Determining the mixed high-dimensional Bell state of a photon pair through the measurement of a single photon

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Recently, a theoretical method and corresponding experiment have shown that the information about entanglement in the state of the qubit photon pair can be retrieved by measuring only one of the photons, even though the state is mixed. In this paper, we investigate this process with the high-dimensional photon pair and more information about the state. We present a method that enables us to determine the state of the photon pair fully. Our method measures only one of the photons while dropping the other one. Importantly, no adjustment of the transformation is required in our method. Moreover, our method is robust to photon loss so that the state can be perfectly determined as long as the photon is not all lost.

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I. INTRODUCTION

In 1991, Zou, Wang, and Mandel (ZWM) proposed a nonlinear interferometer of which the interference pattern of a photon is controlled by the phase shift of the other undetected photon [1,2]. The ZWM interferometer allows one to estimate the transformations of the undetected photon, which inspires various applications, e.g., quantum imaging [3,4]. Recently, Lahiri *et al.* proposed another application that retrieves the information of entanglement in the photonic qubit pair by measuring only one of the photons and adjusting the transformation of the other undetected photon [5]. The method has been demonstrated with a photonic experiment by the same group [6]. In this paper, we investigate a generalization of that interesting method with high-dimensional photon pairs.

In both theoretical and experimental works [5,6], the authors consider a ZWM interferometer with the photon sources that emit identical qubit-qubit entanglement photon pairs (seeing Fig. 1). The photon pair of each source propagates in two modes. A transformation is applied to a propagation mode of the first source, and then this mode is aligned with one of the second source. Such alignment induces coherence between the other modes of the two sources. Based on this scheme, the authors proposed a method to verify entanglement of the photon pair. The method is adjusting the transformation of the undetected photon and then measuring the other photon.

The method paves an avenue for verifying entanglement without detecting both photons, which has been known to be only possible in special situations, e.g., pure bipartite state [7–10] or the state of qubit and pure-dephasing channel [11,12]. The authors also left two questions: Can the method be generalized to the high-dimensional case [5], and what information other than entanglement can be learned [6]?

In this paper, we consider the scheme of Lahiri *et al.* [5,6], with photon sources that generate the photon pair in the mixed high-dimensional Bell state. In this case, we propose a method that can fully determine the state of the photon pair. The same as the method of Lahiri *et al.* [5,6], our method only detects one photon and drops the photon in the aligned mode, so it does not require coincidence measurement or post-selection. Therefore, this paper provides positive answers to the two questions by Lahiri *et al.* [5,6].

Interestingly, different from the method of Lahiri *et al.* [5,6], our method implements a fixed transformation to the undetected photon, that is, no adjustment of transformation is required. Moreover, we analyze two typical errors in the interferometer: the losses that happened to the photon of the first source and the uncertainty of the implemented transformation. Interestingly, we found that our method is robust to photon loss by perfectly maintaining the ability to determine the state of photon pairs as long as the photon is not all lost.

This article is organized as follows. In Sec. II we analyze the behavior of the optical circuit by Lahiri *et al.* with sources that generate a high-dimensional entanglement state. In Sec. III we demonstrate how and why our method can fully recover the mixed two-particle state with measurement of only one particle. In Sec. IV we analyze the behavior of our method with two typical imperfections. The summarization and discussion is given in Sec. V.

II. ZWM INTERFEROMETER WITH MIXED HIGH-DIMENSIONAL ENTANGLEMENT PHOTON PAIR

Let us consider the two identical d-dimensional entanglement photon sources that generate photon pairs in the mixed d-dimensional Bell state,

$$\rho = \sum_{\mu,\nu}^{0,\ldots,d-1} \mathscr{I}_{\mu\nu}|\mu,\mu\rangle\langle\nu,\nu|, \tag{1}$$

where $\mathscr{I}_{\mu\nu}$ are complex numbers satisfying $\mathscr{I}_{\mu\nu} = \mathscr{I}_{\nu\mu}^*$ and $\sum_\mu \mathscr{I}_{\mu\mu} = 1$. Therefore, such state, which has a specific form of rank d, is parametrized by $d-1$ real numbers $\mathscr{I}_{\mu\nu}$

