Flavor-vacuum entanglement in boson mixing

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Mixing transformations in quantum field theory are nontrivial, since they are intimately related to the unitary inequivalence between Fock spaces for fields with definite mass and fields with definite flavor. Considering the superposition of two neutral scalar (spin-0) bosonic fields, we investigate some features of the emerging condensate structure of the flavor vacuum. In particular, we quantify the flavor vacuum entanglement in terms of the von Neumann entanglement entropy of the reduced state. Furthermore, in a suitable limit, we show that the flavor vacuum has a structure akin to the thermal vacuum of thermo field dynamics, with a temperature dependent on both the mixing angle and the particle mass difference.

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I. INTRODUCTION

The analysis of quantum correlations in the context of particle physics (and in particular for neutrino and meson systems) is currently gaining more attention. The initial observation that flavor mixing and oscillations can be associated with (single-particle) entanglement [1,2] served as a basis for several studies [3–11] in which violations of Bell, Leggett-Garg, and Mermin-Svetchlichny inequalities, nonlocality, gravity- or acceleration degradation effects and other similar occurrences have been investigated both theoretically and experimentally.

Most of the above studies have been carried out within the framework of quantum mechanics (QM). The extension to quantum field theory (QFT) was later considered in Ref. [12], thus leading to the discovery of nontrivial properties of the mixing transformation [13]. Indeed, while in QM such a transformation acts as a simple rotation between flavor and mass states [14], its QFT counterpart behaves as a rotation nested into a noncommuting Bogoliubov transformation [13,15]. As a result, the vacua for fields with definite mass and fields with definite flavor become orthogonal to each other, the latter acquiring the structure of a SU(2) coherent state [16] and turning into a condensate of massive particle-antiparticle pairs [13,17]. This gives rise to a deeper understanding of particle mixing, since the Fock spaces for flavor and mass fields are found to describe *unitarily inequivalent* representations. Corrections to the standard QM predictions also appear in the oscillation probability formula, as shown in Ref. [18].

Originally developed for neutrinos propagating in flat space-time, the above considerations have been later extended to bosonic fields [19,20] as well as to nontrivial space-time backgrounds (see Refs. [21,22] for more details). Recently, further evidence for the complex structure of the flavor vacuum has been provided in Ref. [23], where it has been established that the Fock space for flavor fields cannot be obtained by the direct product of the spaces for massive fields. Therefore, entanglement connected with flavor mixing appears to be an exquisite concept, boiling down to the nonfactorizability of the flavor states in terms of those with definite masses. In other words, it is possible to come across flavor entanglement already at the level of the vacuum. More generally, the phenomenon of nonvanishing entanglement even for free fields is ultimately related to the quotient space structure of the tensor Hilbert spaces [24].

In fact, the investigation of the properties of the QFT vacuum is one of the most significant (albeit difficult) tasks in a wide variety of physical scenarios. For instance, the Bardeen-Cooper-Schrieffer ground state plays a pivotal role in condensed matter, being a condensate of Cooper pairs which underpins the phenomenon of superconductivity [25]. In the same fashion, vacuum is crucially important to explain the spontaneous symmetry breaking [26] and the ensuing appearance of Nambu-Jona Lasinio [27], Goldstone [28,29], and pseudo-Goldstone bosons [30], both in low- and high-energy regimes. On the other hand, in QFT, vacuum energy is notoriously responsible for the existence of the Casimir effect [31,32], which has been largely studied both in flat [33–37] and curved [38–43] space-time in recent years. Moreover, the study of vacua in the presence of gravity leads to the loss of the absolute concept of particle and the emergence of distinctive phenomena such as the black-hole radiation [44] and the akin Unruh effect [45], which find application in many research areas [46].

Starting from the above premises, in the present work we analyze some relevant and yet unexplored features of the flavor vacuum condensate, with a particular focus on its entanglement structure. For this purpose, we consider the case of mixing of neutral bosonic fields. Besides its intrinsic importance (i.e., as in the case of meson mixing), this choice allows

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us to avoid unnecessary technicalities related to the spin structure, so as to render the physical insight of our study as transparent as possible. We quantify entanglement by computing the von Neumann entropy of the reduced density matrix of vacuum in the limit of small mass difference and/or mixing angle. Specifically, since we are dealing with a pure state of a bipartite system, the von Neumann entropy is the unique consistent measure of bipartite entanglement. At a minimal level, a meaningful quantifier of bipartite entanglement (either pure or mixed) must satisfy three basic conditions: (i) it must vanish on separable states; (ii) it must not increase under local operations and classical communication; and (iii) it must be invariant under local unitary operations. Any quantifier of bipartite entanglement that complies with the above properties is an entanglement monotone. In turn, a *bona fide* entanglement monotone is further promoted to a full entanglement measure if some additional requirements are fulfilled, such as, for instance, the reduction to the von Neumann entropy on pure states. Prominent examples of mixed-state entanglement measures endowed with operational meaning and that reduce to the von Neumann entropy on pure states include the entanglement of formation, the entanglement cost and the squashed entanglement. For further details, the interested reader can consult the comprehensive reviews [47].

As a next step forward, we compare the properties of the flavor vacuum with those of the thermal vacuum of thermo field dynamics (TFD) [48,49], which exhibits the paradigmatic structure occurring in the case of black holes [50] and the Unruh effect [45], namely, the doubling of Fock space and the ensuing correlation between two different sets of modes (the particle states inside and outside the horizon for the Hawking-Unruh effect, the physical and auxiliary modes in TFD). It must be stressed that entanglement in TFD has been previously discussed in Ref. [51] for both equilibrium and nonequilibrium states. From the comparison between our results and those on TFD entanglement, we find that the condensate in the flavor vacuum is in general richer than that in TFD because it exhibits all types of contributions, both thermal and nonthermal. Finally, we determine a suitable limit in which nonthermal contributions are subdominant, thereby allowing us to recognize a TFD-like structure in the flavor vacuum, with an effective temperature proportional to the mixing angle and the mass difference between the two fields.

