

Quantum dynamics under continuous projective measurements: Non-Hermitian description and the continuum-space limit

Varun Dubey^{1,*}, Cédric Bernardin^{2,†} and Abhishek Dhar^{1,‡}

¹*International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru 560089, India*

²*Université Côte d'Azur, CNRS, LJAD Parc Valrose, 06108 NICE Cedex 02, France*



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The problem of the time of arrival of a quantum system in a specified state is considered in the framework of the repeated measurement protocol and in particular the limit of continuous measurements is discussed. It is shown that for a particular choice of system-detector coupling, the Zeno effect is avoided and the system can be described effectively by a non-Hermitian effective Hamiltonian. As a specific example we consider the evolution of a quantum particle on a one-dimensional lattice that is subjected to position measurements at a specific site. By solving the corresponding non-Hermitian wave-function evolution equation, we present analytic closed-form results on the survival probability and the first arrival time distribution. Finally we discuss the limit of vanishing lattice spacing and show that this leads to a continuum description where the particle evolves via the free Schrödinger equation with complex Robin boundary conditions at the detector site. Several interesting physical results for this dynamics are presented.

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I. INTRODUCTION

The question of the time of arrival of a quantum particle at a detector is a difficult problem since this requires one to make measurements on a system which for quantum systems lead to a change in the system's evolution dynamics. The most dramatic manifestation of this fact is the so-called Zeno effect where a system under constant observation is never detected [1]. In a more general setting, consider a quantum system that is prepared in an initial state $|\phi\rangle$ and then evolves under a unitary dynamics. One is interested in the time of return of the system to the initial state or its arrival to a state $|\chi\rangle$ that is orthogonal to $|\phi\rangle$. A well-defined protocol to answer this question is to consider a sequence of projective measurements done on the system at regular intervals of time, say τ , whose output is 1 (a “yes”) if the system is in the desired final state and is otherwise 0 (a null measurement). The experiment stops when we record a 1 and, if this happens on the n th measurement, we say that the time of detection is $t = n\tau$. Until the time of detection, the system's evolution thus consists of a unitary evolution that is conditioned by null measurements (nondetection in the target state) at regular intervals of time and hence repeated projections into the space complementary to the target space.

In the second of a series of three papers on the quantum time of arrival problem [2], Allcock discussed the repeated projection protocol in the context of the time of arrival of a particle which is released at some point $x < 0$ on the real line and eventually detected by a measuring device at the

origin $x = 0$. Allcock considered a sequence of measurements at discrete time intervals which project the particle to the domain $x < 0$ till the time of detection. It was argued that this process could be effectively modeled by including an imaginary potential $-iV_0\theta(-x)$, where $\theta(x)$ is the Heaviside function, in the free Schrödinger evolution of the particle. The strength of the potential, V_0 , is related to the intervals between measurements, τ , as $V_0 \sim \tau^{-1}$. The limit $\tau \rightarrow 0$ gives the Zeno effect and it can be shown that unitary evolution takes place in the projected sub-space with hard-wall boundary conditions [3]. However, for the case of finite τ , we are not aware of a formal derivation of the equivalence of the repeated projection dynamics with the non-Hermitian Hamiltonian proposed by Allcock [4–6]. In more recent work [7–9] the first detection problem using the repeated measurement scheme has been investigated in a lattice model for a quantum particle satisfying the discrete Schrödinger equation. It was numerically demonstrated that a description of the repeated measurement dynamics by means of an effective non-Hermitian Hamiltonian is in fact quite accurate for small values of τ (more precisely for $\tau \ll \gamma^{-1}$ where γ is the transition rate between lattice sites). In an independent set of work [10,11], for systems with discrete Hilbert spaces, the arrival time problem using the repeated measurements protocol has been treated using a renewal approach similar to what one uses for the classical Polya first passage problem. Using this approach, a number of interesting physical results on first passage in the lattice Schrödinger problem have been obtained in Refs. [12–16]. In particular it has been shown that the return probability on an infinite one-dimensional (1D) lattice is less than one and we do not have the Polya recurrence of the 1D random walk. The distribution of first detection times has been shown to scale as $1/t^3$. Different aspects of the equivalence to the non-Hermitian Hamiltonian have been

*varun.dubey@icts.res.in

†Cedric.BERNARDIN@unice.fr

‡abhishek.dhar@icts.res.in

explored in Refs. [9,17,18]. Other interesting results include discussions of the non-Hermitian description in many-body systems [19,20] and in the context of experiments [21,22].

The present work makes several contributions. First we show that, for systems with a discrete Hilbert space, the equivalence between the repeated projection dynamics and the non-Hermitian Hamiltonian can be established rigorously in a particular limit where we let the measurement time interval $\tau \rightarrow 0$ while taking a large value (of order $1/\tau^{1/2}$) for the strength of the coupling strength between the system and detector. This choice is motivated by similar ones used in the description of the trajectories of continuously monitored quantum systems [23]. Secondly we study the example of a quantum particle on different 1D lattices (finite, semi-infinite and infinite) and, using a Fourier-Laplace transform of the effective equations of motion, obtain closed-form expressions for the survival probability and detection time distribution. We compute the spectrum of the effective non-Hermitian Hamiltonian and show that the results from the Fourier-Laplace solution also follow from a spectral analysis. Finally, we consider the lattice-to-continuum space limit and show that the evolution is described by a Schrödinger equation with complex Robin boundary condition at the location of the detector. Apart from the survival probability and the full spectrum, the form of the “surviving” wave function is also discussed. Such boundary conditions have earlier been discussed formally in the context of the time of arrival problem [24–26].

The plan of the paper is as follows. In Sec. II we describe the limiting procedure which gives the exact mapping between the repeated projective measurements and the non-Hermitian Hamiltonian. In Sec. III we consider the example of a quantum particle on three 1D lattices and write the form of the effective non-Hermitian Hamiltonian for these cases. The exact results for survival probability and detection time distribution using the Fourier-Laplace transform approach and the Green’s function approach are presented in Sec. IV. In Sec. V we discuss the continuum space limit and finally conclude with a summary of the results in Sec. VI.

II. DESCRIPTION OF DYNAMICS UNDER REPEATED PROJECTIVE MEASUREMENTS BY AN EFFECTIVE NON-HERMITIAN HAMILTONIAN: EXACT MAPPING

Consider a quantum system whose states belong to the Hilbert space \mathcal{H} equipped with some inner product $\langle\phi|\psi\rangle$ [27]. We assume that $\mathcal{H} = \mathcal{S} \oplus \mathcal{D}$ can be written as the sum of two orthogonal complementary subspaces, where \mathcal{S} is a “system” space and \mathcal{D} a “detector” space. Let $|i\rangle$ be an orthonormal basis of \mathcal{S} and $|\alpha\rangle$ an orthonormal basis of \mathcal{D} . The set of states $\{|i\rangle, |\alpha\rangle\}$ together form a complete orthonormal basis of \mathcal{H} . Hence we have

$$\sum_i |i\rangle\langle i| + \sum_\alpha |\alpha\rangle\langle\alpha| = I, \quad (1)$$

where I is the identity operator. The most general Hamiltonian describing such a system is given by

$$H = H^{(\mathcal{S})} + H^{(\mathcal{D})} + H^{(\mathcal{SD})},$$

$$H^{(\mathcal{S})} = \underbrace{\sum_{i,j} H_{ij}^{(\mathcal{S})} |i\rangle\langle j|}_{\text{System}}, \quad H^{(\mathcal{D})} = \underbrace{\sum_{\alpha,\beta} H_{\alpha\beta}^{(\mathcal{D})} |\alpha\rangle\langle\beta|}_{\text{Detector}},$$

$$H^{(\mathcal{SD})} = \underbrace{\sum_{i,\alpha} [H_{i\alpha}^{(\mathcal{SD})} |i\rangle\langle\alpha| + H_{\alpha i}^{(\mathcal{SD})} |\alpha\rangle\langle i|]}_{\text{System-Detector}}. \quad (2)$$

Let $|\psi(t)\rangle \in \mathcal{H}$ denote the state of the total system at time t . The unitary evolution of this state with the above Hamiltonian H is given by

$$|\psi(t)\rangle = U_t |\psi(0)\rangle, \quad U_t = \exp(-itH/\hbar). \quad (3)$$

Consider instantaneous projective measurements made to find if the system is in the detector subspace \mathcal{D} . This corresponds to the projection operator

$$P = \sum_\alpha |\alpha\rangle\langle\alpha|, \quad (4)$$

while the complementary operator corresponding to projection into the system subspace \mathcal{S} is given by

$$Q = I - P = \sum_i |i\rangle\langle i|. \quad (5)$$

Now imagine an experiment where initially the system starts in the space \mathcal{S} and begins to evolve unitarily. We keep making measurements, at regular intervals of time τ , to find if the system has arrived in the detector subspace \mathcal{D} . If the result of the measurement is negative, the system (now projected back into \mathcal{S}) continues its unitary evolution, until the next measurement and the process is repeated. The experiment stops when we get a positive result indicating arrival into \mathcal{D} . Thus the evolution of the state vector consists of a sequence of steps in \mathcal{S} , each consisting of a unitary evolution followed by the projection Q . Looking at the state of the system conditioned on survival (nondetection), let us denote by $|\psi(n\tau)\rangle$ the state just after the n th measurement. Then it can be shown [8] that

$$|\psi(n\tau)\rangle = \tilde{U}_\tau^n |\psi(0)\rangle, \quad \tilde{U}_\tau = QU_\tau Q. \quad (6)$$

The probability, $S(n\tau)$, that the system stays undetected after the n th measurement is given by the norm of the state

$$S(n\tau) = \langle\psi(n\tau)|\psi(n\tau)\rangle. \quad (7)$$

The normalized state after the n th measurement is given by

$$|\tilde{\psi}(n\tau)\rangle = \frac{|\psi(n\tau)\rangle}{\sqrt{S(n\tau)}}. \quad (8)$$

It was numerically demonstrated in Refs. [7–9] that for the case where the time between measurements τ is small (compared to typical time scales in the unitary evolution) the above dynamics is accurately described by a continuous time evolution with a non-Hermitian effective Hamiltonian. However, to obtain a description with continuous time, we need the limit $\tau \rightarrow 0$ and this is problematic because it leads to the Zeno effect. Here we discuss a systematic approach where this problem is overcome and the effective non-Hermitian description can be obtained in a rigorous way. For this we let the system-detector couplings scale with the detection interval

τ as

$$H_{i\alpha}^{(SD)} = \sqrt{\frac{\gamma_{i\alpha}}{\tau}}, \quad H_{\alpha i}^{(SD)} = \sqrt{\frac{\gamma_{\alpha i}}{\tau}}. \quad (9)$$

Further, denoting $H_{ij}^{(S)} = \gamma_{ij}$ and $H_{\alpha\beta}^{(D)} = \gamma_{\alpha\beta}$ we note the form of the Hamiltonian H

$$H = \underbrace{\sum_{i,j} \gamma_{ij} |i\rangle\langle j|}_{H^S} + \underbrace{\sum_{\alpha,\beta} \gamma_{\alpha\beta} |\alpha\rangle\langle\beta|}_{H^D} + \underbrace{\sum_{i,\alpha} \left[\sqrt{\frac{\gamma_{i\alpha}}{\tau}} |i\rangle\langle\alpha| + \sqrt{\frac{\gamma_{\alpha i}}{\tau}} |\alpha\rangle\langle i| \right]}_{H^{SD}}. \quad (10)$$

We assume from now on that H is measured in units of \hbar and so has units of frequency, then the γ coefficients (assumed to be positive) so do too. Assuming that \tilde{U}_τ can be expanded in a power series we get

$$\begin{aligned} \tilde{U}_\tau &= Qe^{-i\tau H}Q \\ &= Q \left[I + (-i\tau H) + \frac{(-i\tau H)^2}{2} + \dots \right] Q. \end{aligned} \quad (11)$$

Note that $|\psi(n\tau)\rangle$ belongs to \mathcal{S} , which means that $Q|\psi(n\tau)\rangle = |\psi(n\tau)\rangle$. The following identities follow from the orthonormality of the basis states Eq. (1) and the definition of Q :

$$\begin{aligned} QH^{(S)}Q &= H^{(S)}, \quad QH^{(D)}Q = QH^{(SD)}Q = 0, \\ Q[H^{(S)}]^2Q &= [H^{(S)}]^2, \quad Q[H^{(D)}]^2Q = 0, \\ Q[H^{(SD)}]^2Q &= \frac{1}{\tau} \sum_{i,j,\alpha} |i\rangle\langle j| \sqrt{\gamma_{i\alpha}\gamma_{\alpha j}}. \end{aligned} \quad (12)$$

