Single-mode multiphoton polarization states under random Pauli noises

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We investigate behaviors of single-mode multiphoton polarization states under lossless random Pauli noises, where "single-mode" means that all the photons are in the same spatiotemporal mode and cannot be distinguished by any degree of freedom. We characterize the different decoherence effects of multiphoton and single-photon polarization states via the normalized linear entropy. Our results show that, contrary to single-photon states, multiphoton states cannot always evolve to the maximally mixed states. In particular, decoherence-free states and subspaces are found for the random Pauli noises. We reveal that the distinct behaviors result from the bosonic bunching effect and permutation symmetry, and hence our study may be generalized to other photon degrees of freedom and even some other bosonic systems.

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I. INTRODUCTION

Photons not only play a key role in testing the foundations of quantum mechanics [1] but also have wide applications in various quantum technologies [2]. Compared with the single photon, multiphotons have more complicated and striking features [3], and hence have broader applications [4]. Here we focus on a kind of multiphoton states called the single-mode multiphoton polarization states. By saying "single-mode" we mean the multiphotons occupy the same spatiotemporal mode and cannot be distinguished by any degree of freedom. Such states have been extensively explored as qudit states [5]. Since photon polarization is mathematically equivalent to dual path but can enable stable interferometers, such states are often employed in experiment to demonstrate the precision phase measurement beating the standard quantum limit, such as N00N states [6], Holland-Burnett state [7], etc.

Decoherence, induced by system noise or unavoidable coupling between a quantum system and the environment, is a significant obstacle to the development of quantum technologies, especially in quantum communication and computation [8]. In particular, decoherence of multiparticle quantum states has attracted much interest due to the distinct properties in contrast to single-particle states, for instance, the existence of a "decoherence-free (DF) subspace" [9] and entanglement sudden death effect [10]. With regard to the photonic system, since a qubit is usually encoded in an individual photon, most investigations of multiphoton decoherence are focused on multiphotons lying in different path modes or time-bin modes, such as verification of DF subspaces [11–15] and observations of various entanglement dynamics [16,17]. Recently, Shaham and Eisenberg [18] investigated the decoherence effect on the single-mode biphoton polarization states; however, their study was made via splitting two photons into two separate path modes. A comprehensive study on the decoherence effect and the role that indistinguishability plays for the single-mode multiphoton polarization states is still lacking.

In this paper we present a detailed investigation of the decoherence effect on the single-mode multiphoton polarization states. We consider an important class of noisy channels, namely, the lossless Pauli channels, where a photon (or more generally, a qubit) suffers a random Pauli operation with an arbitrary probability and the number of photons is conserved during the dynamics. The family of Pauli channels represents a wide range of noise processes such as bit flip, dephasing, and depolarizing channels [8]. The Pauli channels are one of the most common models in quantum communication [19–21] and quantum error correction [22,23]. Such channels have also been widely applied to study entanglement and correlation dynamics [24–26]. We investigate behaviors of the single-mode multiphoton polarization states under the Pauli channels in comparison with those of single-photon states via the normalized linear entropy. Decoherence-free states and subspaces are found for the random Pauli channels. We reveal that the underlying physical property of the noise-robust feature is the indistinguishability of the photons in a single mode, leading to the bunching effect and permutation symmetry. Similarly, the indistinguishability of particles, including bosons and fermions, has been proven to protect some states from environmental noises [27–29].

The paper is arranged as follows. In Sec. II we give a brief introduction to the Pauli channels. Then in Sec. III we present the investigations of the multiphoton polarization states under the Pauli channels, and give some examples to show our results. Discussions and conclusions are made in Sec. IV.

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II. THE PAULI CHANNELS

A general method to characterize the evolution of a quantum system under decoherence is the operator-sum representation [8] that describes the dynamics of the interested system without having to consider properties of the environment and the interactions. The action of the lossless Pauli channels on the density operator ρ is described as

$$\mathcal{E}(\rho) = \sum_{k=0}^{3} p_k \hat{E}_k \rho \hat{E}_k^\dagger, \tag{1}$$

where $\hat{E}_0$ is the identity operator $\hat{I}$, and $\{\hat{E}_k\}$ $(k = 1, 2, 3)$ are the three Pauli operators $\{\hat{X}, \hat{Y}, \hat{Z}\}$. The four operators act randomly with probabilities p_k, respectively, which satisfy $0 \leqslant p_k \leqslant 1$ and $\sum_k p_k = 1$. Hence we can see that the Pauli channels are trace-preserving maps without including rotations.