The paper is organized as follows: in Sec. II we review the main aspects of boson field mixing, focusing on the case of neutral scalar fields. In Sec. III we quantify the entanglement content of the flavor vacuum by computing the reduced von Neumann entropy, and we discuss the condensate structure of this state in connection with the thermal vacuum of TFD. Conclusions and outlook are provided in Sec. IV. Throughout the work, we use natural units $c=\hbar=1$ while keeping the Boltzmann constant k_B explicit.

II. MIXING OF BOSONIC FIELDS

In this section, we review the crucial aspects associated with the mixing of two scalar (spin-0) neutral fields. Clearly, the same considerations can be extended to the case of three boson generations. Toward this end, we closely follow Refs. [19,20] and write down the mixing relations between fields with definite mass and flavor; that is,[1]

$$\phi_A(x) = \cos\theta\,\phi_1(x) + \sin\theta\,\phi_2(x), \tag{1}$$

$$\phi_B(x) = -\sin\theta\,\phi_1(x) + \cos\theta\,\phi_2(x), \tag{2}$$

with an analogous set of equations for the conjugate momenta $\pi(x) = \partial_t\phi(x)$. The subscripts A and B indicate the fields in the flavor basis, whereas 1 and 2 indicate those in the mass basis. Consequently, the expansions for ϕ_1 and ϕ_2 take the form

$$\phi_j(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{k,j}}}\left(a_{k,j}e^{-i\omega_{k,j}t} + a^\dagger_{-k,j}e^{i\omega_{k,j}t}\right)e^{i\mathbf{k}\cdot\mathbf{x}},\quad j=1,2, \tag{3}$$

where $\omega_{k,j} = (\mathbf{k}^2 + m_j^2)^{1/2}$ and $a_{k,j}$ ($a^\dagger_{k,j}$) are the bosonic annihilation (creation) operators of field quanta with momentum k and mass m_j.

By requiring that the fields and the conjugate momenta obey the canonical commutation relation (CCR) at equal times,

$$[\phi_i(x), \pi_j(x')]_{t=t'} = i\delta_{ij}\delta(\mathbf{x}-\mathbf{x}'), \tag{4}$$

it follows that the only nontrivial commutator between the ladder operators is

$$[a_{k,i}, a^\dagger_{k',j}] = \delta_{ij}\delta(\mathbf{k}-\mathbf{k}'). \tag{5}$$

Let us now observe that Eqs. (1) and (2) and those for momenta can also be rewritten as

$$\phi_\sigma(x) = G_\theta^{-1}(t)\phi_j(x)G_\theta(t), \tag{6}$$

$$\pi_\sigma(x) = G_\theta^{-1}(t)\pi_j(x)G_\theta(t), \tag{7}$$

where $(\sigma, j) = \{(A,1),(B,2)\}$ and

$$G_\theta(t) = \exp\{\theta[S_+(t) - S_-(t)]\} \tag{8}$$

is the mixing generator and an element of SU(2), whose algebra is built with the following operators:

$$S_+(t) = -i\int d^3x\,\pi_1(x)\phi_2(x), \tag{9}$$

$$S_-(t) = -i\int d^3x\,\pi_2(x)\phi_1(x), \tag{10}$$

$$S_3(t) = -\frac{i}{2}\int d^3x[\pi_1(x)\phi_1(x) - \pi_2(x)\phi_2(x)]. \tag{11}$$

In light of the above equations, the flavor fields can then be expressed as

$$\phi_\sigma(x) = \int \frac{d^3k}{\sqrt{2(2\pi)^3\omega_{k,j}}}\left(a_{k,\sigma}(t)e^{-i\omega_{k,j}t} + a^\dagger_{-k,\sigma}(t)e^{i\omega_{k,j}t}\right)e^{i\mathbf{k}\mathbf{x}},\quad \sigma = A, B, \tag{12}$$

[1] Strictly speaking, in the case of bosons we should refer to the mixing of some other quantum number, such as the strangeness or the isospin, rather than the flavor. However, with an abuse of notation, in the following we keep on denoting such intrinsic properties as "flavor" and the corresponding mixed fields as "flavor fields." Furthermore, we work in a simplified two-flavor model.

and, with the aid of Eqs. (1) and (2), we recognize the Bogoliubov transformation between the flavor and mass ladder operators

$$a_{k,A}(t) = \cos\theta a_{k,1} + \sin\theta[U_k^*(t)a_{k,2} + V_k(t)a_{-k,2}^\dagger], \tag{13}$$

$$a_{k,B}(t) = \cos\theta a_{k,2} - \sin\theta[U_k(t)a_{k,1} - V_k(t)a_{-k,1}^\dagger]. \tag{14}$$

The above relations exhibit the structure of rotations nested into Bogoliubov transformations with coefficients $U_k(t)$ and $V_k(t)$ given by

$$U_k(t) = |U_k|e^{i(\omega_{k,2}-\omega_{k,1})t}, \quad V_k(t) = |V_k|e^{i(\omega_{k,1}+\omega_{k,2})t}, \tag{15}$$

$$|U_k| = \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} + \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}}\right),$$
$$|V_k| = \frac{1}{2}\left(\sqrt{\frac{\omega_{k,1}}{\omega_{k,2}}} - \sqrt{\frac{\omega_{k,2}}{\omega_{k,1}}}\right). \tag{16}$$

