Eternal adiabaticity in quantum evolution

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We iteratively apply a recently formulated adiabatic theorem for the strong-coupling limit in finite-dimensional closed and open quantum systems. This allows us to improve approximations to a perturbed dynamics, beyond the standard approximation based on quantum Zeno dynamics and adiabatic elimination. The effective generators describing the approximate evolutions are endowed with the same block structure as the unperturbed part of the generator, and exhibit adiabatic evolutions. This iterative adiabatic theorem reveals that adiabaticity holds *eternally*, that is, the system evolves within each eigenspace of the unperturbed part of the generator, with an error bounded by $O(1/\gamma)$ *uniformly in time*, where γ is the strength of the unperturbed part of the generator. We prove that the iterative adiabatic theorem reproduces Bloch's perturbation theory in the unitary case, and is therefore a full generalization to open systems. We furthermore prove the equivalence of the Schrieffer-Wolff and des Cloizeaux approaches in the unitary case and generalize both to arbitrary open systems, showing that they share the eternal adiabaticity, and providing explicit error bounds. Finally we discuss the physical structure of the effective adiabatic generators and show that ideal effective generators for open systems do not exist in general.

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I. INTRODUCTION

Modeling physical systems is important in physics and science. Identifying a good effective generator of a system is crucial in the analysis of the physical dynamics of the system. A separation of timescales is most often a key in doing that. It allows us to focus on a subset of relevant energy levels of the system. High-frequency components can be "adiabatically eliminated," and the evolution of the system is well described by an effective generator acting only on the relevant subspace.

Such effective modeling can be justified by an adiabatic theorem [1,2]. Consider first a closed quantum system with a dynamics dominated by a strong part of its Hamiltonian, and the leakage out of the eigenspaces of the strong Hamiltonian is suppressed due to the separation of timescales. This ensures that the evolution of the system is well approximated by the adiabatic evolution within the eigenspaces. In the limit of an infinitely strong separation of timescales, the leakage is completely suppressed and the system is perfectly confined within each eigenspace. It is known as a version of the quantum Zeno effect [3–6]. The adiabatic evolution within the eigenspaces (quantum Zeno dynamics [6,7]) is described by a Hamiltonian projected on the individual eigenspaces (Zeno Hamiltonian). If on the other hand the separation of timescales is strong but finite, the system can slowly transit between eigenspaces. An effective Hamiltonian including such processes can be systematically constructed via the technique known as adiabatic elimination [8,9], and refines the approximation by the Zeno Hamiltonian.

In practice, many quantum systems are noisy, and it is important to extend the theory to Lindbladian generators. It is difficult to give the vast literature on this area the deserved attention, and we only provide some exemplary references for such generalizations of the adiabatic theorem [10–15], of strong coupling limits [16–21], of quantum Zeno dynamics [22–25], and of adiabatic elimination [26–34].

All the above theories for effective generators are, however, usually valid for finite time ranges only. Known error bounds on adiabatic approximations, i.e., bounds on the distance between the true evolution and an adiabatic evolution within the eigenspaces, grow in time [2,21,35,36], and the adiabaticity of the evolution is not guaranteed by the standard adiabatic theorems in the long term. Accordingly we would need a stronger separation of timescales to realize the adiabatic evolution for a longer time.

In this paper we show that adiabaticity actually holds *eternally*. The system remains within each eigenspace of the strong part of its generator with an error remaining $O(1/\gamma)$ *for arbitrarily long times and arbitrary perturbations*, where γ characterizes the strength of the strong Hamiltonian relative to the perturbation. The reason why the standard adiabatic theorems appear to assure the adiabaticity only for finite times is because the adiabatic generators used in the adiabatic theorems to approximate the true evolutions, e.g., by Zeno Hamiltonians, are not fine enough. One can find an adiabatic generator that adapts better to the evolution of the system while provoking no leakage out of the eigenspaces. It well approximates the true evolution with an error bounded by $O(1/\gamma)$ *uniformly in time*.

Let us summarize the main results of the present work. We consider an evolution $e^{t(\gamma B+C)}$ of a finite-dimensional quantum system with a "strong" generator B and a "weak" generator C. These generators can be Hamiltonians or Lindbladians. In this work we focus on static systems with time-independent generators. In Ref. [21] we have developed an adiabatic theorem for the strong-coupling limit $\gamma \to +\infty$ for open systems. Here we intend to improve the adiabatic approximation by applying the adiabatic theorem iteratively (Sec. II). This leads us to a good choice of adiabatic generator $\gamma B + D$, with $D = D(\gamma)$ endowed with the same block structure as B, thus provoking no leakage out of the eigenspaces of B, and at the same time allowing us to bound the distance

$$e^{t(\gamma B+C)} - e^{t[\gamma B+D(\gamma)]} = O(1/\gamma) \tag{1.1}$$

uniformly in time (Sec. III).

An immediate consequence of (1.1) is that for large γ and for an arbitrary perturbation C the evolution of the system clings *forever* to each eigenspace of the strong generator B with an overall leakage $O(1/\gamma)$, namely,

$$\sup_{t\geqslant 0} \|(1-P_\ell)e^{t(\gamma B+C)}P_\ell\| = O(1/\gamma), \tag{1.2}$$

for all ℓ, where P_ℓ is the spectral projection onto the ℓth eigenspace of B. This follows from the block structure of D, which yields $(1-P_\ell)e^{t(\gamma B+D)}P_\ell = 0$.

The ℓth block D_ℓ of the adiabatic generator D in (1.1) acting on the ℓth eigenspace of the strong generator B is given by $D_\ell = P_\ell \Omega_\ell P_\ell$, where Ω_ℓ is a solution of the quadratic operator equation

$$\frac{1}{\gamma}S_\ell \Omega_\ell^2 - \left(1 + \frac{1}{\gamma}CS_\ell\right)\Omega_\ell + S_\ell\Omega_\ell N_\ell + CP_\ell = 0, \tag{1.3}$$

with $\Omega_\ell = \Omega_\ell P_\ell$, and N_ℓ is the spectral nilpotent of the ℓth eigenspace of B, while S_ℓ is the reduced resolvent of B at its ℓth eigenvalue [37] (their details are provided in the following section). This implies that $U_\ell = P_\ell - S_\ell\Omega_\ell/\gamma$ satisfies another quadratic equation

$$U_\ell - S_\ell U_\ell N_\ell + \frac{1}{\gamma}S_\ell(CU_\ell - U_\ell C U_\ell) - P_\ell = 0, \tag{1.4}$$

with $U_\ell P_\ell = U_\ell$ (Appendix C), and in the absence of the nilpotent N_ℓ in the unitary case this equation is nothing but the well-known Bloch equation [38,39]. The iterated adiabatic theorem thus reproduces Bloch's perturbation theory developed for closed systems [38–42], and it is here generalized to open systems. Although we also provide perturbative expansions (Sec. IV), our key focus is the adiabatic generator $D = \sum_\ell D_\ell$, whose components $D_\ell(\gamma)$ are a resummation of a full-order perturbative series. We show the nonperturbative solvability of the Bloch equation and the region where the relevant solution exists and is unique (Appendix D) using the Newton-Kantorovich theorem [43]. This allows us to explicitly bound the eternal adiabaticity (1.1) (Sec. VI and Appendix E).

Next we turn our attention to the structure of the effective generator. Behind eternal adiabaticity, we have similarity

$$\gamma B + C = U(\gamma B + D)U^{-1} \tag{1.5}$$

between the adiabatic generator $\gamma B + D$ and the original generator $\gamma B + C$, with $U = \sum_\ell U_\ell = 1 + O(1/\gamma)$ (see Sec. V). It is known, however, that even in the unitary case there is a lot of gauge freedom in the choice of good adiabatic generators. This fact encourages us to take an axiomatic approach to define an *ideal* effective adiabatic generator, as initiated for the unitary case in Ref. [44]:

(1) An effective adiabatic generator C_{eff} should be endowed with the same block structure as B, i.e., $[C_{\text{eff}}, P_\ell] = 0$, provoking no leakage out of the eigenspaces of B.

(2) The effective adiabatic generator $\gamma B + C_{\text{eff}}$ should be similar to the original generator $\gamma B + C$, sharing the same spectrum.

(3) The similarity transformation U should be small, i.e., close to the identity $U = 1 + O(1/\gamma)$.

(4) The effective adiabatic generator $\gamma B + C_{\text{eff}}$ should be physical, i.e., Hermiticity-preserving (HP), trace-preserving (TP), and conditionally completely positive (CP) (with a positive-semidefinite Kossakowski matrix) [45], generating a completely positive evolution [46,47].

While the first three axioms suffice to show eternal adiabaticity, the fourth is desirable to get a direct physical interpretation of the generator. It is known in the literature that, due to an asymmetry in the construction, the adiabatic generator D from Bloch's perturbation theory is not skew-Hermitian (or not anti-Hermitian) in general even in the unitary case with skew-Hermitian B and C [38–42,48,49]. In the unitary case, on the other hand, des Cloizeaux showed that one can turn the non-skew-Hermitian $\gamma B + D$ into a skew-Hermitian $\gamma B + K$ by an additional similarity transformation keeping the block structure [40,41]. This is an example of an ideal effective generator.

A skew-Hermitian effective generator on a particular eigenspace (without caring about the block structure of the other eigenspaces) can also be obtained from the original $\gamma B + C$ via the Schrieffer-Wolff transformation in the unitary case [48–50]. The connection between Schrieffer-Wolff's, adiabatic elimination, and des Cloizeaux's perturbative approaches has been noted before [51], and another higher-order adiabatic elimination based on a Lippmann-Schwinger-type equation was derived [52,53].