By use of these in the expansion of \tilde{U}_τ , we then get

$$\tilde{U}_\tau = I - i\tau H^S - \tau V^S + \mathcal{O}(\tau^2), \quad (13)$$

where

$$V_{ij}^S = \frac{1}{2} \sum_{\alpha} \sqrt{\gamma_{i\alpha}\gamma_{\alpha j}}. \quad (14)$$

We now take the continuum limit $\tau \rightarrow 0$, $n \rightarrow \infty$ while keeping $t = n\tau$ finite. Let $|\psi(t)\rangle = |\psi(n\tau)\rangle$. Then, using

Eq. (13), we write

$$\begin{aligned} i \frac{\partial |\psi(t)\rangle}{\partial t} &= i \lim_{\tau \rightarrow 0} \frac{|\psi(n\tau + \tau)\rangle - |\psi(n\tau)\rangle}{\tau} \\ &\Rightarrow i \frac{\partial |\psi(t)\rangle}{\partial t} = i \lim_{\tau \rightarrow 0} \frac{\tilde{U}_\tau - I}{\tau} |\psi(n\tau)\rangle. \end{aligned}$$

and we get a new Schrödinger's equation with an effective non-Hermitian Hamiltonian H^{eff}

$$i \frac{\partial |\psi(t)\rangle}{\partial t} = H^{\text{eff}} |\psi(t)\rangle, \quad H^{\text{eff}} = H^S - iV^S. \quad (15)$$

Thus we have shown that the repeated-measurement dynamics, in the special limit where the time interval between measurements τ goes to zero while the system-detector interaction diverges like $\tau^{-1/2}$, is given by the non-Hermitian Hamiltonian in Eq. (15). Note that the dynamics occurs in the restricted system subspace \mathcal{S} .

Note that in the limit $\tau \rightarrow 0$ the survival probability at time t is obtained from Eq. (7):

$$S(t) = \langle \psi(t) | \psi(t) \rangle, \quad (16)$$

with $|\psi(t)\rangle$ solution of the effective Schrödinger's equation Eq. (15). The distribution of first detection times is given by

$$F(t) = -\frac{dS}{dt} = 2\langle \psi(t) | V^S | \psi(t) \rangle. \quad (17)$$

Note that the form of V^S in Eq. (14) ensures that the above decay rate is non-negative as expected physically. In fact we can show that for any eigenvalue λ of $-iH^{\text{eff}}$ we must have $\text{Re}(\lambda) \leq 0$. This follows on noting that, if $|\psi\rangle$ is the eigenvector corresponding to λ , then we have $\lambda \langle \psi | \psi \rangle = -i \langle \psi | H^S | \psi \rangle - \langle \psi | V^S | \psi \rangle$. Then, using the fact that H^S is Hermitian and V^S is positive, it follows that $\text{Re}(\lambda) \leq 0$.

III. EXAMPLE OF A QUANTUM PARTICLE ON DIFFERENT ONE-DIMENSIONAL LATTICE

We now consider some simple physical examples of the Hamiltonian Eq. (10) corresponding to the case of a single particle hopping on a 1D lattice $\Lambda \subset \mathbb{Z}$ containing the site 0 and subjected to the action of a potential $v(n)$. A detector performs a position measurement at the site 0 after every time interval of length τ . The basis vectors $|i\rangle$ now correspond to the position eigenstates and we have $P = |0\rangle\langle 0|$. We consider three different choices of lattices which we specify now along with the corresponding Hamiltonians.

(1) Finite lattice $\Lambda = \{0, \dots, N\}$ of size $N \geq 2$:

$$H = -\gamma_0 \sum_{n=2}^N [|n\rangle\langle n-1| + |n-1\rangle\langle n-2|n\rangle\langle n|] + (2 + \beta)\gamma_0 |1\rangle\langle 1| - \sqrt{\frac{2\alpha\gamma_0}{\tau}} (|0\rangle\langle 1| + |1\rangle\langle 0|). \quad (18)$$

(2) Semi-infinite lattice $\Lambda = \mathbb{N}$:

$$H = -\gamma_0 \sum_{n=2}^{\infty} [|n\rangle\langle n-1| + |n-1\rangle\langle n-2|n\rangle\langle n|] + (2 + \beta)\gamma_0 |1\rangle\langle 1| - \sqrt{\frac{2\alpha\gamma_0}{\tau}} (|0\rangle\langle 1| + |1\rangle\langle 0|). \quad (19)$$

(3) Infinite lattice $\Lambda = \mathbb{Z}$:

$$H = -\gamma_0 \sum_{n \in \mathbb{Z} \setminus \{0\}} [(1 - \delta_{n,1})|n\rangle\langle n-1| + (1 - \delta_{n,-1})|n\rangle\langle n+1| - 2|n\rangle\langle n|] + \sum_{n \in \{\pm 1\}} \left[-\sqrt{\frac{2\alpha\gamma_0}{\tau}}(|0\rangle\langle n| + |n\rangle\langle 0|) + \beta\gamma_0|n\rangle\langle n| \right]. \quad (20)$$

We introduce the dimensionless parameter α which can be regarded as the strength of the detection. We have assumed that the potential $v(n)$ is constant and equal to $2\gamma_0$ except on the two sites $-1, +1$ where it is equal to $(\beta + 2)\gamma_0$. Hence β is also a dimensionless parameter measuring the strength of the potential near the detector. These parameters are encapsulated in the complex number $w = \alpha + i\beta$. Denote then by

$$H_\Lambda = \frac{H^{\text{eff}}}{\gamma_0},$$

the corresponding scaled effective Hamiltonian. This means that we measure time in units of γ_0^{-1} . By using Eqs. (14), (15) we get the following effective Hamiltonians corresponding to the system-detector Hamiltonians in Eqs. (18), (19), (20):

(1) Finite lattice $\Lambda_N = \{1, \dots, N\}$ of size $N \geq 2$:

$$H_{\Lambda_N} = -\sum_{n=2}^N [|n\rangle\langle n-1| + |n-1\rangle\langle n| - 2|n\rangle\langle n|] + (2 - iw)|1\rangle\langle 1|. \quad (21)$$

(2) Semi-infinite lattice \mathbb{N} :

$$H_{\mathbb{N}} = -\sum_{n=2}^{\infty} [|n\rangle\langle n-1| + |n-1\rangle\langle n| - 2|n\rangle\langle n|] + (2 - iw)|1\rangle\langle 1|. \quad (22)$$

(3) Infinite lattice \mathbb{Z} :

$$H_{\mathbb{Z}} = -\sum_{n \in \mathbb{Z} \setminus \{0\}} [(1 - \delta_{n,1})|n\rangle\langle n-1| + (1 - \delta_{n,-1})|n\rangle\langle n+1| - 2|n\rangle\langle n|] - iw \sum_{n, n' \in \{-1, 1\}} |n\rangle\langle n'|. \quad (23)$$

Observe that these operators are not Hermitian (nor normal) and therefore *a priori* not diagonalizable. In fact for, e.g., $N = 2$ we can explicitly show that for specific values of w the operator H_{Λ_2} is not diagonalizable. We discuss the spectral properties of the Hamiltonian $H_{\mathbb{N}}$ and $H_{\mathbb{Z}}$ in Appendix A.

IV. CALCULATION OF SURVIVAL PROBABILITY

Having mapped the measurement problem to that of non-unitary evolution given by Eq. (15), the methods developed in Ref. [28] can be employed to calculate the survival probability. The three cases considered in the previous section are discussed in the following subsections.

A. The lattice Λ_N

The Hamiltonian H_{Λ_N} can be rewritten as

$$H_{\Lambda_N} = -\Delta_N - iw|1\rangle\langle 1|,$$

where Δ_N is the discrete Laplacian on Λ_N . We first prove that in this case the survival probability goes to 0 with an exponential decay at large times. We recall that the spectrum of the Laplacian ($\epsilon_q = -2(1 - \cos(q))$), where $q = s\pi/(N+1)$, $s = 1, 2, \dots, N$. Hence it is contained in $\{z \in \mathbb{C} ; \text{Re}(z) < 0, \text{Im}(z) = 0\}$. Let λ be an eigenvalue of $-iH_{\Lambda_N}$ associated to some eigenvector $|\psi\rangle$ then we have that

$$\lambda\langle\psi|\psi\rangle = i\langle\psi|\Delta_N|\psi\rangle - w|\langle 1|\psi\rangle|^2.$$

If $\langle 1|\psi\rangle \neq 0$, $\text{Re}(\lambda) < 0$ because $\text{Re}(w) > 0$. If $\langle 1|\psi\rangle = 0$ then $i\Delta_N|\psi\rangle = \lambda|\psi\rangle$, hence $|\psi\rangle$ is an eigenvector of Δ_N associated to the eigenvalue $-\lambda$. Since the (explicit) eigenvectors of Δ_N are not such that $\langle 1|\psi\rangle = 0$, the latter case never holds. We conclude that the spectrum of $-iH_{\Lambda_N}$ is contained in $\{z \in \mathbb{C} ; \text{Re}(z) < 0\}$. By using the Jordan-Chevalley decomposition of $-iH_{\Lambda_N}$ we conclude that for any $|\psi(0)\rangle$,

$$S(t) = \langle\psi(t)|\psi(t)\rangle = O(e^{-\mu t})$$

for some real $\mu > 0$. Hence the survival probability $S(t)$ goes to 0 as $t \rightarrow \infty$ exponentially fast. We will see in the two next subsections that when the lattice is infinite, it is no longer the case. For the case where $-iH_{\Lambda_N}$ has a non-degenerate spectrum, let λ_m be the eigenvalue with the smallest negative real part. Then the decay constant is given by $\mu = -\text{Re}[\lambda_m]$. The “surviving” wave function at large times would be given by $|\psi(t)\rangle \sim e^{\lambda_m t}|\chi_m\rangle$ where $|\chi_m\rangle$ is the eigenvector of $-iH_{\Lambda_N}$ corresponding to the eigenvalue λ_m .

From Eq. (17) we see that the first passage time distribution is given by $F(t) = -\frac{\partial S}{\partial t}(t) = 2\alpha|\psi_1(t)|^2$, where we use the notation $\psi_i = \langle i|\psi\rangle$. We now show that one can write a formal solution for $\psi_1(t)$, by using the information on the eigenspectrum of the Hermitian part of the effective Hamiltonian, which in this case is the lattice Laplacian. We first write the equation of motion:

$$i\frac{\partial|\psi\rangle}{\partial t} = -\Delta_N|\psi\rangle - iw|\langle 1|\psi\rangle|1\rangle. \quad (24)$$

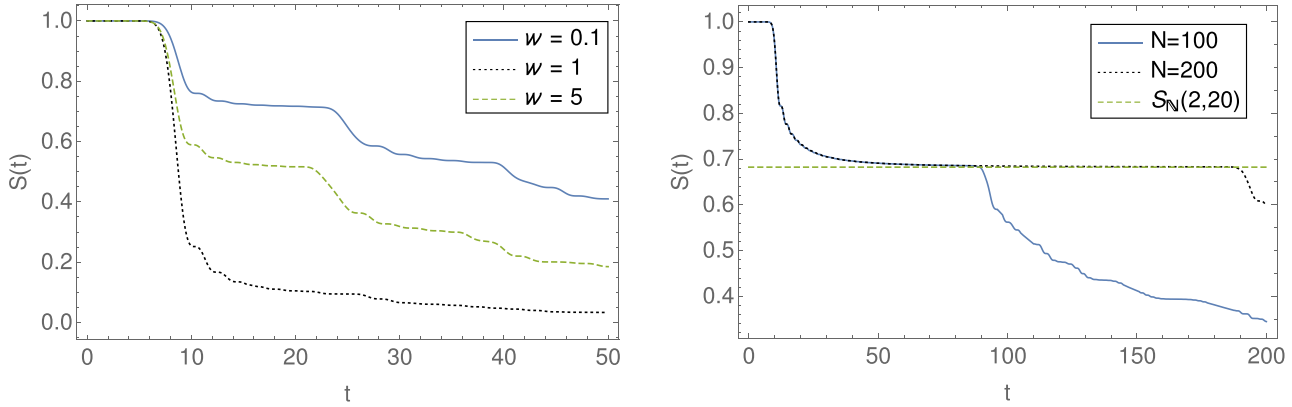


FIG. 1. In the graph on the left, the survival probability $S(t)$ is plotted for $N = 15$, $\psi_i(0) = \delta_{i,15}$ and for different values of w . In the graph to the right, the survival probability $S(t)$ is plotted for lattice sizes $N = 100$ and $N = 200$. In both cases $\psi_i(0) = \delta_{i,20}$, $w = 2$. The dashed line is the value of survival probability S_∞ obtained from Eq. (40) for the \mathbb{N} Lattice.