Accordingly, the flavor vacuum is provided with an SU(2) coherent-state structure [16]

$$|00(t)\rangle_{A,B} = G_\theta^{-1}(t)|00\rangle_{1,2}, \tag{17}$$

with a condensation density given by

$${}_{A,B}\langle 00(t)|a_{k,j}^\dagger a_{k,j}|00(t)\rangle_{A,B} = \sin^2\theta|V_k|^2, \quad j = 1, 2. \tag{18}$$

In the infinite-volume limit [19,20], the two sets of vacua become orthogonal, namely, ${}_{1,2}\langle 00|00(t)\rangle_{A,B} \stackrel{V\to\infty}{\longrightarrow} 0 \ \forall\, t$, giving rise to physically inequivalent Fock spaces (i.e., unitarily inequivalent representations of the canonical commutation relations for fields).

Let us now cast the mixing generator in terms of mass-definite annihilators and creators. Straightforward calculations lead to

$$|00(t)\rangle_{A,B} = \exp\left\{-\theta\int d^3k[U_k^*(t)a_{k,1}^\dagger a_{k,2} - U_k(t)a_{k,1}a_{k,2}^\dagger + V_k(t)a_{k,1}^\dagger a_{-k,2}^\dagger - V_k^*(t)a_{k,1}a_{-k,2}]\right\}|00\rangle_{1,2}. \tag{19}$$

Without harming the generality of our results, henceforth we perform calculations for $t = 0$. Thus, we have $|00(t = 0)\rangle_{A,B} \equiv |00\rangle_{A,B}$ with[2]

$$|00\rangle_{A,B} = \exp\left\{-\theta\int d^3k[U_k(a_{k,1}^\dagger a_{k,2} - a_{k,1}a_{k,2}^\dagger) + V_k(a_{k,1}^\dagger a_{-k,2}^\dagger - a_{k,1}a_{-k,2})]\right\}|00\rangle_{1,2}. \tag{20}$$

Equation (20) allows us to immediately identify the generator of the rotation (the operator in the brackets which multiplies U_k) and the one responsible for the Bogoliubov transformation (the operator in the brackets which multiplies V_k), in complete agreement with the fermion case [15].

For later convenience, we can now further manipulate Eq. (20) by considering the case of a discrete set of modes.

[2]Notice that, for $t = 0$, $U_k(t) = |U_k|$ and $V_k(t) = |V_k|$ [see Eq. (15)].

Apart from an irrelevant numerical factor, Eq. (20) becomes

$$|00\rangle_{A,B} = \exp\left\{-\theta\sum_k[U_k(a_k^\dagger b_k - a_k b_k^\dagger) + V_k(a_k^\dagger b_{-k}^\dagger - a_k b_{-k})]\right\}|00\rangle_{1,2}. \tag{21}$$

where we have introduced the shorthand notation $a_{k,1} \equiv a_k$ and $a_{k,2} \equiv b_k$. In addition, since the transformations (1) and (2) are valid for any value of the rotation angle, we can focus on the case of small values of θ to the $O(\theta^2)$ order, at which the first nontrivial contribution is expected. In this framework, we can use Zassenhaus formula

$$e^{\lambda(X+Y)} = e^{\lambda X}e^{\lambda Y}e^{-\frac{\lambda^2}{2}[X,Y]}, \tag{22}$$

up to $O(\lambda^2)$ to approximate the flavor vacuum in Eq. (21). In fact, if we identify $\lambda \to -\theta$, $X \to \sum_k U_k(a_k^\dagger b_k - a_k b_k^\dagger)$ and $Y \to \sum_k V_k(a_k^\dagger b_{-k}^\dagger - a_k b_{-k})$, we are led to

$$\begin{aligned}|00\rangle_{A,B} &= e^{-\theta\sum_k U_k(a_k^\dagger b_k - a_k b_k^\dagger)}e^{-\theta\sum_k V_k(a_k^\dagger b_{-k}^\dagger - a_k b_{-k})} \\ &\quad\times e^{-\frac{\theta^2}{2}\sum_k U_k V_k(a_k^\dagger a_{-k}^\dagger - b_k^\dagger b_{-k}^\dagger - a_k a_{-k} + b_k b_{-k})}|00\rangle_{1,2} \\ &= e^{-\frac{\theta^2}{2}\sum_k U_k V_k(a_k^\dagger a_{-k}^\dagger - b_k^\dagger b_{-k}^\dagger)}e^{-\theta\sum_k U_k(a_k^\dagger b_k - a_k b_k^\dagger)} \\ &\quad\times e^{-\theta\sum_k V_k(a_k^\dagger b_{-k}^\dagger - a_k b_{-k})}|00\rangle_{1,2},\end{aligned} \tag{23}$$

where in the second step we have made use of a simplification allowed only in the given approximation for θ. Furthermore, by applying the identity (22) to the two operators in Eq. (23) that depend on U_k and V_k, we get

$$\begin{aligned}|00\rangle_{A,B} &= e^{-\frac{\theta^2}{2}\sum_k U_k V_k(a_k^\dagger a_{-k}^\dagger - b_k^\dagger b_{-k}^\dagger)}e^{-\theta\sum_k U_k a_k^\dagger b_k}e^{\theta\sum_k U_k a_k b_k^\dagger} \\ &\quad\times e^{\frac{\theta^2}{2}\sum_k U_k^2(a_k a_k^\dagger - b_k b_k^\dagger)} \\ &\quad\times e^{-\theta\sum_k V_k a_k^\dagger b_{-k}^\dagger}e^{\theta\sum_k V_k a_k b_{-k}}e^{-\frac{\theta^2}{2}\sum_k V_k^2(a_k a_k^\dagger + b_k^\dagger b_k)}|00\rangle_{1,2} \\ &\simeq e^{-\frac{\theta^2}{2}\sum_k U_k V_k(a_k^\dagger a_{-k}^\dagger - b_k^\dagger b_{-k}^\dagger)}e^{-\frac{\theta^2}{2}\sum_k\left(V_k^2 b_k^\dagger b_k + U_k^2 b_k b_k^\dagger\right)} \\ &\quad\times e^{-\theta\sum_k V_k a_k^\dagger b_{-k}^\dagger}|00\rangle_{1,2}.\end{aligned} \tag{24}$$

