Sequential measurement-device-independent entanglement detection by multiple observers

Chirag Srivastava, Shiladitya Mal, Aditi Sen(De), and Ujjwal Sen
Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad 211 019, India

(Received 7 February 2021; accepted 22 February 2021; published 10 March 2021)

Violation of a Bell inequality certifies that the underlying state must be entangled in a device-independent way, although there exist some entangled states which do not violate such an inequality. However, for every entangled state, it is possible to find a Hermitian operator called an entanglement witness that can detect entanglement through some local measurements in a device-dependent method, but implementation of wrong measurements may lead to fake detection of entanglement. To avoid such difficulties, measurement-device-independent entanglement witness (MDI-EW) based on a semiquantum nonlocal game was proposed, which is not only robust against wrong measurements but also against a specific kind of lossy detectors. We employ here a measurement-device-independent entanglement witness to detect entanglement in a scenario where half of an entangled pair is possessed by a single observer while the other half is with multiple observers performing unsharp measurements, sequentially, independently, and preserving entanglement as much as possible. Interestingly, we find that the numbers of successful observers who can detect entanglement, measurement-device-independently, both with equal and unequal sharpness parameters of the noisy measurements, are greater than that obtained with standard and Bell-inequality-based entanglement detection methods, reflecting its robustness. The entanglement contents of the sequentially shared states are also analyzed. Unlike other scenarios, our investigations also reveal that in this measurement-device-independent situation, states having entanglement in proximity to maximal remain entangled until there are two sequential observers, even if they measure sharply.

DOI: 10.1103/PhysRevA.103.032408

I. INTRODUCTION

The existence of entangled states [1] is one of the most nonclassical features of the quantum mechanical description of nature, which, e.g., can lead to violation of Bell inequality [2], testable in the laboratory. The violation implies that quantum theory cannot be replaced by a local realistic model, compatible with classical theory, and this feature of quantum theory enables various quantum information processing tasks like generation of true randomness [3–6], secure key distribution [7–11], and possibly quantum communication [12–16].

Over the years, it has been established that one of the efficient ways to detect entanglement in the laboratory is through entanglement witnesses (EWs), which can be implemented through local measurements performed on the individual systems that constitute the composite system [17–20]. However, implementing an EW requires proper characterization of the measurement devices and some prior information of the shared states. On the other hand, violation of Bell inequality certifies entanglement device-independently while paying a cost; viz., there exist entangled states, where entanglement cannot be probed via violation of a Bell inequality [21–26]. See Ref. [27] in this regard.

It is known that corresponding to every Bell inequality, there is a nonlocal game, and for each nonlocal game, one can construct a Bell inequality [28]. Extending the Bell scenario, Buscemi has recently proposed a "nonlocal semiquantum game," where every entangled state yields a higher payoff compared to all separable states [29]. In this game, two observers share a bipartite state and on top of that, instead of classical inputs, like in a standard Bell scenario, a "referee" gives them quantum inputs. Each party then measures jointly on the respective inputs and their part of the shared state. Outputs of the observers together with inputs are used to constitute the payoff function, which shows an advantage for any entangled state over all separable states. Note that except for the quantum inputs, other devices are untrusted in this scenario. Such a semiquantum nonlocal game can also be extended to the multiparty scenario [29].

Since all entangled states are "nonlocal" according to the semiquantum game [29], i.e., since the payoff function provides a higher value for any entangled state than all separable states, it can be a witness for detecting entanglement. We refer to such a situation—a higher value of the payoff function than all separable states—as "Buscemi nonlocality," and contrast it with the previous notion of "Bell nonlocality" [2], which referred to a violation of Bell inequality. It is known that there exist EWs for every entangled state [17–20,30]. Given such an EW, in Ref. [31], Branciard *et al.* constructed a new EW, based on Buscemi's game, which does not depend on the internal functioning of measurement devices, i.e., it is a measurement-device-independent EW (MDI-EW), and it will not announce any separable state as entangled even under implementation of wrong measurements. However, in practical situations, detectors may suffer from malfunctioning, which may lead to loss of some outcomes. The performance of standard EWs or violations of Bell inequality under the presence of losses in the outcome of measurements has been investigated.

It has been shown that lossy detectors can wrongly indicate a separable state as entangled [32,33]. Recently, it was also shown that under certain losses, even the MDI-EWs may show a separable state to be entangled [34].

In quantum information processing tasks, it is important to distribute resource states among several parties. In the literature, there are various protocols to do that. In an unconventional scenario [35], Silva *et al.* showed that when an entangled pair is shared between a single observer (say, Alice) at one side and several other observers (say, Bobs) at the other, acting sequentially and unsharply, no more than two Bobs can exhibit violation of the Bell-CHSH inequality with Alice [36]. See also Ref. [37], and for experimental verification, see Refs. [38,39]. Note that by "acting unsharply," we mean that the performed measurements are noisy in a particular way, viz., the intended measurements are admixed with white noise. Later, the concept of sequential unsharp action has been extended to other contexts, like Bell-type inequalities with more than two settings at each site [40], quantum steering [41,42], and entanglement witnesses [43]. However, it has recently been shown that there are some information processing tasks, notably in the context of self-testing instruments, sharpness parameters, and randomness generation, where advantages can be exhibited invoking a sequential measurement scenario which cannot be obtained in the standard situation. See, e.g., Refs. [44–50].

In the present work, we focus on an information gain to disturbance trade-off within a sequential measurement scenario. Moreover, we investigate how the power of MDI-EWs can be reflected in the case of this unconventional resource distribution pioneered by Silva *et al.* [35]. In Ref. [43], it was found that at most 12 Bobs can detect entanglement sequentially, when standard EWs were employed. We also, likewise, consider pure entangled states as the initial state and find the maximal number of Bobs allowed in this protocol. The behavior of entanglement content of the subsequent shared states is also observed. We find that the maximal number of Bobs who can identify entanglement with a single Alice can go up to 14 in a measurement-device-independent way when the shared initial state has entanglement more than or equal to 93.5% of the singlet. We also study the case when all the Bobs measure with a common sharpness parameter. For an initially shared maximally entangled state, the maximum number of Bobs who can sequentially detect entanglement while using a common sharpness parameter is six, which is greater than when the same task is considered with standard EWs.

Let us mention here that in the case of detecting entanglement, using unsharp versions of EWs [43], if any of the Bobs measures sharply, i.e., projectively, then there is no possibility of detecting entanglement by any subsequent Bob, as there is no residual entanglement between Alice and the subsequent Bob. Therefore, if the Bobs have to detect entanglement sequentially, then all of them except the last one must measure unsharply [51]. On the other hand, at each step, a very unsharp measurement may rule out the possibility of detecting entanglement, and this can be interpreted as another face of the well-known trade-off between information gain and disturbance [52–54]. Hence, every Bob has to measure with a *threshold* sharpness parameter, so that Alice and he can detect entanglement in the sequential process. Interestingly, in the context of MDI-EWs, we find that even if the first Bob measures sharply, then the second Bob also can detect entanglement, which was not the case when standard EWs were employed [43].