The generalization of Schrieffer-Wolff transformations to open systems was investigated in Ref. [54], where the author focused on the stationary subspace, i.e., the eigenspace of B belonging to the eigenvalue 0, and assumed that the generator B is diagonalizable, with no nilpotent. Physicality was analyzed up to the third order for some specific settings.

Here, based on our generalization of Bloch's equation, we provide a nonperturbative generalization of the Schrieffer-Wolff and des Cloizeaux approaches to the open-system case (Secs. VII and VIII). We construct a very natural and symmetric similarity transformation from the solutions of Bloch's equation which fulfills the first three axioms of an ideal effective generator and reduces to the des Cloizeaux approach in the unitary case. Our formalism can be applied to general generators, which are not necessarily diagonalizable and can admit nilpotents, and deals with all the eigenspaces, including the nonstationary ones, respecting the block structure. We prove that the adiabatic generators are both HP and TP for general open systems (Sec. IX).

After providing a general framework, we will look at a few examples in Sec. X: a dissipative Λ system, for which an analytical expression for the nonperturbative (full-order) adiabatic generator is available (Sec. X A), and a system admitting a nilpotent in the strong part B (Sec. X B). We find that our effective generator is not always completely positive (that is, the fourth axiom is not always fulfilled).

Could there be another approach (choice of gauge) which fulfills all axioms? Surprisingly we show that this is generally impossible by providing a counterexample (Sec. X C) in which axioms one and two imply breaking axiom four. If one wishes to require that an effective generator for an open system should have the complete physical structure (HP, TP, and CP), as a trade-off axioms one and/or two in the above list should be abandoned. There are attempts to develop a general perturbation theory along those lines [31–34].

We will conclude the paper in Sec. XI and provide some details in Appendices A–E.

Here we take the view that the eternal adiabaticity is the most striking feature, as it highlights a certain robustness of quantum evolutions against perturbations. This aspect is further elaborated in Ref. [55], where we explore connections to KAM stability.

II. ITERATED ADIABATIC THEOREM

We iteratively apply the adiabatic theorem developed in Ref. [21] to improve the adiabatic approximation. The goal is to find a good approximation of $e^{t(\gamma B+C)}$ by $e^{t(\gamma B+D)}$ with an operator D endowed with the same block structure as B, causing no leakage from each eigenspace of B. We will show that there exists such a generator D that ensures that the error of $e^{t(\gamma B+D)}$ to $e^{t(\gamma B+C)}$ remains $O(1/\gamma)$ for arbitrarily long times t. Essentially, one can think of this approach as a type of perturbation theory within the exponential function.

Although we ultimately have physical operators (Hamiltonians and Lindbladians) in mind, most of the results of this paper are valid for *arbitrary* square matrices B and C, *without requiring any structural assumptions* on them.

Let

$$B=\sum_\ell (b_\ell P_\ell+N_\ell) \tag{2.1}$$

be the *canonical form* or the *spectral representation* of B (recall the *Jordan normal form*) [37]. Here $\{b_\ell\}$ is the *spectrum* of B, which is the set of distinct eigenvalues of B (labeled such that $b_k\neq b_\ell$ for $k\neq\ell$), $\{P_\ell\}$ are the corresponding eigenprojections, called the *spectral projections* of B, satisfying

$$P_kP_\ell=\delta_{k\ell}P_k,\qquad \sum_\ell P_\ell=1, \tag{2.2}$$

for all k and ℓ, and $\{N_\ell\}$ are the corresponding *nilpotents* of B, satisfying

$$P_kN_\ell=N_\ell P_k=\delta_{k\ell}N_k,\qquad N_\ell^{n_\ell}=0, \tag{2.3}$$

for all k and ℓ, and for some integers $1\leqslant n_\ell\leqslant \operatorname{rank}P_\ell$. Notice that the spectral projections, which determine the partition of the space through the *resolution of identity* (2.2), are not Hermitian in general, $P_\ell\neq P_\ell^\dagger$. We set

$$B_\ell=BP_\ell=b_\ell P_\ell+N_\ell. \tag{2.4}$$

First, we focus on a particular eigenspace of B belonging to eigenvalue b_ℓ, and find a suitable D_ℓ that describes the adiabatic evolution of the system in the eigenspace for large γ. The following iteration works for any choice of D_ℓ satisfying

$$D_\ell=P_\ell D_\ell P_\ell, \tag{2.5}$$

and hence having the same block structure as B. However, later we will find out that there are particularly good choices of D_ℓ.

We wish to estimate the difference between $e^{t(\gamma B+C)}P_\ell$ and $e^{t(\gamma B+D_\ell)}P_\ell$. It can be estimated by writing it as an integral:

$$\begin{aligned}&(e^{t(\gamma B+C)}-e^{t(\gamma B+D_\ell)})P_\ell\\&\quad=-\int_0^t ds\,\frac{\partial}{\partial s}(e^{(t-s)(\gamma B+C)}e^{s(\gamma B+D_\ell)})P_\ell\\&\quad=\int_0^t ds\,e^{(t-s)(\gamma B+C)}(C-D_\ell)P_\ell e^{s(\gamma B+D_\ell)}.\end{aligned} \tag{2.6}$$

The key quantity from Ref. [21] is the reduced resolvent S_ℓ, defined by

$$S_\ell=\sum_{k\neq\ell}(b_k-b_\ell+N_k)^{-1}P_k \tag{2.7}$$

(see Refs. [2,37] for the unitary case). Notice that the inverse $(b_k-b_\ell+N_k)^{-1}$ always exists, because $b_k\neq b_\ell$ for $k\neq\ell$ in the spectral decomposition (2.1). Notice also that in the nonunitary case we need to include the nilpotents N_k in the definition of the reduced resolvent S_ℓ, while they are absent in the unitary case. The reduced resolvent S_ℓ satisfies

$$P_\ell S_\ell=S_\ell P_\ell=0, \tag{2.8}$$

$$(B-b_\ell)S_\ell=S_\ell(B-b_\ell)=1-P_\ell. \tag{2.9}$$

In addition, the key formula for the adiabatic theorem is given by

$$\begin{aligned}&\int_0^t ds\,e^{(t-s)(\gamma B+C)}AP_\ell e^{s(\gamma B+D_\ell)}\\&\quad=\int_0^t ds\,e^{(t-s)(\gamma B+C)}P_\ell AP_\ell e^{s(\gamma B+D_\ell)}\\&\qquad+\frac{1}{\gamma}e^{t(\gamma B+C)}S_\ell AP_\ell-\frac{1}{\gamma}S_\ell AP_\ell e^{t(\gamma B+D_\ell)}\\&\qquad-\frac{1}{\gamma}\int_0^t ds\,e^{(t-s)(\gamma B+C)}\mathcal{K}_\ell(A)P_\ell e^{s(\gamma B+D_\ell)},\end{aligned} \tag{2.10}$$

where

$$\mathcal{K}_\ell(A)=CS_\ell A-S_\ell AD_\ell-\gamma S_\ell AN_\ell, \tag{2.11}$$

for an arbitrary operator A. See Appendix A for the derivation of this key formula. Then the difference (2.6) can be immediately estimated by applying the key formula (2.10) for $A=C-D_\ell\equiv A_\ell^{(0)}$. In particular, if D_ℓ is chosen to be $D_\ell=P_\ell CP_\ell$, then $P_\ell A_\ell^{(0)}P_\ell=P_\ell(C-D_\ell)P_\ell=0$ and the first integral on the right-hand side of (2.10) identically vanishes. Moreover, if there is no nilpotent $N_\ell=0$ in the relevant eigenspace, then $\mathcal{K}_\ell$ is independent of γ, and we get

$$\begin{aligned}(e^{t(\gamma B+C)} - e^{t(\gamma B+P_\ell C P_\ell)})P_\ell \\ = \frac{1}{\gamma}(e^{t(\gamma B+C)}S_\ell C P_\ell - S_\ell C P_\ell e^{t(\gamma B+P_\ell C P_\ell)}) \\ - \frac{1}{\gamma}\int_0^t ds\, e^{(t-s)(\gamma B+C)}[C, S_\ell C P_\ell]P_\ell e^{s(\gamma B+P_\ell C P_\ell)}.\end{aligned} \tag{2.12}$$

This provides an adiabatic theorem [21]: when B is Lindbladian or Hamiltonian, so that the semigroup it generates is uniformly bounded in time, then the evolution is confined within the eigenspace specified by its spectral projection P_ℓ, with an error $O(1/\gamma)$ for any finite t. The adiabatic evolution within the eigenspace is described by the generator $D_\ell = P_\ell C P_\ell$. However, the error would accumulate by the last integral as time t goes on, and the above adiabatic theorem (2.12) does not ensure the adiabaticity of the evolution for long times of $O(\gamma)$. See, e.g., Fig. 2 in Sec. X.

Still, with a careful choice of the generator D_ℓ, one can ensure the adiabaticity to hold *eternally*, for arbitrarily long times. We are going to show this by iteratively refining the generator D_ℓ, and so pushing the validity of the adiabatic approximation to times of higher and higher order of γ.