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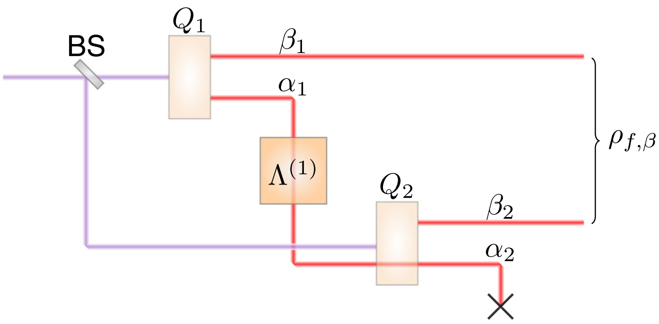

FIG. 1. Schematic of the ZWM interferometer with entanglement photon pairs. Two identical sources, Q_1 and Q_2, individually and never simultaneously emit identical d-dimensional photon pairs with state ρ. The photon pair generated by each source Q_j propagates in two modes α_j and β_j. A transformation $\Lambda^{(1)}$ is applied to mode α_1 of the first source, and then the mode is aligned with mode α_2 of the second source. Finally, the photon in mode α_2 is lost, and measurements (not shown) are only implemented to the photon in modes β_1 and β_2 of the two sources, which is a single-photon $2d$-dimensional system.

for $\mu = \nu = 0, \dots, d-2$ and $d(d-1)/2$ complex numbers $\mathscr{I}_{\mu\nu}$ for $\mu = 1, \dots, d-1$, $\nu = 0, \dots, d-2$, and $\nu < \mu$. Let us consider each source Q_j as an ideal one that converts a pumping photon $a^\dagger_{\mu_j}|0\rangle$ into photon pairs $a^\dagger_{\mu_j,\alpha}a^\dagger_{\mu_j,\beta}|0\rangle$ that propagate in mode α_j and β_j, where $a^\dagger$ is the photon creation operator.

To generate the state as Eq. (1), the pumping photon is in the state,

$$\rho_P = \sum_{\mu,\nu}^{0,\dots,d-1} \mathscr{I}_{\mu\nu} a^\dagger_\mu |0\rangle\langle 0| a_\nu.$$

This pumping photon is first split into two propagation modes by a beam splitter (BS), which works as $a^\dagger_\mu \to \sum_j^{1,2} b_j a^\dagger_{\mu_j}$. In each mode $j = 1, 2$, the corresponding source Q_j converts that photon into photon pairs that propagate in mode α_j and β_j, $a^\dagger_{\mu_j} \to a^\dagger_{\mu_j,\alpha}a^\dagger_{\mu_j,\beta}$.

The photon in mode α_1 generated by the first source Q_1 is transformed by an operation $\Lambda^{(1)}$ and then aligned with the one in mode α_2 of the second source Q_2,

$$a^\dagger_{\mu_1,\alpha} \to \sum_{\omega}^{1,\dots,d,\xi_0,\dots,\xi_e} \Lambda^{(j)}_{\omega\mu} a^\dagger_{\omega_2,\alpha},$$

where $\xi_0, \dots, \xi_e$ denote modes of the lost photon. To simplify the expression, we define an identity operator $\Lambda^{(2)}$ that transforms the photon in mode α_2 generated by the second source, where the identity operator satisfies $\Lambda^{(2)}_{\omega\mu} = 1$ for $\omega = \mu$ and $\Lambda^{(2)}_{\omega\mu} = 0$ for $\omega \neq \mu$.

Taking account of those processes, the final state is

$$\rho_f = \sum_{\mu,\nu}\sum_{j,k} \mathscr{I}^{jk}_{\mu\nu} \sum_{\omega,\chi} \Lambda^{(j)}_{\omega\mu}\Lambda^{(k)*}_{\chi\nu} a^\dagger_{\omega_2,\alpha} a^\dagger_{\mu_j,\beta}|0\rangle\langle 0| a_{\nu_k,\beta} a_{\chi_2,\alpha},$$

where $\mathscr{I}^{jk}_{\mu\nu} = \mathscr{I}_{\mu\nu} b_j b^*_k$. The photon in mode α_2 is undetected, so we are only concerned about the state of the photon in modes β_1 and β_2. After partially tracing out the photon in mode α_2 and using the ket expression $|j, \mu\rangle = a^\dagger_{\mu_j,\beta}|0\rangle$, we obtain the state of photon in modes β_1 and β_2 of the two sources as a single-photon state,

$$\rho_{f,\beta} = \sum_{\mu,\nu}\sum_{j,k} \left(\mathscr{I}^{jk}_{\mu\nu} \sum_\lambda \Lambda^{(j)}_{\lambda\mu}\Lambda^{(k)*}_{\lambda\nu} \right) |j, \mu\rangle\langle k, \nu|,$$

which is a $2d$-dimensional system encoded in hybrid modes (2-dimensional spatial modes $|j\rangle$ and d-dimensional inherent modes $|\mu\rangle$).