It is to be noted, however, that the possibility of a certain number of Bobs being able to detect entanglement does not imply that all of them will be able to perform any task that utilizes entanglement as a resource. The latter possibility exists but needs to be separately checked for every task. This is much like in entanglement detection in the standard (nonsequential, single-Bob) scenario, where the existence of entanglement between Alice and Bob often provides the possibility of performing a nonclassical task, say, quantum dense coding [55], but whether the same is actually possible needs to be checked directly.

We also note here that the setup of sequential measurements by several Bobs and single Alice can in a way tell us about the robustness of the underlying entanglement detection scheme, i.e., the ability to witness the entanglement even after the measurements (to detect entanglement) of earlier observer or observers. Our results compare the performance of entanglement witnesses (EWs) and MDI-EWs under a given task. Under the same assumptions, it was shown that an EW can be used to detect entanglement at most by 12 sequential observers to detect entanglement with a separated observer, but with the corresponding MDI-EW, a maximum of 14 such observers can detect entanglement with the same separated observer. Therefore, apart from guaranteeing entanglement independent of any measurements, MDI-EWs also perform better than EWs in sequential detection of entanglement. We believe that this gives us an operational way in which MDI-EWs can be fundamentally differentiated from EWs.

Let us mention here two potential applications of using sequential measurements in MDI-EWs. One is that generation of randomness can also be devised in the MDI scenario with sequential measurements. Second, MDI quantum key distribution has already been investigated in the literature. See, e.g., Ref. [56]. That task can also be devised within a sequential scenario, where Alice can establish secure keys with many Bobs.

The remaining part of the paper is organized as follows. In Sec. II, we briefly discuss MDI-EWs and the unsharp measurement formalism adopted for the purpose of our work. In Sec. III, the scenario of entanglement sharing in the context of MDI-EW is discussed. In Sec. IV, we find the maximum number of sequential and independent single-laboratory observers, who are able to detect the bipartite entanglement shared with the common distant-laboratory observer using MDI-EWs. In Sec. V, we analyze the change in entanglement content due to an unsharp measurement required for the MDI-EW procedure, in the states shared between the common observer and the sequential observers. In Sec. VI, the case of sequential observers measuring with equal sharpness is considered, and finally we end with conclusion in Sec. VII.

II. ESSENTIALS

Let us begin by discussing the necessary ingredients required to detect bipartite entanglement shared between Alice at the one side and multiple Bobs at the other side in a measurement-device-independent scenario.

A. Measurement-device-independent entanglement witness

An entanglement witness operator, W, is defined as a Hermitian operator such that for all states $\sigma_{AB} \in \mathcal{S}$, $\mathrm{tr}(\sigma_{AB}W) \geqslant 0$, while there exists at least one entangled state, ρ_{AB}, in the same bipartition, such that $\mathrm{tr}(\rho_{AB}W) < 0$, where $\mathcal{S}$ is the set of separable states in the bipartition, $A : B$ [17–20,30], but such witness operators have at least two disadvantages. First, to implement them, one requires characterized devices as well as some prior information about the state to be detected, and second, in the case of lossy measurements, the expectation value of witness operators for separable states may turn out to be negative, leading to a false positive detection of entanglement [32]. To avoid such uncertainties, Branciard *et al.* introduced the concept of measurement-device-independent entanglement witnesses [31]. Specifically, given an EW, the semiquantum nonlocal game of Buscemi [29] is used to obtain as MDI-EW.

A complete set of density matrices can be used to span the space of Hermitian operators. Let an entanglement witness operator, W, act on the tensor-product Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, with the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$ being two each. Consider any two complete sets of density matrices, $\{\tau_s\}_{s=0}^3$ acting on $\mathcal{H}_A$ and $\{\omega_t\}_{t=0}^3$ acting on $\mathcal{H}_B$, such that W can be expanded as

$$W = \sum_{st} \beta_{st} \tau_s^{\mathrm{T}} \otimes \omega_t^{\mathrm{T}}, \tag{1}$$

where the superscript, T, over the states denotes their transposes and β_{st} are real coefficients. Now, consider the scenario where two parties, Alice and Bob, possess a shared state ρ_{AB} operating on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Further, Alice and Bob receive quantum inputs, from a "referee," in the form of a state from a set of qubit states, $\{\tau_s\}_{s=0}^3$ and $\{\omega_t\}_{t=0}^3$, respectively. They then perform a joint measurement on their respective parts of the shared state and the state obtained from the referee, with the referee providing the state randomly from the respective sets. The conditional probability that Alice and Bob obtain the classical outcomes a and b respectively, given that the input states to them are respectively τ_s and ω_t, is denoted by $P(a, b|\tau_s, \omega_t)$.

Let us now consider a situation where one chooses joint measurements that have only two outcomes, i.e., either 0 or 1. The "MDI-EW function" for the state ρ_{AB} is then shown to be given by [31]

$$I(\rho_{AB}) = \sum_{s,t} \beta_{st} P(1, 1|\tau_s, \omega_t). \tag{2}$$

Here,

$$P(1, 1|\tau_s, \omega_t) = \mathrm{tr}[(|\Phi^+\rangle\langle\Phi^+| \otimes |\Phi^+\rangle\langle\Phi^+|)(\tau_s \otimes \rho_{AB}^w \otimes \omega_t)], \tag{3}$$

where outcome 1 indicates the successful projection of the joint measurements by any observer on her or his respective part of the shared state and an input state onto the maximally entangled state, $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. It can be shown after doing a bit of algebra that [31]

$$I(\rho_{AB}) = \frac{\mathrm{tr}(\rho_{AB}W)}{4}. \tag{4}$$

Therefore, entangled states detected by standard entanglement witness operator, W, are also detected by the MDI-EW function I, i.e., $I < 0$ and $\langle W \rangle < 0$ occur for the same set of states. Additionally, in case of the MDI-EW function, whatever be the measurements performed by each party, $I \geqslant 0$ for all separable states [31]. Hence, the function I guarantees entanglement of state independent of measurements that are being performed. The above construction of MDI-EW can easily be generalized to higher dimensions and to cases of states with a higher number of parties [31].

