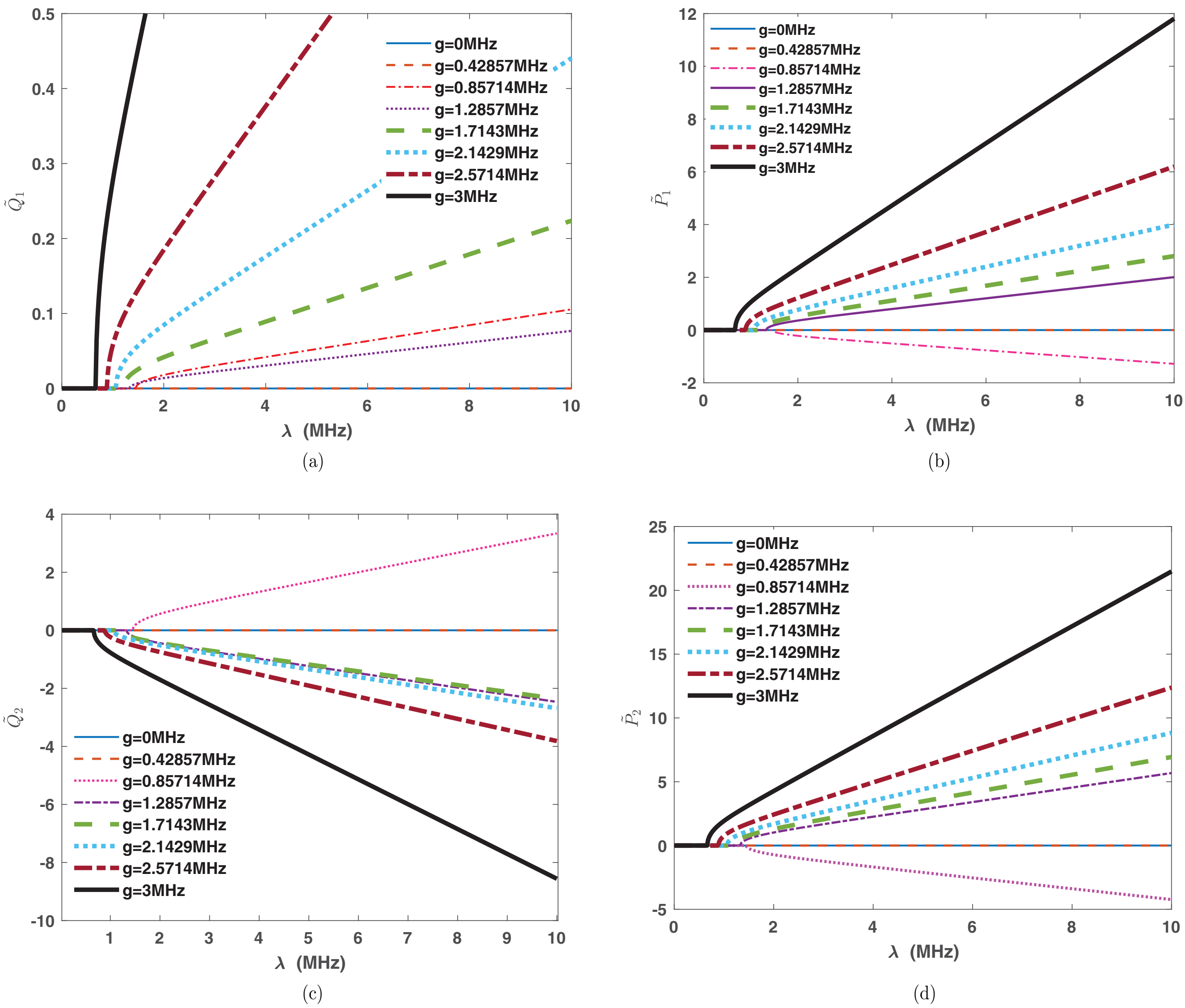

FIG. 4. Field quadratures versus $\lambda = \lambda_1 = \frac{1}{2}\lambda_2$, for various $g(E_p)$ s. All other entries are the same as in the previous figures. Identification of the curves is similar to Fig. 1.

nian is explicitly constructed. Moreover, it is also presupposed that the cavity in which the interaction occurs, is a leaky one so that a proper Liouvillian photonic dissipation is taken into consideration. Consequently, the steady-state solutions to the seven dynamical variables, namely, field quadratures and atomic pseudospins, within the rotating wave approximation, are calculated and discussed. At this point, it is demonstrated that the field-field coupling provides, through the magnitude of the pump field, a mechanism for controlling the symmetries of such dynamical variables. Along these lines, we show that the present system of atom-field exhibits the optical normal (trivial) and *supperradiant* phases. The occurrence of the two optical phases, as we explicitly demonstrate, depends upon the system's parameters. Another point of interest is the stability of the steady-state solutions, which is also discussed in detail. The connection between the stability and governing parameters is also the subject of the present treaty. Although a full discussion of the systems steady behavior is provided in the body of the paper, in what follows we highlight the more important points of our calculations. In attaining these points all governing parameters (such as field dissipations, field frequencies, etc.), except the atom-field and field-field couplings, are fixed at the values given in the text.