In this stage, we consider the transformation $\Lambda^{(1)}$ as a unitary operation of the whole system including the d-dimensional inherent modes of the photon and the loss modes. The unitary transformation $\Lambda^{(1)}$ satisfies

$$\sum_\lambda \Lambda^{(1)}_{\lambda\mu}\Lambda^{(1)*}_{\lambda\nu} = \begin{cases} 1 & \text{for } \mu = \nu, \\ 0 & \text{for } \mu \neq \nu. \end{cases} \tag{2}$$

Therefore, the state of the photon in modes β_1 and β_2 can be simplified as

$$\rho'_{f,\beta} = \sum_\mu \sum_j \mathscr{I}^{jj}_{\mu\mu}|j, \mu\rangle\langle j, \mu| + \left[\sum_{\mu\nu} \left(\mathscr{I}^{12}_{\mu\nu}\Lambda^{(1)}_{\nu\mu}\right)|1, \mu\rangle\langle 2, \nu| + \sum_{\mu\nu} \left(\mathscr{I}^{21}_{\mu\nu}\Lambda^{(1)}_{\mu\nu}\right)|2, \mu\rangle\langle 1, \nu| \right]. \tag{3}$$

The final state of the single photon in modes β_1 and β_2, $\rho_{f,\beta}$ in Eq. (3), depends on both the photon pair state and the transformation through parameters $\mathscr{I}_{\mu\nu}$ and $\Lambda^{(1)}_{\lambda\mu}$. Therefore, through measuring that photon, one is able to estimate the transformation with the known photon pair state, or retrieve the information about the photon pair state with known transformation. The former corresponds to the applications of the ZWM interferometer such as imaging, and the latter corresponds to an application like characterizing mixed state entanglement.

III. DETERMINING THE STATE OF PHOTON PAIR THROUGH MEASURING SINGLE PHOTON

A. Theoretical proposal

In this section, we report our main result to fully determine the high-dimensional photon pair state by measuring the single photon in modes β_1 and β_2 with known transformation. Here, we consider the ideal case that implemented transformation $\Lambda^{(1)}$ is exactly the intended one $\bar{\Lambda}^{(1)}$, $\Lambda^{(1)} = \bar{\Lambda}^{(1)}$, and the transformation $\Lambda^{(1)}$ is lossless, $\Lambda^{(1)}_{l\mu} = 0$ for $\mu = 0, \dots, d-1$ and $l = l_0, \dots, l_e$.

According to Eq. (3), the final state depends on all the parameters $\mathscr{I}_{\mu\nu}$ that determine the state of the photon pair via properly choosing the transformation $\Lambda^{(1)}_{\nu\mu} \neq 0$. Let us assume the state of the photon to be determined is $\tilde{\rho} = \sum_{\mu\nu}^{0,\dots,d-1} \tilde{\mathscr{I}}_{\mu\nu}|\mu, \mu\rangle\langle\nu, \nu|$, where the parameters $\tilde{\mathscr{I}}_{\mu\nu}$ are determined according to the final state $\rho'_{f,\beta}$ and intended transformations $\bar{\Lambda}^{(1)}$. The real parameters are

$$\tilde{\mathscr{I}}_{\mu\mu} = \sum_j \langle j, \mu|\rho'_{f,\beta}|j, \mu\rangle, \tag{4}$$

for $\mu = 0, \ldots, d-1$. The complex parameters are

$$\tilde{\mathscr{I}}_{\mu\nu} = \frac{\langle 1, \mu|\rho'_{f,\beta}|2, \nu\rangle}{\widetilde{b_1 b_2^*}\bar{\Lambda}^{(1)}_{\mu\nu}},$$

for $\mu = 1, \ldots, d-1$, $\nu = 0, \ldots, d-2$, and $\nu < \mu$. Here, the unknown parameter $\widetilde{b_1 b_2^*}$ can be estimated as