B. Unsharp measurement and modified MDI-EW

It can be seen that evaluation of the MDI-EW function, $I(\rho_{AB})$, given by Eq. (2), requires a two-outcome projective measurement, with projectors

$$\mathcal{P}^+ = |\Phi^+\rangle\langle\Phi^+|,$$
$$\mathcal{P}^- = \mathbb{I}_4 - |\Phi^+\rangle\langle\Phi^+|, \tag{5}$$

where $\mathcal{P}^+$ and $\mathcal{P}^-$ are assumed to correspond to outcomes 1 and 0 respectively. Note that $\mathcal{P}^+$ corresponds to the projector of one of the Bell states, precisely $|\Phi^+\rangle$, whereas $\mathcal{P}^-$ corresponds to the projector onto the span of the remaining three Bell states, being therefore the sum of the projectors of those three Bell states. Now in real experiments, the measurements may not be perfect; that is, the measurement apparatuses may be noisy. In particular, we assume that the projectors corresponding to each of the four Bell states are mixed with a white noise, $\frac{\mathbb{I}_4}{4}$, with equal weight. Therefore, we consider an unsharp version of the above projective measurement, described by "effect" operators $\{\mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-\}$, relative to $\{\mathcal{P}^+, \mathcal{P}^-\}$, given by

$$\mathcal{E}_\lambda^+ = \lambda\mathcal{P}^+ + \frac{1-\lambda}{4}\mathbb{I}_4,$$
$$\mathcal{E}_\lambda^- = \lambda\mathcal{P}^- + 3\frac{1-\lambda}{4}\mathbb{I}_4, \tag{6}$$

where $\mathcal{E}_\lambda^+$ and $\mathcal{E}_\lambda^-$ correspond to outcomes 1 and 0 respectively, and $0 \leqslant \lambda \leqslant 1$. $\mathbb{I}_d$ denotes the identity operator on $\mathbb{C}^d$.

Though there are information processing tasks which consider less restrictive measurements and classical communication between Bobs to obtain any number of successful detection of entanglement [35,45], our goal is slightly different. First, unsharp measurements of the type considered in our paper, i.e, Eq. (6), arise very naturally in the case of a nonideal set up. E.g., if the apparatus is of the Stern-Gerlach type, then the unsharpness characterises the overlap between spatial wave packets associated to the up and down spins. Second, our motivation is to understand the way in which unsharp measurements lead to a nontrivial information-gain versus disturbance trade-off, which put limits on the number of Bobs.

The parameter, λ, determines the sharpness quotient of the measurement, i.e., if $\lambda = 1$, then one obtains the ideal measurement given by Eq. (5). Now, it is interesting to study the MDI-EW given in Eq. (2) by considering the fact that observers perform unsharp measurements. An unsharp version of the projective measurement, given in Eq. (6), will cause a lesser reduction in the amount of shared entanglement than the projective measurement itself. Therefore, the maximum number of pairs of observers who can detect an initially shared entanglement, may be more if unsharp measurements are performed. But, on the other hand, the measurements have to be sharp enough [close enough to the projective measurements, given in Eq. (5)] to detect the entanglement present in the state. This will lead to a restriction on the number of pairs who can detect entanglement starting from an initially entangled state, because whenever measurements are performed to collect information, they reduce the shared entanglement. Therefore, it can be of practical as well as of theoretical interest to study the maximum number of pairs of such observers. For simplicity, we assume that Alice can perform her measurement perfectly but the sequential Bobs have noisy measurement apparatuses. Given the situation that Alice and Bob respectively receive states τ_s and ω_t as inputs and Alice measures in $\{\mathcal{P}^+, \mathcal{P}^-\}$ on her part of ρ_{AB} and τ_s while Bob performs an unsharp measurement with $\{\mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-\}$ on his part of ρ_{AB} and ω_t, the conditional probability that Alice and Bob both obtain outcome 1 is then given by

$$P_\lambda(1, 1|\tau_s, \omega_t) = \text{tr}[(\mathcal{P}^+ \otimes \mathcal{E}_\lambda^+)(\tau_s \otimes \rho_{AB} \otimes \omega_t)]. \quad (7)$$

Therefore, the modified MDI-EW function for the case when one of the parties performs an unsharp measurement, with $\{\mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-\}$, on his part of the state ρ_{AB} and the input from the referee, reads

$$I_\lambda(\rho_{AB}) = \sum_{s,t} \beta_{st} P_\lambda(1, 1|\tau_s, \omega_t). \quad (8)$$

1. *Postmeasurement state*

In our sequential-measurement scenario, the postmeasurement state plays an important role and hence lets us identify the rule for assigning the postmeasurement state to a measurement outcome. Suppose an unsharp joint measurement with $\{\mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-\}$ is performed at one side of the shared state and on the quantum input, denoted by η. According to the von Neumann–Lüders transformation rule [57], up to a unitary, if the + outcome occurs, the postmeasurement state is given by

$$\left(\mathbb{I} \otimes \sqrt{\mathcal{E}_\lambda^+}\right) \eta \left(\mathbb{I} \otimes \sqrt{\mathcal{E}_\lambda^+}\right). \quad (9)$$

C. MDI-EW for Werner states

Consider the (bipartite) Werner states, given by

$$\rho_{AB}^w = q\, |\Psi^-\rangle\langle\Psi^-| + \frac{1-q}{4}\mathbb{I}_4, \quad (10)$$

where q is the mixing probability of the singlet, $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, in ρ_{AB}^w. The MDI-EW function, $I(\rho_{AB}^w)$, for this state can be represented as [31]

$$I(\rho_{AB}^w) = \frac{5}{8}\sum_{s=t} P(1, 1|\tau_s, \omega_t) - \frac{1}{8}\sum_{s\neq t} P(1, 1|\tau_s, \omega_t), \quad (11)$$

where s, t take values 0, 1, 2, and 3, and

$$\tau_s = \sigma_s \frac{\mathbb{I}_2 + \vec{\sigma}\cdot\vec{n}}{2}\sigma_s, \quad \omega_t = \sigma_t \frac{\mathbb{I}_2 + \vec{\sigma}\cdot\vec{n}}{2}\sigma_t, \quad (12)$$

with $\sigma_0 = \mathbb{I}_2$, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being the usual Pauli matrices, and $\vec{n} = \frac{1}{\sqrt{3}}(1, 1, 1)$.

The expression for the MDI-EW function in Eq. (11) can be simplified and written in terms of the state parameter q, as

$$I(\rho_{AB}^w) = \frac{1-3q}{16}. \quad (13)$$

This will be useful in our later calculations.