(1) From the phase diagram on the steady values of atomic population ($\tilde{S}_z$) as a function of field-field (arising from nonlinearity of the medium) and atom-field couplings, the regions of trivial and supper-radiant (as well as stability) phases are distinguished. The boundaries of such regions are specified in Fig. 1. We suffice it say that a change in the controllable pump field can well induce an optical phase transition.

(2) The field-field nonlinearity provides a mechanism to activate the superradiant phase for a vast variety of atomic systems. That is, even if the atom-field coupling is weak, by adjusting the pumping field, the system indeed makes such phase transition (please see Figs. 1, 2 and 4).

(3) Since it is of more interest to study the behavior of fields quadratures, we provide Fig. 3 in which these quantities are illustrated as functions of field-field couplings, for

a variety of atom-field couplings. A glance at this figure reveals that one can choose a set of couplings for which the first field is in the *magnetic* supper-radiant phase, while the second one falls into *electromagnetic* supper-radiant phase. This point introduces a mechanism for optical switching, from normal to supper-radiant phase in which either electric or magnetic field is present. It is also evident from Fig. 3 that for a set of couplings, the system undergoes phase transitions of the first order while for different sets it is of second order.

In view of the above points, the calculations in the present paper shed lights on understanding the effect of nonlinearity on the behavior of optical phases in atom-field systems.

ACKNOWLEDGMENT

This work has been in part supported by a grant from the Research Council of Shiraz University, under Contract No. 99GRD1M1110.