Taking a Laplace transform $\tilde{\psi}(s) = \int_0^\infty dt e^{-st} |\psi(t)\rangle$, we get

$$-|\psi(0)\rangle + s\tilde{\psi}(s) = i\Delta_N\tilde{\psi}(s) - w\langle 1|\tilde{\psi}(s)\rangle|1\rangle. \quad (25)$$

Defining the Green's function $G(s) = [s - i\Delta_N]^{-1}$, we get the following formal solution:

$$|\tilde{\psi}(s)\rangle = G(s)|\psi(0)\rangle - w\langle 1|\tilde{\psi}(s)\rangle G(s)|1\rangle. \quad (26)$$

Assuming an initial wave function, $|\psi(0)\rangle = |\ell\rangle$, localized at a site ℓ we take a projection of the above equation on the state $|1\rangle$ to get the Laplace transform $\mathcal{L}\psi_1$ of ψ_1 :

$$[\mathcal{L}\psi_1](s) := \langle 1|\tilde{\psi}(s)\rangle = \frac{G_{1\ell}(s)}{1 + wG_{11}(s)}, \quad (27)$$

where the matrix elements $G_{ij}(s) = \langle i|G(s)|j\rangle$ can be written in terms of the eigenfunctions $\phi_q(j) = \sqrt{2/(N+1)}\sin(qj)$ of the Laplacian Δ_N , and corresponding eigenvalues $\epsilon_q = -2(1 - \cos q)$ with $q = s\pi/(N+1)$, $s = 1, 2, \dots, N$. One gets

$$G_{ij}(s) = \sum_q \frac{\phi_q(i)\phi_q(j)}{s - i\epsilon_q}, \quad (28)$$

and hence an explicit expression for the Laplace transform $(\mathcal{L}\psi_1)(s)$. By using an inverse Laplace transform formula we can then get an explicit formula for the first passage time distribution $F(t)$ which remains however difficult to exploit, even qualitatively. Similar expressions have recently been discussed in Ref. [17] who also point the analogy with the renewal approach for the repeated measurement problem. In the next section we will make use of the analogous formula as Eq. (27) to get explicit results on first passage distribution on the infinite lattice. For the finite lattice we now present some numerical results on the form of the survival probability.

Numerical results. The numerical results are obtained by a direct solution of the non-Hermitian Schrödinger equation. As shown above in this case we have $[S(t)]_{t \rightarrow \infty} = 0$ and we always eventually detect the particle. In Fig. 1 (left panel) we plot the decay of the survival probability on a system for which $N = 15$ for three different values of w . The survival probability cascades to 0 over time. We observe a nonmonotonic dependence with $S(t)$ decaying slowly for both very small $w = 0.1$ as well large $w = 5$. The several plateaus in

the survival probability can be understood as arising from ballistic propagation of the particle with a group velocity ≈ 2 , such that as the wave packet hits the detector, the survival probability plunges considerably. The velocity 2 corresponds to the maximum group velocity $d\epsilon_q/dq$. In Fig. 1 (right panel) we show the decay of survival probability for relatively larger systems. In all cases, $\psi_i(0) = \delta_{i,20}$ and $w = 2.0$. In this case the first short plateau in $S(t)$ corresponds to the time taken for the wave packet to reach the detector while the second longer plateau corresponds to the wave packet traveling from the origin to the right end and back. Hence this second time scale can be seen to be proportional to the lattice size N . The dashed line indicates the infinite time survival probability $S_{\mathbb{N}}(w, 20)$ on the semi-infinite lattice \mathbb{N} which is computed exactly in the next section [see Eq. (40)]. Similar finite size effects have been discussed recently in Ref. [9].

B. \mathbb{N} Lattice

The Schrödinger equation corresponding to the effective Hamiltonian $H_{\mathbb{N}}$ given by Eq. (22) is

$$i\frac{\partial\psi_n}{\partial t} = \begin{cases} (2 - iw)\psi_1 - \psi_2, & n = 1, \\ 2\psi_n - \psi_{n-1} - \psi_{n+1}, & n \geq 2. \end{cases} \quad (29)$$

By introducing the fictitious Dirichlet boundary condition $\psi_0(t) = 0$ the Eq. (29) can be written as

$$i\frac{\partial\psi_n}{\partial t} = 2\psi_n - \psi_{n-1} - \psi_{n+1} - iw\delta_{n,1}\psi_1, \quad n \geq 1. \quad (30)$$

A similar equation for $n \in \mathbb{Z}$ was studied in Ref. [28] and we follow their approach of solution. We first decompose the wave function in the following orthonormal sin-basis:

$$\psi_n(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi dk \hat{\psi}(k, t) \sin(nk)$$

with

$$\hat{\psi}(k, t) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^\infty \psi_n(t) \sin(nk).$$

Observe that the Dirichlet boundary condition is automatically satisfied. Assume $\psi_n(0) = \delta_{n,n_0}$ so that $\hat{\psi}(k, 0) =$

$\sqrt{\frac{2}{\pi}} \sin(n_0 k)$. Then it follows from Eq. (30) that

$$i \frac{\partial \hat{\psi}}{\partial t}(k, t) - 2(1 - \cos(k)) \hat{\psi}(k, t) = -iw \sqrt{\frac{2}{\pi}} \psi_1(t) \sin(k). \quad (31)$$

The Laplace transforms of $\hat{\psi}(k, t)$ and $\psi_1(t)$ are by definition given by

$$\tilde{\psi}(k, s) = [\mathcal{L}\hat{\psi}(k, t)](s) = \int_0^\infty dt \exp(-st) \hat{\psi}(k, t), \quad (32)$$

$$\begin{aligned} [\mathcal{L}\psi_1](s) &= \int_0^\infty dt \exp(-st) \psi_1(t) \\ &= \sqrt{\frac{2}{\pi}} \int_0^\pi dk \tilde{\psi}(k, s) \sin(k). \end{aligned} \quad (33)$$

Taking the Laplace transform of Eq. (31) and by use of the above equations

$$\tilde{\psi}(k, s) = i \sqrt{\frac{2}{\pi}} \frac{\sin(n_0 k) - w \sin(k) [\mathcal{L}\psi_1](s)}{is - 2(1 - \cos(k))}. \quad (34)$$

The two relations Eqs. (33) and (34) between $\tilde{\psi}(k, s)$ and $[\mathcal{L}\psi_1](s)$ above give

$$[\mathcal{L}\psi_1](s) = i \frac{\frac{2}{\pi} \int_0^\pi dk \frac{\sin(k) \sin(n_0 k)}{is - 2(1 - \cos(k))}}{1 + i \frac{2w}{\pi} \int_0^\pi dk \frac{\sin^2 k}{is - 2(1 - \cos(k))}}. \quad (35)$$

Note that this also follows from Eq. (27) on taking the limit $N \rightarrow \infty$. The integrals involved in Eq. (35) can be evaluated by means of contour integration for any complex number s such that $\text{Re}(s) > 0$, see Appendix B 1. We get then

$$|[\mathcal{L}\psi_1](s)|^2 = \begin{cases} \frac{(x + \sqrt{x^2 - 1})^{2n_0}}{1 + |w|^2(x + \sqrt{x^2 - 1})^2 + 2\text{Im}(w)(x + \sqrt{x^2 - 1})}, & x < -1, \\ \frac{1}{1 + |w|^2 + 2[\text{Re}(w)\sqrt{1 - x^2} + \text{Im}(w)x]}, & -1 \leq x \leq 1, \\ \frac{(x - \sqrt{x^2 - 1})^{2n_0}}{1 + |w|^2(x - \sqrt{x^2 - 1})^2 + 2\text{Im}(w)(x - \sqrt{x^2 - 1})}, & x > 1. \end{cases} \quad (39)$$

Denote by $S_{\mathbb{N}}(w, n_0)$ the survival probability given by Eq. (37) corresponding to the initial condition $|\psi(0)\rangle = |n_0\rangle$, where the detection parameter is $\alpha = \text{Re}(w)$ and the impurity parameter is $\beta = \text{Im}(w)$. From Eq. (37) and Eqs. (38) and (39), after simplification, we obtain then that

$$S_{\mathbb{N}}(w, n_0) = 1 - \frac{\text{Re}(w)}{\pi |w|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos \theta}{\frac{1}{2}(|w| + \frac{1}{|w|}) + \cos(\theta - \varphi)} - \frac{2\text{Re}(w)}{\pi} \int_0^1 du \frac{u^{2n_0 - 2}(1 - u^2)(1 + |w|^2 u^2)}{(1 + |w|^2 u^2)^2 - (2\text{Im}(w)u)^2}, \quad (40)$$

where $\varphi = \text{Arg}(w)$. Figure 2 shows the variation of $S_{\mathbb{N}}(w, n_0)$ for some choices of w and starting positions n_0 . In the left panel we see that for $n_0 = 1$, the survival probability goes to 0 with increasing strength α of detection. For $n_0 > 1$ the survival probability attains a minimum. This can be compared with the results obtained in [28] (Fig. 3). As noted earlier [see comment after Eq. (30)], Refn. [28] solves the case with an imaginary potential at the origin of a \mathbb{Z} -lattice and so the similarity of our result to theirs is expected. In the right panel, we see that $S_{\mathbb{N}}$ is symmetric in β for all starting positions n_0 for a fixed α .

In Eq. (40), only the second integral is depending on n_0 . Both the integrals can be explicitly computed but the formula obtained do not have a simple form. For large n_0

that

$$[\mathcal{L}\psi_1](s) = -i \frac{i^{n_0} \left[-\left(\frac{s}{2} + i\right) + \sqrt{\left(\frac{s}{2} + i\right)^2 + 1} \right]^{n_0}}{1 + w \left[-\left(\frac{s}{2} + i\right) + \sqrt{\left(\frac{s}{2} + i\right)^2 + 1} \right]}, \quad (36)$$

where \sqrt{z} denotes the principal square root of $z \in \mathbb{C} \setminus \mathbb{R}_-^*$ (using the nonpositive real axis as a branch cut). We have hence obtained an explicit expression of the Fourier-Laplace transform $\tilde{\psi}$ of the wave function ψ by plugging Eq. (36) into Eq. (34).