To improve the approximation, we iteratively apply the key formula (2.10), to the last integral on its right-hand side. After n iterations we get

$$\begin{aligned}(e^{t(\gamma B+C)} - e^{t(\gamma B+D_\ell)})P_\ell \\ = \int_0^t ds\, e^{(t-s)(\gamma B+C)}\left(\sum_{j=0}^{n}\frac{(-1)^j}{\gamma^j}P_\ell A_\ell^{(j)}P_\ell\right)e^{s(\gamma B+D_\ell)} \\ + \frac{1}{\gamma}e^{t(\gamma B+C)}\left(\sum_{j=0}^{n-1}\frac{(-1)^j}{\gamma^j}S_\ell A_\ell^{(j)}P_\ell\right) \\ - \frac{1}{\gamma}\left(\sum_{j=0}^{n-1}\frac{(-1)^j}{\gamma^j}S_\ell A_\ell^{(j)}P_\ell\right)e^{t(\gamma B+D_\ell)} \\ + \frac{(-1)^n}{\gamma^n}\int_0^t ds\, e^{(t-s)(\gamma B+C)}A_\ell^{(n)}P_\ell e^{s(\gamma B+D_\ell)},\end{aligned} \tag{2.13}$$

where

$$A_\ell^{(0)} = C - D_\ell, \quad A_\ell^{(n)} = \mathcal{K}_\ell(A_\ell^{(n-1)}) = \mathcal{K}_\ell^n(A_\ell^{(0)}). \tag{2.14}$$

As proved in Appendix B, if

$$\gamma > \max\{1, [\|S_\ell\|(\|C\| + \|D_\ell\| + \|N_\ell\|)]^{n_\ell}\}, \tag{2.15}$$

then the last contribution in (2.13) decays out exponentially as $n \to +\infty$ and the series

$$G_\ell = \sum_{j=0}^{\infty}\frac{(-1)^j}{\gamma^j}A_\ell^{(j)} = \sum_{j=0}^{\infty}\frac{(-1)^j}{\gamma^j}\mathcal{K}_\ell^j(C - D_\ell) \tag{2.16}$$

converges. Here and in the following, we will consider only unitary invariant norms. Thus, in the limit $n \to +\infty$ one gets

$$\begin{aligned}(e^{t(\gamma B+C)} - e^{t(\gamma B+D_\ell)})P_\ell \\ = \int_0^t ds\, e^{(t-s)(\gamma B+C)}P_\ell G_\ell P_\ell e^{s(\gamma B+D_\ell)} \\ + \frac{1}{\gamma}(e^{t(\gamma B+C)}S_\ell G_\ell P_\ell - S_\ell G_\ell P_\ell e^{t(\gamma B+D_\ell)}).\end{aligned} \tag{2.17}$$

This equation holds for any choice of D_ℓ with the same block structure as B as in (2.5), and for any sufficiently large γ. We now seek a D_ℓ such that

$$P_\ell G_\ell P_\ell = 0, \tag{2.18}$$

so that the integral in (2.17), which would grow in time and make the error bound larger and larger, vanishes, giving

$$\begin{aligned}(e^{t(\gamma B+C)} - e^{t(\gamma B+D_\ell)})P_\ell \\ = \frac{1}{\gamma}(e^{t(\gamma B+C)}S_\ell G_\ell P_\ell - S_\ell G_\ell P_\ell e^{t(\gamma B+D_\ell)}).\end{aligned} \tag{2.19}$$

Such a D_ℓ actually exists, as proved in the next section.

III. ADIABATIC BLOCH EQUATION

The adiabatic generator D_ℓ fulfilling the condition (2.18) and thus giving (2.19) is given by

$$D_\ell = P_\ell \Omega_\ell = P_\ell \Omega_\ell P_\ell, \tag{3.1}$$

where Ω_ℓ is a solution of the quadratic equation

$$\frac{1}{\gamma}S_\ell\Omega_\ell^2 - \left(1 + \frac{1}{\gamma}CS_\ell\right)\Omega_\ell + S_\ell\Omega_\ell N_\ell + CP_\ell = 0, \tag{3.2}$$

with

$$\Omega_\ell(1 - P_\ell) = 0. \tag{3.3}$$

Because this equation is derived from the iterated adiabatic theorem, and because it generalizes the well-known Bloch wave operator equation [38,39] as shown in Appendix C, we call the quadratic equation (3.2) with (3.3) for Ω_ℓ the *adiabatic Bloch equation*.

With such a particular choice of D_ℓ, we have that $S_\ell G_\ell P_\ell = S_\ell \Omega_\ell = S_\ell \Omega_\ell P_\ell$, and Eq. (2.19) reduces to

$$\begin{aligned}(e^{t(\gamma B+C)} - e^{t(\gamma B+D_\ell)})P_\ell \\ = \frac{1}{\gamma}(e^{t(\gamma B+C)}S_\ell\Omega_\ell P_\ell - S_\ell\Omega_\ell P_\ell e^{t(\gamma B+D_\ell)}).\end{aligned} \tag{3.4}$$

This is valid for arbitrary operators B and C, not necessarily Hamiltonians or Lindbladians.

A. Derivation of the adiabatic Bloch equation

Let us start by looking at the condition (2.18). For large enough γ, the series (2.16) converges, the inverse $(1 + \gamma^{-1}\mathcal{K}_\ell)^{-1}$ exists, and we get

$$G_\ell = (1 + \gamma^{-1}\mathcal{K}_\ell)^{-1}(C - D_\ell). \tag{3.5}$$

By the block structure of D_ℓ in (2.5) and by using $S_\ell P_\ell = 0$, one gets $\mathcal{K}_\ell(D_\ell) = 0$, where

$$G_\ell = (1 + \gamma^{-1}\mathcal{K}_\ell)^{-1}(C) - D_\ell. \tag{3.6}$$

Since $\mathcal{K}_\ell(A)P_\ell = \mathcal{K}_\ell(AP_\ell)$ and $D_\ell = P_\ell D_\ell P_\ell$, the condition $P_\ell G_\ell P_\ell = 0$ is equivalent to

$$P_\ell G_\ell P_\ell = P_\ell(1 + \gamma^{-1}\mathcal{K}_\ell)^{-1}(CP_\ell) - D_\ell = 0, \tag{3.7}$$

which in turn implies

$$(1 + \gamma^{-1}\mathcal{K}_\ell)^{-1}(CP_\ell) - D_\ell = R_\ell, \tag{3.8}$$

with $R_\ell = (1 - P_\ell)R_\ell P_\ell$. Then, by setting $\Omega_\ell = D_\ell + R_\ell = \Omega_\ell P_\ell$, it reads

$$(1 + \gamma^{-1}\mathcal{K}_\ell)^{-1}(CP_\ell) = \Omega_\ell. \tag{3.9}$$

By inverting,

$$CP_\ell = \Omega_\ell + \frac{1}{\gamma}\mathcal{K}_\ell(\Omega_\ell), \tag{3.10}$$

that is, by the definition (2.11) of $\mathcal{K}_\ell$,

$$CP_\ell = \Omega_\ell + \frac{1}{\gamma}CS_\ell\Omega_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell D_\ell - S_\ell\Omega_\ell N_\ell. \tag{3.11}$$

Since $\Omega_\ell R_\ell = 0$, we can write $\Omega_\ell D_\ell = \Omega_\ell^2$. Therefore, we get the quadratic equation (3.2) for Ω_ℓ with (3.3).

It follows from the Newton-Kantorovich theorem [43] that for large enough γ the adiabatic Bloch equation (3.2) with (3.3) has a unique solution within a certain range. See Appendix D. From such a solution Ω_ℓ, we obtain the wanted D_ℓ by (3.1).

B. Simplifying G_ℓ

The solution of the adiabatic Bloch equation (3.2) with (3.3) allows us to simplify the expression for G_ℓ. To this end, let us look at the components of G_ℓ other than $P_\ell G_\ell P_\ell$, which vanishes by (2.18). From (3.6) and (3.9), we get

$$\begin{aligned}(1 - P_\ell)G_\ell P_\ell &= (1 - P_\ell)(1 + \gamma^{-1}\mathcal{K}_\ell)^{-1}(CP_\ell)\\ &= (1 - P_\ell)\Omega_\ell P_\ell,\end{aligned} \tag{3.12}$$

where we have used $(1 - P_\ell)D_\ell = 0$ and $\mathcal{K}_\ell(A)P_\ell = \mathcal{K}_\ell(AP_\ell)$. Therefore, we get $S_\ell G_\ell P_\ell = S_\ell\Omega_\ell P_\ell$ and Eq. (2.19) reduces to (3.4).

In summary, our key equation is the adiabatic Bloch equation (3.2) with (3.3). It admits a unique solution Ω_ℓ within a certain range for large enough γ (Appendix D). A good choice of D_ℓ describing the adiabatic evolution within the relevant eigenspace is given by (3.1), with which the difference between the adiabatic evolution and the true evolution is estimated as (3.4).