In performing the second passage, we have omitted an unimportant constant factor and we have made use of the current approximation to streamline the shape of the total operator. We take advantage of the form (24) of the flavor vacuum in the next section when we compare the condensate structure of this state to the one of the thermal vacuum of thermo field dynamics.

III. ENTANGLEMENT OF THE FLAVOR VACUUM

Let us now quantify the entanglement between the massive particle states in the flavor vacuum. As said earlier, we consider the case $t = 0$. For notational convenience, it comes in handy to rewrite Eq. (20) as

$$|00\rangle_{A,B} \equiv |\psi\rangle = \exp\left\{-\theta\int d^3k[U_k(a_k^\dagger b_k - a_k b_k^\dagger) + V_k(a_k^\dagger b_{-k}^\dagger - a_k b_{-k})]\right\}|00\rangle, \tag{25}$$

where $|00\rangle \equiv |00\rangle_{1,2}$ and we have used the same notation as in Eq. (21) for ladder operators. Interestingly, we observe that the above transformation is the result of a simultaneous coexistence of a beam splitter and a two-mode squeezing transformation.

As a preliminary analysis, we can perform a first-order approximation in θ and in the case of small mass difference, namely, when $m_1 = m$, $m_2 = m + \delta m$ and hence $\varepsilon \equiv (m_2 - m_1)/m_1 = \delta m/m \ll 1$. This is in line with many works regarding flavor mixing, and in particular with Ref. [15], in which the neutrino flavor vacuum is written up to second order in θ and ε. Here, we make the same considerations with the purpose of seeking an analogous result.

Starting from Eq. (25), it is immediate to derive that

$$|\psi\rangle = \exp\left\{ - \theta \int d^3k \left[(a_k^\dagger b_k - a_k b_k^\dagger) - \frac{\varepsilon}{2}\frac{m^2}{\omega_k^2}(a_k^\dagger b_{-k}^\dagger - a_k b_{-k}) \right]\right\}|00\rangle, \tag{26}$$

and thus expand the exponential operator as

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{n!}\left\{ - \theta \int d^3k [(a_k^\dagger b_k - a_k b_k^\dagger) - \frac{\varepsilon}{2}\frac{m^2}{\omega_k^2}(a_k^\dagger b_{-k}^\dagger - a_k b_{-k})] \right\}^n |00\rangle. \tag{27}$$

Since terms of the order $O(\varepsilon^2)$ are neglected, we observe that it is possible to identify two recursive formulas in the expression (27), one in front of the operator $a^\dagger b^\dagger$ and another one next to $(a^\dagger a^\dagger - b^\dagger b^\dagger)$. The exact computation yields

$$|\psi\rangle = \left\{ \mathbf{1} + \frac{\varepsilon}{2}\left[\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\theta^{2n+2}4^n}{(2n+2)!} \right] \times \int d^3k \frac{m^2}{\omega_k^2}[a_k^\dagger a_{-k}^\dagger - b_k^\dagger b_{-k}^\dagger] + \frac{\varepsilon}{2}\left[\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}4^n}{(2n+1)!} \right] \times \int d^3k \frac{m^2}{\omega_k^2} a_k^\dagger b_{-k}^\dagger \right\}|00\rangle. \tag{28}$$

It is straightforward to realize that the series in Eq. (28) converge to two simple analytic functions:

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\theta^{2n+2}4^n}{(2n+2)!} = -\frac{\sin^2\theta}{2}, \tag{29}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n}\theta^{2n+1}4^n}{(2n+1)!} = \frac{\sin 2\theta}{2}. \tag{30}$$

As a result, in the limit of small mass difference the flavor vacuum takes the form

$$|\psi\rangle = \left\{ \mathbf{1} - \frac{\varepsilon}{4}\sin^2\theta \int d^3k \frac{m^2}{\omega_k^2}[a_k^\dagger a_{-k}^\dagger - b_k^\dagger b_{-k}^\dagger] + \frac{\varepsilon}{4}\sin 2\theta \int d^3k \frac{m^2}{\omega_k^2} a_k^\dagger b_{-k}^\dagger \right\}|00\rangle. \tag{31}$$

Before proceeding with the computation of von Neumann entropy, we pause to compare the above form of the flavor vacuum with the thermal vacuum of thermo field dynamics defined in the Appendix [see in particular Eq. (A9)]. The similarity between these two states has its roots in the fact that, in TFD, there is the need to double the physical Hilbert space by introducing a dual space (and hence an auxiliary field) whose excitations are holes from the point of view of the physical field (a detailed mathematical explanation of these notions can be found in the Appendix). As a consequence, the thermal vacuum appears as a condensate of excitations of the physical and auxiliary fields, which thus resembles the structure of the flavor vacuum being a condensate of particle-antiparticle pairs with different masses.

