Gaussian state entanglement witnessing through lossy compression

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We study the possibility of witnessing Gaussian entanglement between two continuous-variable systems with the help of two spatially separated qubits. Its key ingredient is a local lossy state transfer from the original systems onto local qubits. The qubits are initially in a pure product state, therefore by detecting entanglement between the qubits we witness entanglement between the two original systems.

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I. INTRODUCTION

Entanglement is considered one of the key resources in quantum information science [1]. It naturally emerges in the majority of many-body systems [2] and can be engineered on various experimental platforms [3–8]. However, despite the fact that entanglement seems to be all around us, its detection is challenging, especially in high-dimensional and continuous-variable systems. Detection of entanglement often requires a partial tomography of the system's state [9], whose full description is determined by the number of measurements which grows exponentially with the dimension (and is infinite in the continuous case).

In this article we focus on the problem of how to extract information about entanglement in the state of a complex bipartite system A. The main idea is to pair A with a simple system B [10]. A is assumed to be difficult to analyze, whereas B allows a full analysis. In other words, we limit the interaction with A to a minimum, whereas we are allowed to perform full tomography on B. The goal is to learn whether A is entangled by studying solely B. At this point we stress that the two subsystems A and B can be defined as separate particles, or as different degrees of freedom of a single particle (e.g., path/polarization, time/polarization, etc., see Ref. [8]).

In particular, we propose a method to detect entanglement between two continuous-variable systems in a Gaussian state (see also Refs. [11–13] for comparison) by transferring their state onto a state of two qubits and then by analyzing the resulting two-qubit state. In order to develop some intuition, we first show how to design a protocol to detect entanglement between two qudits and then we generalize it to the continuous-variable case. The qubits are initially prepared in a separable state. Hence, any entanglement arising between them must stem from the initial entanglement between the more complex systems. Notably, a somewhat reverse idea of coupling and performing operations on continuous-variable systems via the simplest discrete-variable system has been introduced in Ref. [14] in the context of the implementation of an interface between a quantum optical field and a qubit.

It is clear that such a state transfer cannot be perfect since the dimension of the system onto which the transfer is made is lower than the dimension of the original system. Therefore, the above process can be considered a lossy compression, which aims to preserve only the relevant information. In this case, we want to keep the information about entanglement and discard anything else.

II. *d*-LEVEL SYSTEMS

Before we analyze entanglement and continuous-variable systems, let us first discuss a single qudit (A) and a single qubit (B). We will introduce a unitary coupling operation which allows us to transfer some properties of the system A to the system B. Later we will generalize the scheme to a pair: Two qudits–two qubits.

A. Single system

As a coupling operator we use a controlled rotation (CROT), i.e., a rotation of the qubit controlled by the state of the qudit. More precisely, CROT is defined for a bipartite system AB composed of a controlling d-level qudit state A, and a target qubit B, the state of which is rotated along the y axis, by

$$U_{\text{CROT}} = \sum_{j=0}^{d-1} |j\rangle\langle j| \otimes \exp(-i\sigma_y \xi_j)$$
$$= \sum_{j=0}^{d-1} |j\rangle\langle j| \otimes (\cos\xi_j \mathbb{1} - i\sin\xi_j\sigma_y), \qquad (1)$$

where the rotation parameter ξ_j depends on the original state of the qudit, $\xi_j = \frac{j\pi}{2(d-1)}$. After the coupling we ignore the subsystem A by tracing it out, and perform an analysis on the qubit B.

The above operation resembles the von Neumann measurement apparatus [15], with the exception that the pointer is not a continuous-variable system, but a single qubit. As a result, the measurement of the observable (in this case $A = \sum_j j|j\rangle\langle j|$) cannot be perfect due to the fact that one can encode at most a single bit of classical information on a single qubit. Nevertheless, we are going to show that after the CROT operation some important information about the qudit's state can be decoded from the qubit's state.