Here we consider the Pauli operations to act on the photonic polarization basis states $\{|H\rangle, |V\rangle\}$, where H and V denote the horizontal and vertical polarization, respectively. Explicitly, according to the linear optical evolution representation [30], the action of the Pauli operations on the creation operators can be expressed as

$$\hat{E}_k \hat{a}_j^\dagger |\text{vac}\rangle = \hat{E}_k \hat{a}_j^\dagger \hat{E}_k^\dagger \hat{E}_k |\text{vac}\rangle = \hat{E}_k \hat{a}_j^\dagger \hat{E}_k^\dagger |\text{vac}\rangle, \tag{2}$$

where $j = H,\ V$, and $|\text{vac}\rangle$ represents the vacuum state. Then we obtain the following evolutions:

$$\begin{aligned}
&\hat{X}\hat{a}_H^\dagger \hat{X}|\text{vac}\rangle = \hat{a}_V^\dagger|\text{vac}\rangle,\ \ \hat{X}\hat{a}_V^\dagger \hat{X}|\text{vac}\rangle = \hat{a}_H^\dagger|\text{vac}\rangle,\\
&\hat{Y}\hat{a}_H^\dagger \hat{Y}|\text{vac}\rangle = i\hat{a}_V^\dagger|\text{vac}\rangle,\ \ \hat{Y}\hat{a}_V^\dagger \hat{Y}|\text{vac}\rangle = -i\hat{a}_H^\dagger|\text{vac}\rangle,\\
&\hat{Z}\hat{a}_H^\dagger \hat{Z}|\text{vac}\rangle = \hat{a}_H^\dagger|\text{vac}\rangle,\ \ \hat{Z}\hat{a}_V^\dagger \hat{Z}|\text{vac}\rangle = -\hat{a}_V^\dagger|\text{vac}\rangle.
\end{aligned} \tag{3}$$

III. BEHAVIORS OF THE SINGLE-MODE MULTIPHOTON POLARIZATION STATES IN THE PAULI CHANNELS

An arbitrary single-mode N-photon polarization pure state can be expanded as

$$\begin{aligned}
|\Psi_N\rangle &= \sum_{n=0}^{N} c_n |(N-n)H, nV\rangle\\
&= \sum_{n=0}^{N} \frac{c_n}{\sqrt{(N-n)!n!}} \hat{a}_H^{\dagger N-n} \hat{a}_V^{\dagger n} |\text{vac}\rangle,
\end{aligned} \tag{4}$$

where the coefficients c_n are arbitrary complex numbers satisfying $\sum_{n=0}^{N} |c_n|^2 = 1$. In the following we study the evolution of the state in the Pauli channels. We should note that although the channels given by Eqs. (1) and (3) are defined for single-photon polarization states, since the N photons in a single mode bunch together and are indistinguishable, they inherently suffer the same Pauli operations. Hence, the evolution of the state $|\Psi_N\rangle$ under the Pauli operations can be written as

$$\begin{aligned}
\hat{X}|\Psi_N\rangle &= \sum_{n=0}^{N} \frac{c_n}{\sqrt{(N-n)!n!}} (\hat{X}\hat{a}_H^\dagger \hat{X})^{N-n} (\hat{X}\hat{a}_V^\dagger \hat{X})^n |\text{vac}\rangle\\
&= \sum_{n=0}^{N} c_n |nH, (N-n)V\rangle,
\end{aligned} \tag{5}$$

$$\begin{aligned}
\hat{Y}|\Psi_N\rangle &= \sum_{n=0}^{N} \frac{c_n}{\sqrt{(N-n)!n!}} (\hat{Y}\hat{a}_H^\dagger \hat{Y})^{N-n} (\hat{Y}\hat{a}_V^\dagger \hat{Y})^n |\text{vac}\rangle\\
&= i^N \sum_{n=0}^{N} (-1)^n c_n |nH, (N-n)V\rangle,
\end{aligned} \tag{6}$$

$$\begin{aligned}
\hat{Z}|\Psi_N\rangle &= \sum_{n=0}^{N} \frac{c_n}{\sqrt{(N-n)!n!}} (\hat{Z}\hat{a}_H^\dagger \hat{Z})^{N-n} (\hat{Z}\hat{a}_V^\dagger \hat{Z})^n |\text{vac}\rangle\\
&= \sum_{n=0}^{N} (-1)^n c_n |(N-n)H, nV\rangle.
\end{aligned} \tag{7}$$