III. SCENARIO

Consider a scenario where, initially, a bipartite entangled state is shared between two spatially separated laboratories, overseen respectively by Alice (A) and the Bobs (B_i, $i = 1, 2, \ldots, n$). A measures projectively on her part and several Bobs (B_i), in the other laboratory, measure sequentially and independently. The aim is to find the *maximum* number of Bobs, n, such that each AB_i pair is able to witness Buscemi nonlocality or MDI entanglement between them. As operations are local and strong enough to fetch information about the entanglement content of a state shared by A and B_i, it is expected that the state shared by A and B_{i+1} (next Bob in sequence) will have less entanglement than that of AB_i. The unsharp measurement has to be strong enough to detect the shared state's entanglement, and at the same time, it has to be weak enough so that the postmeasured state shared between Alice and the next Bob retains as much entanglement as possible, so that the remnant resource can be used subsequently. This observation tells us that there may exist an upper bound on the maximum number of Bobs, such that each of them can detect entanglement by combining their and Alice's statistics. Note that A can do her part of measurements at any time, i.e., independent of any of the B_i's measurement, as operators from $\mathcal{H}_A$ and those from $\mathcal{H}_B$ commute with each other.

A. Subsequent shared states due to unsharp measurement

Let us consider the cases where Alice, A, and the first Bob, B_1, share a pure entangled state. It will be seen in the following paragraph that subsequent weak measurements by each observer produce a mixed state with a mixture of initial entangled state, shared by the AB_1 pair, and white noise.

Suppose that Alice and the first Bob share the state $|\Psi\rangle = \alpha|01\rangle - \sqrt{1-\alpha^2}|10\rangle$, for $0 < \alpha \leqslant \frac{1}{\sqrt{2}}$, or equivalently, $\rho_{AB_1}^{w_\alpha}$, being given by

$$\rho_{AB_1}^{w_\alpha} = q_1 |\Psi\rangle\langle\Psi| + \frac{1-q_1}{4}\mathbb{I}_4, \quad (14)$$

with $q_1 = 1$. Now, let B_1 measure the unsharp POVM (positive operator valued measurement) with effect operators $\{\mathcal{E}_{\lambda_1}^+, \mathcal{E}_{\lambda_1}^+\}_{BB'}$, and sharpness parameter λ_1, on his part of the shared state and input system B' (from the referee) in state ω_t. The quantum input ω_t has four random choices, say, for each $t = 0, 1, 2$, and 3, occurring with equal probability. As each Bob measures independently, the average state ρ_{AB_2} that A and the next Bob B_2 share is given by

$$\mathrm{tr}_{B'}\Big\{\frac{1}{4}\sum_{t=0}^{3}\Big[\big(\mathbb{I}_2\otimes\sqrt{\mathcal{E}^{+}_{\lambda_1}}\big)\big(\rho^{w_\alpha}_{AB_1}\otimes\omega_t\big)\big(\mathbb{I}_2\otimes\sqrt{\mathcal{E}^{+}_{\lambda_1}}\big) + \big(\mathbb{I}_2\otimes\sqrt{\mathcal{E}^{-}_{\lambda_1}}\big)\big(\rho^{w_\alpha}_{AB_1}\otimes\omega_t\big)\big(\mathbb{I}_2\otimes\sqrt{\mathcal{E}^{-}_{\lambda_1}}\big)\Big]\Big\}, \tag{15}$$

simplifying which again turns out to be a state of form in Eq. (14), viz.,

$$\rho^{w_\alpha}_{AB_2} = q_2|\Psi\rangle\langle\Psi| + \frac{1-q_2}{4}\mathbb{I}_4,$$

with $q_2 = f(\lambda_1)q_1$, where

$$f(\lambda) = \frac{1}{2}\Big[1 + \frac{\sqrt{(1+3\lambda)(1-\lambda)} + \sqrt{(3-3\lambda)(3+\lambda)}}{4}\Big]. \tag{16}$$

The above structure is iterative, and therefore the state that the AB_i duo possesses reads

$$\rho^{w_\alpha}_{AB_i} = q_i|\Psi\rangle\langle\Psi| + \frac{1-q_i}{4}\mathbb{I}_4,$$

where

$$q_i = f(\lambda_{i-1})q_{i-1}, \tag{17}$$

with $f(\lambda_i)$ being given in Eq. (16).

B. Modified MDI-EW for nonmaximally entangled states mixed with white noise

In this subsection, we show how MDI-EW can be modified for unsharp measurements on a shared state $\rho^{w_\alpha}_{AB}$. Note that the Werner states, ρ^{w}_{AB}, are a mixture of a singlet with white noise. For this class, the MDI-EW, $I(\rho^{w}_{AB})$, is an optimal witness [30,31].

Further, as the MDI-EW is independent of measurements, the bound for separable states remains zero even when one of the parties perform unsharp measurements. Hence, the modified measurement-device-independent entanglement witness for states, $\rho^{w_\alpha}_{AB}$, with B doing an unsharp measurement is given by

$$I_\lambda(\rho^{w_\alpha}_{AB}) = \frac{5}{8}\sum_{s=t}P_\lambda(1,1|\tau_s,\omega_t) - \frac{1}{8}\sum_{s\neq t}P_\lambda(1,1|\tau_s,\omega_t), \tag{18}$$

where $P(1,1|\tau_s,\omega_t)$ for state ρ^{w}_{AB} in Eq. (11) is just replaced by $P_\lambda(1,1|\tau_s,\omega_t)$ for state $\rho^{w_\alpha}_{AB}$. We find that

$$I_\lambda(\rho^{w_\alpha}_{AB}) = -\frac{\lambda q\alpha\sqrt{1-\alpha^2}}{4} + \frac{1-\lambda q}{16}. \tag{19}$$

It can be seen that for $\lambda = 1$ and $\alpha = \frac{1}{\sqrt{2}}$, Eq. (19) reduces to Eq. (13). It gives a lower bound on the sharpness parameter, which we refer to as the "threshold sharpness parameter," $\lambda^{th} = \frac{1}{q(1+4\alpha\sqrt{1-\alpha^2})}$, such that $I_\lambda(\rho^{w_\alpha}_{AB}) < 0$, $\forall\lambda > \lambda^{th}$. Note that for the maximal resourceful state, i.e., the singlet, $\lambda^{th} = \frac{1}{3}$, which is the lowest for any entangled state of the form $\rho^{w_\alpha}_{AB}$.

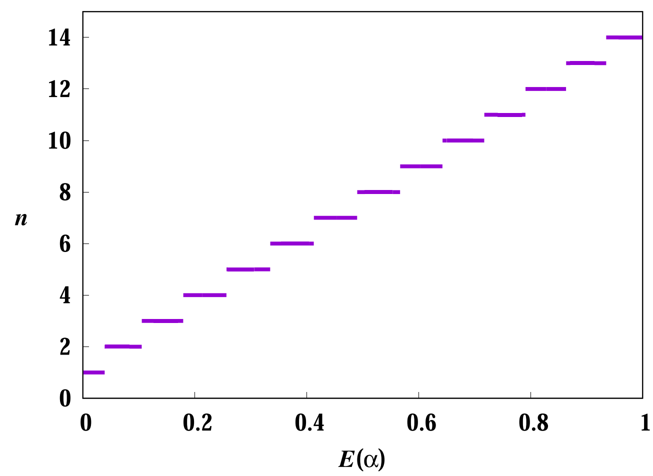

FIG. 1. Sequential witnessing of entanglement in a measurement-device-independent scenario. We plot here the maximum number n in the MDI scenario vs the entanglement, $E(\alpha)$, of the initial shared state $|\Psi\rangle$. The vertical axis is dimensionless, while the horizontal one is in ebits.