IV. PERTURBATIVE SOLUTION OF THE ADIABATIC BLOCH EQUATION

Let us look for a perturbative solution of the adiabatic Bloch equation (3.2) with (3.3) in the form

$$\Omega_\ell = \Omega_\ell^{(0)} + \frac{1}{\gamma}\Omega_\ell^{(1)} + \frac{1}{\gamma^2}\Omega_\ell^{(2)} + \cdots = \sum_{j=0}^{\infty}\frac{1}{\gamma^j}\Omega_\ell^{(j)}. \tag{4.1}$$

Substituting it into the adiabatic Bloch equation (3.2) and comparing order by order, we obtain

$$\Omega_\ell^{(0)} - S_\ell\Omega_\ell^{(0)}N_\ell = CP_\ell, \tag{4.2}$$

$$\Omega_\ell^{(j)} - S_\ell\Omega_\ell^{(j)}N_\ell = -CS_\ell\Omega_\ell^{(j-1)} + S_\ell\sum_{i=0}^{j-1}\Omega_\ell^{(j-i-1)}\Omega_\ell^{(i)}. \tag{4.3}$$

By solving this iterative equation, we get that $\Omega_\ell^{(j)} = \Omega_\ell^{(j)}P_\ell$ and the perturbative expressions for $D_\ell^{(j)} = P_\ell\Omega_\ell^{(j)}$ read

$$D_\ell^{(0)} = P_\ell CP_\ell, \tag{4.4}$$

$$D_\ell^{(1)} = -P_\ell CS_\ell\langle C\rangle P_\ell, \tag{4.5}$$

$$D_\ell^{(2)} = P_\ell CS_\ell\langle CS_\ell\langle C\rangle\rangle P_\ell - P_\ell CS_\ell^2\langle\langle C\rangle P_\ell C\rangle P_\ell, \tag{4.6}$$

$$\begin{aligned}D_\ell^{(3)} = &-P_\ell CS_\ell\langle CS_\ell\langle CS_\ell\langle C\rangle\rangle\rangle P_\ell + P_\ell CS_\ell\langle CS_\ell^2\langle\langle C\rangle P_\ell C\rangle\rangle P_\ell\\ &+ P_\ell CS_\ell^2\langle\langle C\rangle P_\ell CS_\ell\langle C\rangle\rangle P_\ell + P_\ell CS_\ell^2\langle\langle CS_\ell\langle C\rangle\rangle P_\ell C\rangle P_\ell\\ &- P_\ell CS_\ell^3\langle\langle\langle C\rangle P_\ell C\rangle P_\ell C\rangle P_\ell,\end{aligned} \tag{4.7}$$

where we set

$$\langle A\rangle = \sum_{n=0}^{n_\ell-1} S_\ell^n A N_\ell^n, \tag{4.8}$$

for an arbitrary operator A. If there is no nilpotent N_ℓ (i.e., $n_\ell = 1$) in the relevant eigenspace, we simply have $\langle A\rangle = A$, and these expressions reproduce the perturbative series obtained in Refs. [38,40], but are here generalized to nonunitary evolution.

Notice that the zeroth-order term $D_\ell^{(0)}$ in (4.4) is nothing but the "Zeno generator" [4,6,7,21], while the first-order term $D_\ell^{(1)}$ yields the "adiabatic elimination" [8,9,26,30]. The higher-order terms refine the approximation beyond the adiabatic elimination.

V. SIMILARITY OF THE GENERATORS

Let us gather the adiabatic generators $D_\ell = P_\ell\Omega_\ell P_\ell$ and define

$$D = \sum_\ell D_\ell. \tag{5.1}$$

The total generator $\gamma B + D$ describing the adiabatic evolution of the system within the eigenspaces is similar to the original generator $\gamma B + C$. That is, the intertwining relations

$$(\gamma B + C)U_\ell = U_\ell(\gamma B + D_\ell) \tag{5.2}$$

hold for all the operators

$$U_\ell = P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell P_\ell, \tag{5.3}$$

and this implies the similarity relation

$$\gamma B + D = U^{-1}(\gamma B + C)U, \tag{5.4}$$

for sufficiently large γ, where

$$U=\sum_\ell U_\ell=1-\frac{1}{\gamma}\sum_\ell S_\ell\Omega_\ell P_\ell. \tag{5.5}$$

Let us prove these facts in this section. We will use the properties

$$U_\ell P_\ell=U_\ell,\quad P_\ell U_\ell=P_\ell. \tag{5.6}$$

A. Intertwining relations

By using the definition of U_ℓ in (5.3), we have

$$\begin{aligned}(\gamma B+C-\gamma b_\ell)U_\ell\\ &=\gamma N_\ell+CP_\ell-\frac{1}{\gamma}(\gamma B+C-\gamma b_\ell)S_\ell\Omega_\ell.\end{aligned} \tag{5.7}$$

Recalling that $(B-b_\ell)S_\ell=1-P_\ell$ in (2.9),

$$=\gamma N_\ell+CP_\ell-(1-P_\ell)\Omega_\ell-\frac{1}{\gamma}CS_\ell\Omega_\ell. \tag{5.8}$$

Using the adiabatic Bloch equation (3.2),

$$\begin{aligned}&=\gamma N_\ell+P_\ell\Omega_\ell-\frac{1}{\gamma}S_\ell\Omega_\ell^2-S_\ell\Omega_\ell N_\ell\\ &=\left(P_\ell-\frac{1}{\gamma}S_\ell\Omega_\ell\right)(P_\ell\Omega_\ell+\gamma N_\ell)\\ &=U_\ell(D_\ell+\gamma N_\ell).\end{aligned} \tag{5.9}$$

Finally, since $U_\ell=U_\ell P_\ell$ and $P_\ell B=P_\ell(b_\ell+N_\ell)$, this gives (5.2).

B. Similarity of the generators

Summing the intertwining relations in (5.2) over ℓ and noting $U_\ell=U_\ell P_\ell$,

$$\begin{aligned}(\gamma B+C)U&=\sum_\ell(\gamma B+C)U_\ell\\ &=\sum_\ell U_\ell(\gamma B+D_\ell)\\ &=\sum_\ell U_\ell(\gamma B+D)\\ &=U(\gamma B+D).\end{aligned} \tag{5.10}$$

This proves the similarity relation (5.4).

The operator U_ℓ reduces to Bloch's wave operator [38,39] in the unitary case, as shown in Appendix C. Here it is generalized to open systems, where B can have nilpotents. One can prove that U_ℓ is a solution of the equation

$$U_\ell-S_\ell U_\ell N_\ell+\frac{1}{\gamma}S_\ell(CU_\ell-U_\ell CU_\ell)-P_\ell=0, \tag{5.11}$$

with

$$U_\ell(1-P_\ell)=0. \tag{5.12}$$

See Appendix C for the derivation. Compared with the original Bloch equation [38], the equation (5.11) contains an additional term that takes care of the nilpotent N_ℓ.

We are mainly interested in the evolutions of physical systems, but the similarity and the generalized Bloch equation discussed here are valid for arbitrary operators B and C, not necessarily Hamiltonians or Lindbladians.

VI. ETERNAL ADIABATICITY

The similarity (5.4) proved in the previous section allows us to reproduce the relation (3.4) immediately. Indeed, the similarity (5.4) of the generators implies the similarity of the evolutions,

$$e^{t(\gamma B+C)}U=Ue^{t(\gamma B+D)}. \tag{6.1}$$

By inserting the definition of U in (5.5), we get

$$\begin{aligned}&e^{t(\gamma B+C)}-e^{t(\gamma B+D)}\\ &\quad=\frac{1}{\gamma}\sum_\ell(e^{t(\gamma B+C)}S_\ell\Omega_\ell P_\ell-S_\ell\Omega_\ell P_\ell e^{t(\gamma B+D)}).\end{aligned} \tag{6.2}$$

This is equivalent to (3.4).

Now, if B and C are physical generators, the spectrum of $\gamma B+C$ is confined in the left half-plane (the real parts of the eigenvalues are nonpositive), and purely imaginary eigenvalues are semisimple (the corresponding eigenspaces are diagonalizable and have no nilpotents). Due to the similarity (5.4), the adiabatic generator $\gamma B+D$ has the same spectrum as $\gamma B+C$. Therefore, $e^{t(\gamma B+D)}$, as well as $e^{t(\gamma B+C)}$, are bounded semigroups, i.e.,

$$\|e^{t(\gamma B+C)}\|\leqslant M,\quad \|e^{t(\gamma B+D)}\|\leqslant M, \tag{6.3}$$

for some $M\geqslant 1$ for all $t\geqslant 0$ and $\gamma\geqslant 0$. This ensures that the distance between the true evolution $e^{t(\gamma B+C)}$ and the adiabatic approximation $e^{t(\gamma B+D)}$, namely the norm of (6.2), is bounded by

$$\|e^{t(\gamma B+C)}-e^{t(\gamma B+D)}\|\leqslant\frac{2M}{\gamma}\sum_\ell\|S_\ell\Omega_\ell P_\ell\|, \tag{6.4}$$

for all $t\geqslant 0$. This means that the adiabatic evolution $e^{t(\gamma B+D)}$ is a good approximation to the true evolution $e^{t(\gamma B+C)}$ with the error remaining $O(1/\gamma)$ *for all times* $t\geqslant 0$. This proves the *eternal adiabaticity* of the evolution, and this is the central result of this paper.

In the operator norm [56]

$$\|A\|=\sup_{\|\sigma\|_1=1}\|A(\sigma)\|_1, \tag{6.5}$$

we have $\|e^{t(\gamma B+C)}\|=1$ for a physical evolution [57], and the distance (6.4) can be explicitly bounded by

$$\|e^{t(\gamma B+C)}-e^{t(\gamma B+D)}\|<\frac{1}{\gamma}\sum_\ell\gamma_\ell\|P_\ell\|, \tag{6.6}$$

for $\gamma\geqslant 2\max_\ell\gamma_\ell$, where

$$\gamma_\ell=4\|S_\ell\|\|C\|\|P_\ell\|\frac{1-(\|S_\ell\|\|N_\ell\|)^{n_\ell}}{1-\|S_\ell\|\|N_\ell\|}. \tag{6.7}$$

See Appendix E for its derivation and its tighter bound valid also for other norms. In the unitary case $\|P_\ell\|=1$, $\|N_\ell\|=0$ and hence $\gamma_\ell=4\|S_\ell\|\|C\|\leqslant 4\|C\|/\eta$, where η is the spectral gap of B.

In this way, the eternal bound (6.6) involves $\|S_\ell\|$ and $\|N_\ell\|$, i.e., the spectral gap and the "nondiagonalizability" of

B. Note also that the bound does not necessarily grow with the dimension of the system, but it is rather determined by the number of distinct eigenvalues of B, that is the number of terms in the sum in (6.6). Recall the spectral decomposition of B in (2.1).