Then according to Eq. (1) we can obtain the density matrix of the state after the Pauli channels as

$$\rho' = \sum_{n,m=0}^{N} \rho'_{nm} |(N-n)H, nV\rangle\langle (N-m)H, mV|, \tag{8}$$

with the matrix element given by

$$\begin{aligned}
\rho'_{nm} &= [p_0 + (-1)^{n+m} p_3] c_n c_m^*\\
&\quad + [p_1 + (-1)^{n+m} p_2] c_{N-n} c_{N-m}^*.
\end{aligned} \tag{9}$$

We can see that the trace of the density matrix is preserved under the Pauli noises, and therefore, the number of photons is conserved.

To quantify the mixedness of a density matrix σ after decoherence, we employ the normalized linear entropy (NLE) defined as [31,32]

$$S_L(\sigma) \equiv \frac{D}{D-1}[1 - \text{Tr}(\sigma^2)], \tag{10}$$

where D is the dimension of the system. The NLE is 0 for pure states and 1 for completely mixed states, namely, the normalized D-dimensional identity I/D. Then recalling that a single-mode N-photon polarization state can be seen as an $(N+1)$-dimensional state under the basis of $\{|(N-k)H, kV\rangle\}$, $(k = 0, \ldots, N)$ [5], we can calculate the NLE of the density matrix ρ' as

$$\begin{aligned}
S_L(\rho') &= \frac{N+1}{N}[1 - \text{Tr}(\rho'^2)]\\
&= \frac{N+1}{N}\left(1 - \sum_{n,m=0}^{N} |\rho'_{nm}|^2\right).
\end{aligned} \tag{11}$$

By substituting Eq. (9) into Eq. (11) we can obtain a general expression for the NLE. We can see that for a fixed Pauli channel the NLE varies against photon number and state expression. Note that in the above analysis we have restricted our analysis to the lossless case, in which the number of photons is conserved under the effect of the noise. However, by employing the photon-number postselection technique [33] widely used in photonic quantum information experiments, namely, by only considering those states with all the photons surviving from photon loss, our results can apply to the case with polarization-independent losses.

In the following, we focus on the case of isotropic noise, namely, the depolarizing channel, where

$$p_1 = p_2 = p_3 = \frac{p}{4}, \quad p_0 = 1 - \frac{3p}{4}. \tag{12}$$

Then by substituting the above equation into Eq. (9) we can write the matrix element of the density matrix after the depolarizing channel as

$$\rho'_{d,nm} = (1-p)c_n c_m^* + \frac{p}{4}[1+(-1)^{n+m}](c_n c_m^* + c_{N-n}c_{N-m}^*). \tag{13}$$

Thus from Eq. (11) we can calculate the NLE as

$$\begin{aligned} S_L(\rho'_d) &= \frac{N+1}{2N}p(2-p)\left\{1+\frac{1}{2}\sum_{n,m=0}^{N}\{[1-(-1)^{n+m}]|c_n c_m|^2 - [1+(-1)^{n+m}]\mathrm{Re}(c_n c_m^* c_{N-n}^* c_{N-m})\}\right\} \\ &= \frac{N+1}{2N}p(2-p)\left\{1-\sum_{n=0}^{N}|c_n c_{N-n}|^2 + \sum_{n=0}^{N-1}\sum_{m=n+1}^{N}\{[1-(-1)^{n+m}]|c_n c_m|^2 - [1+(-1)^{n+m}]\mathrm{Re}(c_n c_m^* c_{N-n}^* c_{N-m})\}\right\}. \end{aligned} \tag{14}$$

We can clearly see that different state coefficients or photon numbers may show distinct behaviors under the same channels. In the following, we investigate two extreme cases to show the differences. One case is for the states most affected by the noises which maximize the normalized linear entropy. Such states could be resources for investigating the noisy dynamics. The other case is for the decoherence-free states that are immune to the noises.