[1] N. E. Rehler and J. H. Eberly, Superradiance, Phys. Rev. A **3**, 1735 (1971).
[2] K. Hepp and E. H. Lieb, On the superradiant phase transition for molecules in a quantized radiation field: The Dicke maser model, Ann. Phys. **76**, 360 (1973).
[3] F. Hioe, Phase transitions in some generalized Dicke models of superradiance, Phys. Rev. A **8**, 1440 (1973).
[4] C. F. Lee and N. F. Johnson, First-Order Superradiant Phase Transitions in a Multiqubit Cavity System, Phys. Rev. Lett. **93**, 083001 (2004).
[5] Y. Li, Z. D. Wang, and C. P. Sun, Quantum criticality in a generalized Dicke model, Phys. Rev. A **74**, 023815 (2006).
[6] O. Viehmann, J. von Delft, and F. Marquardt, Superradiant Phase Transitions and the Standard Description of Circuit QED, Phys. Rev. Lett. **107**, 113602 (2011).
[7] C. Ciuti and P. Nataf, Comment on "Superradiant Phase Transitions and the Standard Description of Circuit QED", Phys. Rev. Lett. **109**, 179301 (2012).
[8] A. Salah, A. Abdel-Rady, A.-N. A. Osman, and S. S. Hassan, Enhancing quantum phase transitions in the critical point of extended Tc-Dicke model via Stark effect, Sci. Rep. **8**, 1 (2018).
[9] M. Gegg, A. Carmele, A. Knorr, and M. Richter, Superradiant to subradiant phase transition in the open system Dicke model: Dark state cascades, New J. Phys. **20**, 013006 (2018).
[10] M. Bamba, K. Inomata, and Y. Nakamura, Superradiant Phase Transition in a Superconducting Circuit in Thermal Equilibrium, Phys. Rev. Lett. **117**, 173601 (2016).
[11] N. Skribanowitz, I. Herman, J. MacGillivray, and M. Feld, Observation of Dicke Superradiance in Optically Pumped HF Gas, Phys. Rev. Lett. **30**, 309 (1973).
[12] M. Scheibner, T. Schmidt, L. Worschech, A. Forchel, G. Bacher, T. Passow, and D. Hommel, Superradiance of quantum dots, Nat. Phys. **3**, 106 (2007).
[13] M. O. Scully and A. A. Svidzinsky, The super of superradiance, Science **325**, 1510 (2009).
[14] J. Klinder, H. Keßler, M. R. Bakhtiari, M. Thorwart, and A. Hemmerich, Observation of a Superradiant Mott Insulator in the Dicke-Hubbard Model, Phys. Rev. Lett. **115**, 230403 (2015).
[15] L.-P. Yang and Z. Jacob, Quantum critical detector: Amplifying weak signals using discontinuous quantum phase transitions, Opt. Express **27**, 10482 (2019).
[16] L.-P. Yang and Z. Jacob, Engineering first-order quantum phase transitions for weak signal detection, J. Appl. Phys. **126**, 174502 (2019).
[17] M.-J. Hwang and M. B. Plenio, Quantum Phase Transition in the Finite Jaynes-Cummings Lattice Systems, Phys. Rev. Lett. **117**, 123602 (2016).
[18] S. Sachdev, *Quantum Phase Transitions* (Cambridge University press, Cambridge, 2011).
[19] D. De Bernardis, T. Jaako, and P. Rabl, Cavity quantum electrodynamics in the nonperturbative regime, Phys. Rev. A **97**, 043820 (2018).
[20] M.-J. Hwang, R. Puebla, and M. B. Plenio, Quantum Phase Transition and Universal Dynamics in the Rabi Model, Phys. Rev. Lett. **115**, 180404 (2015).
[21] A. Baksic and C. Ciuti, Controlling Discrete and Continuous Symmetries in "Superradiant" Phase Transitions with Circuit QED Systems, Phys. Rev. Lett. **112**, 173601 (2014).
[22] J. Larson and E. K. Irish, Some remarks on 'superradiant' phase transitions in light-matter systems, J. Phys. A: Math. Theor. **50**, 174002 (2017).
[23] C. Gao and Z. Liang, Steady-state phase diagram of quantum gases in a lattice coupled to a membrane, Phys. Rev. A **99**, 013629 (2019).
[24] C. J. Zhu, L. L. Ping, Y. P. Yang, and G. S. Agarwal, Squeezed Light Induced Symmetry Breaking Superradiant Phase Transition, Phys. Rev. Lett. **124**, 073602 (2020).
[25] J. Fan, Z. Yang, Y. Zhang, J. Ma, G. Chen, and S. Jia, Hidden continuous symmetry and nambu-goldstone mode in a two-mode Dicke model, Phys. Rev. A **89**, 023812 (2014).
[26] D. S. Shapiro, W. V. Pogosov, and Y. E. Lozovik, Universal fluctuations and squeezing in a generalized Dicke model near the superradiant phase transition, Phys. Rev. A **102**, 023703 (2020).
[27] D. Braak, Integrability of the Rabi Model, Phys. Rev. Lett. **107**, 100401 (2011).
[28] X.-Y. Lü, G.-L. Zhu, L.-L. Zheng, and Y. Wu, Entanglement and quantum superposition induced by a single photon, Phys. Rev. A **97**, 033807 (2018).
[29] A. Patra, B. L. Altshuler, and E. A. Yuzbashyan, Driven-dissipative dynamics of atomic ensembles in a resonant cavity: Nonequilibrium phase diagram and periodically modulated superradiance, Phys. Rev. A **99**, 033802 (2019).
[30] L. Bakemeier, A. Alvermann, and H. Fehske, Quantum phase transition in the Dicke model with critical and noncritical entanglement, Phys. Rev. A **85**, 043821 (2012).
[31] C. Emary and T. Brandes, Quantum Chaos Triggered by Precursors of a Quantum Phase Transition: The Dicke Model, Phys. Rev. Lett. **90**, 044101 (2003).
[32] M. Hayn, C. Emary, and T. Brandes, Superradiant phase transition in a model of three-level-λ systems interacting with two bosonic modes, Phys. Rev. A **86**, 063822 (2012).

