Femtosecond synchronization of clocks

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The precise-timing community is currently working on the task of synchronizing moving clocks in the field to an accuracy of femtoseconds. We argue that its recent claims of partial success must be examined in the context of relativity, and show that one must establish the context very carefully when claiming any sort of synchronization at this level. In particular, irrespective of the sense of the synchronization, we show that realistic movements of the clocks will destroy the synchronization moments after it is established.

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I. INTRODUCTION

The ideas and practice of promulgating high-precision time across a network are evolving as demands on that precision increase. To be of use, a network's clocks must be synchronized, and modern requirements pursue ever higher accuracies. Synchronization has two core components: a theoretical one that asks what synchronization means, and a practical one that attempts to overcome environmental noise in having two clocks communicate to establish the synchronization.

Recent papers have discussed this practical component, via experiments in synchronizing stationary clocks to femtosecond accuracy in the presence of simulated motion [1–3]. In these field experiments, it was impractical to give the clocks any real relative motion. Instead, that motion was typically simulated by mounting a retro-reflector on a quadcopter, and using that to create a changing optical-path length for signals exchanged by the ground-fixed clocks, as shown in Fig. 1. The synchronization procedure interpreted this changing path length as relative opening or closing speeds of the clocks of tens of meters per second. A careful procedure employing optical frequency combs resulted in a synchronization at the level of 1 fs.

Overcoming experimental difficulties in the field is one side of the coin, but this level of reported synchronization must still be examined in a theoretical context; specifically, to ascertain how compatible it might be with relativity. The motion simulation employed above does not incorporate the relativistic concept that *true* relative motion will affect the clocks' tick rates in a nontrivial way; for example, in the simplest of cases, each clock will say that the other ticks slowly. Hence a true synchronization can never be performed, even in principle. In practice, one clock might be given preeminence over the others, which are "geared" to keep time with that master clock. But even in this case, if their relative speeds are liable to change unpredictably, the clocks can never be synchronized better than to some practical level. We will show that for real relative speeds of tens of meters per second, these different tick rates will destroy the femtosecond synchronization almost immediately after that synchronization has been achieved.

To be specific, we pose three questions about three identical clocks that are in relative motion. Clock 1 is fixed to the ground on Earth's equator. Clock 2 is nearby at the same altitude but is moving over the ground: clock 1 says that clock 2 is moving east at $V = 10$ m/s relative to clock 1. This is a very slow speed for an airborne clock, but it is a conservative example, since higher speeds will only increase the time disparities. This scenario is being monitored by a third clock at spatial infinity, the "primary clock." Including such a clock might seem strange, but it is precisely this clock that defines a sort of universal time that turns out to be proportional to the UTC time that is central to modern global precision timing.

Clock 1 ticks out a time interval of $\Delta\tau_1 = 5$ s. We ask the following:

(1) How much time Δt does the clock at spatial infinity say has passed?

(2) How much time $\Delta\tau_2$ does the clock at spatial infinity say has passed on clock 2?

(3) How much time $\Delta\tau_2'$ does clock 1 say has passed on clock 2?

II. THE NECESSARY RELATIVITY

To relate the displays on the two Earth clocks to the display on the primary clock at spatial infinity, it is sufficient to model the flow of time at a point near Earth's surface by the *weak-field metric* of general relativity [4], using nominally spherical-polar coordinates:

$$d\tau^2 = (1 + 2\Phi/c^2)\,dt^2 - (1 - 2\Phi/c^2)\,\frac{1}{c^2}[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2]. \quad (1)$$

Here τ is proper time: the time that elapses on a clock moving from point (r, θ, ϕ) to $(r + dr, \theta + d\theta, \phi + d\phi)$ during a lapse dt of time recorded by the primary clock at spatial infinity. $\Phi(r, \theta, \phi)$ is Earth's gravitational potential, which is sufficient to be set equal to $-GM/r$ (i.e., a spherical Earth), since any refinements on that potential only produce small

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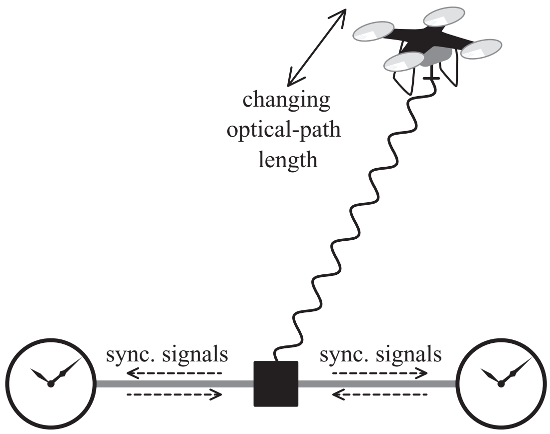

FIG. 1. The synchronization setup of [1–3]. The clocks, fixed to the ground, have minimal separation, and communicate via the schematic black box. This box bounces optical signals from a retroreflector on a quadcopter about 2 km away, which moves to simulate a changing distance between the clocks.

changes to the numbers that follow. The radial coordinate r originates at Earth's center; θ is colatitude, and ϕ is longitude.