VII. CONJUGATE ADIABATIC BLOCH EQUATION

One might have noticed the asymmetry in the perturbative expressions (4.6) and (4.7) for the second- and higher-order terms. This asymmetry stems from the asymmetry in the derivation of the adiabatic theorem. We can think of an alternative way of estimating the difference between an adiabatic evolution and the true evolution. Instead of (2.6), we can proceed as

$$\begin{aligned} &P_\ell(e^{t(\gamma B+C)} - e^{t(\gamma B+D_\ell)}) \\ &\quad = -P_\ell \int_0^t ds\, \frac{\partial}{\partial s}(e^{s(\gamma B+D_\ell)}e^{(t-s)(\gamma B+C)}) \\ &\quad = \int_0^t ds\, e^{s(\gamma B+D_\ell)}P_\ell(C-D_\ell)e^{(t-s)(\gamma B+C)}. \end{aligned} \tag{7.1}$$

Notice the difference in the order of the operators compared to (2.6). The components are the same but they are ordered in the opposite order. We can repeat the same steps followed above, starting from this reverted expression (7.1). We can derive an adiabatic theorem, we can iteratively apply the adiabatic theorem to improve the adiabatic approximation, and we can prove the eternal adiabaticity. All the formulas originating from (7.1) are similar to those obtained above, but the orders of operators are exactly reverted.

Let us collect the main formulas. We get a new set of adiabatic Bloch equations

$$\frac{1}{\gamma}\tilde{\Omega}_\ell^2 S_\ell - \tilde{\Omega}_\ell\left(1+\frac{1}{\gamma}S_\ell C\right) + N_\ell\tilde{\Omega}_\ell S_\ell + P_\ell C = 0, \tag{7.2}$$

with

$$(1-P_\ell)\tilde{\Omega}_\ell = 0, \tag{7.3}$$

from the iterated adiabatic theorem based on the reversed equation (7.1). Compare them with (3.2) and (3.3). Now, by choosing as eternal adiabatic generator

$$\tilde{D}_\ell = \tilde{\Omega}_\ell P_\ell = P_\ell\tilde{\Omega}_\ell P_\ell, \tag{7.4}$$

we get

$$\begin{aligned} &e^{t(\gamma B+C)} - e^{t(\gamma B+\tilde{D})} \\ &\quad = \frac{1}{\gamma}\sum_\ell (P_\ell\tilde{\Omega}_\ell S_\ell e^{t(\gamma B+C)} - e^{t(\gamma B+\tilde{D})}P_\ell\tilde{\Omega}_\ell S_\ell), \end{aligned} \tag{7.5}$$

where

$$\tilde{D} = \sum_\ell \tilde{D}_\ell. \tag{7.6}$$

This is the counterpart of (6.2). The similarity between $\gamma B + \tilde{D}$ and $\gamma B + C$ also holds. We have the intertwining relations

$$\tilde{U}_\ell(\gamma B + C) = (\gamma B + \tilde{D}_\ell)\tilde{U}_\ell, \tag{7.7}$$

for

$$\tilde{U}_\ell = P_\ell - \frac{1}{\gamma}\tilde{\Omega}_\ell S_\ell, \tag{7.8}$$

and the similarity relation

$$\gamma B + \tilde{D} = \tilde{U}(\gamma B + C)\tilde{U}^{-1}, \tag{7.9}$$

with

$$\tilde{U} = \sum_\ell \tilde{U}_\ell = 1 - \frac{1}{\gamma}\sum_\ell \tilde{\Omega}_\ell S_\ell. \tag{7.10}$$

These correspond to (5.2) and (5.4), respectively. Note that $\tilde{U}_\ell$ satisfies

$$P_\ell\tilde{U}_\ell = \tilde{U}_\ell, \quad \tilde{U}_\ell P_\ell = P_\ell, \tag{7.11}$$

similarly to (5.6). The equation for $\tilde{U}_\ell$ is given by

$$\tilde{U}_\ell - N_\ell\tilde{U}_\ell S_\ell + \frac{1}{\gamma}(\tilde{U}_\ell C - \tilde{U}_\ell C\tilde{U}_\ell)S_\ell - P_\ell = 0. \tag{7.12}$$

Compare it with (5.11).

In the unitary case, C and S_ℓ are skew-Hermitian, P_ℓ is Hermitian, and there is no nilpotent N_ℓ. Comparing the Bloch equation for Ω_ℓ in (3.2) and the one for $\tilde{\Omega}_\ell$ in (7.2), one realizes that $\tilde{\Omega}_\ell = -\Omega_\ell^\dagger$, and hence, $\tilde{U}_\ell = U_\ell^\dagger$. This alternative approach is therefore a conjugate version of the original approach in the unitary case.

VIII. GENERALIZED SCHRIEFFER-WOLFF TRANSFORMATION FOR OPEN SYSTEMS

In the unitary case, where B and C are both skew-Hermitian with no nilpotent in B, the asymmetry in the perturbative expressions (4.6) and (4.7) leads to a non-skew-Hermitian D, in spite of the skew-Hermiticity of B and C. This fact is known in the literature [38–41,49,51]. This does not spoil the validity of the approximation and the eternal adiabaticity, but it would be nicer if we could have an effective generator that has the correct structure as a physical generator (i.e., skew-Hermitian in the unitary case) and works equally well as D as an approximation.

In the unitary case, it is known that the perturbative series (4.4)–(4.7) can be made symmetric and the skew-Hermiticity of the adiabatic generator D can be amended via an additional similarity transformation [38,40]. We can generalize it for open systems. It provides us with a generalization of the Schrieffer-Wolff transformation [48–50] for open systems [54].

Let us first show that

$$\tilde{P}_\ell = U_\ell(\tilde{U}_\ell U_\ell)^{-1}\tilde{U}_\ell \tag{8.1}$$

is the projection onto the direct sum of the eigenspaces of $\gamma B + C$ belonging to the eigenvalues perturbed from the unperturbed eigenvalue γb_ℓ of γB. Here $(\tilde{U}_\ell U_\ell)^{-1}$ is the inverse of $\tilde{U}_\ell U_\ell$ on P_ℓ, defined by

$$(\tilde{U}_\ell U_\ell)^{-1} = \left(1+\frac{1}{\gamma^2}\tilde{\Omega}_\ell S_\ell^2\Omega_\ell\right)^{-1}P_\ell. \tag{8.2}$$

Note the properties $\Omega_\ell = \Omega_\ell P_\ell$ in (3.3), $\tilde{\Omega}_\ell = P_\ell\tilde{\Omega}_\ell$ in (7.3), $U_\ell P_\ell = U_\ell$, $P_\ell U_\ell = P_\ell$ in (5.6), and $P_\ell\tilde{U}_\ell = \tilde{U}_\ell$, $\tilde{U}_\ell P_\ell = P_\ell$ in (7.11). Thus

$$\tilde{U}_\ell U_\ell = P_\ell\tilde{U}_\ell U_\ell P_\ell, \quad (\tilde{U}_\ell U_\ell)^{-1} = P_\ell(\tilde{U}_\ell U_\ell)^{-1}P_\ell \tag{8.3}$$

reside in the subspace P_ℓ. Now $\tilde{P}_\ell$ is clearly a projection, satisfying $\tilde{P}_\ell^2 = \tilde{P}_\ell$. In addition, $\tilde{P}_\ell$ commutes with $\gamma B + C$. Indeed,

$$\begin{aligned}(\gamma B + C)\tilde{P}_\ell &= (\gamma B + C)U_\ell(\tilde{U}_\ell U_\ell)^{-1}\tilde{U}_\ell \\ &= U_\ell(\gamma B + D_\ell)(\tilde{U}_\ell U_\ell)^{-1}\tilde{U}_\ell \\ &= U_\ell(\tilde{U}_\ell U_\ell)^{-1}(\gamma B + \tilde{D}_\ell)\tilde{U}_\ell \\ &= U_\ell(\tilde{U}_\ell U_\ell)^{-1}\tilde{U}_\ell(\gamma B + C) \\ &= \tilde{P}_\ell(\gamma B + C),\end{aligned} \tag{8.4}$$

where we have used the intertwining relations (5.2) and (7.7) for the second and fourth equalities, respectively, and for the third equality we have used

$$(\gamma B + D_\ell)(\tilde{U}_\ell U_\ell)^{-1} = (\tilde{U}_\ell U_\ell)^{-1}(\gamma B + \tilde{D}_\ell), \tag{8.5}$$

which follows from

$$\tilde{U}_\ell U_\ell(\gamma B + D_\ell) = \tilde{U}_\ell(\gamma B + C)U_\ell = (\gamma B + \tilde{D}_\ell)\tilde{U}_\ell U_\ell. \tag{8.6}$$

Observe also that $\tilde{P}_\ell \to P_\ell$ as $\gamma \to +\infty$, and the eigenvalues of $\tilde{P}_\ell(\gamma B + C)\tilde{P}_\ell$ are close to γb_ℓ for large γ. These facts imply that $\tilde{P}_\ell$ is the projection onto the direct sum of the eigenspaces of $\gamma B + C$ corresponding to the eigenprojection P_ℓ of B.