We now turn to the computation of the survival probability. Recalling Eq. (16), the survival probability after time t is $S(t) = \sum_{n=1}^\infty |\psi_n(t)|^2$. From Eq. (29), one has

$$i \frac{\partial}{\partial t} |\psi_n|^2 = \begin{cases} -2i \text{Re}(w) |\psi_1|^2 + 2i \text{Im}(\psi_1 \psi_2^*), & n = 1, \\ 2i \text{Im}(-\psi_{n-1} \psi_n^* + \psi_n \psi_{n+1}^*), & n \geq 2. \end{cases}$$

Since ψ_n goes to 0 as $n \rightarrow \infty$, the equations can be summed to obtain $dS/dt = -2\text{Re}(w) |\psi_1|^2$. Integrating, we get the survival probability

$$S_\infty = \lim_{t \rightarrow \infty} S(t) = 1 - 2\text{Re}(w) \int_0^\infty dt |\psi_1(t)|^2. \quad (37)$$

For square integrable function ψ_1 , we have [29], Chapter 4, that

$$\int_0^\infty dt |\psi_1(t)|^2 = \lim_{\epsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} ds |[\mathcal{L}\psi_1](s)|^2. \quad (38)$$

For $s = \epsilon + 2i(x - 1)$ in the limit $\epsilon \rightarrow 0^+$, by Eq. (36), we have that

a simple expansion can be done for the latter at any order, see Appendix B 1. Therefore for $n_0 \gg 1$, we have, e.g., that

$$\begin{aligned} S_{\mathbb{N}}(w, n_0) &\approx 1 - \frac{\text{Re}(w)}{\pi |w|} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \frac{\cos \theta}{\frac{1}{2}(|w| + \frac{1}{|w|}) + \cos(\theta - \varphi)} \\ &\quad - \frac{1}{n_0^2} \frac{\text{Re}(w)}{\pi} \frac{(1 + |w|^2)}{(1 + |w|^2)^2 - 4[\text{Im}(w)]^2}. \end{aligned} \quad (41)$$

In fact we can also get an explicit formula for the function ψ_1 and therefore for the survival probability $S(t) = 1 - 2\text{Re}(w) \int_0^t dt' |\psi_1(t')|^2$ at time t . For that it is sufficient to

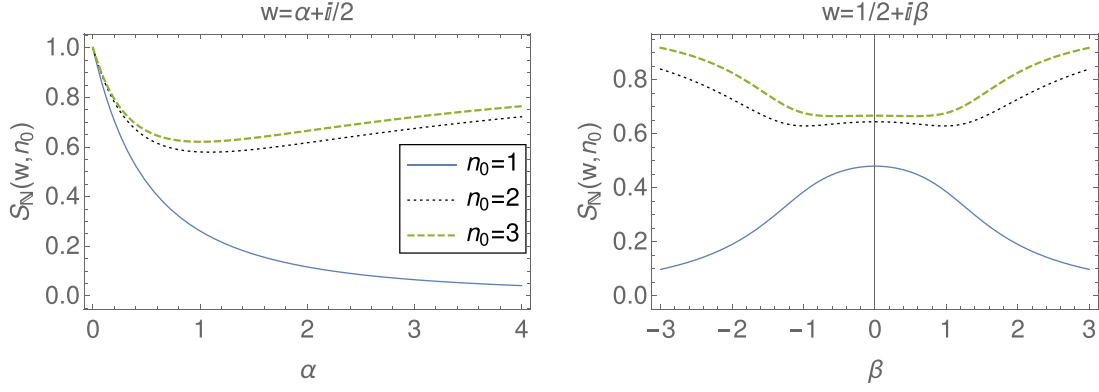


FIG. 2. Plots showing variation of $S_{\mathbb{N}}$ [Eq. (40)] with w for various starting positions n_0 . The left plot is for $w = \alpha + i/2$ and the right plot for $w = 1/2 + i\beta$.

recognize in Eq. (36) the Laplace transform of some explicit function. In Appendix B 1 we identify this function by using properties of the Bessel functions of the first kind:

$$J_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{-i(kx-t)\sin(x)}. \quad (42)$$

We find that

$$\psi_1(t) = -2i^{n_0+1} e^{-2it} \sum_{k=0}^{\infty} (k+n_0)(-w)^k \frac{J_{k+n_0}(2t)}{2t}, \quad (43)$$

which immediately gives us the explicit form of the first detection distribution $F(t) = 2\text{Re}[w]|\psi_1(t)|^2$. At large times using the asymptotic form of $J_k(2t)$ we get

$$\psi_1(t) \asymp \frac{(-i)^{n_0+1} e^{-i2t}}{2\sqrt{\pi}t^3} \left[e^{i(2t-\frac{\pi}{4}+\frac{i n_0 \pi}{2})} f(n_0, w) + e^{-i(2t-\frac{\pi}{4}+\frac{i n_0 \pi}{2})} f(n_0, w^*) \right], \quad (44)$$

where

$$f(n_0, w) = \frac{n_0 - i(n_0 - 1)w}{(1 - iw)^2}.$$

Hence we get the long time decay form $F(t) \sim 1/t^3$.

C. \mathbb{Z} Lattice

The calculations here are along the same lines as for the \mathbb{N} lattice case. For the effective Hamiltonian $H_{\mathbb{Z}}$ given by Eq. (23) the Schrödinger equation is equivalent to

$$i \frac{\partial \psi_n}{\partial t} = 2\psi_n - \psi_{n+1} - \psi_{n-1} - iw(\delta_{n,-1} + \delta_{n,1})(\psi_{-1} + \psi_1), \quad n \in \mathbb{Z} \setminus \{0\} \quad (45)$$

with $\psi_0(t) = 0$. Let us define

$$\begin{aligned} \hat{\psi}^{\pm}(k, t) &= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \psi_{\pm n}(t) \sin(nk) \\ \Rightarrow \psi_{\pm n}(t) &= \sqrt{\frac{2}{\pi}} \int_0^{\pi} dk \hat{\psi}^{\pm}(k, t) \sin(nk), \end{aligned}$$

which both satisfy the same equation

$$\begin{aligned} i \frac{\partial \hat{\psi}^{\pm}}{\partial t}(k, t) - 2(1 - \cos(k))\hat{\psi}^{\pm}(k, t) \\ = -iw \sqrt{\frac{2}{\pi}} [\psi_1(t) + \psi_{-1}(t)] \sin(k). \end{aligned} \quad (46)$$

We assume that $\psi_n(0) = \delta_{n,n_0}$ with $n_0 \geq 1$ so that $\hat{\psi}^+(k, 0) = \sqrt{\frac{2}{\pi}} \sin(n_0 k)$ and $\hat{\psi}^-(k, 0) = 0$. The Laplace transforms of $\hat{\psi}^{\pm}(k, t)$ and $\psi_{\pm 1}(t)$ are by definition

$$\tilde{\psi}^{\pm}(k, s) = [\mathcal{L}\hat{\psi}^{\pm}(k, t)](s) = \int_0^{\infty} dt \exp(-st) \hat{\psi}^{\pm}(k, t), \quad (47)$$

$$\begin{aligned} [\mathcal{L}\psi_{\pm 1}](s) &= \int_0^{\infty} dt \exp(-st) \psi_{\pm 1}(t) \\ \Rightarrow [\mathcal{L}\psi_{\pm 1}](s) &= \sqrt{\frac{2}{\pi}} \int_0^{\pi} dk \tilde{\psi}^{\pm}(k, s) \sin(k). \end{aligned} \quad (48)$$

Taking the Laplace transform of Eq. (46) and by use of the above equations

$$\begin{aligned} \tilde{\psi}^+(k, s) &= i \sqrt{\frac{2}{\pi}} \frac{\sin(n_0 k) - w \sin(k) \mathcal{L}[\psi_{-1} + \psi_1](s)}{is - 2(1 - \cos(k))}, \\ \tilde{\psi}^-(k, s) &= -i \sqrt{\frac{2}{\pi}} \frac{w \sin(k) \mathcal{L}[\psi_{-1} + \psi_1](s)}{is - 2(1 - \cos(k))}. \end{aligned} \quad (49)$$

Then we get

$$[\mathcal{L}(\psi_{-1} + \psi_1)](s) = i \frac{\frac{2}{\pi} \int_0^{\pi} dk \frac{\sin(k) \sin(n_0 k)}{is - 2(1 - \cos(k))}}{1 + i \frac{4w}{\pi} \int_0^{\pi} dk \frac{\sin^2 k}{is - 2(1 - \cos(k))}} \quad (50)$$

from which follows the explicit expression for the Fourier-Laplace transform of the wave function. This is the same expression as Eq. (35) where w has been replaced by $2w$.

Proceeding as in the derivation of Eq. (37), in this case the expression for the survival probability is

$$S_{\infty} = \lim_{t \rightarrow \infty} S(t) = 1 - 2\text{Re}(w) \int_0^{\infty} dt |\psi_{-1}(t) + \psi_1(t)|^2. \quad (51)$$

If $S_{\mathbb{Z}}(w, n_0)$ denotes the survival probability S_{∞} corresponding to this initial condition $\psi_n(0) = \delta_{n,n_0}$ we get that

$$S_{\mathbb{Z}}(w, n_0) = S_{\mathbb{N}}(2w, n_0). \quad (52)$$

D. Green's function approach

It is of interest to recover the value Eq. (40) of the survival probability and the value Eq. (43) of $\psi_1(t)$ by considering the spectral analysis of $H_{\mathbb{N}}$ performed in Appendix A 3. A similar analysis can be performed on the lattice \mathbb{Z} as in Appendix A 4. We recall that $w = \alpha + i\beta$ and we denote $\xi = \frac{\beta - i\alpha}{1 + \beta - i\alpha} = \frac{w}{i+w}$. In general we expect a non-Hermitian Hamiltonian to have distinct right and left eigenvectors. However, for the cases that we consider the Hamiltonian is a symmetric matrix and hence the left eigenvector would be simply the transpose of the right eigenvector. In Appendix A 3 we provide explicitly the complete orthonormal basis of eigenfunctions of the Hamiltonian $H_{\mathbb{N}}$. This basis is composed of scattering states Eq. (A11) $\{\eta^k; k \in (0, \pi)\}$ (associated with the eigenvalue $E_s(k) = 2(1 - \cos(k))$), and, if $|1 - \xi^{-1}| < 1 \Leftrightarrow |w| > 1$, supplemented by a bound state η^b Eq. (A12), associated to the eigenvalue $E_b(\xi) = 1 + \xi^{-1}$.

From this spectral analysis we have that the solution $\psi(t)$ of Eq. (29) with initial condition $\psi(0)$ can be expressed as

$$\psi(t) = G(t)\psi(0), \quad (53)$$

where the entries of the infinite matrix $G(t)$ are given by

$$G_{mm} = \int_0^\pi dk \eta_n^k \eta_m^k e^{-iE_s(k)t} + \Theta(|w| - 1) \eta_n^b \eta_m^b e^{-iE_b(\xi)t}, \quad (54)$$

where Θ denotes the Heaviside step function. In particular, starting with an initial condition $|\psi(0)\rangle = |n_0\rangle$ we have that $\psi_1(t) = G_{1,n_0}(t)$. By using the explicit forms of $\{\eta^k; k \in (0, \pi)\}$, $E_s(k)$ in Eq. (A11) and using that $\xi/(1 - \xi) = -iw$ we get

$$\begin{aligned} \eta_1^k \eta_{n_0}^k e^{-iE_s(k)t} &= -\frac{i}{\pi} e^{-2i(1 - \cos(k))t} \sin(k) \\ &\times \left[\frac{e^{in_0k}}{1 - iw e^{ik}} - \frac{e^{-in_0k}}{1 - iw e^{-ik}} \right]. \end{aligned}$$

Assuming $|w| < 1$ we can transform the previous expression in series since $|iw e^{\pm ik}| < 1$, exchange the sum with the integrals, and by using symmetries we get

$$\begin{aligned} \int_0^\pi dk \eta_n^k \eta_m^k e^{-iE_s(k)t} &= -\frac{i e^{-2it}}{\pi} \sum_{p=0}^{\infty} (iw)^p I_{n_0+p} \\ I_n &= \int_{-\pi}^{\pi} dk e^{2i \cos(k)t} \sin(k) e^{ink}. \quad (55) \end{aligned}$$

By performing an integration by parts in the definition of the Bessel function J_p [see Eq. (42)] we get

$$I_n = \frac{\pi i^n}{t} n J_n(2t)$$

and plugging this into Eq. (55) we recover Eq. (43).

If $|w| > 1$ we have to take into account the existence of a bound state but for the rest we proceed in a similar way by

rewriting for $|z| = 1$

$$\frac{1}{1 - iwz} = \frac{i}{wz} \frac{1}{1 + \frac{i}{wz}} = \frac{i}{wz} \sum_{p=0}^{\infty} (-1)^p \left(\frac{i}{wz}\right)^p.$$

Then we get the same expression as before Eq. (43) for $\psi_1(t)$. The expression for $|w| = 1$ is obtained from the previous expression by a continuity extension.

V. CONTINUUM LIMIT

In this section we discuss the continuum limit of the infinite models defined in Sec. III when the lattice spacing ϵ goes to 0.