$$\widetilde{b_1 b_2^*} = \frac{\langle 1, \mu'|\rho'_{f,\beta}|2, \mu'\rangle}{\tilde{\mathscr{I}}_{\mu'\mu'}\bar{\Lambda}^{(1)}_{\mu'\mu'}}$$

for a μ'. Therefore, we have the complex parameters,

$$\tilde{\mathscr{I}}_{\mu\nu} = \frac{\langle 1, \mu|\rho'_{f,\beta}|2, \nu\rangle\tilde{\mathscr{I}}_{\mu'\mu'}\bar{\Lambda}^{(1)}_{\mu'\mu'}}{\langle 1, \mu'|\rho'_{f,\beta}|2, \mu'\rangle\bar{\Lambda}^{(1)}_{\mu\nu}}. \tag{5}$$

In the ideal case $\bar{\Lambda}^{(1)}_{\mu\nu}$, the parameters $\tilde{\mathscr{I}}_{\mu\nu} = \mathscr{I}_{\mu\nu}$ according to Eqs. (4) and (5), so the state is fully determined, $\tilde{\rho} = \rho$.

B. A possible choice of measurements

Now, let us consider the measurement operators that enable one to obtain the desired information. The estimation process above requires one to know the final state $\rho'_{f,\beta}$ and the transformation $\bar{\Lambda}^{(1)}_{\mu\nu}$. Generally, to determine a $2d$-dimensional state, $\rho'_{f,\beta}$, requires a tomography process which uses $4d^2$ measurements. However, for the specific form of the state with $\langle j, \mu|\rho'_{f,\beta}|j, \nu\rangle = 0$ for $\mu \neq \nu$, the number of measurements can be simplified. We now give a method to determine the state $\tilde{\rho}$ with only $d^2 + 3$ measurements.

First, one applies projective measurements with projectors onto state $|j, \mu\rangle$, for $j = 1, 2$ and $\mu = 0, \ldots, d-1$. The probability of getting the result corresponding to the projector onto the state $|j, \mu\rangle$ is

$$p(j, u) = \langle j, \mu|\rho'_{f,\beta}|j, \mu\rangle = \mathscr{I}_{\mu\mu}|b_j|^2.$$

Since $|b_1|^2 + |b_2|^2 = 1$, the real parameter $|b_1|^2$ can be estimated from two probabilities $p(j, \mu'')$ with $j = 1, 2$ as $|\tilde{b}_1|^2 = p(1, \mu'')/[p(1, \mu'') + p(2, \mu'')]$. Therefore, the real parameters $\mathscr{I}_{\mu\mu}$ can be estimated as

$$\tilde{\mathscr{I}}_{\mu\mu} = \frac{p(1, \mu)}{|\tilde{b}_1|^2} = \frac{p(1, \mu)[p(1, \mu') + p(2, \mu')]}{p(1, \mu')}. \tag{6}$$

This stage requires $d + 1$ measurements with projectors onto d states $|1, \mu\rangle$ for $\mu = 0, \ldots, d-1$ and another one $|2, \mu''\rangle$ for an arbitrary μ''.

After that, one implements two kinds of projective measurements with projectors onto states $(|1, \mu\rangle + |2, \nu\rangle)/\sqrt{2}$ and $(|1, \mu\rangle + i|2, \nu\rangle)/\sqrt{2}$. The corresponding probabilities are

$$q(\mu, \nu) = \mathrm{Re}[\langle 1, \mu|\rho'_{f,\beta}|2, \nu\rangle] + \tfrac{1}{2}[p(1, \mu) + p(2, \nu)],$$

and

$$r(\mu, \nu) = -\mathrm{Im}[\langle 1, \mu|\rho'_{f,\beta}|2, \nu\rangle] + \tfrac{1}{2}[p(1, \mu) + p(2, \nu)],$$