In this vein, we emphasize that a first attempt to perform a comparison between the flavor and thermal vacua was carried out in Ref. [52] and later in Ref. [15], arguing that the two states cannot be exactly matched, since the would-be entropy operator defined for mixed fields does not possess the same properties as the one introduced in TFD [see Eq. (A11)]. However, here we tackle this problem from a different perspective; indeed, we directly look at the inherent structure of the two vacua. Even though the result of Ref. [15] remains in general valid since the thermal state (A9) does not contain terms of the form $a_k^\dagger a_{-k}^\dagger$, $b_k^\dagger b_{-k}^\dagger$ which instead appear in the flavor vacuum (31), for small values of the mixing angle we can approximate Eq. (31) to leading order as

$$|\psi\rangle = \left\{ \mathbf{1} + \frac{\varepsilon}{2}\theta \int d^3k \frac{m^2}{\omega_k^2} a_k^\dagger b_{-k}^\dagger \right\}|00\rangle. \tag{32}$$

Then, from comparison with the TFD vacuum in Eq. (A13),

$$|0(\vartheta)\rangle \simeq \mathbf{1} + \sum_k a_k^\dagger \tilde{a}_k^\dagger \vartheta_k |0\rangle,$$

one can recognize in the state (32) a thermal-like vacuum with an inverse temperature $\beta = (k_B T)^{-1}$ given by [up to a scaling factor due to the conversion of the k integral into a discrete sum, as viewed in Eq. (21) and following]

$$\beta \simeq \frac{1}{\omega_k} \ln\left(\frac{2\omega_k^2}{\varepsilon^2} \right). \tag{33}$$

Clearly, for θ and/or $\varepsilon = 0$, we have $\beta \to \infty$, or equivalently $T \to 0$. This is somehow expected since, for vanishing mixing, the flavor and mass vacua coincide with each other, which corresponds in TFD language to the case where the doubling of the degrees of freedom (and thus the temperature) disappears.

Let us now come back to the quantification of von Neumann entropy of the flavor vacuum. To this aim, we focus on the $k = 0$ mode, for which the condensation density reaches its maximal value in the case of boson mixing [see Eq. (18)] [20]. Therefore, we expect that the effects of the mixing transformation are maximally nontrivial in this case. However, for this purpose we need to go beyond the linear-order approximation in the mass difference, as it can be easily shown that the linear term in ε gives a vanishing contribution. Then, by resorting to Eq. (27) and retaining only the factors that do not exceed $O(\varepsilon^2)$, we find a recurrence series of the form

$$|\psi\rangle = \left\{\mathbf{1} + \left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{4}\right)\left[\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}4^n}{(2n+1)!}\right]a^\dagger b^\dagger + \left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{4}\right)\left[\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\theta^{2n+2}4^n}{(2n+2)!}\right][(a^\dagger)^2 - (b^\dagger)^2] \right.$$
$$+ \frac{\varepsilon^2}{4}\left[\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+2}4^n}{(2n+2)!}\right][(a^\dagger)^2(b^\dagger)^2 - \mathbf{1}] + \frac{3\varepsilon^2}{4}\left\{\sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}\theta^{2n+3}4^n}{(2n+3)!}\sum_{m=0}^{n}(3\sqrt{2}+\sqrt{6})^m\right]\right\}[(a^\dagger)^3 b^\dagger - a^\dagger (b^\dagger)^3]$$
$$\left. + \frac{3\varepsilon^2}{4}\left\{\sum_{n=0}^{\infty}\left[\frac{(-1)^n\theta^{2n+4}4^n}{(2n+4)!}\sum_{m=0}^{n}(3\sqrt{2}+\sqrt{6})^m\right]\right\}[(a^\dagger)^4 - 6\sqrt{2}(a^\dagger)^2(b^\dagger)^2 + (b^\dagger)^4]\right\}|00\rangle. \tag{34}$$

The series appearing in Eq. (34) converge to trigonometric functions in θ. More precisely, we have

$$|\psi\rangle = \left[1 - \frac{\varepsilon^2}{8}\sin^2\theta\right]|00\rangle - \left(\frac{\varepsilon}{4} - \frac{\varepsilon^2}{8}\right)\sin^2\theta(|20\rangle - |02\rangle) + \left(\frac{\varepsilon}{4} - \frac{\varepsilon^2}{8}\right)\sin 2\theta|11\rangle$$
$$+ \frac{\varepsilon^2}{4}\left[\frac{\sin^2\theta}{2} - 18\sqrt{2}\chi(\theta)\right]|22\rangle + \frac{3\varepsilon^2}{4}\chi(\theta)(|40\rangle + |04\rangle) + \frac{3\varepsilon^2}{4}\xi(\theta)(|31\rangle - |13\rangle), \tag{35}$$

where

$$\chi(\theta) = \frac{(6+2\sqrt{3})(1-\cos 2\theta) - \sqrt{2}\left[1 - \cos\left(2^{\frac{5}{4}}\sqrt{3+\sqrt{3}}\theta\right)\right]}{32(3+\sqrt{3})(3\sqrt{2}+\sqrt{6}-1)}, \tag{36}$$

$$\xi(\theta) = \frac{2^{\frac{3}{4}}\sin\left(2^{\frac{5}{4}}\sqrt{3+\sqrt{3}}\theta\right) - 2\sqrt{3+\sqrt{3}}\sin 2\theta}{16\sqrt{3+\sqrt{3}}(3\sqrt{2}+\sqrt{6}-1)}. \tag{37}$$

The state in Eq. (35) is not normalized, since $\langle\psi|\psi\rangle \neq 1$. For this reason, before computing the projector $\rho = |\psi\rangle\langle\psi|$, we must divide the above state by a suitable normalization factor, which is explicitly given by

$$N = \sqrt{\langle\psi|\psi\rangle} = \sqrt{1 - \frac{\varepsilon^2}{4}\left[\sin^2\theta - \frac{\sin^4\theta}{2} - \frac{\sin^2 2\theta}{4}\right]}. \tag{38}$$