A. Half-line case

We consider the Hamiltonian $H_{\mathbb{N}}$ given by Eq. (22):

$$H_{\mathbb{N}} = - \sum_{n=1}^{\infty} [|n\rangle \langle n+1| + |n+1\rangle \langle n| - 2|n\rangle \langle n| - iw|1\rangle \langle 1|].$$

We recall that Schrödinger equation take the form

$$i \frac{\partial \psi_n}{\partial t} = \begin{cases} (2 - iw)\psi_1 - \psi_2, & n = 1, \\ 2\psi_n - \psi_{n-1} - \psi_{n+1}, & n \geq 2. \end{cases}$$

These are equivalent to the equations

$$i \frac{\partial \psi_n}{\partial t} = 2\psi_n - \psi_{n+1} - \psi_{n-1}, \quad n \geq 1, \quad (56)$$

with the boundary condition $\psi_0 = iw\psi_1$ which can be rewritten in the form

$$\psi_0 - \frac{iw}{1 - iw} (\psi_1 - \psi_0) = 0. \quad (57)$$

We now define a small lattice spacing parameter ϵ and define the continuous wave function Ψ as [30]

$$\Psi(x, t) = \lim_{\epsilon \rightarrow 0} \epsilon^{-1/2} \psi_{\lfloor x/\epsilon \rfloor}(t \epsilon^{-2}). \quad (58)$$

Then in the limit $\epsilon \rightarrow 0$ Eqs. (56) and (57) reduce to

$$i \frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2}, \quad \text{with Robin b.c.} \quad \left[\Psi + \zeta \frac{\partial \Psi}{\partial x} \right]_{x=0} = 0, \quad (59)$$

where ζ is a finite complex number defined as

$$\zeta = \lim_{\epsilon \rightarrow 0} \epsilon \frac{iw}{iw - 1}. \quad (60)$$

We note that to obtain the desired limit in Eq. (60) we need to set $\alpha = \text{Re}(w) = \mathcal{O}(\epsilon)$ and $\beta = \text{Im}w = -1 + \mathcal{O}(\epsilon)$. Hence from the definition of ζ and the fact that $\alpha = \text{Re}(w) > 0$ we observe that $\text{Im}(\zeta) < 0$ while $\text{Re}(\zeta)$ can have either sign. Robin boundary conditions for real ζ have been studied and they arise in models of generic reflecting walls [31]. Here we have ζ as a complex number and it replaces w as the single parameter quantifying the interaction with the detector.

We now proceed with the solution of the above boundary value problem with a specified initial wave function $\Psi_0(x) = \Psi(x, 0)$. The eigenspectrum of the system in Eq. (59) is discussed in Appendix A 1, where it is shown that for

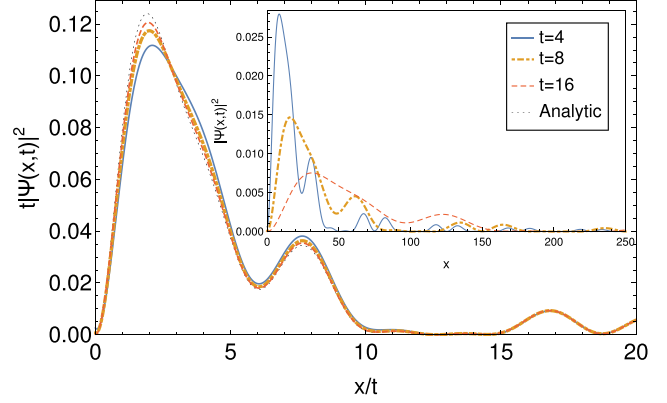
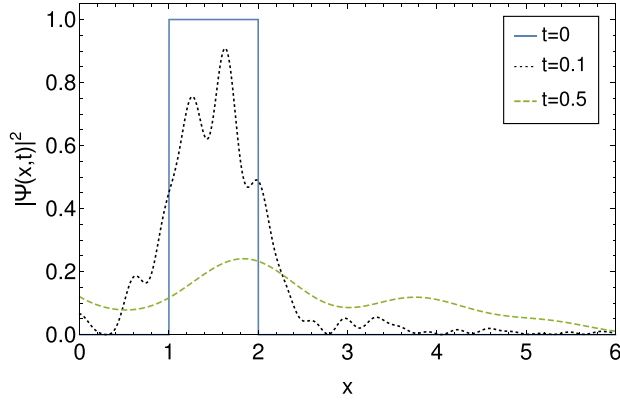


FIG. 3. Plot showing $|\Psi(x, t)|^2$ at times $t = 0, 0.1, 0.5$, obtained from the solution in Eq. (61) with the square initial condition $\Psi(x, 0) = 1$ for $1 < x < 2$ and zero elsewhere. The parameter value $\zeta = 0.2 - 0.5i$ was taken.

$\text{Re}(\zeta) < 0$, the spectrum only has scattering eigenfunctions while for $\text{Re}(\zeta) > 0$, one has in addition a bound state that is localized near the origin. In this section we consider this latter case, the discussion would be similar for the other case. The scattering functions $\{\eta^k; k > 0\}$ and the bound state η^b defined, respectively, by Eqs. (A1) and (A2) in Appendix A 1 form a complete basis of eigenfunctions of the Laplacian operator on the half line with the complex Robin boundary condition at the origin, see Eq. (59). Moreover they satisfy the orthogonality relations Eq. (A3). Then we can write the general time-dependent solution as

$$\Psi(x, t) = \int_0^\infty dk c(k) \eta^k(x) e^{-ik^2 t} + c_b \eta^b(x) e^{it/\zeta^2},$$

where

$$c(k) = \int_0^\infty dx \Psi_0(x) \eta^k(x), \quad c_b = \int_0^\infty dx \Psi_0(x) \eta^b(x). \quad (61)$$

The coefficients $c(k)$ as well as the oscillating integrals above are well defined for any initial wave function Ψ_0 which is sufficiently smooth since then the $c(k)$'s have a fast decay in k . If $\Psi_0 \in \mathbb{L}^2([0, \infty))$ is only square integrable the previous formula has to be understood by using an approximation of the initial condition by smooth initial conditions. The approximation scheme propagates in time thanks to the decay Eq. (67) of the \mathbb{L}^2 norm.

From Eq. (61) and using the explicit formula for the basis states, the wave function at long times is given by (the bound state contribution vanishes):

$$\begin{aligned} \Psi(x, t) &= \int_0^\infty dk c(k) \eta^k(x) \exp(-ik^2 t) \\ &= \frac{i}{\sqrt{2\pi}} \int_{-\infty}^\infty dk c(k) \frac{(1 - i\zeta k)}{\sqrt{1 + \zeta^2 k^2}} e^{it(k\frac{x}{t} - k^2)}, \end{aligned}$$

where we used the fact that $c(-k) = -c(k)$. Using the standard stationary phase approximation we find a stationary point at $k^* = x/(2t)$ and this then gives the limiting form $\Psi \rightarrow \Psi_\infty$,

where

$$\Psi_\infty(x, t) = -\frac{1}{\sqrt{2t}} c\left(\frac{x}{2t}\right) \frac{(1 - i\zeta \frac{x}{2t})}{\sqrt{1 + \zeta^2 \frac{x^2}{4t^2}}} \exp\left[i\left(\frac{x^2}{4t} - \frac{\pi}{4}\right)\right]. \quad (62)$$

For $x \lesssim \sqrt{t}$ we need the form of $c(k)$ at $k \rightarrow 0$. Using the explicit form of the wave functions in Appendix A 1 we find immediately that

$$c(k) \sim \frac{2ik}{\sqrt{2\pi}} m_{\psi_0}, \quad \text{where } m_{\psi_0} = \int_0^\infty dx (x - \zeta) \Psi_0(x). \quad (63)$$

Hence we have for $x \lesssim \sqrt{t}$

$$\Psi_\infty(x, t) \approx m_{\psi_0} \frac{x}{\sqrt{4\pi t^3}} \exp\left[i\left(\frac{x^2}{4t} + \frac{3\pi}{4}\right)\right], \quad (64)$$

which has a universal structure apart from the factor m_{ψ_0} .

As an explicit numerical example, we now consider the evolution of an initial wave function of the form

$$\Psi_0(x) = 1, \quad \text{for } 1 < x < 2,$$

and zero elsewhere. In this case the basis expansion coefficients are given

$$\begin{aligned} c(k) &= \frac{k\zeta \cos(3k/2) - \sin(3k/2) \sin(k/2)}{\sqrt{2\pi}(1 + k^2 \zeta^2)} \frac{\sin(k/2)}{k/2}, \\ c_b &= \sqrt{2\zeta} e^{-2/\zeta} (-1 + e^{1/\zeta}). \end{aligned}$$

We choose $\zeta = 0.2 - 0.5i$. In Fig. 3 we show the evolution of the wave function at early (a) and late times [inset of Fig. 3(b)]. For the scaled wave function in Fig. 3(b) we see an excellent agreement with the analytic form in Eq. (62).

The survival probability $S(t)$ is given by

$$S(t) = \int_0^\infty dx |\Psi(x, t)|^2 \quad (65)$$

and after some straightforward manipulations one can show, using Eq. (59) that

$$S(t) = 1 + 2 \frac{\text{Im}(\zeta)}{|\zeta|^2} \int_0^t dt' |\Psi(0, t')|^2, \quad (66)$$

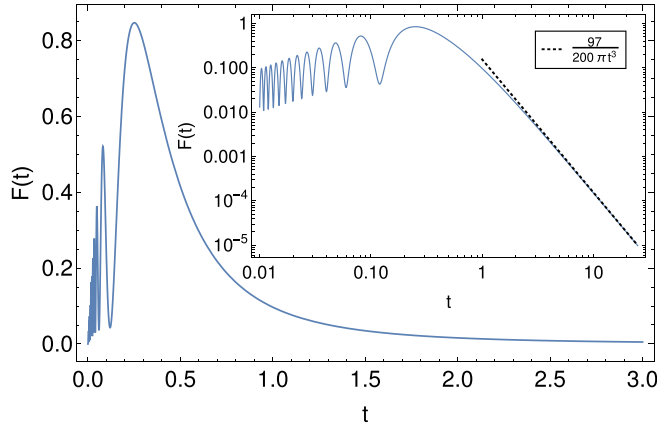


FIG. 4. The first passage time distribution $F(t)$ for the initial wave packet and parameter values considered in Fig. 3. The inset shows the decay $F(t) \sim t^{-3}$ at large times, with a pre-factor given by Eq. (69).

which is the continuous counterpart of Eq. (37). For purely real ζ , i.e., $\alpha = 0$, which corresponds to non-measurement, the system Eq. (59) determines unitary evolution on half line. We note that the first passage time distribution is given by

$$F(t) = -\frac{dS(t)}{dt} = -2\frac{\text{Im}(\zeta)}{|\zeta|^2}|\Psi(0, t)|^2 > 0. \quad (67)$$

From the asymptotic scaling form in Eq. (62) we find that

$$S_\infty = \lim_{t \rightarrow \infty} S(t) = \int_0^\infty dk \left| \frac{1 - i\zeta k}{1 + i\zeta k} \right| |c(k)|^2. \quad (68)$$

We are not able to find more explicit forms for $F(t)$ or $S(t)$. However we can obtain the asymptotic long time form of $F(t)$. We need the wave function at the origin, $\Psi(0, t)$. This can be obtained from the scaling solution in Eq. (64) by use of the boundary condition $\Psi(0, t) = -\zeta \left[\frac{\partial \Psi}{\partial x} \right]_{x=0}$. This gives us, for $t \rightarrow \infty$, $\Psi(0, t) = [m_{\Psi_0} \zeta / \sqrt{4\pi t^3}] e^{-i3\pi/4}$. Hence we get

$$\lim_{t \rightarrow \infty} t^3 F(t) = -\frac{\text{Im}(\zeta)}{2\pi} |m_{\Psi_0}|^2. \quad (69)$$

In Fig. 4 we plot $F(t)$ for the same parameters and initial wave function used in Fig. 3. At large times, we verify the above asymptotic form given in Eq. (69).

We observe that the right-hand side of Eq. (69) may vanish for some special initial condition Ψ_0 . This means that the asymptotic decay of the first passage time distribution is not universal. An exhaustive study can be performed showing that it is always possible to start with a special initial wave function Ψ_0 such that the asymptotic decay of the first passage time distribution will be of order $t^{-(2s+1)}$ for some integer $s \geq 1$. The detailed study is performed in Appendix B 2.