IV. WITNESSING BUSCEMI NONLOCALITY SEQUENTIALLY WITH INITIALLY SHARED ENTANGLED PURE STATE

We now move on to study the maximum number (n) of Bobs who can act independently and sequentially to witness shared entanglement with a single observer, Alice, in the measurement-device-independent scenario. Note that this maximum is achieved only when all of the Bobs measure with their respective threshold sharpness parameters. The initial state shared between the two laboratories is assumed to be pure entangled state, $|\Psi\rangle = \alpha|01\rangle - \sqrt{1-\alpha^2}|10\rangle$, where $0 < \alpha \leqslant 1/\sqrt{2}$. The entanglement in this state is measured by the von Neumann entropy of the reduced density matrix (entanglement entropy) and is given by $E(\alpha) = -\alpha^2\log_2\alpha^2 - (1-\alpha^2)\log_2(1-\alpha^2)$.

In Fig. 1, we depict the maximum number of Bobs that can detect entanglement in an MDI-way for a given entanglement content $E(\alpha)$. In particular, we find that if the initial shared state is close to the maximally entangled state, viz. if $E(\alpha) \gtrapprox 0.9349$, the maximum number of Bobs, n, which can keep the state entangled, reaches 14, the highest in the given scenario. In a similar study using standard EWs [43], which can be termed a "device-dependent (DD) scenario," the maximum number again remains fixed for a certain finite range of initial shared entanglement (of the AB_1 pair). However, it is interesting to notice that in the DD case, the maximum number of Bobs that can identify entanglement is 12, which is less than that in the MDI scenario considered here. For any initially shared pure entangled state, the maximum number of Bobs in the device-dependent scenario of Ref. [43], denoted by n^{DD}, is either less or equal to that in the MDI scenario considered here; i.e., $n^{DD} \leqslant n$ for any given initial entanglement.

The lower value of n^{DD} than n for arbitrary initially shared pure entanglement deserves a comment. This is arguably due to the fact that the number of successful Bobs detecting entanglement with a single Alice depends on the choice of the witness, and in particular, on how the measurement disturbs the shared state. In the standard DD scenario considered in Ref. [43], the sharp limit of the unsharp measurements are rank-1 projective measurements, while the MDI scenario considered here involves quantum inputs, and the sharp measurement limit on the portion of the shared state in possession of the Bobs becomes a POVM of nonunit rank. A nonunit rank measurement has a general tendency to affecting the entanglement of the shared state less, and potentially affects the MDI procedure when we are far from the beginning Bob in the sequence of Bobs. It is to be remembered that the later Bobs are required to make sharper measurements to detect entanglement. Another point to mention in this respect is that the MDI scenario uses quantum inputs at both the laboratories possessing the bipartite state, and the subsequent measurements at both the laboratories are on $\mathbb{C}^2 \otimes \mathbb{C}^2$. In contrast, the DD scenario of Ref. [43] considers single-qubit measurements at both the laboratories. Consequently, a comparison between the two scenarios is made difficult by another roadblock.

Comparing the results in Ref. [43] with ours, we can comment that the MDI scenario for witnessing entanglement is more robust in the context of unsharp measurements than that of the device-dependent witness. As we mentioned earlier, MDI-EW was also shown to be robust against standard EW in the case of lossy detectors [31].

V. REDUCTION IN ENTANGLEMENT BY UNSHARP MEASUREMENT: LIMIT ON SUCCESSFUL DETECTION OF ENTANGLEMENT SEQUENTIALLY

In this section, we study the entanglement content of bipartite states shared by Alice and each of the sequential Bobs. We investigate the reduction in bipartite entanglement, occurring due to the unsharp measurement performed at one side. For this purpose, we calculate the negativity [58–61], N, which for the state, $\rho_{AB_i}^{w_\alpha}$, is given by

$$N(\rho_{AB_i}^{w_\alpha}) = \max\left\{\frac{q_i(1+4\alpha\sqrt{1-\alpha^2})-1}{4}, 0\right\}. \tag{20}$$

For simplicity of notation, we will use N_i instead of $N(\rho_{AB_i}^{w_\alpha})$, $i = 1, 2, \dots, n$. The change in the negativity, denoted by $\Delta N_i(\lambda_i)$, due to an unsharp measurement by B_i which is "valid" for $0 \leqslant \lambda_i \leqslant 1$ and "required" to satisfy $\lambda_i > \lambda_i^{th}$ to witness entanglement in $\rho_{AB_i}^{w_\alpha}$, is defined as the difference in the negativities of the states before and after this measurement, i.e.,

$$\Delta N_i(\lambda_i) = N_i - N_{i+1}. \tag{21}$$

Here, by "valid," we mean that the parameters λ_i in the measurement are to be chosen in the given range ($0 \leqslant \lambda_i \leqslant 1$) for the measurement to be to be quantum mechanically allowed, and by "required," we mean that the sharpness parameters λ_i are to be chosen such that entanglement present can be detected. Surely, $\Delta N_i(\lambda_i)$ is a positive quantity, as local measurements can only keep or decrease entanglement. The negativity of the state that observers A and B_{i+1} share can be obtained by the above equation if one knows the negativity of the state that A and B_i share, and the change in negativity due to the measurement by B_i. This procedure is repeated by subsequent Bobs, until the negativity of the state shared between some B_{i+1} and A, after an unsharp measurement by B_i, reduces to zero.

The change in the negativity of $\rho_{AB_i}^{w_\alpha}$, due to a "valid" and "required" measurement by B_i, can be evaluated to be

$$\begin{aligned}\Delta N_i(\lambda_i) &= \frac{1+4N_i}{4}(1-f(\lambda_i)), \quad N_{i+1} \neq 0;\\ &= N_i, \quad N_{i+1} = 0.\end{aligned} \tag{22}$$

It can be easily checked that $\Delta N_i(\lambda_i) > 0$ for $1/3 > \lambda_i \geqslant 1$ for any $N_i \neq 0$. Therefore, subsequent measurements to witness the shared entanglement result only in lowering of entanglement content. This is expected, as negativity is an entanglement monotone and therefore its value either decreases or remains the same under local quantum operations and classical communication, as mentioned earlier. One can also observe that it is an increasing function of λ_i, and therefore sharper measurements correlate with greater decrease in the entanglement content.