Accordingly, the normalized density matrix reads

$$\rho = \frac{|\psi\rangle\langle\psi|}{N^2} = \left[1 - \frac{\varepsilon^2}{8}\sin^4\theta - \frac{\varepsilon^2}{16}\sin^2 2\theta\right]|00\rangle\langle 00|$$
$$+ \left(\frac{\varepsilon}{4} - \frac{\varepsilon^2}{8}\right)\sin 2\theta(|00\rangle\langle 11| + |11\rangle\langle 00|)$$
$$- \left(\frac{\varepsilon}{4} - \frac{\varepsilon^2}{8}\right)\sin^2\theta(|00\rangle\langle 20| - |00\rangle\langle 02| + |20\rangle\langle 00| - |02\rangle\langle 00|) + \frac{\varepsilon^2}{16}\sin^2 2\theta|11\rangle\langle 11|$$
$$+ \frac{\varepsilon^2}{4}\left[\frac{\sin^2\theta}{2} - 18\sqrt{2}\chi(\theta)\right](|00\rangle\langle 22| + |22\rangle\langle 00|)$$
$$+ \frac{3\varepsilon^2}{4}\chi(\theta)(|00\rangle\langle 40| + |00\rangle\langle 04| + |40\rangle\langle 00| + |04\rangle\langle 00|)$$
$$+ \frac{\varepsilon^2}{16}\sin^4\theta(|20\rangle\langle 20| - |20\rangle\langle 02| - |02\rangle\langle 20| + |02\rangle\langle 02|)$$
$$+ \frac{3\varepsilon^2}{4}\xi(\theta)(|00\rangle\langle 31| - |00\rangle\langle 13| + |31\rangle\langle 00| - |13\rangle\langle 00|)$$
$$- \frac{\varepsilon^2}{16}\sin^2\theta\sin 2\theta(|20\rangle\langle 11| - |02\rangle\langle 11| + |11\rangle\langle 20| - |11\rangle\langle 02|). \tag{39}$$

By partial tracing ρ with respect to either one of the two subsystems, say $\mathcal{S}_1$, we obtain the reduced density matrix

$$\rho_r^{(2)} = \mathrm{Tr}_{\mathcal{S}_1}\rho = \left[1 - \frac{\varepsilon^2}{16}\sin^2 2\theta - \frac{\varepsilon^2}{16}\sin^4\theta\right]|0\rangle\langle 0| + \left(\frac{\varepsilon}{4} - \frac{\varepsilon^2}{8}\right)\sin^2\theta(|0\rangle\langle 2| + |2\rangle\langle 0|) + \frac{\varepsilon^2}{16}\sin^2 2\theta|1\rangle\langle 1|$$
$$+ \frac{3\varepsilon^2}{4}\chi(\theta)(|0\rangle\langle 4| + |4\rangle\langle 0|) + \frac{\varepsilon^2}{16}\sin^4\theta|2\rangle\langle 2|, \tag{40}$$

with the same result for $\rho_r^{(1)} = \mathrm{Tr}_{S_2}\rho$. From Eq. (40), we can compute the von Neumann entropy up to $O(\varepsilon^2)$. Notice that the matrix $\rho_r^{(1)} = \rho_r^{(2)} \equiv \rho_r$, given by

$$\rho_r = \begin{pmatrix} 1 - \frac{\varepsilon^2}{16}\sin^2 2\theta - \frac{\varepsilon^2}{16}\sin^4\theta & 0 & \left(\frac{\varepsilon}{4} - \frac{\varepsilon^2}{8}\right)\sin^2\theta & \frac{3\varepsilon^2}{4}\chi(\theta) \\ 0 & \frac{\varepsilon^2}{16}\sin^2 2\theta & 0 & 0 \\ \left(\frac{\varepsilon}{4} - \frac{\varepsilon^2}{8}\right)\sin^2\theta & 0 & \frac{\varepsilon^2}{16}\sin^4\theta & 0 \\ \frac{3\varepsilon^2}{4}\chi(\theta) & 0 & 0 & 0 \end{pmatrix}, \tag{41}$$

has eigenvalues

$$\lambda_i = \left\{O(\varepsilon^3), O(\varepsilon^3), 1 - \frac{\varepsilon^2}{32}(1 - \cos 4\theta), \frac{\varepsilon^2}{16}\sin^2 2\theta\right\}. \tag{42}$$

Therefore, to order ε^2, von Neumann entanglement entropy reads

$$S_V = -\sum_i \lambda_i \log_2 \lambda_i = -\left[1 - \frac{\varepsilon^2}{32}(1 - \cos 4\theta)\right]\log_2\left[1 - \frac{\varepsilon^2}{32}(1 - \cos 4\theta)\right] - \frac{\varepsilon^2}{16}\sin^2 2\theta \log_2\left(\frac{\varepsilon^2}{16}\sin^2 2\theta\right). \tag{43}$$

As expected, in the limit of either $\varepsilon \to 0$ or $\theta \to 0$, we recover $S_V = 0$, as it should correctly be, because in these cases the flavor vacuum reduces to the tensor product $|00\rangle_{1,2} = |0\rangle_1 \otimes |0\rangle_2$. In Fig. 1 we report the behavior of von Neumann entropy as a function of both ε and θ. To better feature the dependence of S_V on the mixing angle, it is also appropriate to exhibit how it varies by keeping ε fixed. This is done in Fig. 2. In particular, we can notice that for arbitrary ε the maximum of S_V always occurs at the perfectly balanced mixing angle $\theta_{\max} = \pi/4$. Increasing ε only results in an upper shift in the magnitude of S_V, with an essentially invariant shape. The value of $\theta_{\max}$ is exactly the one that could have been initially predicted, since $\pi/4$ corresponds to maximal mixing.