In Ref. [49] it is pointed out that the Schrieffer-Wolff transformation for the unitary case is nothing but the "direct rotation" $(\tilde{P}_\ell P_\ell)^{1/2}$ connecting P_ℓ and $\tilde{P}_\ell$ [49, Definition 2.2]. A natural generalization of the Schrieffer-Wolff transformation for open systems, namely, a natural generalization of the direct rotation, is thus provided by

$$W_\ell = (\tilde{P}_\ell P_\ell)^{1/2} = U_\ell(\tilde{U}_\ell U_\ell)^{-1/2}, \tag{8.7}$$

where $(\tilde{U}_\ell U_\ell)^{-1/2}$ is the square root of $(\tilde{U}_\ell U_\ell)^{-1}$ defined in (8.2). We use the primary square root such that $(\tilde{P}_\ell P_\ell)^{1/2} \to P_\ell$ and $(\tilde{U}_\ell U_\ell)^{-1/2} \to P_\ell$ in the limit $\gamma \to +\infty$ (see, e.g., Refs. [58, Chap. 1] and [59, Sec. 6.4] for primary matrix function). The equivalence of the last two expressions in (8.7) can be verified by looking at their squares, $U_\ell(\tilde{U}_\ell U_\ell)^{-1/2}U_\ell(\tilde{U}_\ell U_\ell)^{-1/2} = U_\ell(\tilde{U}_\ell U_\ell)^{-1} = U_\ell(\tilde{U}_\ell U_\ell)^{-1}\tilde{U}_\ell P_\ell = \tilde{P}_\ell P_\ell$, where we have used $P_\ell U_\ell = P_\ell$ and $\tilde{U}_\ell P_\ell = P_\ell$. This W_ℓ connects P_ℓ and $\tilde{P}_\ell$ as

$$W_\ell = W_\ell P_\ell = \tilde{P}_\ell W_\ell, \tag{8.8}$$

which can be verified trivially on the basis of the definitions of $\tilde{P}_\ell$ and W_ℓ in (8.1) and (8.7), respectively. Then,

$$\gamma B_\ell + K_\ell = W_\ell^{-1}(\gamma B + C)W_\ell \tag{8.9}$$

provides an effective generator which has the same block structure as B, where W_ℓ^{-1} is a pseudoinverse satisfying

$$W_\ell^{-1}W_\ell = P_\ell, \quad W_\ell W_\ell^{-1} = \tilde{P}_\ell, \tag{8.10}$$

which is explicitly given by

$$W_\ell^{-1} = (P_\ell\tilde{P}_\ell)^{1/2} = (\tilde{U}_\ell U_\ell)^{-1/2}\tilde{U}_\ell. \tag{8.11}$$

This W_ℓ^{-1} brings $\tilde{P}_\ell$ back to P_ℓ as

$$W_\ell^{-1}\tilde{P}_\ell = P_\ell W_\ell^{-1} = W_\ell^{-1}. \tag{8.12}$$

In the unitary case, $P_\ell = P_\ell^\dagger$ and $\tilde{U}_\ell = U_\ell^\dagger$ (see Sec. VII), and the polar decomposition of U_ℓ reads $U_\ell = V_\ell|U_\ell|$, where $|U_\ell| = (U_\ell^\dagger U_\ell)^{1/2}$ and V_ℓ is some unitary. Thus, in the unitary case, W_ℓ in (8.7) and W_ℓ^{-1} in (8.11) are reduced to $W_\ell = V_\ell P_\ell$ and $W_\ell^{-1} = P_\ell V_\ell^\dagger$, respectively, and (8.9) reads

$$\gamma B_\ell + K_\ell = P_\ell V_\ell^\dagger(\gamma B + C)V_\ell P_\ell, \tag{8.13}$$

so that K_ℓ is guaranteed to be skew-Hermitian. This reproduces the Schrieffer-Wolff formalism [49, Definition 3.1], and the transformation (8.9) is a generalization of the Schrieffer-Wolff transformation for open systems.

Recalling the intertwining relations (5.2) and (7.7), we can rewrite the Schrieffer-Wolff transformation (8.9) as

$$\begin{aligned}\gamma B_\ell + K_\ell &= (\tilde{U}_\ell U_\ell)^{-1/2}\tilde{U}_\ell(\gamma B + C)U_\ell(\tilde{U}_\ell U_\ell)^{-1/2} \\ &= (\tilde{U}_\ell U_\ell)^{1/2}(\gamma B + D_\ell)(\tilde{U}_\ell U_\ell)^{-1/2} \\ &= (\tilde{U}_\ell U_\ell)^{-1/2}(\gamma B + \tilde{D}_\ell)(\tilde{U}_\ell U_\ell)^{1/2}.\end{aligned} \tag{8.14}$$

It is clear from the first expression of (8.14) that the perturbative series of $K_\ell = \sum_{j=0}^\infty K_\ell^{(j)}/\gamma^j$ is symmetric also for open systems. The first few orders are given by

$$K_\ell^{(0)} = P_\ell C P_\ell, \tag{8.15}$$

$$K_\ell^{(1)} = -\tfrac{1}{2}P_\ell C S_\ell \overrightarrow{\langle C\rangle} P_\ell - \tfrac{1}{2}P_\ell \overleftarrow{\langle C\rangle} S_\ell C P_\ell, \tag{8.16}$$

$$\begin{aligned}K_\ell^{(2)} = {}&\tfrac{1}{2}P_\ell C S_\ell \overrightarrow{\langle C S_\ell \langle C\rangle\rangle} P_\ell + \tfrac{1}{2}P_\ell \overleftarrow{\langle\langle C\rangle S_\ell C\rangle} S_\ell C P_\ell \\ &-\tfrac{1}{2}P_\ell C S_\ell^2 \overrightarrow{\langle\langle C\rangle P_\ell C\rangle} P_\ell - \tfrac{1}{2}P_\ell \overleftarrow{\langle C P_\ell \langle C\rangle\rangle} S_\ell^2 C P_\ell,\end{aligned} \tag{8.17}$$

$$\begin{aligned}K_\ell^{(3)} = {}&-\tfrac{1}{2}P_\ell C S_\ell \overrightarrow{\langle C S_\ell \langle C S_\ell \langle C\rangle\rangle\rangle} P_\ell - \tfrac{1}{2}P_\ell \overleftarrow{\langle\langle\langle C\rangle S_\ell C\rangle S_\ell C\rangle} S_\ell C P_\ell \\ &+ \tfrac{1}{2}P_\ell C S_\ell \overrightarrow{\langle C S_\ell^2 \langle\langle C\rangle P_\ell C\rangle\rangle} P_\ell + \tfrac{1}{2}P_\ell \overleftarrow{\langle\langle C P_\ell \langle C\rangle\rangle S_\ell^2 C\rangle} S_\ell C P_\ell \\ &+ \tfrac{1}{2}P_\ell C S_\ell^2 \overrightarrow{\langle\langle C\rangle P_\ell C S_\ell \langle C\rangle\rangle} P_\ell + \tfrac{1}{2}P_\ell \overleftarrow{\langle\langle C\rangle S_\ell C P_\ell \langle C\rangle\rangle} S_\ell^2 C P_\ell \\ &+ \tfrac{1}{2}P_\ell C S_\ell^2 \overrightarrow{\langle\langle C S_\ell \langle C\rangle\rangle P_\ell C\rangle} P_\ell + \tfrac{1}{2}P_\ell \overleftarrow{\langle C P_\ell \langle\langle C\rangle S_\ell C\rangle\rangle} S_\ell^2 C P_\ell \\ &- \tfrac{1}{2}P_\ell C S_\ell^3 \overrightarrow{\langle\langle\langle C\rangle P_\ell C\rangle P_\ell C\rangle} P_\ell - \tfrac{1}{2}P_\ell \overleftarrow{\langle C P_\ell \langle C P_\ell \langle C\rangle\rangle\rangle} S_\ell^3 C P_\ell \\ &- \tfrac{1}{8}N_\ell \overleftarrow{\langle C\rangle} S_\ell^2 \overrightarrow{\langle C\rangle} P_\ell \overleftarrow{\langle C\rangle} S_\ell^2 \overrightarrow{\langle C\rangle} P_\ell - \tfrac{1}{8}P_\ell \overleftarrow{\langle C\rangle} S_\ell^2 \overrightarrow{\langle C\rangle} P_\ell \overleftarrow{\langle C\rangle} S_\ell^2 \overrightarrow{\langle C\rangle} N_\ell \\ &+ \tfrac{1}{4}P_\ell \overleftarrow{\langle C\rangle} S_\ell^2 \overrightarrow{\langle C\rangle} N_\ell \overleftarrow{\langle C\rangle} S_\ell^2 \overrightarrow{\langle C\rangle} P_\ell,\end{aligned} \tag{8.18}$$

where

$$\overrightarrow{\langle A\rangle}=\sum_{n=0}^{n_\ell-1}S_\ell^nAN_\ell^n,\quad \overleftarrow{\langle A\rangle}=\sum_{n=0}^{n_\ell-1}N_\ell^nAS_\ell^n. \tag{8.19}$$

The first bracket $\overrightarrow{\langle A\rangle}$ is the same as the one introduced in (4.8), but an arrow is put here to stress the order of the operators. Concatenated brackets like $\overrightarrow{\langle CS_\ell\overrightarrow{\langle CS_\ell\overrightarrow{\langle C\rangle}\rangle}\rangle}$ are simply denoted with a single arrow like $\overrightarrow{\langle CS_\ell\langle CS_\ell\langle C\rangle\rangle\rangle}$. Concatenation of brackets with different orientations of arrows does not appear. In the unitary case, this series reduces to the perturbative series obtained in Refs. [40,41].