Note that the unsharp measurement parameter, λ_i, for each i, should be equal to the threshold unsharpness parameter, λ_i^{th}, for the purpose of witnessing the shared entanglement sequentially in the optimal scenario (to obtain the maximum number of B_i who can sequentially witness the shared entanglement with A), discussed in the previous section. The threshold unsharpness parameter further depends on the negativity of the shared state, $\rho_{AB_i}^{w_\alpha}$, via the relation

$$\lambda_i^{th} = \frac{1}{4N_i+1}. \tag{23}$$

Therefore, in the optimal scenario, the change in negativity of state, $\rho_{AB_i}^{w_\alpha}$, with negativity, N_i, when $N_{i+1} \neq 0$, turns out to be

$$\Delta N_i(\lambda_i^{th}) = \tfrac{1}{8}[1+4N_i - \sqrt{N_i(1+N_i)} - \sqrt{3N_i(1+3N_i)}]. \tag{24}$$

Note that given a state with negativity N_i, $\Delta N_i(\lambda_i^{th})$ is always positive, and a strictly increasing function of the discrete variable i (for $N_{i+1} \neq 0$), which guarantees that $N_j = 0$ can be reached at some finite number of B_j.

VI. EQUISTRENGTH UNSHARP MEASUREMENTS

The optimal scenario where subsequent observers are allowed to measure with the threshold value of sharpness parameter can be experimentally challenging as well as costly. It can be challenging because each subsequent Bob needs to tune the apparatus precisely to attain the maximum number of Bobs, which can be costly if they need to use separate apparatuses for the different sharpness parameter. In this section, therefore, we put some more restrictions on the Bobs. Specifically, independence of the sequential observers is lifted, to the extent that they are required to measure with equal sharpness parameter, $\lambda \in (\frac{1}{3}, 1]$. In Fig. 2, the maximum number, n, of such sequential observers with the same sharpness parameter, λ, is plotted for fixed entanglement contents of the initial state, namely $E(\alpha) = 0.935$ and 1 ebit. In the former case, the maximum n over all parameter range of λ, denoted by say, $n_{\max}$, is found to be five, whereas in the latter, the same maximum is six. Note that for any given initial entanglement, n is the maximum number of Bobs measuring at any

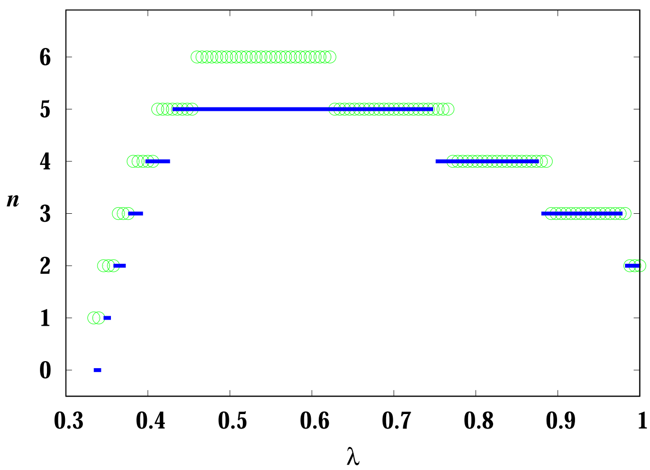

FIG. 2. How weak can effective and equally strong Bobs be to maximize their number? We plot here the maximum number of observers, n, making unsharp measurements, against the common sharpness parameter, λ. The Bobs are required to do unsharp measurements of equal strength. The green circles represent situation for which the shared initial state has $E(\alpha) = 1$, while the blue line is for initial states having $E(\alpha) = 0.935$. Both axes represent dimensionless quantities, while E is in ebits.

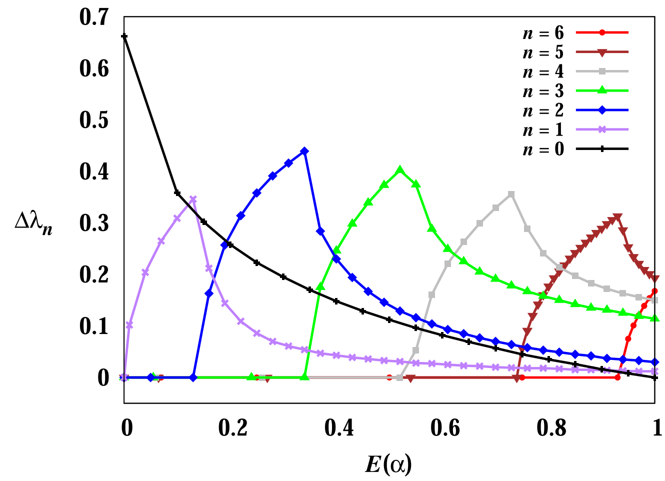

FIG. 3. Variation of the range of the common sharpness parameter with respect to the initial entanglement in the equistrength unsharp measurements scenario. The range of common sharpness parameter, $\Delta\lambda_n$, is plotted against the initial entanglement, $E(\alpha)$, of the shared state with n as the parameter. The ordinate is dimensionless while abscissa is in ebits.

common sharpness parameter, λ, whereas $n_{\max}$ denotes the maximum n over all λ. We can see that for initially shared nearly maximally entangled states, $n_{\max} = 6$. Again, this is better when compared to the same task in a device-dependent scenario, where a maximum of five observers can witness the entanglement with equal unsharp measurements for initially shared nearly maximally entangled states [43].

In the DD case, observers performing the sharpest measurement (measurement with sharpness parameter equal to 1) cause shared entanglement between two laboratories to vanish. That is, only the first Bob can detect the entanglement with Alice and the rest Bobs cannot. On the other hand, here we observe that even after the sharpest measurement, the shared entanglement will exist for some initial shared entangled states. This can be seen in Fig. 2; e.g., when the initial state possesses nearly maximal or maximal entanglement, two Bobs can detect entanglement with sharpness parameter being equal to unity. The fact that entanglement can be nonzero even after a sharp measurement suggests a better robustness of the MDI-EW compared to the device-dependent witness operator.