The weak-field metric (1) is a standard of timing literature. For example, it (and refinements on it) forms the basis of how UTC time is defined [5], and it can be employed very straightforwardly to generate the well-known 39 μs/day slower tick required in the manufacture of a GPS satellite's clock. In this paper, we apply (1) to the related task of answering the three questions above. I am not aware of anything like those three questions appearing in timing literature (at least as a set, but perhaps not individually either), since that literature tends not to dwell on questions of simultaneity such as we study here.

Picture a clock fixed to the ground at (r, θ, ϕ). The primary clock at spatial infinity records Earth rotating at angular rate ω. In the primary's time interval dt, the Earth-fixed clock moves in the "Earth-centered inertial frame" by amounts

$$dr = 0, \quad d\theta = 0, \quad d\phi = \omega\, dt. \tag{2}$$

The time that the Earth-fixed clock ticks is its proper time $d\tau$ in (1). Dividing that equation by dt^2 gives

$$\frac{d\tau^2}{dt^2} = 1 + 2\Phi/c^2 - (1 - 2\Phi/c^2)\,\frac{r^2\omega^2}{c^2}\sin^2\theta. \tag{3}$$

In the primary clock's frame, Earth's spin imparts a velocity $v = v(r, \theta)$ to the clock fixed to Earth's surface:

$$v^2 = r^2\omega^2\sin^2\theta. \tag{4}$$

(This equation is in fact slightly inaccurate since r is not precisely a radial *distance*; but this approximation is adequate for what follows.) Equation (3) is then

$$\frac{d\tau^2}{dt^2} = 1 + 2\Phi/c^2 - (1 - 2\Phi/c^2)\,v^2/c^2. \tag{5}$$

To gain an impression of the relative sizes of the terms in (5), note that for a representative point on the equator of a spherical Earth of radius $R = 6370$ km and spinning at $\omega = 2\pi/(24\text{ h})$ (a *solar* day length is sufficient here), the following two dimensionless quantities are

$$2\Phi/c^2 \approx \frac{-2GM}{Rc^2} \simeq -1.4\times 10^{-9},$$
$$v^2/c^2 = R^2\omega^2/c^2 \simeq 2.4\times 10^{-12}. \tag{6}$$

These both have magnitudes much less than one. Thus, to first order, the square root of (5) is

$$d\tau/dt \simeq 1 + (\Phi - v^2/2)/c^2. \tag{7}$$

For the parameters in (6), Eq. (7) becomes

$$d\tau/dt \simeq 1 - 0.7\times 10^{-9} - 1.2\times 10^{-12}$$
$$\simeq 1 - 0.7\times 10^{-9}. \tag{8}$$

(To the level of accuracy used here, the constant 0.7×10^{-9} is called L_G in the literature [5].) The gravity potential contributes about 99.8% of this departure from unity, with the clock's motion (from Earth's rotation) contributing only 0.2%.

Clock 1 is fixed to the geoid, and so the answer to question 1 in Sec. I follows quickly from (8). The time Δt elapsed on the primary clock is, from (8),

$$\Delta t = \Delta\tau_1 \times dt/d\tau_1 \simeq 5\text{ s} \times (1 + 0.7\times 10^{-9})$$
$$= 5\text{ s} + 3.5\text{ ns}. \tag{9}$$

The primary clock says that an extra 3.5 ns passed while clock 1 ticked 5 s.

To answer question 2, we apply (7) to each of the clocks 1 and 2, remembering that t is the time on the primary clock; τ_1, τ_2 are the times on clocks 1 and 2; and v_1, v_2 are their velocities in the frame of the primary clock. It is useful to answer the question in a slightly more general sense, where clocks 1 and 2 might be at different potentials Φ_1, Φ_2. When the primary clock has ticked a time dt, it says that the times elapsed on clocks 1 and 2 are $d\tau_1, d\tau_2$. Equation (7) then says that the ratio of those clocks' tick rates is

$$\frac{d\tau_2}{d\tau_1} = \frac{d\tau_2/dt}{d\tau_1/dt} \simeq \frac{1 + (\Phi_2 - v_2^2/2)/c^2}{1 + (\Phi_1 - v_1^2/2)/c^2}$$
$$\simeq \left(1 + \frac{\Phi_2 - v_2^2/2}{c^2}\right)\left(1 - \frac{\Phi_1 - v_1^2/2}{c^2}\right)$$
$$\simeq 1 + \frac{1}{c^2}\left[\Phi_2 - \Phi_1 + \frac{v_1^2 - v_2^2}{2}\right]. \tag{10}$$