[33] R. N. Bisset, R. M. Wilson, D. Baillie, and P. B. Blakie, Ground-state phase diagram of a dipolar condensate with quantum fluctuations, Phys. Rev. A **94**, 033619 (2016).
[34] J. Fan, L. Yu, G. Chen, and S. Jia, Phase-factor-dependent symmetries and quantum phases in a three-level cavity QED system, Sci. Rep. **6**, 25192 (2016).
[35] G. Chen, J. Li, and J.-Q. Liang, Critical property of the geometric phase in the Dicke model, Phys. Rev. A **74**, 054101 (2006).
[36] F. Plastina, G. Liberti, and A. Carollo, Scaling of Berry's phase close to the Dicke quantum phase transition, EPL (Europhysics Letters) **76**, 182 (2006).
[37] J. Larson, Absence of Vacuum Induced Berry Phases without the Rotating Wave Approximation in Cavity QED, Phys. Rev. Lett. **108**, 033601 (2012).
[38] A. Abdel-Rady, S. S. Hassan, A.-N. A. Osman, and A. Salah, Quantum phase transition and Berry phase of the Dicke model in the presence of the Stark-shift, Int. J. Mod. Phys. B **31**, 1750091 (2017).
[39] C. A. Estrada Guerra, J. Mahecha-Gómez, and J. G. Hirsch, Quantum phase transition and Berry phase in an extended Dicke model, arXiv:2006.06597.
[40] S. Diehl, E. Rico, M. A. Baranov, and P. Zoller, Topology by dissipation in atomic quantum wires, Nat. Phys. **7**, 971 (2011).
[41] M. V. Medvedyeva, T. Prosen, and M. Žnidarič, Influence of dephasing on many-body localization, Phys. Rev. B **93**, 094205 (2016).
[42] H. Flayac and V. Savona, Single photons from dissipation in coupled cavities, Phys. Rev. A **94**, 013815 (2016).
[43] L. Garbe, P. Wade, F. Minganti, N. Shammah, S. Felicetti, and F. Nori, Dissipation-induced bistability in the two-photon Dicke model, Sci. Rep. **10**, 1 (2020).
[44] M. Naderi and M. Soltanolkotabi, Influence of nonlinear quantum dissipation on the dynamical properties of the F-deformed Jaynes-Cummings model in the dispersive limit, Eur. Phys. J. D **39**, 471 (2006).
[45] F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum computation and quantum-state engineering driven by dissipation, Nat. Phys. **5**, 633 (2009).
[46] J. Raftery, D. Sadri, S. Schmidt, H. E. Türeci, and A. A. Houck, Observation of a Dissipation-Induced Classical to Quantum Transition, Phys. Rev. X **4**, 031043 (2014).
[47] A. A. Zhukov, D. S. Shapiro, W. V. Pogosov, and Y. E. Lozovik, Dynamical lamb effect versus dissipation in superconducting quantum circuits, Phys. Rev. A **93**, 063845 (2016).
[48] Z.-J. Ying, L. Cong, and X.-M. Sun, Quantum phase transition and spontaneous symmetry breaking in a nonlinear quantum Rabi model, J. Phys. A: Math. Theor. **53**, 345301 (2020).
[49] D. Walls, Quantum theory of nonlinear optical phenomena, J. Phys. A: Gen. Phys. **4**, 813 (1971).
[50] N. Bogoliubov, R. Bullough, and J. Timonen, Exact solution of generalized tavis-cummings models in quantum optics, J. Phys. A: Math. Gen. **29**, 6305 (1996).
[51] M. Abbasi and M. Golshan, Thermal entanglement of a two-level atom and bimodal photons in a Kerr nonlinear coupler, Physica A **392**, 6161 (2013).
[52] M. Lindberg and C. M. Savage, Steady-state two-level atomic population inversion via a quantized cavity field, Phys. Rev. A **38**, 5182 (1988).
[53] H. Krauter, C. A. Muschik, K. Jensen, W. Wasilewski, J. M. Petersen, J. I. Cirac, and E. S. Polzik, Entanglement Generated by Dissipation and Steady State Entanglement of Two Macroscopic Objects, Phys. Rev. Lett. **107**, 080503 (2011).
[54] M. O. Scully, Collective Lamb Shift in Single Photon Dicke Superradiance, Phys. Rev. Lett. **102**, 143601 (2009).
[55] F. Haake, M. I. Kolobov, C. Fabre, E. Giacobino, and S. Reynaud, Superradiant Laser, Phys. Rev. Lett. **71**, 995 (1993).
[56] J. G. Bohnet, Z. Chen, J. M. Weiner, D. Meiser, M. J. Holland, and J. K. Thompson, A steady-state superradiant laser with less than one intracavity photon, Nature (London) **484**, 78 (2012).
[57] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Long-distance quantum communication with atomic ensembles and linear optics, Nature (London) **414**, 413 (2001).
[58] H. J. Kimble, The quantum internet, Nature (London) **453**, 1023 (2008).
[59] W.-T. Liao and S. Ahrens, Gravitational and relativistic deflection of x-ray superradiance, Nat. Photonics **9**, 169 (2015).
[60] C. Emary and T. Brandes, Chaos and the quantum phase transition in the Dicke model, Phys. Rev. E **67**, 066203 (2003).
[61] J. M. Fink, R. Bianchetti, M. Baur, M. Göppl, L. Steffen, S. Filipp, P. J. Leek, A. Blais, and A. Wallraff, Dressed Collective Qubit States and the Tavis-Cummings Model in Circuit QED, Phys. Rev. Lett. **103**, 083601 (2009).
[62] T. E. Tessier, I. H. Deutsch, A. Delgado, and I. Fuentes-Guridi, Entanglement sharing in the two-atom Tavis-Cummings model, Phys. Rev. A **68**, 062316 (2003).
[63] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1999).
[64] J. Ye and C. L. Zhang, Superradiance, Berry phase, photon phase diffusion, and number squeezed state in the u(1) Dicke (Tavis-Cummings) model, Phys. Rev. A **84**, 023840 (2011).
[65] L. D. Carr, *Understanding Quantum Phase Transitions* (CRC Press, Boca Raton, 2010).
[66] D. Ferrari, G. Celardo, G. P. Berman, R. Sayre, and F. Borgonovi, Quantum biological switch based on superradiance transitions, J. Phys. Chem. C **118**, 20 (2014).