Note that for any initially shared pure entangled state, $n = n_{\max}$ is reached for intermediate values (not very high and not very low) of sharpness measurement parameter, λ; i.e., $n = n_{\max}$ is never achieved for values of λ close to $1/3$ or close to 1 (see Fig. 2). Specifically, as one moves away from the intermediate values of λ on the either side, i.e., either higher or lower values, maximum number of observers, n, measuring unsharply with equal strengths, either decrease or remain the same. This suggests that in order to achieve the maximum of n over all the values of sharpness parameter, the observers should not set their sharpness parameter too high or too low. Such observation can be explained by the results reported in Sec. V. If the first observer, B_1, sharing a state with A having negativity N_1, measures unsharply with a parameter λ_1, then the state that is at the disposal of A and B_2 surely possesses, on average, a lower value of negativity, N_2, compared to N_1, i.e., $N_2 < N_1$. This can be seen from the relation given in Eq. (22). Since the negativity is decreasing with subsequent measurements, the new threshold parameter, λ_2, is greater than λ_1 [see Eq. (23)]. Therefore, if B_1 fixes the sharpness parameter to be at the threshold value at which he can detect the shared entanglement, i.e., at λ_1^{th}, then only he can witness the entanglement while others cannot, as the threshold sharpness parameter to detect entanglement will be greater for subsequent Bobs. This explains the occurrence of the least number of B_i at lower values of sharpness parameter. On the other hand, if B_1 chooses to measure sharply, i.e., $\lambda_1 = 1$, then the state is disturbed to the maximum possible, as discussed in Sec. V, and therefore, a lower number of B_is can only witness entanglement sequentially with the same λ.

Let us now study the dependence of the length or the range of common sharpness parameter, denoted by $\Delta\lambda_n$, on the initial entanglement, $E(\alpha)$, of the shared state, with n being the parameter. For $n = n_{\max}$, $\Delta\lambda_n$ decreases with decrease in $E(\alpha)$, and for the rest of the values of n, $\Delta\lambda_n$ increases with decrease in $E(\alpha)$. See Fig. 3.

VII. CONCLUSION

Entangled states have already been established as a resource in several quantum information processing tasks. Therefore, detection of entanglement in laboratory setups is an important task. If partial knowledge of an entangled state is available, employing entanglement witnesses for entanglement detection is, in principle, possible with trusted devices. On the other hand, violation of Bell inequality certifies entanglement in a device-independent way but at the cost that not all entangled states violate a Bell inequality. To bridge this gap, a measurement-device-independent entanglement witness (MDI-EW) has recently been introduced which yields a higher payoff for every entangled state compared to separable states by invoking a semiquantum nonlocal game. Here we employed a MDI-EW to detect entanglement in a entanglement distribution scenario where half of a pure entangled state is measured by a single observer, while the other half is measured by several observers sequentially and independently. We found that the number of observers who successfully detect entanglement with the other party is larger than in the similar sequential scenarios considered for violation of Bell inequality and for device-dependent entanglement witness operators. More interestingly, we observed that without employing unsharp measurements, one can still have detection of entanglement up to two observers, which was not the case for the two other entanglement identification schemes. Both these results established that the MDI-EW method is more robust than the other methods of entanglement detection. The sequential sharing of entanglement was studied both in the cases when all the observers are free to choose their optimal unsharp measurements and when all of them are constrained to choose a specific unsharp measurement. Our results show that sequential sharing of quantum states in a measurement-device-independent way can be beneficial for quantum information processing tasks.

ACKNOWLEDGMENTS

The authors acknowledge support from the Department of Science and Technology, Government of India through the QuEST grant (Grants No. DST/ICPS/QUST/Theme-1/2019/23 and No. DST/ICPS/QUST/Theme-3/2019/120).

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[2] J. S. Bell, On the Einstein Podolsky Rosen paradox, Physics **1**, 195 (1964).

[3] R. Colbeck, Quantum and relativistic protocols for secure multi-party computation, Ph.D. thesis, Cambridge University, Cambridge, UK, 2009 (unpublished).

[4] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, Nature (London) **464**, 1021 (2010).

[5] A. Acín and L. Masanes, Certified randomness in quantum physics, Nature (London) **540**, 213 (2016).

[6] P. Bierhorst, E. Knill, S. Glancy, Y. Zhang, A. Mink, S. Jordan, A. Rommal, Y.-K. Liu, B. Christensen, S. W. Nam, M. J. Stevens, and L. K. Shalm, Experimentally generated randomness certified by the impossibility of superluminal signals, Nature (London) **556**, 223 (2018).

[7] C. H. Bennett and G. Brassard, Quantum cryptography: Public key distribution and coin tossing, in *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, New York, 1984), p. 175.

[8] A. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).

[9] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-Independent Security of Quantum Cryptography Against Collective Attacks, Phys. Rev. Lett. **98**, 230501 (2007).

[10] A. Acín, D. Cavalcanti, E. Passaro, S. Pironio, and P. Skrzypczyk, Necessary detection efficiencies for secure quantum key distribution and bound randomness, Phys. Rev. A **93**, 012319 (2016).

[11] A. Boaron, G. Boso, D. Rusca, C. Vulliez, C. Autebert, M. Caloz, M. Perrenoud, G. Gras, F. Bussières, M.-J. Li, D. Nolan, A. Martin, and H. Zbinden, Secure Quantum Key Distribution Over 421 km of Optical Fiber, Phys. Rev. Lett. **121**, 190502 (2018).

[12] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum State Via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).

[13] L. Hardy, Disentangling nonlocality and teleportation, arXiv:quant-ph/9906123.

[14] M. Żukowski, Bell theorem for the nonclassical part of the quantum teleportation process, Phys. Rev. A **62**, 032101 (2000).

[15] S. Ghosh, G. Kar, A. Roy, and U. Sen, Entanglement versus noncommutativity in teleportation, Phys. Rev. A **65**, 032309 (2002).

[16] A. Kumar, I. Chakrabarty, A. K. Pati, A. Sen(De), and U. Sen, Quantum no-go theorems in causality respecting systems in the presence of closed timelike curves: Tweaking the Deutsch condition, EPL **122**, 10007 (2018).

[17] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).

[18] B. M. Terhal, Bell inequalities and the separability criterion, Phys. Lett. A **271**, 319 (2000).

[19] D. Bruß, J. I. Cirac, P. Horodecki, F. Hulpke, B. Kraus, M. Lewenstein, and A. Sanpera, Reflections upon separability and distillability, J. Mod. Opt. **49**, 1399 (2002).

[20] O. Gühne and G. Tóth, Entanglement detection, Phys. Rep. **474**, 1 (2009).

[21] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A **40**, 4277 (1989).

[22] V. Scarani and N. Gisin, Spectral decomposition of Bell's operators for qubits, J. Phys. A **34**, 6043 (2001).

[23] J. Barrett, Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a Bell inequality, Phys. Rev. A **65**, 042302 (2002).

[24] M. Żukowski, Č. Brukner, W. Laskowski, and M. Wieśniak, Do All Pure Entangled States Violate Bell's Inequalities for Correlation Functions? Phys. Rev. Lett. **88**, 210402 (2002).