As an example let us consider a d-level system being in the state

$$|\psi(p)\rangle_A = \sum_{k=0}^{d-1} \sqrt{\binom{d-1}{k} p^k (1-p)^{d-1-k}}\, |k\rangle, \tag{2}$$

parametrized by a single unknown parameter p. The probability amplitudes are given by the Bernoulli distribution. Applying the coupling operation (1) on the qudit-qubit pair $|\psi(p)\rangle_A \otimes |0\rangle_B$, we get

$$\begin{aligned}|\Psi\rangle_{AB} &= U_{\mathrm{CROT}}[|\psi(p)\rangle_A \otimes |0\rangle_B] \\ &= \sum_{j=0}^{d-1} \sqrt{\binom{d-1}{j} p^j (1-p)^{d-1-j}}\, |j\rangle_A \\ &\quad \otimes (\cos\xi_j |0\rangle_B + \sin\xi_j |1\rangle_B).\end{aligned} \tag{3}$$

The reduced density matrix of the system B is given by

$$\begin{aligned}\rho_B = \mathrm{Tr}_A |\Psi\rangle_{AB}\langle\Psi| = \sum_{j=0}^{d-1} \binom{d-1}{j} p^j (1-p)^{d-1-j} \\ \times \begin{pmatrix} \cos^2\xi_j & \frac{1}{2}\sin 2\xi_j \\ \frac{1}{2}\sin 2\xi_j & \sin^2\xi_j \end{pmatrix},\end{aligned} \tag{4}$$

allowing us to extract information about the parameter p by, for example, a measurement along σ_z, $\mathrm{Tr}(\rho_B\sigma_z)$. In the limit of infinite dimensions d,

$$\rho_B \overset{d\to\infty}{=} \begin{pmatrix} \cos^2\left(\frac{\pi p}{2}\right) & \frac{1}{2}\sin(\pi p) \\ \frac{1}{2}\sin(\pi p) & \sin^2\left(\frac{\pi p}{2}\right) \end{pmatrix}, \tag{5}$$

ρ_B becomes a pure state $|\Psi\rangle_B = \cos\left(\frac{\pi p}{2}\right)|0\rangle + \sin\left(\frac{\pi p}{2}\right)|1\rangle$. Hence, $p = (1/\pi)\arccos \mathrm{Tr}(\rho_{\mathrm{B}}\sigma_z)$.

B. Entangled systems

Let us now suppose the system A is composed of a pair of d-dimensional qudits in the state $|\psi\rangle_A = \sum_{j,l=0}^{d-1} a_{jl}|jl\rangle$, which we want to couple with a pair of qubits B. In order to do this, we use the coupling operator $U_{\mathrm{CROT}}^{\otimes 2}$ for each pair of subsystems, such that the CROT operator couples the first (second) qudit to its respective qubit.

If both qubits are initially in the state $|0\rangle$, then an application of $U_{\mathrm{CROT}}^{\otimes 2}$ to the total system $|\psi\rangle_A|00\rangle_B$ results in

$$\begin{aligned}|\Psi\rangle_{AB} &= U_{\mathrm{CROT}}^{\otimes 2}(|\psi\rangle_A \otimes |00\rangle_B) \\ &= \sum_{j,l=0}^{d-1} a_{jl}|jl\rangle_A \otimes [\cos\xi_j\cos\xi_l|00\rangle_B \\ &\quad + \cos\xi_j\sin\xi_l|01\rangle_B + \sin\xi_j\cos\xi_l|10\rangle_B \\ &\quad + \sin\xi_j\sin\xi_l|11\rangle_B].\end{aligned} \tag{6}$$

In general, the state $|\Psi\rangle_{AB}$ can be highly four-partite entangled, which results in separable subsystems. Therefore, if we want to transfer entanglement from the system A to B, we are obligated to do a conditional (projective) measurement on the system A. One of the good candidates is the local projection onto the state $|{+}{+}\rangle_A = |+\rangle|+\rangle$ with $|+\rangle = 1/\sqrt{d}\sum_{k=0}^{d-1}|k\rangle$. After successful projection, the resulting state reads

$$\begin{aligned}\mathcal{N}|{+}{+}\rangle_A \sum_{j,l=0}^{d-1} a_{jl}[\cos\xi_j\cos\xi_l|00\rangle_B \\ + \cos\xi_j\sin\xi_l|01\rangle_B + \sin\xi_j\cos\xi_l|10\rangle_B \\ + \sin\xi_j\sin\xi_l|11\rangle_B],\end{aligned} \tag{7}$$

where $(1/\mathcal{N})^2$ is the probability of projecting the system A of two qudits onto $|+\rangle|+\rangle$.