The generators $\gamma B+C$, $\gamma B+D$, $\gamma B+\tilde{D}$, and $\gamma B+K$ with

$$K=\sum_\ell K_\ell \tag{8.20}$$

are similar to each other, and they share the same spectrum,

$$\begin{aligned}\gamma B+C&=U(\gamma B+D)U^{-1}\\&=\tilde{U}^{-1}(\gamma B+\tilde{D})\tilde{U}\\&=W(\gamma B+K)W^{-1},\end{aligned} \tag{8.21}$$

where $U=\sum_\ell U_\ell$ and $\tilde{U}=\sum_\ell\tilde{U}_\ell$ are introduced in (5.5) and (7.10), respectively, and

$$W=\sum_\ell W_\ell,\quad W^{-1}=\sum_\ell W_\ell^{-1}. \tag{8.22}$$

Thanks to the similarity relation and its closeness to the identity $W-1=O(1/\gamma)$, the distance between the approximate adiabatic evolution $e^{t(\gamma B+K)}$ and the true evolution $e^{t(\gamma B+C)}$ remains $O(1/\gamma)$ eternally. In the norm induced by the operator trace norm, we have $\|e^{t(\gamma B+C)}\|=1$ for the physical evolution [57], and the distance can be bounded in the same way as the one for $e^{t(\gamma B+D)}$ given in (6.6). That is,

$$\|e^{t(\gamma B+C)}-e^{t(\gamma B+K)}\|<\frac{1}{\gamma}\sum_\ell\gamma_\ell\|P_\ell\|, \tag{8.23}$$

for $\gamma\geqslant 2\max_\ell\gamma_\ell$, with γ_ℓ defined in (6.7). See Appendix E for its derivation and its tighter bound valid also for other norms.

IX. PHYSICAL PROPERTIES OF THE ADIABATIC GENERATORS D, $\tilde{D}$, AND K

As already mentioned, the adiabatic generator D is generally not skew-Hermitian even for unitary evolution with skew-Hermitian generators B and C. This is easily anticipated from the asymmetry in the perturbative series in (4.4)–(4.7). This asymmetry can be fixed by the transformation discussed in the previous section. The adiabatic generator K obtained by the generalized Schrieffer-Wolff transformation is symmetric, and it is guaranteed to be skew-Hermitian for unitary evolution.

In the nonunitary case, the structure of a physical generator is much more subtle than in the unitary case [46,47]. It should be Hermiticity-preserving (HP), trace-preserving (TP), and conditionally completely positive (CP) (with a positive-semidefinite Kossakowski matrix) [45] as a generator acting on density operators. These impose a delicate structure on the generator, leading to the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) form [46,47].

In this section we are going to show that D, $\tilde{D}$, and K obtained for physical (i.e., HP, TP, and CP) generators B and C acting on density operators are both HP and TP in the general nonunitary case (including the unitary case). On the other hand, CP is not guaranteed in the nonunitary case, even for the symmetric K, as we will see in the next section.

A. D, $\tilde{D}$, and K are TP

Note first that the spectrum $\{b_\ell\}$ of a physical generator B acting on density operators is contained in the closed left half-plane $\mathrm{Re}\,b_\ell\leqslant 0$, and B always has $b_0=0$ in its spectrum. In addition, purely imaginary eigenvalues $b_\ell\in i\mathbb{R}$ including $b_0=0$ are semisimple, that is $P_\ell BP_\ell=b_\ell P_\ell$ are diagonalizable with no nilpotents. See, e.g., Refs. [60,61], in particular Propositions 6.1–6.3 and Theorem 6.1 of Ref. [60].

Since B is assumed to be a physical generator, it is TP, i.e., $\mathrm{tr}[B(\sigma)]=0$ for any operators σ acting on the Hilbert space. Since this can be written as $\mathrm{tr}[B(\sigma)]=(\mathbb{1}|B(\sigma))=(\mathbb{1}|B|\sigma)=0$, with $(\varrho|\sigma)=\mathrm{tr}(\varrho^\dagger\sigma)$ being the Hilbert-Schmidt inner product of operators ϱ and σ acting on the Hilbert space, the TP of B as a generator is represented by

$$(\mathbb{1}|B=0. \tag{9.1}$$

Projecting it by P_ℓ from the right, we get

$$(\mathbb{1}|BP_\ell=(\mathbb{1}|(b_\ell P_\ell+N_\ell)=0. \tag{9.2}$$

This condition is trivial for $\ell=0$, since $b_0=0$ and there is no nilpotent $N_0=0$ in this sector. For nonvanishing eigenvalues b_ℓ, let us multiply $N_\ell^{n_\ell-1}$ from the right of (9.2). It yields $(\mathbb{1}|N_\ell^{n_\ell-1}=0$, since $N_\ell^{n_\ell}=0$, $P_\ell N_\ell=N_\ell$, and $b_\ell\neq 0$. Then, by multiplying $N_\ell^{n_\ell-2}$ from the right of (9.2) again, we realize that $(\mathbb{1}|N_\ell^{n_\ell-2}=0$. After $n_\ell-1$ such iterations, we reach

$$(\mathbb{1}|N_\ell=0. \tag{9.3}$$

This further implies

$$(\mathbb{1}|P_\ell=0\quad\text{for}\quad b_\ell\neq 0. \tag{9.4}$$

Finally, since $\sum_\ell P_\ell=1$, we need to have

$$(\mathbb{1}|P_0=(\mathbb{1}|, \tag{9.5}$$

namely, P_0 too is TP.

Now, let us look at the adiabatic Bloch equation (7.2) for $\tilde{\Omega}_\ell$. Putting $(\mathbb{1}|$ on the left of the adiabatic Bloch equation, we get

$$(\mathbb{1}|\tilde{\Omega}_\ell\left(1+\frac{1}{\gamma}S_\ell C-\frac{1}{\gamma}\tilde{\Omega}_\ell S_\ell\right)=0, \tag{9.6}$$

where we have used (9.3)–(9.5) and $(\mathbb{1}|C=0$. This implies

$$(\mathbb{1}|\tilde{\Omega}_\ell=0 \tag{9.7}$$

for large enough γ, since $1+\frac{1}{\gamma}S_\ell C-\frac{1}{\gamma}\tilde{\Omega}_\ell S_\ell$ is invertible. Therefore we have

$$(\mathbb{1}|\tilde{U}_\ell=(\mathbb{1}|\left(P_\ell-\frac{1}{\gamma}\tilde{\Omega}_\ell S_\ell\right)=(\mathbb{1}|P_\ell \tag{9.8}$$

and

$$(\mathbb{1}|(\tilde{U}_\ell U_\ell)^\alpha = (\mathbb{1}|\left(1+\frac{1}{\gamma^2}\tilde{\Omega}_\ell S_\ell^2\Omega_\ell\right)^\alpha P_\ell = (\mathbb{1}|P_\ell \tag{9.9}$$

for $\alpha = -1$ and $-1/2$. Recall the definition of the pseudoinverse $(\tilde{U}_\ell U_\ell)^{-1}$ in (8.2). Then it immediately follows that D, $\tilde{D}$, and K are TP. For instance, using the similarity in (8.14), the adiabatic generator D is proved to be TP as

$$\begin{aligned}(\mathbb{1}|D &= \sum_\ell (\mathbb{1}|[(\tilde{U}_\ell U_\ell)^{-1}\tilde{U}_\ell(\gamma B + C)U_\ell - \gamma B_\ell] \\ &= (\mathbb{1}|P_0 C U_0 = 0.\end{aligned} \tag{9.10}$$

TP of $\tilde{D}$ and K can be proved in the same way.

B. $D, \tilde{D}$, and K are HP

Let us next prove that D, $\tilde{D}$, and K are HP. To this end it is convenient to introduce an orthogonal basis of Hermitian matrices $\{\tau_0, \tau_1, \dots, \tau_{d^2-1}\}$ for a d-dimensional system. Here $\tau_0 = \mathbb{1}$ is the $d\times d$ identity matrix, and the $d \times d$ matrices τ_i $(i = 1, \dots, d^2-1)$ are Hermitian $\tau_i = \tau_i^\dagger$ and traceless $\operatorname{tr}\tau_i = 0$, which are orthogonal to each other with respect to the Hilbert-Schmidt inner product $(\tau_i|\tau_j) = \operatorname{tr}(\tau_i^\dagger\tau_j) = 2\delta_{ij}$ $(i, j = 1, \dots, d^2-1)$. The matrix representation $\mathsf{B}_{ij} = (\tau_i|B|\tau_j) = (\tau_i|B(\tau_j))$ $(i, j = 0, 1, \dots, d^2-1)$ of B in such a basis is the generator of the evolution of the coherence vector $r_i = (\tau_i|\varrho)$ $(i = 0, 1, \dots, d^2-1)$ representing the density operator ϱ of the system. Notice that the coherence vector $(r_0, r_1, \dots, r_{d^2-1})$ corresponding to a Hermitian density operator ϱ is a real vector. Therefore, the matrix elements B_{ij} of a physical generator B should be all real, since B should preserve the Hermiticity of density operator ϱ and hence the reality of the coherence vector. In other words, the reality of B_{ij} is equivalent to HP of B. Let us call the spectral projections and nilpotents of the real matrix B in this representation P_ℓ and N_ℓ, respectively.