A. States that maximize the normalized linear entropy

The first case we consider is the states that maximize the NLE under any fixed depolarizing channel. These states can be found straightforwardly by maximizing the value of the NLE given by Eq. (14). However, when N is large the maximization process may become complicated. Alternatively, since we can see from Eq. (14) that the effects of the state coefficients on the NLE are the same for any nonzero value of p, we can simplify this problem to the extreme case of $p=1$. In this case a possible maximal value of the NLE could reach 1 at the $(N+1)$-dimensional identity $I_{N+1}/(N+1)$. Thus we can solve the state coefficients by letting the density matrix elements equal to those of the identity. Explicitly, making $p=1$ in Eq. (13) we can write the conditions for the nondiagonal elements

$$\rho'_{d,nm}(p=1) = \tfrac{1}{4}[1+(-1)^{n+m}](c_n c_m^* + c_{N-n}c_{N-m}^*) = 0, \tag{15}$$

and the diagonal elements

$$\rho'_{d,nn}(p=1) = \frac{1}{2}(|c_n|^2 + |c_{N-n}|^2) = \frac{1}{N+1}. \tag{16}$$

The solutions can be obtained as follows.

(i) For $0 \leqslant n, m \leqslant N$, $n \neq m$, and $n+m$ is even, the coefficients satisfy

$$c_n c_m^* + c_{N-n}c_{N-m}^* = 0. \tag{17}$$

(ii) For $0 \leqslant n \leqslant N$, the coefficients satisfy

$$|c_n|^2 + |c_{N-n}|^2 = \frac{2}{N+1}. \tag{18}$$

Note that the above two equations may not be satisfied for any photon number N. If the two equations can be satisfied for a value of N, they constitute the necessary and sufficient conditions for the state $|\Psi_N\rangle$ given by Eq. (4) to reach the maximal value of the NLE under a fixed depolarizing channel. Here we should note that the normalization condition is omitted, and so is it in the discussion below. The maximal value of the NLE is given by

$$S_L(\rho'_d)_{\max} = 2p - p^2. \tag{19}$$

The corresponding density matrix is written as

$$\rho'_{d,\max} = (1-p)\rho_{d,\max} + \frac{p}{N+1}I_{N+1}, \tag{20}$$

where $\rho_{d,\max}$ represents the initial state satisfying the conditions of Eqs. (17) and (18). We can see that in this case the behaviors of the multiphoton states are the same for any photon number N irrespective of the dimensions.

If Eqs. (17) and (18) cannot be satisfied for a value of N, the condition has to be found by maximizing the value of the NLE directly. Consequently, in this case the maximally mixed state cannot be reached under the depolarizing channel, which means the robustness of multiphoton states compared to single-photon states.

B. Decoherence-free states

The DF states mean the states that are not disturbed by decoherence [9], which are of importance in encoding strategies to deal with decoherence. Thus in the second case we consider such type of states that are immune to the random Pauli noises. Again we first consider the depolarizing channel. We can find the DF states from the density matrix elements given by Eq. (13) and find the solution for the equation of $\rho'_{d,nm}(p=0) = \rho'_{d,nm}$ regardless of the value of p. Explicitly, this equation can be written into the following two subequations.

(i) If $n+m$ is odd, with $0 \leqslant n, m \leqslant N$, we have

$$c_n c_m^* = (1-p)c_n c_m^*. \tag{21}$$

(ii) If $n+m$ is even, with $0 \leqslant n, m \leqslant N$, including $n=m$, we have

$$c_n c_m^* = (1-p)c_n c_m^* + \frac{p}{2}(c_n c_m^* + c_{N-n}c_{N-m}^*). \tag{22}$$

Then the solution can be expressed as follows.

(i) If $n+m$ is odd, with $0 \leqslant n, m \leqslant N$, the solution is

$$c_n c_m^* = 0. \tag{23}$$

(ii) If $n+m$ is even, with $0 \leqslant n, m \leqslant N$, including $n=m$, the solution is

$$c_n c_m^* = c_{N-n} c_{N-m}^*. \tag{24}$$

The above solution gives the sufficient and necessary conditions for a state $|\Psi_N\rangle$ given by Eq. (4) to be immune to the depolarizing noise. Moreover, we can infer that one necessary condition for a DF state existing is that N should be an even number.

Although the above conditions are obtained for the depolarizing channel, by substituting the conditions into Eq. (9), we can know a state satisfying the conditions is a DF state under any random Pauli noises as well. Consequently, Eqs. (23) and (24) are also the sufficient (but not the necessary) conditions to find a DF state immune to any random Pauli noises.