At this point it is worth considering an example. Let A be in the maximally entangled state corresponding to $a_{jl} = \delta_{jl}/\sqrt{d}$. Then, after the coupling operation, the overlap of the resulting state $|\Psi\rangle_B$ with the maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ decreases with d, but asymptotically approaches $\pi^2/(\pi^2+4) \approx 0.712$ as $d \to \infty$. Also we must note that the probability of finding the first system in the desired state depends on its dimension d. For a general pure state of two qudits $|\psi\rangle_A = \sum_{j=0}^{d-1} a_{ij}|ij\rangle$, it equals $|\sum_{i,j=0}^{d-1} a_{ij}|^2/d^2$, which for the maximally entangled state, $a_{ij} = 1/\sqrt{d}$, scales as $1/d$. Note also that the success probability is optimal for full-rank maximally entangled states. In particular, for an entangled state of rank k, namely $|\psi\rangle_A = \frac{1}{\sqrt{d}}\sum_{j=0}^{k-1}|jj\rangle$, the probability equals $\frac{k}{d^2}$.

Notice that in the special case of $d = 2$, the operation swaps the state of the system A to the system B,

$$\begin{aligned}|\Psi\rangle_B \overset{d=2}{=} \mathcal{N}\sum_{j,l=0}^{1} a_{jl}\Big[\cos\left(\frac{j\pi}{2}\right)\cos\left(\frac{l\pi}{2}\right)|00\rangle \\ + \cos\left(\frac{j\pi}{2}\right)\sin\left(\frac{l\pi}{2}\right)|01\rangle \\ + \sin\left(\frac{j\pi}{2}\right)\cos\left(\frac{l\pi}{2}\right)|10\rangle \\ + \sin\left(\frac{j\pi}{2}\right)\sin\left(\frac{l\pi}{2}\right)|11\rangle\Big] \\ = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.\end{aligned} \tag{8}$$

For a general d the resulting state $|\Psi\rangle_B$ is separable if the input state $|\psi\rangle_A$ is separable. This is because a factorization of the amplitudes $a_{jl} = a'_j a''_l$ allows us to factorize the resulting state $|\Psi\rangle_B$,

$$|\Psi\rangle_B \overset{|\psi_{\rm prod}\rangle_A}{=} \sum_{j=0}^{d-1} a'_j(\cos\xi_j|0\rangle + \sin\xi_j|1\rangle) \otimes \sum_{l=0}^{d-1} a''_l(\cos\xi_l|0\rangle + \sin\xi_l|1\rangle). \quad (9)$$

Let us now consider a general mixed state of two qudits, i.e., $\rho_A = \sum_{i,j,k,l=0}^{d-1} \rho_{ij,kl}|i\rangle\langle j| \otimes |k\rangle\langle l|$. If we denote ρ_{AB} as the state of the total system after the coupling,

$$\rho_{AB} = U_{\rm CROT}^{\otimes 2}(\rho_A \otimes |00\rangle_B\langle 00|)\big(U_{\rm CROT}^{\otimes 2}\big)^\dagger, \quad (10)$$

projecting the subsystem A onto $|++\rangle$ results in subsystem B becoming

$$\rho_B = \sum_{i,j,k,l=0}^{d-1}\sum_{m,n,p,q=0}^{1} \rho_{ij,kl}\; a_i^m a_k^n a_j^p a_l^q\; |m\rangle\langle p| \otimes |n\rangle\langle q|, \quad (11)$$

where

$$a_\beta^\alpha = \begin{cases}\cos\frac{\beta\pi}{2(d-1)}, & \text{for } \alpha = 0,\\ \sin\frac{\beta\pi}{2(d-1)}, & \text{for } \alpha = 1.\end{cases} \quad (12)$$

Note that for $d = 2$, we have $\rho_B = \rho_A$, the same as in the case of pure states.