B. Real-line case

The Schrödinger Eq. (45) can be rewritten in the form

$$\begin{aligned} i\frac{\partial \psi_n}{\partial t} &= 2\psi_n - \psi_{n+1} - \psi_{n-1}, \quad n \in \mathbb{Z} \\ \psi_0 &= iw(\psi_{-1} + \psi_1). \end{aligned} \quad (70)$$

Assuming that $iw = \frac{1}{2} + \frac{\epsilon}{\zeta}$ where ζ is a complex number and defining the continuous wave function Ψ as

$$\Psi(x, t) = \lim_{\epsilon \rightarrow 0} \epsilon^{-1/2} \psi_{[x/\epsilon]}(t\epsilon^{-2}),$$

we get in the limit $\epsilon \rightarrow 0$ the above equation reduces to the Schrödinger equation with a complex Robin boundary condition at the origin

$$i\frac{\partial \Psi}{\partial t} = -\frac{\partial^2 \Psi}{\partial x^2}, \quad 2\Psi|_{x=0} + \zeta \left[\frac{\partial \Psi}{\partial x} \Big|_{x=0^+} - \frac{\partial \Psi}{\partial x} \Big|_{x=0^-} \right] = 0. \quad (71)$$

The solution of Eq. (71) can be obtained in the following way. Let us define

$$\begin{aligned} \Psi^s(x, t) &= \frac{\Psi(x, t) + \Psi(-x, t)}{2}, \\ \Psi^a(x, t) &= \frac{\Psi(x, t) - \Psi(-x, t)}{2}, \end{aligned}$$

the symmetric and antisymmetric part of the wave function Ψ . Both functions are uniquely determined by their restrictions to the half line $[0, \infty)$. Equation (71) implies that on $[0, \infty)$, Ψ^s is solution of the Schrödinger Eq. (59) with a complex Robin boundary condition at the origin and that on $[0, \infty)$, Ψ^a is solution of the Schrödinger equation with the Dirichlet boundary condition $\Psi^a(0, t) = 0$. To solve the latter, we observe that it is in fact sufficient to solve the free Schrödinger equation on the real line with the initial antisymmetric wave function Ψ^a since this property will be preserved by the free propagator and in particular the solution will vanish on 0 at any time $t > 0$. It follows that the solution of Eq. (71) is given by

$$\begin{aligned} \Psi(x, t) &= \int_0^\infty dk [\hat{c}(k)\hat{\eta}^k(x) + a(k)\sigma^k(x)] e^{-ik^2 t} \\ &\quad + \hat{c}_b \hat{\eta}^b(x) e^{i\frac{t}{\zeta^2}}, \end{aligned}$$

where

$$\begin{aligned} \hat{c}(k) &= \int_{-\infty}^\infty dx \Psi_0(x) \hat{\eta}^k(x), \\ a(k) &= \int_{-\infty}^\infty dx \Psi_0(x) \sigma^k(x), \\ \hat{c}_b &= \int_{-\infty}^\infty dx \Psi_0(x) \hat{\eta}^b(x), \end{aligned} \quad (72)$$

with the scattering states $\hat{\eta}^k, \sigma^k$ and the bound state $\hat{\eta}^b$ defined by Eqs. (A5) and (A6). The latter are nothing but the eigenfunctions of the Laplacian operator on the real line with complex Robin boundary condition at the origin as it appears in Eq. (71), see Appendix A 2.

The survival probability $S(t)$ is given by

$$S(t) = \int_{-\infty}^\infty dx |\Psi(x, t)|^2 \quad (73)$$

and after some straightforward manipulations one can show, using Eq. (71) that

$$S(t) = 1 + 4\frac{\text{Im}(\zeta)}{|\zeta|^2} \int_0^t dt' |\Psi(0, t')|^2. \quad (74)$$

Since $\sigma^k(0) = 0$ and $\hat{\eta}^k, \hat{\eta}^b$ are even, we conclude that the first passage time distribution $F = -\partial S/\partial t$ is up to a multiplicative constant the same as the one for the half-line, but starting from the wave function Ψ_0^s (instead of Ψ) restricted to the half-line. We observe in particular that if we start with an antisymmetric wave function $\Psi = \Psi^a$ then the particle is never detected.

VI. CONCLUSION

Subjecting a quantum particle to repeated projective measurements at regular intervals of time (τ) to ascertain its observation by a detector is one of the standard methods to study the quantum time of arrival problem. Since the work of Allcock [2], it is believed that an equivalent way to study the first detection problem is to introduce an imaginary absorbing potential. In the present work we discussed a precise limiting procedure of taking $\tau \rightarrow 0$ and at the same time allowing the coupling strength between system and detector to be large (as $1/\tau^{1/2}$). We summarize here our main results:

(1) Formulating the problem for general quantum systems with a discrete Hilbert space, we rigorously showed the equivalence between the repeated measurement protocol and the non-Hermitian description.

(2) For a quantum particle on a 1D lattice with a detector at one site we then solved the corresponding Schrödinger equation with a complex potential to obtain closed form analytic results for the survival probability and the distribution of first detection time of a particle starting from an arbitrary initial lattice site. Various asymptotic cases were discussed.

(3) We then studied the limit of lattice spacing going to 0 to obtain a formulation for the continuum case. For the semi-infinite lattice with a detector at one end we find that the effective description is in terms of free Schrödinger evolution with complex Robin boundary conditions at the detector site. Again in this case we provide analytic results for several objects of interest. The long time asymptotic form of the surviving wave function was obtained. We find that while the detection time probability density generically decays as $1/t^3$, it is possible to construct special initial states for which the decay is faster. A similar dependence of decay exponent on initial states was observed in a lattice study [15] and it will be interesting to relate these results.

We note that this non-Hermitian description on the \mathbb{N} -lattice remains unchanged even if we consider the detector to be localized on all sites on the negative axis. Taking the continuum space limit then gives us the case considered by Allcock [2]. However, we see that with our model of system-detector interactions, the non-Hermitian description is in terms of one with complex Robin boundary conditions instead of one with an imaginary potential [26]. An interesting direction for future work would be to consider other

forms of the system-detector interaction that could lead to different forms of effective non-Hermitian Hamiltonians. From the point of view of experiments a more realistic setup is to consider repeated interactions of the system with a probe (see, e.g., Refs. [32,33]) where the interaction is such that the state of the probe changes only when a particle is present in a particular location. Projective measurements are then done on the probe. We expect that intense probe pulses sent at frequent intervals would approximate our setup but this needs further investigation. An extension of our results to higher dimensions would be another interesting question to explore.

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APPENDIX A: SPECTRUM ANALYSIS

1. Spectrum on the half-line

The operator appearing in Eq. (59) has a complete set of basis functions comprising of a continuum of scattering states given by

$$\eta^k(x) = \frac{i[(1 - ik\xi)e^{ikx} - (1 + ik\xi)e^{-ikx}]}{\sqrt{2\pi(1 + \xi^2k^2)}}, \quad k > 0, \quad (A1)$$

and, for $\text{Re}[\xi] > 0$, a bound state given by

$$\eta^b(x) = \sqrt{\frac{2}{\xi}} e^{-x/\xi}. \quad (A2)$$

Here \sqrt{z} denotes the principal square root of $z \in \mathbb{C} \setminus \mathbb{R}_-$ using the nonpositive real axis as a branch cut. They satisfy the boundary condition at $x = 0$ and the orthonormality conditions:

$$\int_0^\infty dx [\eta^b(x)]^2 = 1, \quad \int_0^\infty dx \eta^k(x)\eta^b(x) = 0, \\ \int_0^\infty dx \eta^k(x)\eta^{k'}(x) = \delta(k - k'). \quad (A3)$$

This is easily proved using the identity $\int_0^\infty dx e^{ikx} = iP(1/k) + \pi\delta(k)$, where P denotes the principal part. One also has the completeness relation:

$$\int_0^\infty dk \eta^k(x)\eta^k(x') = \delta(x - x') \quad \text{for } \text{Re}[\xi] < 0, \quad \int_0^\infty dk \eta^k(x)\eta^k(x') + \eta^b(x)\eta^b(x') = \delta(x - x') \quad \text{for } \text{Re}[\xi] > 0. \quad (A4)$$

To prove this, we note that

$$\int_0^\infty dk \eta^k(x)\eta^k(x') = -\frac{1}{2\pi} \int_0^\infty dk \frac{1}{(1 + \xi^2k^2)} [(1 - ik\xi)e^{ikx} - (1 + ik\xi)e^{-ikx}][(1 - ik\xi)e^{ikx'} - (1 + ik\xi)e^{-ikx'}]$$

$$\begin{aligned}
&= \delta(x - x') - \frac{1}{2\pi} \int_0^\infty dk \left[\frac{(1 - ik\xi)^2 e^{ik(x+x')}}{(1 + \xi^2 k^2)} + \frac{(1 + ik\xi)^2 e^{-ik(x+x')}}{(1 + \xi^2 k^2)} \right] \\
&= \delta(x - x') - \frac{1}{2\pi} \int_{-\infty}^\infty dk \frac{(1 - ik\xi) e^{ik(x+x')}}{(1 + ik\xi)} = \delta(x - x') - \frac{1}{\pi} \int_{-\infty}^\infty dk \frac{e^{ik(x+x')}}{1 + ik\xi},
\end{aligned}$$

where in the last step we used the fact that $\int_{-\infty}^\infty dk e^{ik(x+x')} = 0$, since $x, x' > 0$. Performing the final integration, we then get the completeness relations in Eq. (A4).

2. Spectrum on the real line

The operator appearing in Eq. (71) has a complete set of basis functions comprised of a continuum of scattering states given by

$$\begin{aligned}
\hat{\eta}^k(x) &= \frac{i[(1 - i\xi k)e^{ik|x|} - (1 + i\xi k)e^{-ik|x|}]}{\sqrt{4\pi(1 + \xi^2 k^2)}}, \quad k > 0, \\
\sigma^k(x) &= \frac{1}{\sqrt{\pi}} \sin(kx), \quad k > 0,
\end{aligned} \tag{A5}$$

and, for $\text{Re}[\xi] > 0$, a bound state given by

$$\hat{\eta}^b(x) = \frac{1}{\sqrt{\xi}} \exp\left(-\frac{|x|}{\xi}\right). \tag{A6}$$

Observe that, up to a multiplicative constant, $\hat{\eta}^k$ and $\hat{\eta}^b$ are, respectively, the symmetrization of η^k and η^b . The eigenfunctions σ^k are odd while $\hat{\eta}^k, \hat{\eta}^b$ are even. All the functions $\hat{\eta}^b, \hat{\eta}^k$ and σ^k satisfy the boundary condition in Eq. (71). Actually σ^k satisfy the boundary condition somewhat trivially since they are smooth and $\sigma^k(0) = 0$. All these eigenfunctions also satisfy the orthonormality conditions:

$$\begin{aligned}
\int_{-\infty}^\infty dx [\hat{\eta}^b]^2 &= 1, \\
\int_{-\infty}^\infty dx \hat{\eta}^k \hat{\eta}^{k'} &= \int_{-\infty}^\infty dx \sigma^k \sigma^{k'} = \delta(k - k'), \quad k, k' > 0, \\
\int_{-\infty}^\infty dx \sigma^k \hat{\eta}^{k'} &= \int_{-\infty}^\infty dx \sigma^k \hat{\eta}^b = \int_{-\infty}^\infty dx \hat{\eta}^k \hat{\eta}^b = 0,
\end{aligned} \tag{A7}$$

and the completeness relation

$$\begin{aligned}
\int_0^\infty dk [\hat{\eta}^k(x)\hat{\eta}^k(x') + \sigma^k(x)\sigma^k(x')] &= \delta(x - x'), \quad \text{for } \text{Re}[\xi] < 0, \\
\int_0^\infty dk [\hat{\eta}^k(x)\hat{\eta}^k(x') + \sigma^k(x)\sigma^k(x')] + \hat{\eta}^b(x)\hat{\eta}^b(x') &= \delta(x - x'), \quad \text{for } \text{Re}[\xi] > 0.
\end{aligned} \tag{A8}$$

The results of Eq. (A7) are easily checked by using $\int_{-\infty}^\infty dx e^{irx} = 2\pi\delta(r)$. The completeness relation Eq. (A8) of these eigenfunctions is a consequence of the completeness relation Eq. (A4) of the $\{\eta^k \ k > 0\}$, η^b when restricted to the half line. Indeed, if $\Psi(x)$ is a square integrable wave function, we can write it as $\Psi = \Psi^s + \Psi^a$ where Ψ^s (respectively, Ψ^a) is the symmetric (respectively, antisymmetric) part of Ψ . We can decompose Ψ^s on $[0, \infty)$ on the basis of the η^k, η^b thanks to the completeness relation Eq. (A4). Since $\Psi^s, \hat{\eta}^k, \hat{\eta}^b$ are symmetric, the decomposition also holds on $(-\infty, \infty)$ with η^k (respectively, η^b) replaced by $\hat{\eta}^k$ (respectively, $\hat{\eta}^b$), up to some multiplicative constants. On the other hand, we can decompose the function Ψ^a on the standard Fourier basis $\Psi^a(x) = (2\pi)^{-1/2} \int_{-\infty}^\infty dk [\mathfrak{F}\Psi^a](k) e^{-ikx}$ and by the oddness of Ψ^a it follows that $\Psi^a(x) = -i\sqrt{2} \int_0^\infty dk [\mathfrak{F}\Psi^a](k) \sigma^k(x)$. This concludes the proof of the completeness.