C. Some examples

We have studied behaviors of single-mode polarization states with arbitrary photon numbers under random Pauli noises and obtained the conditions for two typical cases. In the following we give some examples with some particular photon numbers to show the above results.

1. N = 1

We first consider the case of a single photon, i.e., $N=1$, and thus the initial state is an arbitrary single-photon polarization state given by

$$|\Psi_1\rangle = c_0|H\rangle + c_1|V\rangle. \tag{25}$$

Then according to Eqs. (8) and (13) the density matrix of the state after the depolarizing channel is given by

$$\rho_{d1}' = (1-p)|\Psi_1\rangle\langle\Psi_1| + p\frac{I_2}{2}. \tag{26}$$

From Eq. (14) the NLE can be calculated as

$$S_L(\rho_{d1}') = p(2-p). \tag{27}$$

Thus we obtain the commonly known statement, i.e., the linear entropy increases monotonically as the depolarizing noise increases [32], regardless of the state form. The maximal value 1 is obtained at $p=1$ where the state evolves to the maximally mixed state $I_2/2$. This result is consistent with Eqs. (17) and (18), since they always hold when $N=1$.

2. N = 2

Then we consider the case of $N=2$. The single-mode two-photon polarization state is a qutrit state expressed as

$$|\Psi_2\rangle = c_0|2H\rangle + c_1|H,V\rangle + c_2|2V\rangle. \tag{28}$$

The NLE of the state $|\Psi_2\rangle$ after the depolarizing channel can be calculated from Eq. (14) as

$$S_L(\rho_{d2}') = p(2-p)\big\{1 - \tfrac{1}{4}(3|c_1|^2-1)^2 - 3[\mathrm{Re}(c_0c_2^*)]^2\big\}, \tag{29}$$

where we use a formula that for arbitrary complex numbers,

$$[\mathrm{Re}(ab^*)]^2 = \tfrac{1}{2}[|ab|^2 + \mathrm{Re}(a^2b^{*2})]. \tag{30}$$

The conditions for the state ρ_{d2}' to reach the maximal value of NLE can be obtained by Eqs. (17) and (18) as

$$\mathrm{Re}(c_0c_2^*) = 0, \tag{31}$$

$$|c_1| = 1/\sqrt{3}, \tag{32}$$

with the maximal value of $p(2-p)$. It is straightforward to testify that the conditions coincide with that obtained through maximizing the NLE given by Eq. (29). When $p=1$ the state becomes the maximally mixed state $I_3/3$.

From the conditions given by Eqs. (23) and (24), we can find the following three DF states:

$$|\Psi_{2,\mathrm{DF1}}\rangle = |H,V\rangle, \tag{33}$$

$$|\Psi_{2,\mathrm{DF2}}\rangle = \frac{1}{\sqrt{2}}(|2H\rangle + |2V\rangle), \tag{34}$$

$$|\Psi_{2,\mathrm{DF3}}\rangle = \frac{1}{\sqrt{2}}(|2H\rangle - |2V\rangle). \tag{35}$$

It is clear that the above three states can make $S_L(\rho_{d2}')=0$ regardless of the value of p. As we stated previously, it is straightforward to testify that the above three states are immune to the general random Pauli noises given by Eq. (1). However, the three states cannot constitute a DF subspace since they acquire different global phases under three Pauli operations, which induce an arbitrary coherent superposition of two of them not to be conserved under the random Pauli noises.

To investigate the underlying physical property of the noise-robust feature, as an example, we compare the noisy dynamics of two indistinguishable and distinguishable photons in states $|H,V\rangle$ and $|H\rangle_1|V\rangle_2$, respectively, where the indices 1 and 2 label the two distinguishable photons. According to Eqs. (5)–(7), for the state $|H,V\rangle$, we have $\hat{X}|H,V\rangle = |H,V\rangle$, $\hat{Y}|H,V\rangle = |H,V\rangle$, $\hat{Z}|H,V\rangle = -|H,V\rangle$. It is clear that the state is invariant up to a global phase under Pauli noises. On the other hand, the evolution of state $|H\rangle_1|V\rangle_2$ under Pauli noises can be written as $\hat{X}|H\rangle_1|V\rangle_2 = |V\rangle_1|H\rangle_2$, $\hat{Y}|H\rangle_1|V\rangle_2 = |V\rangle_1|H\rangle_2$, $\hat{Z}|H\rangle_1|V\rangle_2 = -|H\rangle_1|V\rangle_2$. We can see that such state cannot be preserved under $\hat{X}$ and $\hat{Y}$ noises. This comparison shows that the permutation symmetry resulted from the indistinguishability in a single mode leads to the noise-robust feature against Pauli noises.