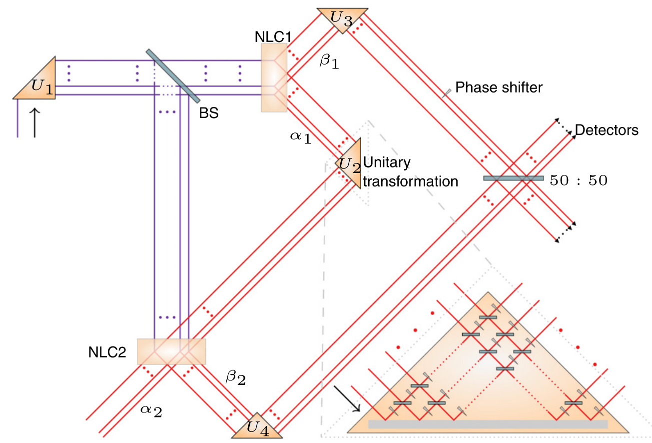

FIG. 2. A possible experimental setup. The pumping source is prepared into the path state through a unitary operation U_1, and then split into two NLCs. The generated photon in mode α_1 is transformed by U_2 and then aligned with the one in mode α_2. The measurements of the final state in modes β_1 and β_2 are completed using unitary operations U_3 and U_4 followed by interference between those two modes at a BS. The inset shows the setup of unitary operation using BS and phase shifters. The optical compensation is not shown here.

respectively. Thus, according to Eq. (5), the complex terms $\mathscr{I}_{\mu\nu}$ can be determined as

$$\tilde{\mathscr{I}}_{\mu\nu} = \frac{\tilde{\mathscr{I}}_{\mu'\mu'}\bar{\Lambda}^{(1)}_{\mu'\mu'}}{\bar{\Lambda}^{(1)}_{\mu\nu}} \times \frac{\left[q(\mu, \nu) - ir(\mu, \nu) - \frac{1-i}{2}[p(1, \mu) + p(2, \nu)]\right]}{\left[q(\mu', \mu') - ir(\mu', \mu') - \frac{1-i}{2}[p(1, \mu') + p(2, \mu')]\right]}. \tag{7}$$

This stage requires $2 + d(d-1)$ measurements, including 2 ones for an arbitrary μ' and $d(d-1)$ ones for $\mu = 1, \ldots, d-1$, $\nu = 0, \ldots, d-2$, and $\nu < \mu$.

It is to be noted that to determine the state of the photon pair, the transformation should be chosen so that $\Lambda^{(1)}_{\mu\nu} \neq 0$ for all $\mu \neq \nu$ and at least one $\mu = \nu = \mu'$. This choice of transformation is not unique. As an example, it can be the Grover diffusion operator [13,14], which is widely used in the quantum-walk-based search algorithm. Here,

$$G = \frac{2}{d}\sum_{\mu\nu}^{0,\ldots,d-1} |\mu\rangle\langle\nu| - \mathbb{1}, \tag{8}$$

where $\mathbb{1}$ is the identity operator; that is, $\Lambda^{(1)}_{\mu\nu}$ equals $\frac{2}{d}$ for $\mu \neq \nu$ and $\frac{2-d}{d}$ for $\mu = \nu$.

C. Possible experimental schemes

To realize our protocol, the high-dimensional photonic system can be chosen as various degrees of freedom of the photon, such as path, transverse spatial modes, and time-frequency bins. Advances of relevant experimental techniques can be found in a recent review [15] and the references therein. Here, we give a possible experimental setup as shown in Fig. 2, which adopts the path degree of freedom.

Two core techniques required by the physical implementation are the entanglement photon source and the unitary operations. In our setup, the path entangled photon pairs are created through the spontaneous parametric down-conversion process in nonlinear crystals (NLC). The d-dimensional unitary transformation of such a path state is realized through the universal scheme using only BSs and phase shifters [16]. Here, we implement four unitary operations, which are U_1 to prepare the state of the pumping source, U_2 to transform the photon in mode α_1, and U_3 and U_4 to project the photon in modes β_1 and β_2, respectively.

The final photon in modes β_1 and β_2 is a $2d$-dimensional single photon that propagates in $2d$ paths, which correspond to states $|j,\mu\rangle$ with $j=1,2$ and $\mu=0,1,\ldots,d-1$. Therefore, detections of the photon in those paths correspond to the measurements with the projector onto states $|j,\mu\rangle$. Our protocol requires other kinds of measurements with projectors onto states $(|1,\mu\rangle+e^{i\phi}|2,\nu\rangle)/\sqrt{2}$. To realize those measurements, one first transforms that state into $|1,0\rangle$ and then detects the photon on the path corresponding to this state. This transformation is realized via two steps. First, two unitary operations U_3 and U_4 are implemented to modes β_1 and β_2, which transform states $|1,\mu\rangle$ and $|2,\nu\rangle$ into $|1,0\rangle$ and $|2,0\rangle$, respectively. After that, a phase shifter is applied to the path corresponding to $|1,0\rangle$, followed by a 50:50 BS that induces interference between two paths corresponding to $|1,0\rangle$ and $|2,0\rangle$.