3. Spectrum on the \mathbb{N} lattice

We want to study the spectrum of the non-hermitian Hamiltonian $H_{\mathbb{N}}$ defined by Eq. (22). We start to look for eigenfunctions $(\psi_n)_{n \geq 1}$ associated to the complex eigenvalue E :

$$\begin{aligned}
(2 - iw)\psi_1 - \psi_2 &= E\psi_1, \\
2\psi_n - \psi_{n-1} - \psi_{n+1} &= E\psi_n, \quad n > 1.
\end{aligned} \tag{A9}$$

This is equivalent to solving the second equation for $n \geq 1$ with the boundary condition

$$\psi_0 + \xi(\psi_1 - \psi_0) = 0, \quad \xi = \frac{-iw}{1 - iw}. \tag{A10}$$

The most general scattering solutions are of the form $\psi_n = Ae^{ikn} + Be^{-ikn}$. This satisfies the second equation of Eq. (A9)

with $E_s(k) = 2(1 - \cos(k)) = 4 \sin^2(k/2)$. Then the boundary condition Eq. (A10) above gives $B = -A \frac{1-\xi+\xi e^{ik}}{1-\xi+\xi e^{-ik}}$.

This gives us all the scattering solutions and we define for each $k \in (0, \pi)$ the eigenfunction η^k associated to the eigenvalue $2(1 - \cos(k))$ by

$$\eta_n^k = \frac{i[(1 - \xi + \xi e^{-ik})e^{ikn} - (1 - \xi + \xi e^{ik})e^{-ikn}]}{\sqrt{2\pi(1 - \xi + \xi e^{ik})(1 - \xi + \xi e^{-ik})}} \quad (\text{A11})$$

for any $n \geq 1$.

We can then look for bound state solutions of the form e^{-kn} with $\text{Re}(k) > 0$. In this case we find $2(1 - \cosh(k))$ and using the boundary condition it gives $1 - \xi^{-1} = e^{-k}$, which has a solution with $\text{Re}(k) > 0$ only for $|1 - \xi^{-1}| < 1$ or equivalently $|w| > 1$. In this case we define the bound state η^b associated to the eigenvalue $E_b = 1 + \xi^{-1}$ by

$$\eta_n^b = \frac{(1 - \xi^{-1})^n}{\sqrt{1 - (1 - \xi^{-1})^2}} \quad (\text{A12})$$

for any $n \geq 1$.

Orthogonality relations and completeness property similar to Eqs. (A3) and (A4) may be established as in the continuum case.

4. Spectrum on the \mathbb{Z} lattice

The spectrum of the non-hermitian Hamiltonian $H_{\mathbb{Z}}$ defined by Eq. (23) can be deduced from the spectrum of $H_{\mathbb{N}}$ derived in the previous section. We recall that the equations of motion are given by Eq. (70). Let $(\psi_n)_{n \neq 0}$ be an eigenstate associated to the eigenvalue E . It satisfies

$$\begin{aligned} (2 - iw)\psi_1 - \psi_2 - iw\psi_{-1} &= E\psi_1, \\ (2 - iw)\psi_{-1} - \psi_{-2} - iw\psi_1 &= E\psi_{-1}, \\ 2\psi_n - \psi_{n-1} - \psi_{n+1} &= E\psi_n, \quad |n| \geq 2. \end{aligned} \quad (\text{A13})$$

Introduce the symmetric part $\psi^s = (\frac{\psi_n + \psi_{-n}}{2})_{n \neq 0}$ and the antisymmetric part $\psi^a = (\frac{\psi_n - \psi_{-n}}{2})_{n \neq 0}$ of ψ . The functions ψ^s satisfy Eq. (A9) for $n \geq 1$ with w replaced by $2w$ and is therefore eigenfunctions of $H_{\mathbb{N}}$ associated to the eigenvalue E . On the other hand, the function ψ^a satisfies for $n \geq 1$ the equations

$$\begin{aligned} 2\psi_1^a - \psi_2^a &= E\psi_1^a, \\ 2\psi_n^a - \psi_{n-1}^a - \psi_{n+1}^a &= E\psi_n^a, \quad n \geq 2, \end{aligned}$$

which can be equivalently written as

$$2\psi_n^a - \psi_{n-1}^a - \psi_{n+1}^a = E\psi_n^a, \quad n \geq 1,$$

with the boundary condition $\psi_0^a = 0$. Since ψ^a is antisymmetric, this is equivalent to say that ψ^a is an antisymmetric eigenstate for the discrete Laplacian on \mathbb{Z} . Observe moreover that ψ^s and ψ^a are orthogonal since $\sum_{n \neq 0} \psi_n^a \psi_n^s = 0$ due to the symmetry properties of the two functions.

By introducing $\zeta = \frac{-2iw}{1-2iw}$ we deduce that the spectrum of $H_{\mathbb{Z}}$ is composed of

(1) Symmetric scattering states $\hat{\eta}^k$ for each $k \in (0, \pi)$ associated to the eigenvalue $2(1 - \cos(k))$ and defined for $|n| \geq 1$

by

$$\hat{\eta}_n^k = \frac{i[(1 - \zeta + \zeta e^{-ik})e^{ik|n|} - (1 - \zeta + \zeta e^{ik})e^{-ik|n|}]}{\sqrt{4\pi(1 - \zeta + \zeta e^{ik})(1 - \zeta + \zeta e^{-ik})}} \quad (\text{A14})$$

(2) Antisymmetric scattering states σ^k for each $k \in (0, \pi)$ associated to the eigenvalue $2(1 - \cos(k))$ and defined for $|n| \geq 1$ by

$$\sigma_n^k = \frac{1}{\sqrt{\pi}} \sin(kn). \quad (\text{A15})$$

(3) A symmetric bound state $\hat{\eta}^b$ associated to the eigenvalue $E_b = 1 + \zeta^{-1}$ if $|1 - \zeta^{-1}| < 1$ or equivalently $|2w| > 1$. It is defined by

$$\hat{\eta}_n^b = \frac{(1 - \zeta^{-1})^{|n|}}{\sqrt{2(1 - (1 - \zeta^{-1})^2)}}. \quad (\text{A16})$$

These eigenstates satisfy orthogonality relations by construction and a completeness property similar to Eqs. (A7) and (A8) may be established as in the continuum case.

APPENDIX B: TECHNICAL COMPUTATIONS TO ESTIMATE THE SURVIVAL PROBABILITY

1. \mathbb{N} Lattice

a. Computation of the expression given in Eq. (35)

We compute here the quantity

$$Z(w, n_0) = i \frac{\frac{2}{\pi} \int_0^\pi dk \frac{\sin(k) \sin(n_0 k)}{is - 2(1 - \cos(k))}}{1 + i \frac{2w}{\pi} \int_0^\pi dk \frac{\sin^2 k}{is - 2(1 - \cos(k))}}, \quad (\text{B1})$$

where $w = \alpha + i\beta$. We will assume that $s \in \mathbb{C}$ satisfies $\text{Im}(s) \neq -2$ and $s \notin i\mathbb{R}$.

The integral above can be evaluated by means of contour integration. We first note that

$$\begin{aligned} \int_0^\pi dk \frac{\sin(k) \sin(n_0 k)}{is - 2(1 - \cos(k))} \\ = \frac{1}{8} [-I_{n_0+1}(s) + I_{n_0-1}(s) + I_{-n_0+1}(s) - I_{-n_0-1}(s)], \end{aligned}$$

where

$$\begin{aligned} I_n(s) &= \int_{-\pi}^\pi dk \frac{\exp(ink)}{is - 2(1 - \cos(k))} \\ &= -i \int_{\mathcal{C}} dz \frac{z^n}{z^2 - (2 - is)z + 1}. \end{aligned}$$

The integration contour \mathcal{C} in the second integral above is the unit circle. It is easily seen that $I_n(s) = I_{-n}(s)$ so that we need only to compute $I_n(s)$ for $n \geq 0$. The poles of the integrand are

$$z_{\pm}(s) = i\theta_{\pm} \left(\frac{s}{2} + i \right)$$

where

$$\theta_{\pm}(z) = -z \pm \sqrt{z^2 + 1}. \quad (\text{B2})$$

We need first to know the positions of these poles with respect to \mathcal{C} .

These two functions are well defined analytic functions in the subdomain Ω , defined as the complex plane \mathbb{C} where the imaginary axis has been removed apart from the open segment of the imaginary axis between $-i$ and i . We claim that $|\theta_+(z)| < 1$ and $|\theta_-(z)| > 1$ for $z \in \Omega_+ = \Omega \cap \{z \in \mathbb{C} : \text{Re}(z) > 0\}$ and $|\theta_+(z)| > 1$ and $|\theta_-(z)| < 1$ for $z \in \Omega_- = \Omega \cap \{z \in \mathbb{C} : \text{Re}(z) < 0\}$. Since the proof of the two claims are similar we only prove the first one. Notice that θ_{\pm} are analytic on the connected set Ω_+ and satisfy

$$\theta_+(z) \cdot \theta_-(z) = -1, \quad \frac{\theta_+(z) + \theta_-(z)}{2} = -z.$$

Consider $z \in \Omega_+$. If $\theta_+(z)$ or $\theta_-(z)$ was belonging to \mathcal{C} then both would be because of the first relation above. It would then follow that $z = -i \sin[\arg(\theta_+(z))]$ which is excluded since $z \in \Omega_+$. A similar *ad absurdum* argument shows that $\theta_-(z)$ and $\theta_+(z)$ do not belong to the imaginary axis. Since θ_{\pm} are analytic on the connected set Ω_+ , so is $\theta_{\pm}(\Omega_+)$. The imaginary axis being excluded from $\theta_{\pm}(\Omega_+)$, the domains $\theta_{\pm}(\Omega_+)$ are included into $\{z \in \mathbb{C}; \text{Re}(z) > 0\}$ or into $\{z \in \mathbb{C}; \text{Re}(z) < 0\}$. Since $\theta_+(1) = \sqrt{2} - 1 > 0$ and $\theta_-(1) = -\sqrt{2} - 1 < 0$ we get $\theta_+(\Omega_+) \subset \{z \in \mathbb{C}; \text{Re}(z) > 0\}$ and $\theta_-(\Omega_+) \subset \{z \in \mathbb{C}; \text{Re}(z) < 0\}$. Similarly, since $\theta_{\pm}(\Omega_+) \cap \mathcal{C} = \emptyset$ the domains $\theta_{\pm}(\Omega_+)$ are included into the interior of the unit disk or into the exterior of the unit disk. The values above of $\theta_{\pm}(1)$ imply that $\theta_+(\Omega_+)$ is included in the interior of unit disk and $\theta_-(\Omega_+)$ is included in the exterior of unit disk. Hence we have that if $\text{Re}(s) > 0$, the pole inside the unit disk is $z_+(s)$.