3. N = 3

For $N=3$, the single-mode three-photon polarization state is expressed as

$$|\Psi_3\rangle = c_0|3H\rangle + c_1|2H,V\rangle + c_2|H,2V\rangle + c_3|3V\rangle. \tag{36}$$

The NLE of the state $|\Psi_3\rangle$ after the depolarizing channel can be obtained via Eq. (14) as

$$S_L(\rho_{d3}') = \tfrac{2}{3}p(2-p)[1 + 2|c_0c_1 - c_2c_3|^2]. \tag{37}$$

Through Eqs. (17) and (18) we can get the condition for maximizing the NLE as

$$c_0c_2^* + c_3c_1^* = 0, \tag{38}$$

$$|c_0|^2 + |c_3|^2 = \tfrac{1}{2}. \tag{39}$$

The conditions can also be obtained by maximizing Eq. (37) directly. The maximal value is $p(2-p)$ and hence the state becomes the maximally mixed state $I_4/4$ when $p=1$.

From Eq. (37) we can see that when $c_0c_1 = c_2c_3$, the NLE reaches the minimum $2p(2-p)/3$, and hence no DF state can be found. This result agrees with the necessary condition we deduced for a DF state existing, namely, N should be an even number.

4. $N = 4$

The single-mode four-photon polarization state is written as

$$|\Psi_4\rangle = c_0|4H\rangle + c_1|3H, V\rangle + c_2|2H, 2V\rangle + c_3|H, 3V\rangle + c_4|4V\rangle. \tag{40}$$

Through Eq. (14), the NLE can be expressed as

$$S_L(\rho'_{d4}) = \tfrac{5}{16}p(2-p)\{3 - (1-2|c_1|^2 - 2|c_3|^2)^2 - 8[\mathrm{Re}(c_1c_3^*)]^2 - 2[2\mathrm{Re}(c_0c_4^*) + |c_2|^2]^2\}, \tag{41}$$

where the formula given in Eq. (30) is employed in the derivation.

By substituting $N = 4$ into Eqs. (17) and (18), the maximizing conditions become

$$c_2^*(c_0 + c_4) = \mathrm{Re}(c_0c_4^*) = \mathrm{Re}(c_1c_3^*) = 0, \tag{42}$$

$$|c_0|^2 + |c_4|^2 = |c_1|^2 + |c_3|^2 = 2|c_2|^2 = \tfrac{2}{5}. \tag{43}$$

With calculations, we can see that these equations cannot hold simultaneously. By maximizing Eq. (41), we can get the following condition for reaching the maximal NLE:

$$|c_1|^2 + |c_3|^2 = \tfrac{1}{2}, \tag{44}$$

$$\mathrm{Re}(c_1c_3^*) = 0, \tag{45}$$

$$2\mathrm{Re}(c_0c_4^*) + |c_2|^2 = 0, \tag{46}$$

with the maximal value of $15p(2-p)/16$.

We can find the following DF states via the condition given by Eqs. (23) and (24):

$$|\Psi_{4,\mathrm{DF1}}\rangle = \alpha\frac{1}{\sqrt{2}}(|4H\rangle + |4V\rangle) + \beta|2H, 2V\rangle, \tag{47}$$

$$|\Psi_{4,\mathrm{DF2}}\rangle = \frac{1}{\sqrt{2}}(|4H\rangle - |4V\rangle), \tag{48}$$

$$|\Psi_{4,\mathrm{DF3}}\rangle = \frac{1}{\sqrt{2}}(|3H, V\rangle + |H, 3V\rangle), \tag{49}$$

$$|\Psi_{4,\mathrm{DF4}}\rangle = \frac{1}{\sqrt{2}}(|3H, V\rangle - |H, 3V\rangle), \tag{50}$$

where α, β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. We can see that the DF states given by Eq. (47) are arbitrary superposition states on two orthogonal basis states $|0\rangle_\mathrm{L}, |1\rangle_\mathrm{L}$ expressed as

$$|0\rangle_\mathrm{L} = \frac{1}{\sqrt{2}}(|4H\rangle + |4V\rangle), \tag{51}$$

$$|1\rangle_\mathrm{L} = |2H, 2V\rangle. \tag{52}$$

The above basis states can build up a two-dimensional DF subspace, and thus a qubit can be encoded into the DF subspace by $\alpha|0\rangle_\mathrm{L} + \beta|1\rangle_\mathrm{L}$. Again we should note that the DF states and subspace also apply for arbitrary random Pauli noises.