IV. CONCLUSIONS

In this paper, we have explored some nontrivial features of the flavor vacuum in the case of mixing of two neutral scalar fields. A first preliminary analysis has shown that we cannot quantify entanglement by resorting to a direct comparison between $|00\rangle_{A,B}$ and the free thermal vacuum introduced in thermo field dynamics, since the nature of the two states is not exactly the same. Note that this is consistent with the result found in Ref. [15] in the context of neutrino mixing. A similar outcome has also been exhibited in Refs. [21], where it has been shown that the flavor vacuum for an accelerated observer is not strictly a thermal state, in contrast with the case of the standard Unruh effect. However, in spite of these differences, we have found that, in a proper limit, the flavor vacuum can be identified with the vacuum of TFD, with a temperature dependent on the mixing angle and the mass difference of the two mixed fields.

To further investigate the condensate structure of the flavor vacuum, we have restricted our attention to the mode $k = 0$, for which the condensation density in $|00\rangle_{A,B}$ is maximal. In this setting, we have quantified the von Neumann entropy for small values of the mass difference. The shape exhibited in Fig. 1 is the one derived from our analysis, and the picture of Fig. 2 shows that there is an angle in correspondence of which von Neumann entropy is maximal for a given value of the small mass difference. It is worth observing that such angle is precisely the one responsible for maximal mixing; namely, $\theta_{\max} = \pi/4$. Clearly, it would be interesting to go beyond the single-mode analysis and derive the full expression for the von Neumann entropy. However, we expect that the result will exhibit some ultraviolet divergence of the same kind as

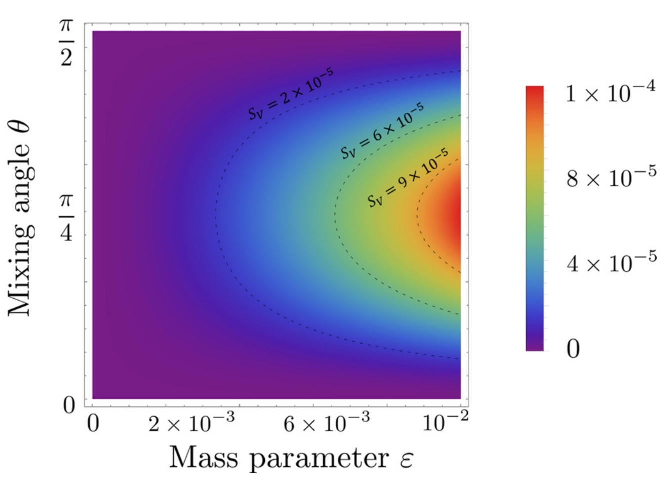

FIG. 1. Behavior of von Neumann entanglement entropy S_V as a function of $\varepsilon \in [0, 10^{-2}]$ and $\theta \in [0, \frac{\pi}{2}]$. As explicitly shown by the level curves, S_V monotonically decreases with decreasing ε.

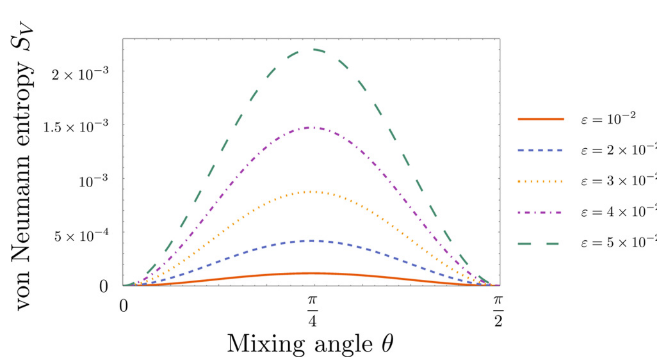

FIG. 2. Behavior of von Neumann entanglement entropy S_V as a function of $\theta \in [0, \frac{\pi}{2}]$ for different values of ε.

in Ref. [53], where the contribution of the flavor vacuum energy to the cosmological constant was shown to diverge only logarithmically, in contrast with the standard asymptotic behavior of free-field vacuum energy.

It is worth remarking that entanglement entropy and particle mixing have already been analyzed together in several papers in recent years [1,2], but only in the context of flavor transitions and for one-particle flavor states. In this respect, it is interesting to compare the field-theoretical approach to flavor entanglement developed in the present work with the exact methods for the quantification of entanglement introduced in the framework of quantum information and quantum optics of nonrelativistic continuous variable systems (see for instance Refs. [54–57] and therein). Such methods revolve around the transformation of the quadrature operators (namely, position, and momentum) under a given operation, which in our case is the mixing transformation. Since at the level of these operators mixing acts as a mere rotation, we would not achieve the desired result, thus preventing us from reaching an accurate evaluation of the entanglement entropy of the flavor vacuum via the procedures adopted for quantum optical systems [54–56].