Additionally, we also show that a separable state of two qudits is mapped onto a separable state of two qubits. Consider the separable state of two qudits $\rho_A^{\rm sep} = \sum_\lambda p_\lambda \rho_1^\lambda \otimes \rho_2^\lambda$, where

$$\rho_1^\lambda = \sum_{i,j=0}^{d-1} \rho_{1,ij}^\lambda |i\rangle\langle j|, \quad (13)$$

$$\rho_2^\lambda = \sum_{k,l=0}^{d-1} \rho_{2,kl}^\lambda |k\rangle\langle l|, \quad (14)$$

and hence

$$\rho_A^{\rm sep} = \sum_\lambda p_\lambda \sum_{i,j,k,l=0}^{d-1} \rho_{1,ij}^\lambda \rho_{2,kl}^\lambda |i\rangle\langle j| \otimes |k\rangle\langle l|. \quad (15)$$

Performing analogous calculations as in the general case for mixed states and taking into account the linearity of all operations, we get a separable state:

$$\begin{aligned}\rho_B &\overset{\rho_A^{\rm sep}}{=} \sum_\lambda p_\lambda \\ &\quad\times \sum_{i,j,k,l=0}^{d-1}\sum_{m,n,p,q=0}^{1} \rho_{1,ij}^\lambda\rho_{2,kl}^\lambda\; a_i^m a_k^n a_j^p a_l^q |m\rangle\langle p| \otimes |n\rangle\langle q| \\ &= \sum_\lambda p_\lambda \left(\sum_{i,j=0}^{d-1}\sum_{m,p=0}^{1} \rho_{1,ij}^\lambda\; a_i^m a_j^p |m\rangle\langle p|\right) \\ &\quad\otimes \left(\sum_{k,l=0}^{d-1}\sum_{n,q=0}^{1} \rho_{2,kl}^\lambda\; a_k^n a_l^q |n\rangle\langle q|\right). \end{aligned} \quad (16)$$

Please note that the condition for ρ_A to be separable so that the resulting ρ_B is also separable is only sufficient, not necessary. There are instances of entangled states ρ_A which are not mapped into entangled ρ_B, hence the scheme effectively works as an entanglement witness.

III. CONTINUOUS-VARIABLE SYSTEMS

We will now generalize our scheme to the case in which the system A is being described by a continuous-variable state. In this regard we limit our considerations to the broad family of Gaussian states.

A. Single system

If the first subsystem has a continuous spectrum, the coupling operator reads

$$U_{\rm CROT} = \int_{-\infty}^{\infty} dx|x\rangle\langle x| \otimes (\cos x\mathbb{1} - i\sin x\sigma_y). \quad (17)$$

Next, consider a Gaussian state

$$|\psi(\sigma,m)\rangle_A = \int dx \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(x-m)^2}{4\sigma^2}}\,|x\rangle \quad (18)$$

specified by two parameters (σ, m). After applying the coupling operation to $|\psi(\sigma,m)\rangle_A \otimes |0\rangle_B$, we get

$$U_{\rm CROT}[|\psi(\sigma,m)\rangle_A \otimes |0\rangle_B] = \int dx \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(x-m)^2}{4\sigma^2}}\,|x\rangle \otimes (\cos x|0\rangle + \sin x|1\rangle). \quad (19)$$

The reduced state of the qubit B is

$$\rho_B = \frac{1}{2}\left[\mathbb{1} + e^{-2\sigma^2}\begin{pmatrix}\cos 2m & \sin 2m\\ \sin 2m & -\cos 2m\end{pmatrix}\right] \quad (20)$$

and can be visualized by a Bloch vector $\vec{b}$ lying in the xz plane. The vector $\vec{b}$ makes an angle $2m$ with the z axis and its norm is $e^{-2\sigma^2}$. The parameters of the original Gaussian state can be recovered from a tomography on the qubit. In particular, $\sigma^2 = -(1/4)\ln||\vec{b}||^2$ and $m = \mathrm{arccot}(b_z/b_x)/2$.