5. $N = 5$

The single-mode five-photon polarization state is

$$|\Psi_5\rangle = c_0|5H\rangle + c_1|4H, V\rangle + c_2|3H, 2V\rangle + c_3|2H, 3V\rangle + c_4|H, 4V\rangle + c_5|5V\rangle. \tag{53}$$

Utilizing Eq. (14), we can write the NLE as

$$S_L(\rho'_{d5}) = \tfrac{3}{5}p(2-p)(1 + 2|c_0c_1 - c_4c_5|^2 + 2|c_0c_3 - c_2c_5|^2 + 2|c_1c_2 - c_3c_4|^2). \tag{54}$$

According to Eqs. (17) and (18), we can write the condition for maximizing the NLE as

$$c_0c_2^* + c_5c_3^* = c_0c_4^* + c_5c_1^* = c_1c_3^* + c_4c_2^* = 0, \tag{55}$$

$$|c_0|^2 + |c_5|^2 = |c_1|^2 + |c_4|^2 = |c_2|^2 + |c_3|^2 = \tfrac{1}{3}, \tag{56}$$

with the maximal value of $p(2-p)$. When $p = 1$ the state evolves to the maximally mixed state $I_6/6$.

As we showed previously, for the odd photon numbers no DF state exists. From Eq. (54) we can see the minimum of the NLE is $3p(2-p)/5$.

6. $N = 6$

The single-mode six-photon polarization state is written as

$$|\Psi_6\rangle = c_0|6H\rangle + c_1|5H, V\rangle + c_2|4H, 2V\rangle + c_3|3H, 3V\rangle + c_4|2H, 4V\rangle + c_5|H, 5V\rangle + c_6|6V\rangle. \tag{57}$$

The NLE given by Eq. (14) is written as

$$S_L(\rho'_{d6}) = \tfrac{7}{24}p(2-p)\{3 - (1-2|c_1|^2 - 2|c_3|^2 - 2|c_5|^2)^2 - 8[\mathrm{Re}(c_0c_6^* + c_2c_4^*)]^2 - 2[2\mathrm{Re}(c_1c_5^*) + |c_3|^2]^2\}, \tag{58}$$

where the formula given in Eq. (30) is employed in the derivation.

By substituting $N = 6$ into Eqs. (17) and (18), the maximizing conditions become

$$c_0c_2^* + c_6c_4^* = c_0c_4^* + c_6c_2^* = c_3(c_1^* + c_5^*) = 0, \tag{59}$$

$$\mathrm{Re}(c_0c_6^*) = \mathrm{Re}(c_1c_5^*) = \mathrm{Re}(c_2c_4^*) = 0, \tag{60}$$

$$|c_0|^2 + |c_6|^2 = |c_1|^2 + |c_5|^2 = |c_2|^2 + |c_4|^2 = 2|c_3|^2 = \tfrac{2}{7}. \tag{61}$$

With calculations, we can see that these equations cannot hold simultaneously. Hence, by maximizing Eq. (58), we obtain the following conditions for reaching the maximal value of NLE:

$$|c_1|^2 + |c_3|^2 + |c_5|^2 = \tfrac{1}{2}, \tag{62}$$

$$\mathrm{Re}(c_0c_6^* + c_2c_4^*) = 0, \tag{63}$$

$$2\mathrm{Re}(c_1c_5^*) + |c_3|^2 = 0, \tag{64}$$

with the maximal value of $7p(2-p)/8$.