[25] A. Sen(De), U. Sen, and M. Żukowski, Functional Bell inequalities can serve as a stronger entanglement witness than conventional Bell inequalities, Phys. Rev. A **66**, 062318 (2002).

[26] P. Hyllus, O. Gühne, D. Bruß, and M. Lewenstein, Relations between entanglement witnesses and Bell inequalities, Phys. Rev. A **72**, 012321 (2005).

[27] A. A. Methot and V. Scarani, An anomaly of non-locality, Quantum Inf. Comput. **7**, 157 (2007)

[28] J. Silman, S. Machnes, and N. Aharon, On the relation between Bell's inequalities and nonlocal games, Phys. Lett. A **372**, 3796 (2008).

[29] F. Buscemi, All Entangled Quantum States are Nonlocal, Phys. Rev. Lett. **108**, 200401 (2012).

[30] O. Gühne, P. Hyllus, D. Bruß, A. Ekert, M. Lewenstein, C. Macchiavello, and A. Sanpera, Experimental detection of entanglement via witness operators and local measurements, J. Mod. Opt. **50**, 1079 (2003).

[31] C. Branciard, D. Rosset, Y. C. Liang, and N. Gisin, Measurement-Device-Independent Entanglement Witnesses for all Entangled Quantum States, Phys. Rev. Lett. **110**, 060405 (2013).

[32] P. Skwara, H. Kampermann, M. Kleinmann, and D. Bruß, Entanglement witnesses and a loophole problem, Phys. Rev. A **76**, 012312 (2007).

[33] K. Sen, S. Das, and U. Sen, Closing the detection loophole in nonlinear entanglement witnesses, Phys. Rev. A **100**, 062333 (2019).

[34] K. Sen, C. Srivastava, S. Mal, A. Sen (De), and U. Sen, Detection loophole in measurement-device-independent entanglement witness, arXiv:2004.09101 [quant-ph].

[35] R. Silva, N. Gisin, Y. Guryanova, and S. Popescu, Multiple Observers can Share the Nonlocality of Half of an Entangled Pair by Using Optimal Weak Measurements, Phys. Rev. Lett. **114**, 250401 (2015).

[36] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).

[37] S. Mal, A. S. Majumdar, and D. Home, Sharing of nonlocality of a single member of an entangled pair of qubits is not possible by more than two unbiased observers on the other wing, Mathematics **4**, 48 (2016).

[38] M. Schiavon, L. Calderaro, M. Pittaluga, G. Vallone, and P. Villoresi, Three-observer Bell inequality violation on a two-qubit entangled state, Quantum Sci. Technol. **2**, 015010 (2017).

[39] M.-J. Hu, Z.-Y. Zhou, X.-M. Hu, C.-F. Li, G.-C. Guo, and Y.-S. Zhang, Observation of non-locality sharing among three observers with one entangled pair via optimal weak measurement, npj Quantum Inf. **4**, 63 (2018).

[40] D. Das, A. Ghosal, S. Sasmal, S. Mal, and A. S. Majumdar, Facets of bipartite nonlocality sharing by multiple observers via sequential measurements, Phys. Rev. A **99**, 022305 (2019).

[41] S. Sasmal, D. Das, S. Mal, and A. S. Majumdar, Steering a single system sequentially by multiple observers, Phys. Rev. A **98**, 012305 (2018).

[42] A. Shenoy H, S. Designolle, F. Hirsch, R. Silva, N. Gisin, and N. Brunner, Unbounded sequence of observers exhibiting Einstein-Podolsky-Rosen steering, Phys. Rev. A **99**, 022317 (2019).

[43] A. Bera, S. Mal, A. Sen (De), and U. Sen, Witnessing bipartite entanglement sequentially by multiple observers, Phys. Rev. A **98**, 062304 (2018).

[44] F. J. Curchod, M. Johansson, R. Augusiak, M. J. Hoban, P. Wittek, and A. Acín, Unbounded randomness certification using sequences of measurements, Phys. Rev. A **95**, 020102(R) (2017)

[45] P. J. Brown and R. Colbeck, Arbitrarily Many Independent Observers can Share the Nonlocality of a Single Maximally Entangled Qubit Pair, Phys. Rev. Lett. **125**, 090401 (2020).

[46] G. Foletto, L. Calderaro, A. Tavakoli, M. Schiavon, F. Picciariello, A. Cabello, P. Villoresi, and G. Vallone, Experimental Certification of Sustained Entanglement and Nonlocality after Sequential Measurements, Phys. Rev. Appl. **13**, 044008 (2020).

[47] K. Mohan, A. Tavakoli, and N. Brunner, Sequential random access codes and self-testing of quantum measurement instruments, New J. Phys. **21**, 083034 (2019).

[48] A. Tavakoli, M. Smania, T. Vertesi, N. Brunner, and M. Bourennane, Self-testing non-projective quantum measurements in prepare-and-measure experiments, Sci. Adv. **6**, 16 (2020).

[49] N. Miklin, J. J. Borkała, and M. Pawłowski, Semi-device-independent self-testing of unsharp measurements, Phys. Rev. Research **2**, 033014 (2020).

[50] S. Roy, A. Bera, S. Mal, A. Sen(De), and U. Sen, Recycling the resource: Sequential usage of shared state in quantum teleportation with weak measurements, Phys. Lett. A **392**, 127143 (2021).

[51] P. Busch, P. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement*, 2nd ed. (Springer, Berlin, 1996).

[52] C. A. Fuchs and A. Peres, Quantum-state disturbance versus information gain: Uncertainty relations for quantum information, Phys. Rev. A **53**, 2038 (1996).

[53] C. A. Fuchs, Information gain vs. state disturbance in quantum theory, Fortschr. Phys. **46**, 535 (1998).

[54] M. Ozawa, Universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement, Phys. Rev. A **67**, 042105 (2003).

[55] C. H. Bennett and S. J. Wiesner, Communication via One- and Two-Particle Operators On Einstein-Podolsky-Rosen States, Phys. Rev. Lett. **69**, 2881 (1992).

[56] H.-K. Lo, M. Curty, and B. Qi, Measurement-Device-Independent Quantum Key Distribution, Phys. Rev. Lett. **108**, 130503 (2012).

[57] P. Busch, Unsharp reality and joint measurements for spin observables, Phys. Rev. D **33**, 2253 (1986).

[58] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, Volume of the set of separable states, Phys. Rev. A **58**, 883 (1998).

[59] J. Lee, M. S. Kim, Y. J. Park, and S. Lee, Partial teleportation of entanglement in a noisy environment, J. Mod. Opt. **47**, 2151 (2000).

[60] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).

[61] M. B. Plenio, Logarithmic Negativity: A Full Entanglement Monotone That is not Convex, Phys. Rev. Lett. **95**, 090503 (2005).