B. Entangled systems

Now, we consider the lossy entanglement transfer from the bipartite Gaussian state onto the two-qubit state. In order to do this, we use the coupling operator of (17) for each respective pair of subsystems, $U_{\rm CROT}^{\otimes 2}$, such that the first (second) complicated subsystem interacts with its respective qubit.

After the coupling operation, we project the system A of the two particles onto a product of Gaussian states

$$|x_1^+x_2^+(\Gamma)\rangle = \int dx_1 \int dx_2 \frac{1}{(2\pi\Gamma^2)^{1/2}} e^{-\frac{(x_1^2+x_2^2)}{4\Gamma^2}}|x_1\rangle|x_2\rangle. \quad (21)$$

This is an analogy to the projection onto a uniform superposition that we used in the two-qudit case. This time the projection is parametrized by a single parameter Γ, which corresponds to the standard deviation. Note that in the limit $\Gamma \to \infty$ the Gaussian function becomes a uniform superposition over the whole space, akin to what was considered in the qudit case. Such a projection can be interpreted as a projection onto a ground state of a harmonic oscillator, for which the parameter Γ can be manipulated by the oscillator's frequency.

As an example we consider two particles in an entangled Gaussian state,

$$|\psi(\sigma,\Sigma)\rangle_A = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \frac{1}{(2\pi\sigma\Sigma)^{1/2}} e^{-\frac{(x_1+x_2)^2}{8\sigma^2}} e^{-\frac{(x_1-x_2)^2}{8\Sigma^2}} |x_1\rangle|x_2\rangle. \tag{22}$$

This state is entangled whenever $\sigma \neq \Sigma$. Since it is a pure state, its entanglement can be measured by the purity of a subsystem, which in this case is given by [16]

$$P = \frac{2\sigma\Sigma}{\sigma^2+\Sigma^2}. \tag{23}$$

After the coupling operation, the state $|\psi(\sigma,\Sigma)\rangle_A \otimes |00\rangle_B$ becomes

$$\begin{aligned} &U_{\rm CROT}^{\otimes 2}[|\psi(\sigma,\Sigma)\rangle \otimes |00\rangle] \\ &= \int dx_1 \int dx_2 \frac{1}{(2\pi\sigma\Sigma)^{1/2}} e^{-\frac{(x_1+x_2)^2}{8\sigma^2}} e^{-\frac{(x_1-x_2)^2}{8\Sigma^2}} |x_1\rangle|x_2\rangle \\ &\otimes[\cos x_1 \cos x_2|00\rangle + \cos x_1 \sin x_2 |01\rangle \\ &+ \sin x_1 \cos x_2 |10\rangle + \sin x_1 \sin x_2 |11\rangle]. \end{aligned} \tag{24}$$

After projecting the system A onto $|x_1^+ x_2^+(\Gamma)\rangle$, the state of system B becomes

$$\mathcal{N}(a_+|00\rangle + a_-|11\rangle), \tag{25}$$

where

$$a_\pm = \frac{\pm e^{-\frac{2\sigma^2\Gamma^2}{\sigma^2+\Gamma^2}} + e^{-\frac{2\Sigma^2\Gamma^2}{\Sigma^2+\Gamma^2}}}{\sqrt{\frac{(\sigma^2+\Gamma^2)(\Sigma^2+\Gamma^2)}{\sigma\Sigma\Gamma^2}}}. \tag{26}$$

The probability of successful projection onto a desired state equals $\frac{4\sigma\Sigma\Gamma^2}{(\sigma^2+\Gamma^2)(\Sigma^2+\Gamma^2)}$, and is maximal for $\Gamma = \sqrt{\sigma\Sigma}$, for which it reads $\frac{4\sigma\Sigma}{(\sigma+\Sigma)^2}$. Notice that due to a similarity with the purity of the subsystem (23), we have the trade-off: The higher the entanglement of the original system (in other words, the lower the purity), the lower is the probability of successfully projecting the state as desired by the protocol. Fortunately, the probability of successful projection drops below 10% only for a strongly entangled Gaussian state with the purity of a subsystem $P < 0.05$. This corresponds to the ratio $\sigma/\Sigma = 38$. If $\sigma/\Sigma = 5$, the original state is still highly entangled ($P \approx 0.38$) and the probability of successful projection is above 50%.