We can find the following DF states via the conditions given by Eqs. (23) and (24):

$$|\Psi_{6,\mathrm{DF1}}\rangle = \alpha\frac{1}{\sqrt{2}}(|5H,V\rangle + |H,5V\rangle) + \beta|3H,3V\rangle, \quad (65)$$

$$|\Psi_{6,\mathrm{DF2}}\rangle = \frac{1}{\sqrt{2}}(|5H,V\rangle - |H,5V\rangle), \quad (66)$$

$$|\Psi_{6,\mathrm{DF3}}\rangle = \gamma\frac{1}{\sqrt{2}}(|6H\rangle + |6V\rangle) + \delta\frac{1}{\sqrt{2}} \times (|4H,2V\rangle + |2H,4V\rangle), \quad (67)$$

$$|\Psi_{6,\mathrm{DF4}}\rangle = \mu\frac{1}{\sqrt{2}}(|6H\rangle - |6V\rangle) + \nu\frac{1}{\sqrt{2}}(|4H,2V\rangle - |2H,4V\rangle), \quad (68)$$

where α, β, γ, δ, μ, and ν are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, $|\gamma|^2 + |\delta|^2 = 1$, and $|\mu|^2 + |\nu|^2 = 1$. It is clear that three DF subspaces $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ exist which can be respectively spanned by the basis of $\{|0\rangle_{\mathrm{L1}}, |1\rangle_{\mathrm{L1}}\}$, $\{|0\rangle_{\mathrm{L2}}, |1\rangle_{\mathrm{L2}}\}$, and $\{|0\rangle_{\mathrm{L3}}, |1\rangle_{\mathrm{L3}}\}$ expressed as

$$|0\rangle_{\mathrm{L1}} = \frac{1}{\sqrt{2}}(|5H,V\rangle + |H,5V\rangle), \quad (69)$$

$$|1\rangle_{\mathrm{L1}} = |3H,3V\rangle, \quad (70)$$

$$|0\rangle_{\mathrm{L2}} = \frac{1}{\sqrt{2}}(|6H\rangle + |6V\rangle), \quad (71)$$

$$|1\rangle_{\mathrm{L2}} = \frac{1}{\sqrt{2}}(|4H,2V\rangle + |2H,4V\rangle), \quad (72)$$

$$|0\rangle_{\mathrm{L3}} = \frac{1}{\sqrt{2}}(|6H\rangle - |6V\rangle), \quad (73)$$

$$|1\rangle_{\mathrm{L3}} = \frac{1}{\sqrt{2}}(|4H,2V\rangle - |2H,4V\rangle). \quad (74)$$

Thus a qubit can be encoded into any one of the three DF subspaces by $\alpha|0\rangle_{L1} + \beta|1\rangle_{L1}$, $\gamma|0\rangle_{L2} + \delta|1\rangle_{L2}$, and $\mu|0\rangle_{L3} + \nu|1\rangle_{L3}$. Again we should note that the DF states and subspaces also apply for arbitrary random Pauli noises.

IV. CONCLUSIONS AND DISCUSSIONS

We have studied behaviors of single-mode multiphoton polarization states under lossless random Pauli noises. Due to the photon bunching effect, the indistinguishable multiphotons suffer the inherently collective random Pauli noises acting on the polarization degree of freedom, and consequently, the decoherence behaviors of multiphoton states are different from that of single-photon states. The decoherence effects are characterized by calculating the mixedness of the states after the Pauli channels via the NLE. In contrast to the single-photon states, multiphoton states cannot always evolve to the maximally mixed states. More interestingly, we have demonstrated DF states and DF subspaces exist when photon numbers are even, which may find applications in a variety of quantum technologies for overcoming the Pauli noises. As examples, we gave detailed calculations for photon numbers ranging from $N = 1$ to 6, and in particular, we revealed that the smallest DF subspace requires four photons.

From the physical point of view, the distinct decoherence behaviors result from the bunching effect of bosons and permutation symmetry. The collective decoherence with permutation symmetry may lead to DF subspaces in a larger Hilbert space as proved in our results. Compared with the previous studies on the DF states and subspaces [9,11–15] based on the case of several individual qubits (photons) experiencing collective decoherence, our approach employs indistinguishable photons lying in the same spatiotemporal mode which experience decoherence collectively inherently. The indistinguishability of the bosons (photons) being all in the same mode leads to the quantum coherence protection against the Pauli noises. Our results also manifest that the indistinguishability of elementary systems can be a resource for quantum information processing [34]. Hence, our investigations can be generalized to other photon degrees of freedom, for example, orbital angular momentum and also other bosonic systems [35,36]. For instance, the decoherence-free state given in Eq. (33) coincides with the noise-free state given in [28], when two bosons are in the same mode and undergoes a preparation noise leading to a Werner state. We hope our approach can stimulate more such investigations in a broader field.

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