We emphasize that the present analysis may have several implications in a broad range of contexts. For instance, once fully extended to noninertial frames [58–60] and, more generally, to gravity scenarios [61–71], one could investigate how the information content of the flavor vacuum is degraded for increasing values of the acceleration or gravity. A similar analysis has been carried out in Ref. [72] for the entanglement between two free modes of a scalar field as seen by an inertial observer detecting one of the modes and a uniformly accelerated observer detecting the other one. Finally, at a phenomenological level, one may rely on quantum simulation experiments, whose relevance in the context of high-energy physics has been confirmed by a growing body of literature. Concerning flavor mixing and oscillations, it has been shown that neutrino phenomenology can be perfectly reproduced by quantum optical systems, such as the binary waveguide arrays with an effective refracting index [73] and ultracold atomic setups as trapped ions [74]. In the former case, it is even possible to recognize a peculiar behavior of those neutrinos that become important during the phenomenon of supernova explosions. Therefore, it is essential to resort to quantum optical simulation as a powerful tool to probe inaccessible energy scales and other theoretical aspects of particle mixing, such as those discussed in the present paper. The study of all these aspects deserve careful attention and will be performed elsewhere.

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APPENDIX: THERMO FIELD DYNAMICS

This Appendix is devoted to review the basics of thermo field dynamics (TFD) [48], which represents one of the approaches to QFT at finite temperature and density. The reason for this study lies in the fact the TFD and the formalism of mixing analyzed in Sec. II exhibit some nontrivial conceptual similarities, the most prominent being the doubling of the degrees of freedom of the original system.

To introduce the TFD from a more quantitative point of view, let us consider the Hamiltonian of a free physical system written in the usual form

$$H = \sum_k \epsilon_k a_k^\dagger a_k. \tag{A1}$$

We now define a completely identical fictitious system, which we name the "tilde" system, sharing the same properties of the physical one. The former can then be written as

$$\tilde{H} = \sum_k \epsilon_k \tilde{a}_k^\dagger \tilde{a}_k. \tag{A2}$$

The above relation allows us to establish a new set of "thermal" operators via the Bogoliubov transformations

$$a_k(\vartheta) = a_k \cosh\vartheta_k - \tilde{a}_k^\dagger \sinh\vartheta_k, \tag{A3}$$

$$\tilde{a}_k(\vartheta) = \tilde{a}_k \cosh\vartheta_k - a_k^\dagger \sinh\vartheta_k, \tag{A4}$$

with θ_k encoding the information about the original and thermal ladder operators. Notice that the transformations (A3) and (A4) leave the total Hamiltonian of the system $H_{\text{tot}} = H - \tilde{H}$ invariant, provided that the Hamiltonians (A1) and (A2) have the same spectrum.

The Bogoliubov transformation (A3) and (A4) can also be recast as

$$a_k(\vartheta) = e^{-iG} a_k e^{iG}, \tag{A5}$$

$$\tilde{a}_k(\vartheta) = e^{-iG} \tilde{a}_k e^{iG}, \tag{A6}$$

with

$$G = i\sum_k \vartheta_k (a_k^\dagger \tilde{a}_k^\dagger - \tilde{a}_k a_k) \tag{A7}$$

being the generator of the transformation, which is also a conserved quantity (namely $[G, H_{\text{tot}}] = 0$) since the total Hamiltonian is conserved under its action. By means of G, we can now build a new vacuum state for the operators $a_k(\vartheta)$ and $\tilde{a}_k(\vartheta)$ defined as [48]

$$|0(\vartheta)\rangle = e^{-iG}|0\rangle, \tag{A8}$$

where $|0\rangle$ is the vacuum associated with a_k and $\tilde{a}_k$. If we explicitly act with the above operator on $|0\rangle$, we are left with

$$|0(\vartheta)\rangle = \prod_k \frac{e^{a_k^\dagger \tilde{a}_k^\dagger \tanh\vartheta_k}}{\cosh\vartheta_k}|0\rangle. \tag{A9}$$

The above expression conveys the idea that the "theta" vacuum is actually a condensate of a and $\tilde{a}$ particles, which somehow resembles the situation we have already encountered with mixing. However, in order to construct an effective comparison with the latter framework, we need to reformulate Eq. (A9) as

$$|0(\vartheta)\rangle = e^{-\frac{\mathcal{K}}{2}} e^{\sum_k a_k^\dagger \tilde{a}_k^\dagger}|0\rangle = e^{-\frac{\tilde{\mathcal{K}}}{2}} e^{\sum_k a_k^\dagger \tilde{a}_k^\dagger}|0\rangle, \tag{A10}$$

where

$$\mathcal{K} = -\sum_k (a_k^\dagger a_k \ln\sinh^2\vartheta_k - a_k a_k^\dagger \ln\cosh^2\vartheta_k) \tag{A11}$$

is called the entropy operator [48], because its vacuum expectation value multiplied by k_B yields precisely the entropy of the physical system. As a matter of fact, to make contact with a feasible physical picture and describe QFT for free fields at finite temperature, it is possible to prove that the arbitrary factors ϑ_k should satisfy the relation

$$\beta\omega_k = -\ln\tanh^2\vartheta_k, \tag{A12}$$

with $\beta = (k_B T)^{-1}$ being the inverse of the emerging temperature T of the thermal vacuum $|0(\vartheta)\rangle$ and $\omega_k = \epsilon_k - \mu$, where μ is the chemical potential. Remarkably, in the limit $\theta_k \to 0$ (i.e., for T small enough), Eq. (A9) reads

$$|0(\vartheta)\rangle \simeq \mathbf{1} + \sum_k a_k^\dagger \tilde{a}_k^\dagger \vartheta_k |0\rangle. \tag{A13}$$

As a final remark, we notice that the thermal vacuum (A9), obtained by augmenting the physical Fock space by a fictitious "tilde" space, has the same condensate structure of the vacuum perceived by stationary (uniformly accelerating) observers outside black holes (in Minkowski space-time). Of course, in that case the dual space can be interpreted in terms of the particle states on the hidden side of the horizon [45,50] and the entanglement arises among modes across the horizon.

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