It follows by the Residues theorem that

$$I_n(s) = -i\pi \frac{[z_+(s)]^n}{\sqrt{\left(\frac{s}{2} + i\right)^2 + 1}}, \quad \text{Re}(s) > 0. \quad (\text{B3})$$

This gives

$$\int_0^\pi dk \frac{\sin(k) \sin(n_0 k)}{is - 2(1 - \cos k)} = -\frac{\pi}{2} [z_+(s)]^{n_0}, \quad \text{Re}(s) > 0. \quad (\text{B4})$$

From Eqs. (B1) and (B4)

$$Z(w, n_0) = -i \frac{[z_+(s)]^{n_0}}{1 - iw[z_+(s)]}, \quad \text{Re}(s) > 0. \quad (\text{B5})$$

b. Integral approximation

Here we estimate the integral

$$\int_0^1 du \frac{u^{2n_0-2}(1-u^2)(1+|w|^2u^2)}{(1+|w|^2u^2)^2 - (2\text{Im}(w)u)^2}.$$

With $g(t)$ defined by

$$g(t) = e^{-t} \frac{(1 - e^{-2t})(1 + |w|^2 e^{-2t})}{(1 + |w|^2 e^{-2t})^2 - (2\text{Im}(w) e^{-t})^2},$$

the integral to be estimated is same as

$$\int_0^\infty dt e^{-(2n_0-2)t} g(t) = \frac{1}{2(n_0 - 1)} \int_0^\infty dt e^{-t} g\left(\frac{t}{2(n_0 - 1)}\right).$$

For t near 0 we have

$$g(t) \sim \frac{2(1 + |w|^2)}{(1 + |w|^2)^2 - 4[\text{Im}(w)]^2} t,$$

and therefore for large n_0 we get

$$\int_0^1 du \frac{u^{2n_0-2}(1-u^2)(1+|w|^2u^2)}{(1+|w|^2u^2)^2 - (2\text{Im}(w)u)^2} \sim \frac{1}{n_0^2} \frac{(1 + |w|^2)}{2(1 + |w|^2)^2 - 8[\text{Im}(w)]^2}. \quad (\text{B6})$$

c. Expression of ψ_1 in terms of Bessel functions of first kind

We obtain here the explicit formula Eq. (43) for ψ_1 whose Laplace transform can be expressed as

$$[\mathcal{L}\psi_1](s) = -i^{n_0+1} \chi\left(\frac{s}{2} + i\right),$$

where

$$\chi(z) = \frac{[\theta_+(z)]^{n_0}}{1 + w\theta_+(z)}$$

with the function θ_{\pm} defined in Eq. (B2). We recall that this function is analytic in the subdomain Ω , defined as the complex plane \mathbb{C} where the imaginary axis has been removed apart from the open segment of the imaginary axis between $-i$ and i . Moreover we have seen that

$$\text{Re}(z) > 0 \quad \text{implies} \quad |\theta_+(z)| < 1, \quad |\theta_-(z)| > 1.$$

Let us first identify $\chi(z)$ as the Laplace transform $[\mathcal{L}f](z)$ of some explicit function f . It is known that if $z \in \mathbb{C}$ satisfies $\text{Re}(z) > 0$ then

$$\int_0^\infty dt e^{-zt} \frac{J_k(t)}{t} = \frac{(\theta_+(z))^k}{k}.$$

Assuming that z is such that $|w\theta_+(z)| < 1$ we have then that

$$\begin{aligned} \chi(z) &= \sum_{k=0}^\infty (-w)^k [\theta_+(z)]^{n_0+k} \\ &= \sum_{k=0}^\infty (k + n_0) (-w)^k \int_0^\infty dt e^{-zt} \frac{J_{k+n_0}(t)}{t} \\ &= \int_0^\infty dt e^{-zt} f(t) dt = [\mathcal{L}f](z) \end{aligned}$$

with

$$f(t) = \sum_{k=0}^\infty (k + n_0) (-w)^k \frac{J_{k+n_0}(t)}{t}.$$

The interversion of the sum and the integral is justified since $J_k(2t) \sim (2\pi k)^{-1/2} (et/k)^k$ for large k . It follows that for any time $t \geq 0$

$$\psi_1(t) = -2i^{n_0+1} e^{-2it} \sum_{k=0}^\infty (k + n_0) (-w)^k \frac{J_{k+n_0}(2t)}{2t},$$

since the functions above have their Laplace transform coinciding on $\{s \in \mathbb{C} ; \text{Re}(s) > \sigma\}$ for some $\sigma > 0$ sufficiently large.

2. First passage time distribution on the half line

Recall Eq. (61) giving the expression of the solution of the Schrödinger equation on the half-line with complex Robin boundary condition and initial condition $\Psi_0(x) = \Psi(x, 0)$. To simplify we assume that Ψ_0 has compact support and has a bounded derivative defined almost everywhere. In particular all the moments of Ψ_0 and $\frac{\partial \Psi_0}{\partial x}$ are well defined. After a change of variables we can write $\Psi(0, t)$ as

$$\Psi(0, t) = \sqrt{\frac{2}{\pi}} \frac{\zeta}{t} \int_0^\infty dy \frac{y}{\sqrt{1 + \frac{\zeta^2}{t} y^2}} c\left(\frac{y}{\sqrt{t}}\right) \exp(-iy^2) + \sqrt{\frac{2}{\zeta}} c_b \exp\left(i \frac{t}{\zeta^2}\right),$$

where

$$c(k) = -\sqrt{\frac{2}{\pi(1 + \zeta^2 k^2)}} k H(k),$$

$$H(k) = \int_0^\infty dx \underbrace{\left[\Psi_0(x) + \zeta \frac{\partial \Psi_0}{\partial x}(x) \right]}_{\Phi(x)} \frac{\sin(kx)}{k}.$$

Up to factors, $H(k)$ is the sine transform of $\Phi(x)$. If $H(k)$ were to vanish identically, then $\Phi(x) \equiv 0$ and $\Psi_0(x)$ would correspond to the bound state. In this situation $\Psi(0, t)$ evolves solely by the second term and the first passage time distribution $F(t)$ defined by Eq. (67) has exponential decay for ζ lying in the fourth quadrant.

Assume $H(k)$ is not identically 0. Substituting for $c(k)$ one has

$$\Psi(0, t) \approx -\frac{2}{\pi} \frac{\zeta}{t^{\frac{3}{2}}} \int_0^\infty dy \frac{y^2}{1 + \frac{\zeta^2}{t} y^2} H\left(\frac{y}{\sqrt{t}}\right) \exp(-iy^2),$$

where the exponentially decaying term is neglected. By expanding $\sin(kx)$, and switching [34] the summation and integration for $\Phi(x)$ one can obtain an absolutely convergent series expansion for $H\left(\frac{y}{\sqrt{t}}\right)$. We get that

$$H\left(\frac{y}{\sqrt{t}}\right) = \sum_{s=0}^\infty \frac{(-1)^s}{t^s} \frac{y^{2s}}{(2s+1)!} \int_0^\infty dx x^{2s+1} \Phi(x) = \sum_{s=0}^\infty \frac{(-1)^s}{t^s} \frac{y^{2s}}{(2s+1)!} [M_{2s+1} - (2s+1)\zeta M_{2s}], \tag{B7}$$

where $M_s = \int_0^\infty x^s \Psi_0(x) dx$, the s th moment of Ψ_0 . In the second line we perform an integration by parts. This allows us to finally write down the series

$$\Psi(0, t) \approx -\frac{2}{\pi} \frac{\zeta}{t^{\frac{3}{2}}} \left[\sum_{s=0}^\infty \frac{(-1)^s}{(2s+1)! t^s} M_{2s+1} \left(1 - \frac{\zeta}{\zeta_s}\right) I_s\left(\frac{\zeta}{\sqrt{t}}\right) \right], \tag{B8}$$

where

$$I_s(z) = \int_0^\infty dy \frac{y^{2s+2}}{1 + z^2 y^2} \exp(-iy^2) \tag{B9}$$

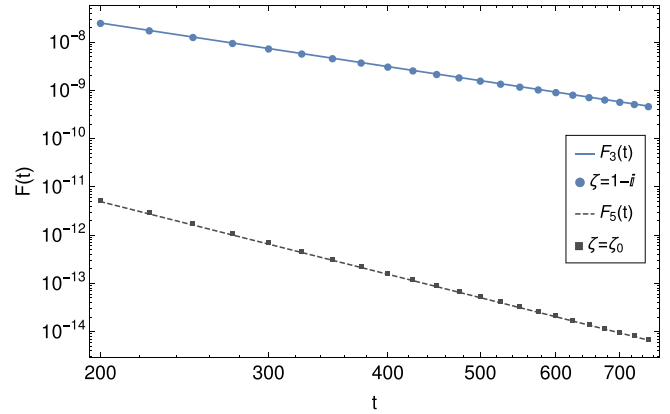


FIG. 5. Plot comparing numerical estimates of $F(t)$ with $F_3(t)$ (resp. $F_5(t)$) for $\zeta = 1 - i$ (resp. $\zeta (= \zeta_0) = 2 - i/2$). The discrete points correspond to numerical values evaluated from Eq. (61).

and

$$\zeta_s = \frac{1}{2s+1} \frac{M_{2s+1}}{M_{2s}} \tag{B10}$$

with the convention that $\zeta_s = \infty$ if $M_{2s} = 0$. Noting that

$$\lim_{z \rightarrow 0} I_0(z) = -\frac{\sqrt{\pi}}{4} e^{\frac{i\pi}{4}} \quad \text{and} \quad \lim_{z \rightarrow 0} I_1(z) = -\frac{3\sqrt{\pi}}{8} e^{-\frac{i\pi}{4}},$$

one has then for large t

$$\Psi(0, t) = \frac{1}{2\sqrt{\pi}} \frac{\zeta}{t^{\frac{3}{2}}} \left[\left(1 - \frac{\zeta}{\zeta_0}\right) M_1 e^{\frac{i\pi}{4}} - \left(1 - \frac{\zeta}{\zeta_1}\right) \frac{M_3}{4t} e^{-\frac{i\pi}{4}} \right] + O(t^{-5/2}). \tag{B11}$$

This proves Eq. (69).

Consider the state

$$\Psi_0(x) = \Theta(x-2) - \frac{\Theta(x-1) + \Theta(x-3)}{2} + i \frac{\Theta(x-1) - \Theta(x-3)}{2} \tag{B12}$$

where $\Theta(x)$ is the Heaviside step function. Then

$$M_1 = \frac{1}{2} + 2i, \quad \zeta_0 = 2 - \frac{i}{2},$$

$$M_3 = \frac{25}{4} + 10i, \quad \zeta_1 = \frac{67}{82} - \frac{17i}{164}.$$

The numerical evaluation of the integral in Eq. (61) can be performed to obtain $\Psi(0, t)$. We choose the range $t \in (200, 750)$ which is sufficient to suppress the bound state contribution in Eq. (61). To begin with let $\zeta = 1 - i$. Then ζ is at a sufficient distance from ζ_0 and the first term in the expansion Eq. (B11) dominates. Therefore one has as $t \rightarrow \infty$ that

$$F_3(t) = -2 \frac{\text{Im}(\zeta)}{|\zeta|^2} |\Psi(0, t)|^2 \sim \frac{5}{8\pi t^3}.$$

We use this analytic estimate to compare with the numerical evaluation via Eq. (61) in Fig. 5. This shows that the analytic estimate is quite good and indeed $F(t) \sim \frac{1}{t^3}$.

Now choose $\zeta = \zeta_0$. Doing so causes the first term in the expansion Eq. (B11) to drop out. The estimate for $F(t)$ is obtained from Eq. (B11) which gives

$$F_5(t) \asymp -\frac{\text{Im}(\zeta_0)}{32\pi t^5} \left| M_3 \left(1 - \frac{\zeta_0}{\zeta_1} \right) \right|^2 = \frac{5105}{1024} \frac{1}{\pi t^5}.$$

Figure 5 shows good agreement of the above estimate with numerical values.

We could now claim that if $\Psi_0(x)$ was so constructed that

$$\zeta_0 = \zeta_1 = \dots = \zeta_{s-1} \neq \zeta_s (\neq 0),$$

then one has

$$F(t; \zeta = \zeta_0) \sim \frac{1}{t^{3+2s}},$$

while for other choices of ζ one has

$$F(t, \zeta \neq \zeta_0) \sim \frac{1}{t^3}.$$

This is under the assumption that the moments $\int_0^\infty x^k \Psi_0(x) dx$ do not vanish up until $k = 2s + 1$. Lastly, we note that for the normalized state

$$\Psi_0(x) = \sqrt{\frac{1}{\zeta_0} + \frac{1}{\zeta_0^*}} \exp\left(-\frac{x}{\zeta_0}\right)$$

with ζ_0 in the fourth quadrant, $\zeta_s = \zeta_0$ for all s follows from Eq. (B10). For the measurement parameter $\zeta = \zeta_0$, the estimate in Eq. (B8) is identically 0 and $F(t)$ falls exponentially, as has already been noted.

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