It is to be noted that the interferometric stability of this setup can be improved by using the interferometer based on calcite beam displacers [17–21] or the integrated optical device [22–25].

IV. IMPERFECTIONS OF TRANSFORMATION

Now, let us analyze the behavior of the method with experimental imperfections of the transformation $\Lambda^{(1)}$ applied to the photon; that is, the implemented transformation $\Lambda^{(1)}$ is different from the desired one $\bar{\Lambda}^{(1)}$. According to Eq. (4), the real terms $\tilde{\mathscr{I}}_{\mu\mu}$ are unaffected. However, according to Eq. (5), the complex terms $\tilde{\mathscr{I}}_{\mu\nu}$ for $\mu\neq\nu$, which depends on both $\bar{\Lambda}^{(1)}_{\mu\nu}$ and $\Lambda^{(1)}_{\mu\nu}$, are affected. More precisely, the complex terms are

$$\tilde{\mathscr{I}}_{\mu\nu}=\frac{\Lambda^{(1)}_{\mu\nu}\bar{\Lambda}^{(1)}_{\mu'\mu'}}{\Lambda^{(1)}_{\mu'\mu'}\bar{\Lambda}^{(1)}_{\mu\nu}}\mathscr{I}_{\mu\nu}, \tag{9}$$

for $\mu\neq\nu$.

To quantify the effect of such difference in complex terms, we compare the reconstructed state $\tilde{\rho}$ with the original one ρ using the distance as

$$D(\tilde{\rho},\rho)=\tfrac{1}{2}\mathrm{Tr}[(\tilde{\rho}-\rho)(\tilde{\rho}-\rho)^{\dagger}].$$

This distance ranges between 0 for a perfect match and 1 for a complete mismatch. Substituting the parameters $\tilde{\mathscr{I}}_{\mu\nu}$ and $\mathscr{I}_{\mu\nu}$, the distance is

$$D(\tilde{\rho},\rho)=\frac{1}{2}\sum_{\mu,\nu\neq\mu}^{0,\ldots,d-1}\left|\frac{\Lambda^{(1)}_{\mu\nu}\bar{\Lambda}^{(1)}_{\mu'\mu'}-\Lambda^{(1)}_{\mu'\mu'}\bar{\Lambda}^{(1)}_{\mu\nu}}{\Lambda^{(1)}_{\mu'\mu'}\bar{\Lambda}^{(1)}_{\mu\nu}}\mathscr{I}_{\mu\nu}\right|^2. \tag{10}$$

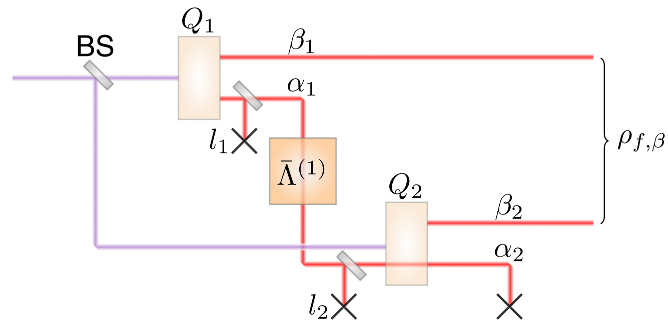

FIG. 3. LWM interferometer with loss. Two BS are placed in mode 1 of source Q_1 before and after transformation.

In this paper, we consider two typical imperfections: photon loss, which happens naturally during photon propagation, and uncertainty of transformation, which is limited by the accuracy of experiment.

A. Photon loss

To analyze the loss of photon in mode α_1 of source Q_1, we consider inserting BS that partially reflects the photon. As shown in Fig. 3, two BS before and after the transformation reflect the photon with rates l_1 and l_2, respectively. Each BS splits the photon $a^{\dagger}_{\mu_1,\alpha}|0\rangle$ into two modes as

$$a^{\dagger}_{\mu_1,\alpha}\rightarrow\sqrt{1-l_b}a^{\dagger}_{\mu_1,\alpha}+e^{i\phi_b}\sqrt{l_b}a^{\dagger}_{\xi_b(\mu),\alpha},$$

where $b=1,2$ denotes two BS, ϕ_b is the relative phase between the reflected photon and transmitted ones, and $\xi_b(\mu)$ denotes the modes of the reflected path. Therefore, different mode loss to different modes, $\xi_b(\mu)\neq\xi_b(\nu)$ for $\mu\neq\nu$, and the loss modes of two BS are different, $\xi_1(\mu)\neq\xi_2(\nu)$ for arbitrary μ and ν.