After the projection, the purity of the qubit subsystem B is

$$P_q = \frac{1}{2}\left[\mathrm{sech}^2\left(\frac{2\Gamma^4(\sigma-\Sigma)(\sigma+\Sigma)}{(\sigma^2+\Gamma^2)(\Sigma^2+\Gamma^2)}\right) + 1\right], \tag{27}$$

which becomes in the limit of $\Gamma \to \infty$,

$$\lim_{\Gamma\to\infty} P_q = \tfrac{1}{2}\{\mathrm{sech}^2[2(\sigma-\Sigma)(\sigma+\Sigma)] + 1\}. \tag{28}$$

In Fig. 1 we present how the purity of a subsystem depends on Γ. We analyze the extreme case $(\sigma-\Sigma) \to \infty$. In this case the purity is $\frac{1}{2}[\mathrm{sech}^2(2\Gamma^2) + 1]$ and decreases with Γ. Already for Γ above 1, the purity is close to 1/2.

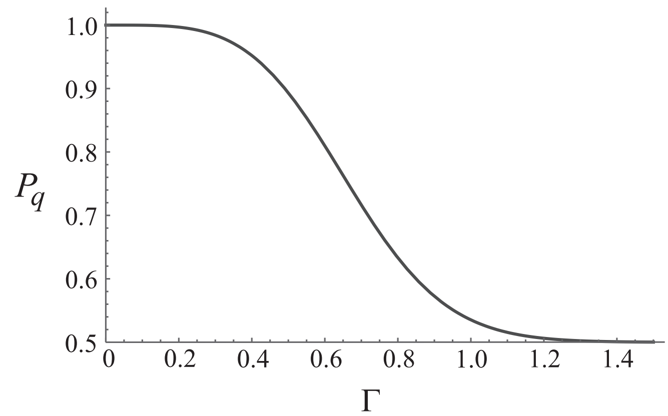

FIG. 1. The purity of a subsystem for the resulting state with $\sigma - \Sigma \to \infty$ in a function of the projection parameter Γ.

C. Comparison to other methods

Given that our method of entanglement witnessing rests on the analysis of the ancillary system B, which is coupled to the original system A, we may compare this to other methods of detecting entanglement. The most obvious one would be to measure the purity of the original state itself, in which case the entanglement is being witnessed whenever the purity of the subsystem of A is less than 1. In Fig. 2(a) we show the purity of the corresponding subsystem (23) which indicates entanglement present in the system A whenever $\Sigma \neq \sigma$. In our method, however, the entanglement of A is inferred from measuring the purity of the subsystem of the ancillary two-qubit state (system B), which is presented in Fig. 2(b). As we see, entanglement is witnessed for any $\Sigma \neq \sigma$. Also, in both cases the situation $\sigma - \Sigma \to \infty$ corresponds to the maximally entangled states.

One can also compare our method with the necessary and sufficient condition for the entanglement of bipartite Gaussian states, which is based on coefficients of the quadrature components of the wave function [17,18]. In particular, the system is entangled whenever $(\sigma^2 - \Sigma^2)^2 \geqslant 0$, which brings the discussion back to the previous case.

D. Possible realization

Here, we give examples for possible implementations of the above scheme. The first concept is in a sense an inverted Stern-Gerlach scenario. We focus on a single system, since the entangled case is a straightforward generalization.

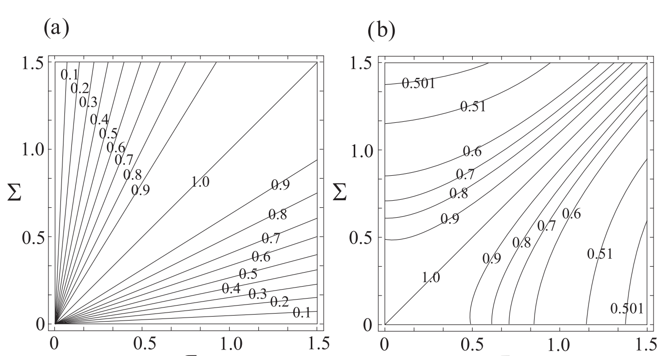

FIG. 2. The purity of a subsystem in a function of σ and Σ for (a) the original and (b) resulting state.