This is the general expression. In question 2, $\Phi_1 = \Phi_2$. The time $\Delta\tau_2$ that the primary clock says has passed on clock 2 is then

$$\Delta\tau_2 = \Delta\tau_1 \times d\tau_2/d\tau_1 \simeq \Delta\tau_1\left(1 + \frac{v_1^2 - v_2^2}{2c^2}\right). \tag{11}$$

Clock 1 is fixed to Earth's equator, and so has a velocity of $v_1 = 465$ m/s east in the primary clock's frame (the approximately inertial frame in which Earth spins). The speeds are low enough here that we can say $v_2 = v_1 + V = 475$ m/s. Equation (11) requires

$$v_1^2 - v_2^2 \simeq v_1^2 - (v_1 + V)^2 \simeq -2v_1 V \quad \text{for } V \ll v_1, \tag{12}$$

and hence yields

$$\Delta\tau_2 \simeq \Delta\tau_1\left(1-\frac{v_1V}{c^2}\right) \simeq 5\text{ s}\times\left(1-\frac{465\times 10}{9\times 10^{16}}\right) \simeq 5\text{ s} - 260\text{ fs}. \tag{13}$$

In summary, the primary clock says that after 5 s + 3.5 ns, clock 1 has ticked 5 s, and clock 2 has ticked 5 s − 260 fs.

These first two questions in Sec. I that we have just answered were posed by the primary clock. In fact, while they were addressed in standard general-relativistic fashion, they are not as straightforward as might first appear. In question 1, the primary clock is essentially saying "I maintain that *simultaneously* with the moment that my clock displays time t, clock 1 displays some time τ_1. Later, I maintain that *simultaneously* with the moment that clock 1 displays $\tau_1 + 5$ seconds, my clock displays $t + \Delta t$. What is Δt?". This question demands a knowledge of what simultaneity means. The primary clock is infinitely far from the gravity field and thus is inertial. Although the entire space-time is not flat, it is static, and this allows the primary clock to define simultaneity via a light-ray argument similar to that found in introductory textbooks on *special* relativity, even though the speed of light rays in the primary clock's frame is generally not c.

In contrast, question 3 requires a notion of simultaneity for clock 1 instead of the primary clock. Clock 1 is *not* quite inertial: it is immersed in Earth's gravity, and shares Earth's rotation. How do we address this question? The subject of rotating observers in relativity is an old one, with arguments over the last century that are still not settled [6]. I have argued in previous work [7,8] that for scenarios not encompassing a large part of Earth's surface, we can ignore Earth's rotation even at the femtosecond level.

But what about gravity? A theory has yet to be found of how to construct a robust notion of simultaneity in curved space-time. Nevertheless, Earth's gravity is *very* weak. A rule-of-thumb space-time curvature near Earth's surface is $c^2/g \simeq$ 1 light-year, where g is gravity's strength near Earth's surface [9]; for example, a thrown rock and a fired bullet both follow trajectories in a three-axis plot of time/height/horizontal distance that have curvatures of c^2/g. So, a plot of time versus one dimension of space can be envisaged as a small piece of the surface of a sphere of radius one light-year: clearly, if such a piece were tens of kilometers on a side (modeling a real-world scenario), it would feel *extremely* flat—probably flatter than the flattest surface ever made by engineers. If g were constant everywhere within a light-year of Earth, we could conclude that over distances much less than a light-year, and time intervals much less than a year, we can use a special-relativistic argument to discuss the simultaneity standard of clock 1.

Of course, g is *not* constant in this way. But the general idea is taken over by introductory textbooks of special relativity: these treat such scenarios as occurring in inertial frames. Then, in a kind of pseudoinertial frame of clock 1, clock 2 is moving at speed V, and so clock 2 ticks slower than clock 1 by the usual special-relativistic gamma factor of

$$\gamma(V) = \frac{1}{\sqrt{1-V^2/c^2}} \simeq 1 + \frac{V^2}{2c^2}. \tag{14}$$

We then have

$$\Delta\tau_2' = \frac{\Delta\tau_1}{\gamma(V)} \simeq \Delta\tau_1\left(1-\frac{V^2}{2c^2}\right) = 5\text{ s}\times\left(1-\frac{10^2}{2\times 9\times 10^{16}}\right) \simeq 5\text{ s} - 2.8\text{ fs}. \tag{15}$$

Clock 1 says that clock 2 has lost 2.8 fs. Clearly, any femtosecond synchronization is destroyed after just a few seconds. Of course, if clock 2 maintains a *fixed* speed V relative to clock 1, then clock 2 can be "geared" up by a factor of $\gamma(V)$ to run at the same rate as clock 1, and the synchronization will then be preserved *from the viewpoint of clock 1 only*. It will not be preserved from the viewpoint of clock 2, which will say that clock 1 is ticking slowly. This well-known mismatch in how the clocks view the world is due to their different standards of simultaneity.