Since one implements a lossless transformation $\bar{\Lambda}^{(1)}_{\mu\nu}$, the actual transformation $\Lambda^{(1)}_{\mu\nu}$ transforms the photon as

$$\begin{aligned}a^{\dagger}_{\mu_1,\alpha}\rightarrow&\sqrt{1-l_1}\sqrt{1-l_2}\sum_{\omega}^{0,\ldots,d-1}\bar{\Lambda}^{(1)}_{\omega\mu}a_{\omega_2,\alpha}\\&+\sqrt{1-l_1}\sqrt{l_2}\sum_{\omega}\bar{\Lambda}^{(1)}_{\omega\mu}a_{\xi_2(\omega),\alpha}+\sqrt{l_1}a_{\xi_1(\mu),\alpha};\end{aligned}$$

that is, the actual transformation $\Lambda^{(1)}$ is

$$\Lambda^{(1)}_{\omega\mu}=\begin{cases}\sqrt{1-l_1}\sqrt{1-l_2}\bar{\Lambda}^{(1)}_{\omega\mu} & \text{for } \omega=0,\ldots,d-1\\ \sqrt{l_1} & \text{for } \omega=\xi_1(\mu)\\ \sqrt{1-l_1}\sqrt{l_2}\bar{\Lambda}^{(1)}_{\mu'\mu} & \text{for } \omega=\xi_2(\mu') \text{ and}\\ & \quad \mu'=0,\ldots,d-1\\ 0 & \text{otherwise}\end{cases}. \tag{11}$$

This transformation is a unitary operation of the $3d$-dimensional system.

With this transformation, the final state of the photon in modes β_1 and β_2 is

$$\rho''_{f,\beta} = \sum_{\mu}\sum_{j} \mathscr{I}^{jj}_{\mu\mu}|j,\mu\rangle\langle j,\mu| + \sqrt{1-l_1}\sqrt{1-l_2}\left[\sum_{\mu\nu}\left(\mathscr{I}^{12}_{\mu\nu}\bar{\Lambda}^{(1)}_{\nu\mu}\right)|1,\mu\rangle\langle 2,\nu| + \sum_{\mu\nu}\left(\mathscr{I}^{21}_{\mu\nu}\bar{\Lambda}^{(1)}_{\mu\nu}\right)|2,\mu\rangle\langle 1,\nu|\right]. \quad (12)$$

It can be seen that the losses induce decrease of the coherent terms of the state; that is, losses of the undetected photon induce decoherence to the state of modes β_1 and β_2.

Let us consider the task to determine the state of the photon pair. Substituting the transformation in Eq. (11) into Eqs. (4) and (5), it can be seen that $\tilde{\mathscr{I}}_{\mu\nu} = \mathscr{I}_{\mu\nu}$ when $l_1 l_2 \neq 0$. Therefore, our estimation protocol is robust to losses of the undetected photon, since these losses do not affect the estimated parameters. This can also be checked by substituting the transformation in Eq. (11) into the distance in Eq. (10), which gives $D = 0$.

Interestingly, losses of the undetected photon can be intuitively understood as the misalignment of modes α_1 and α_2 of two sources. Therefore, our method is robust to the misalignment, even though the method is based on the alignment.

B. Uncertainty of transformation

Another imperfection is the uncertainty of the transformation, which comes from the accuracy in implementing the transformation. Specifically, we consider that one intended to implement the Grover diffusion operation, $\bar{\Lambda}^{(1)} = G$. Such transformation is an evolution of the photon under Hamiltonian H for time π, $G = e^{-iH\pi}$, where

$$H = \frac{1}{d}\sum_{\mu\nu}|\mu\rangle\langle\nu| - \mathbb{1}. \quad (13)$$

When there is an uncertainty δ_t of evolution time, the actual transformation $\Lambda^{(1)}$ for time $\pi + \delta_t$ becomes

$$G(\delta_t) = e^{-i(H+\delta_h\mathbb{1})(\pi+\delta_t)} = \frac{1}{d}(1+e^{i\delta_t})\sum_{\mu\nu}|\mu\rangle\langle\nu| - e^{i\delta_t}\mathbb{1}. \quad (14)$$