Consider a spin-1/2 particle, say, a silver atom, propagating along the z axis. The continuous-variable state of interest is encoded in the transversal degree of freedom, say, a spatial state along the x axis $|\psi(x)\rangle$. The spin of the particle is initially pointing up along the z axis. In addition, consider a region in which there is a nonzero magnetic field $\vec{B} = B(x)\hat{y}$ pointing along the y axis with a gradient along the x axis. We assume that in the region in which $|\psi(x)\rangle$ is supported one can use the approximation $B(x) \approx B_0 x$. This magnetic field region starts at $z = z_0$ and ends at $z = z_1$ $(0 < z_0 < z_1)$. Outside of this region there is no magnetic field. The particle starts at $z = 0$ and moves towards the magnetic field region with velocity v. It spends the time $t = (z_1 - z_0)/v$ within the magnetic field region. The magnetic field causes a position-dependent rotation of spin about the y axis,

$$|\uparrow_z\rangle \rightarrow \cos\alpha(x)|\uparrow_z\rangle + \sin\alpha(x)|\downarrow_z\rangle, \tag{29}$$

where $\alpha(x) \propto B_0 x(z_1 - z_0)/v$. This conditional rotation can be associated with the CROT operation. This way the state $|\psi(x)\rangle$ is lossy transferred onto the spin state.

Another possible implementation is the interaction of different degrees of freedom of photons. A natural choice for the degree of freedom of the d-level system is using path encoding as it easily allows us to manipulate, say, the polarization state of the photon depending on the path state using wave plates for building up the CROT operation as given in Eq. (1). This general concept can be combined with a plethora of different degrees of freedom. Wavelength division multiplexers allow coupling frequency-bin encoded qudits to qubits using this scheme. Similarly, when using, e.g., orbital angular momentum (OAM) for the qudit system, the OAM encoding can first be translated to path encoding using a mode sorter [19]. Recently, a controlled-$\hat{X}$ gate between the radial degree of freedom of light and its OAM has been shown [20], providing another perfect test bed for our coupling.

In a recent work, a high-finesse cavity has been used to couple a ^{87}Rb atom to the coherent state of a light field reflected at the cavity for creating Schrödinger cat states [21]. This technique may also allow us to couple the state of the light field to the atom using the CROT operation as given in Eq. (17). Our proposal could hence facilitate probing for the entanglement of two light fields.

Furthermore, our proposal can also be used if both systems A and B are actually qudits. For example, entanglement of a system of two high-dimensional trapped ions could be probed by properly designing the interaction with two other ions using a suitably modified CROT operation such that the qudits of system B make use of only a two-level submanifold of the ion. This greatly simplifies their read-out as they can now be treated as qubits. This procedure is applicable also to other high-dimensional systems such as, say, superconducting transmon qudits.

Finally, we would like to mention that our approach also works for multiqubit systems, in which the entanglement between two specific subsets of particles is to be analyzed. The entanglement between a set A_1 of qubits and a set A_2 of qubits can be studied by first compressing the multiqubit states ρ_{A_1} and ρ_{A_2} into the single qubits B_1 and B_2, respectively, using a CROT operation. Afterwards, the verification of entanglement of those two single-qubit systems implies entanglement between the initial multiqubit systems.

IV. CONCLUSIONS

In this paper we address the problem of detecting entanglement properties of a complex system by analyzing an auxiliary system coupled to the original one. In order to do this we define a coupling operator which transforms the auxiliary system so that after the operation the measured properties of the coupled system provide relevant information about the nature of the original one. Since the auxiliary system is chosen to be of lower dimensionality than the original one, the transfer of information through the coupling operator cannot be exact, hence we can consider the operation a lossy compression. In the process, however, we are being offset by the reduction of the number of measurements required to analyze the entanglement properties of the measured system. Moreover, the scheme works also when we intend to detect entanglement between two continuous-variable systems in a Gaussian state, which in principle can be partially encoded in a simple two-qubit state.

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