III. CONCLUDING COMMENTS

The three questions posed in Sec. I lead to time intervals that differ from 5 s by 3.5 ns, −260 fs, and −2.8 fs respectively: a variety of time differences whose meanings must be understood in each context.

Why involve the primary clock in questions 1 and 2? That clock, when "geared" to tick at a slightly different rate, is realized by our world's UTC time that runs precise timing scenarios such as location by GPS. UTC is a coordinate time for our Earth, in the sense that it has been imposed on all observers, moving or not, by a primary clock, and ignores questions of simultaneity for different observers. Two clocks showing the same UTC do not generally agree on simultaneity to a very fine level. Consider observers Alice and Bob at different locations on Earth's surface. Suppose that when Alice's UTC clock displays 12:00, she is able to say "According to my definition of 'now,' Bob's UTC is also 12:00." Then in general, when Bob's clock displays 12:00, he will say "According to my definition of 'now,' Alice's UTC is *not quite* 12:00."

Whether this UTC time is needed by real airborne vehicles depends on the scenario; it might be that a "local time in the field" is sufficient for the task. But as mentioned two paragraphs up, this local time must still be that of one of the clocks that has been given the special status of a primary clock, and again we will have simultaneity mismatches. And even then, as discussed in question 3 above, speed changes of one of the airborne vehicles will quickly destroy their femtosecond synchronization. Latitude changes will act similarly, since two clocks with the same ground speed but separated in latitude over our rotating Earth will have different speeds in the inertial frame in which Earth spins: a degree of latitude separation equates to such a speed difference of typically several meters per second. Of course, the synchronization might be reestablished periodically, but if the time between resynchronizations becomes very short, the very idea of synchronization loses its meaning, because the secondary clocks are then effectively never running independently of the primary.

UTC time is an example of a common time *coordinate* that can be displayed by the three clocks above. That is, clock 1 can be geared by a factor of $\Delta t/\Delta\tau_1$, so that it matches the primary clock's display t according to the simultaneity standard of the primary clock; and similarly for clock 2. But this time coordinate does not embody the simultaneity standards of clocks 1 and 2. Put another way, the relative clock rate in question 3 above was not $\Delta\tau_1/\Delta\tau_2$ (which would be trivially related to the answers to questions 1 and 2); instead, it was $\Delta\tau_1/\Delta\tau_2'$. The clock rates are fundamentally dependent on who is doing the measuring. Hence, the clocks cannot be geared in a way that will allow each of them to say "Right now, every clock's display matches mine." Expressed differently, even in special relativity, the rates of clocks do not follow a transitive law. Given three clocks A, B, C, if A says "B runs at $\frac{1}{2}$ of my rate," and B says "C runs at $\frac{1}{3}$ of my rate," then A *cannot* say "C runs at $\frac{1}{6}$ of my rate." Hence, no gearing exists that will make the three clocks agree that they all display the same time simultaneously.

UTC time is not actually displayed at any given moment by any one clock, but instead results from postprocessing data from many clocks around the world, and hence is not completely known at any moment "now." Might the same idea be applied to a set of clocks in the field, to analyze the data they collect after the fact, by correcting their time stamps using "location and speed stamps" that they logged with each of those time stamps? This could be done *if* one of the clocks were designated as a primary. (In the case of UTC, this primary is effectively the clock at spatial infinity.) But the last paragraph's closing comment about the impossibility of gearing applies here. The primary clock's time becomes a time coordinate for the secondary clocks. The time stamps of those secondaries will be (postprocessing) altered to agree with the primary clock, but the secondaries will not regard their time stamps as a true time, equal values of which denote simultaneity for each secondary. Events with identical postprocessed time stamps were only simultaneous for the primary. That might well be sufficient for the purposes of the network, but it might not; it depends on the purpose of the network. For everyday transactions that use UTC, femtosecond precision is not needed, and so the fact that UTC only really denotes simultaneity for the primary clock causes no problems. But if we are to synchronize clocks to the level of femtoseconds, the game changes completely, and we must be fully aware of the purpose of the time used in the network. Will we need to care if the individual clocks disagree about simultaneity, or not?

The examples above show that relativistic ideas of simultaneity are central to any practical application of timing at the femtosecond level. I think that many analyses of precise timing would benefit from an application of the nuances of time that have been described in this paper. When we deal at the level of femtoseconds and beyond, a proper understanding and application of relativistic ideas is essential.

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