In this case, the distance between the estimated state and the actual one in Eq. (1) becomes

$$D(\delta_t) = \frac{1}{2}\left|\frac{de^{i\delta_t} - d}{2 - 2de^{i\delta_t} + 2e^{i\delta_t}}\right|^2 \sum_{\mu,\nu\neq\nu}^{0,\dots,d-1}|\mathscr{I}_{\mu\nu}|^2. \quad (15)$$

Thus, the estimation is sensitive to the uncertainty of transformation. In Fig. 4, we plot the relation between distance and time uncertainty δ_t for different dimensions. It can be seen that the effect of the time uncertainty on distance decreases for the higher-dimensional system.

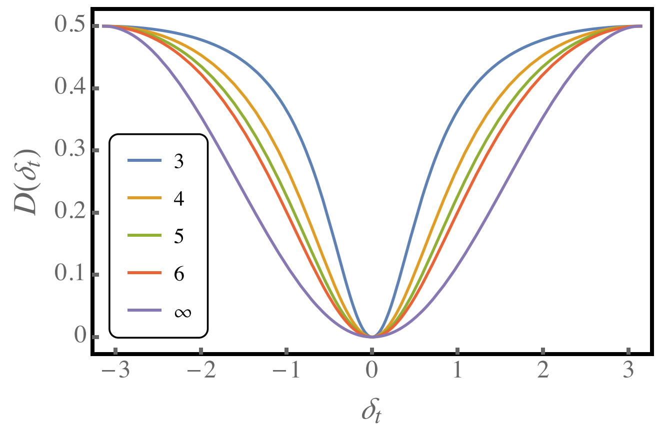

FIG. 4. Relation between distance $D(\delta_t)$ and uncertainty of time δ_t for different dimensions (represented by different colors). Here, we assume $\sum_{\mu,\nu\neq\nu}^{0,\dots,d-1}|\mathscr{I}_{\mu\nu}|^2 = 1$.

V. CONCLUSION AND DISCUSSION

In summary, we have proposed a method to fully determine the mixed state of the high-dimensional photon pair by measuring one of the photons. Our method, as a generalization and modification of the one with the qubit photon pair [5,6], generalizes the applications of the LWM interferometer. Therefore, we solve two concerns by the authors of Refs. [5,6], that we deal with higher dimensional systems and acquire more information than entanglement. Our method can be demonstrated with recent experimental technologies in the generation of high-dimensional entanglement photons [15,17–28].

An important feature of our method is that it does not require adjustment of the implemented transformation. Considering that experimental noises might be introduced during adjustment of transformation, our method avoids the risk of suffering such noise. After analyzing two typical experimental imperfections, we found our method to be robust to the photon loss, while sensitive to the uncertainty of transformation. Interestingly, the misalignment of the modes of two sources equals the photon loss of the first source. Therefore, our method, which is robust to photon loss, does not require perfect alignment.

As proposed in Refs. [5,6], a natural question is to consider the case with a mixed multiphoton entanglement state. For a case that the multiphoton state is $\sum_{\mu,\nu}^{0,\dots,d-1}\mathscr{I}_{\mu\nu}|\mu,\mu,\dots,\mu\rangle\langle\nu,\nu,\dots,\nu|$, the state of two of the photons is ρ that is considered in this paper. Therefore, implementing our method to an arbitrary pair of those photons can fully recover this multiphoton state. For a more complex multiphoton state, it is worth further investigation. Here, we conjecture that our method should help that investigation by considering the multiphoton state as a high-dimensional composite system. For example, a d-dimensional four-photon state can be treated as a state of two d^2-dimensional systems, which include two photons.

The method to characterize the entanglement photon source here is restricted to a particular scenario. It still needs further investigation to determine whether there are practical applications of this method in quantum information technologies. However, the method here to characterize a device without full tomography is a topic of great interest [5,6,29,30].

Note that the high-dimensional system in our method cannot be chosen as photon-number degree of freedom, which is also a high-dimensional system of great interest [30]. Therefore, we do not take account of the error of multiphoton pair generation and non-number-resolving detectors. To adopt our method, one should make sure that the photon sources generate perfect single-photon pairs with very lower multipair production. An interesting topic worth further investigation is to analyze the LWM interferometer with the photon-number state.

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