We note that all the nonreal eigenvalues of a real matrix occur in conjugate pairs. In addition, the spectral projections and the nilpotents of the real matrix $\mathsf{B} = \mathsf{B}^*$ satisfy

$$\mathsf{P}_\ell = \mathsf{P}_{\bar{\ell}}^*, \quad \mathsf{N}_\ell = \mathsf{N}_{\bar{\ell}}^*, \tag{9.11}$$

where $*$ of a matrix represents the elementwise complex conjugation and $\bar{\ell}$ refers to its complex conjugate eigenvalue $b_{\bar{\ell}} = b_\ell^*$. Indeed, the spectral projection P_ℓ can be constructed by

$$\mathsf{P}_\ell = \int_{\mathcal{C}_\ell}\frac{dz}{2\pi i}(z-\mathsf{B})^{-1}, \tag{9.12}$$

where $\mathcal{C}_\ell$ is a contour running anticlockwise around the eigenvalue b_ℓ on the complex z plane [37]. Since $\mathsf{B} = \mathsf{B}^*$ is real and $\mathcal{C}_\ell$ is flipped to $-\mathcal{C}_{\bar{\ell}}$ (running clockwise around the complex conjugate eigenvalue $b_\ell^* = b_{\bar{\ell}}$) by complex conjugation, we get $\mathsf{P}_\ell^* = -\int_{-\mathcal{C}_{\bar{\ell}}}\frac{dz}{2\pi i}(z-\mathsf{B}^*)^{-1} = \int_{\mathcal{C}_{\bar{\ell}}}\frac{dz}{2\pi i}(z-\mathsf{B})^{-1} = \mathsf{P}_{\bar{\ell}}$, and $\mathsf{N}_\ell^* = [(\mathsf{B}-b_\ell)\mathsf{P}_\ell]^* = (\mathsf{B}^*-b_\ell^*)\mathsf{P}_\ell^* = (\mathsf{B}-b_{\bar{\ell}})\mathsf{P}_{\bar{\ell}} = \mathsf{N}_{\bar{\ell}}$. This proves (9.11). This symmetry is inherited by the reduced resolvents,

$$\mathsf{S}_\ell = \sum_{k\neq\ell}(b_k - b_\ell + \mathsf{N}_k)^{-1}\mathsf{P}_k = \mathsf{S}_{\bar{\ell}}^*. \tag{9.13}$$

Now, let us look at the adiabatic Bloch equation (3.2) in this representation,

$$\frac{1}{\gamma}\mathsf{S}_\ell\Omega_\ell^2 - \left(\mathsf{I}+\frac{1}{\gamma}\mathsf{C}\mathsf{S}_\ell\right)\Omega_\ell + \mathsf{S}_\ell\Omega_\ell\mathsf{N}_\ell + \mathsf{C}\mathsf{P}_\ell = 0. \tag{9.14}$$

Note that the matrix representation C of C is also a real matrix, since C is assumed to be physical. Taking the complex conjugation of this adiabatic Bloch equation (9.14) yields

$$\frac{1}{\gamma}\mathsf{S}_{\bar{\ell}}\Omega_\ell^{*2} - \left(\mathsf{I}+\frac{1}{\gamma}\mathsf{C}\mathsf{S}_{\bar{\ell}}\right)\Omega_\ell^* + \mathsf{S}_{\bar{\ell}}\Omega_\ell^*\mathsf{N}_{\bar{\ell}} + \mathsf{C}\mathsf{P}_{\bar{\ell}} = 0, \tag{9.15}$$

which implies

$$\Omega_\ell^* = \Omega_{\bar{\ell}}. \tag{9.16}$$

By looking at the conjugate adiabatic Bloch equation (7.2) for $\tilde{\Omega}_\ell$, we also confirm that $\tilde{\Omega}_\ell^* = \tilde{\Omega}_{\bar{\ell}}$. The operators U_ℓ and $\tilde{U}_\ell$ are also endowed with the same symmetry $\mathsf{U}_\ell^* = \mathsf{U}_{\bar{\ell}}$, $\tilde{\mathsf{U}}_\ell^* = \tilde{\mathsf{U}}_{\bar{\ell}}$, and so are the adiabatic generators. For instance,

$$\begin{aligned}\mathsf{D}_\ell^* &= (\tilde{\mathsf{U}}_\ell^*\mathsf{U}_\ell^*)^{-1}\tilde{\mathsf{U}}_\ell^*(\gamma\mathsf{B}^*+\mathsf{C}^*)\mathsf{U}_\ell^* - \gamma\mathsf{B}^*\mathsf{P}_\ell^* \\ &= (\tilde{\mathsf{U}}_{\bar{\ell}}\mathsf{U}_{\bar{\ell}})^{-1}\tilde{\mathsf{U}}_{\bar{\ell}}(\gamma\mathsf{B}+\mathsf{C})\mathsf{U}_{\bar{\ell}} - \gamma\mathsf{B}\mathsf{P}_{\bar{\ell}} \\ &= \mathsf{D}_{\bar{\ell}}.\end{aligned} \tag{9.17}$$

Therefore,

$$\mathsf{D} = \sum_\ell \mathsf{D}_\ell = \sum_\ell \mathsf{D}_\ell^* = \mathsf{D}^*. \tag{9.18}$$

The reality of $\tilde{\mathsf{D}}$ and K can be shown in the same way, and hence, D, $\tilde{D}$, and K are HP.

X. EXAMPLES

Let us look at some examples.

A. Dissipative Lambda system

We consider a five-level system, whose level structure is depicted in Fig. 1. The Hamiltonian is given by

$$H_\Lambda = \begin{pmatrix}\omega & 0 & 0 & 0 & 0\\ 0 & -\delta/2 & 0 & g_1^*/2 & 0\\ 0 & 0 & \delta/2 & g_2^*/2 & 0\\ 0 & g_1/2 & g_2/2 & \Delta & 0\\ 0 & 0 & 0 & 0 & 2\Delta\end{pmatrix}. \tag{10.1}$$

Levels $|1\rangle$, $|2\rangle$, and $|3\rangle$ constitute a Λ configuration, and there is strong decay from $|4\rangle$ to $|2\rangle$ with decay rate κ_0 and weak decay from $|0\rangle$ to $|1\rangle$ and from $|0\rangle$ to $|2\rangle$ with decay rate κ. We are interested in the situation where $\Delta, \kappa_0 \gg \omega, |\delta|, |g_{1,2}|, \kappa$.

For this kind of Λ system, one often attempts to derive an effective generator for the subspace $\{|0\rangle, |1\rangle, |2\rangle\}$, which is energetically well separated from the higher energy levels $|3\rangle$ and $|4\rangle$. The Λ system is a standard setup to discuss adiabatic elimination, and approximations beyond the adiabatic elimination have been studied on these platforms in the literature [9,51]. Here we can deal with the Λ system in the presence of noise, and get an effective generator which well approximates the evolution of the open system for all times.

Let us normalize the physical parameters Δ, ω, δ, $g_{1,2}$, κ, and κ_0 by some unit of frequency g_0, and set $\gamma = \Delta/g_0$, which

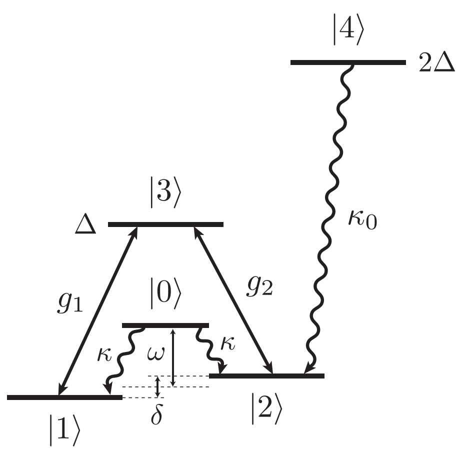

FIG. 1. A dissipative five-level system. Levels $|1\rangle$, $|2\rangle$, and $|3\rangle$ constitute a Λ configuration, and there is strong decay from $|4\rangle$ to $|2\rangle$ with decay rate κ_0 and weak decay from $|0\rangle$ to $|1\rangle$ and from $|0\rangle$ to $|2\rangle$ with decay rate κ.

is considered to be much greater than $\tilde{\omega} = \omega/g_0$, $\tilde{\delta} = \delta/g_0$, $\tilde{g}_{1,2} = g_{1,2}/g_0$, $\tilde{\kappa} = \kappa/g_0$, while $\tilde{\kappa}_0 = \kappa_0/\Delta = O(1)$. We apply our formalism to Markovian generators of the GKLS form

$$
\begin{aligned}
B &= -i[H_0, \bullet] - \frac{1}{2}\tilde{\kappa}_0(L_0^\dagger L_0 \bullet + \bullet L_0^\dagger L_0 - 2L_0 \bullet L_0^\dagger),\\
C &= -i[H_I, \bullet] - \frac{1}{2}\tilde{\kappa} \sum_{i=1,2}(L_i^\dagger L_i \bullet + \bullet L_i^\dagger L_i - 2L_i \bullet L_i^\dagger),
\end{aligned}
\tag{10.2}
$$

with

$$
H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad L_0 = |2\rangle\langle 4|,
$$

$$
H_I = \begin{pmatrix} \tilde{\omega} & 0 & 0 & 0 & 0\\ 0 & -\tilde{\delta}/2 & 0 & \tilde{g}_1^*/2 & 0\\ 0 & 0 & \tilde{\delta}/2 & \tilde{g}_2^*/2 & 0\\ 0 & \tilde{g}_1/2 & \tilde{g}_2/2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{cases} L_1 = |1\rangle\langle 0|,\\ L_2 = |2\rangle\langle 0|. \end{cases}
\tag{10.3}
$$

By abuse of notation, we will omit tildes $\tilde{\omega} \to \omega$, $\tilde{\delta} \to \delta$, $\tilde{g}_{1,2} \to g_{1,2}$, $\tilde{\kappa} \to \kappa$, and $\tilde{\kappa}_0 \to \kappa_0$ in the following analysis.

According to the perturbative formulas in (8.15)–(8.18), we get the jth-order term $K^{(j)} = \sum_\ell K_\ell^{(j)}$ of the adiabatic generator $K = \sum_{j=0}^\infty K^{(j)}/\gamma^j$ in the GKLS form [62]

$$
\begin{aligned}
K^{(j)} &= -i[H^{(j)}, \bullet]\\
&\quad - \frac{1}{2}\sum_i \Gamma_i^{(j)}\big(L_i^{(j)\dagger}L_i^{(j)} \bullet + \bullet L_i^{(j)\dagger}L_i^{(j)} - 2L_i^{(j)} \bullet L_i^{(j)\dagger}\big).
\end{aligned}
\tag{10.4}
$$

The lowest-order term $K^{(0)}$ is the Zeno generator, given by

$$
H^{(0)} = \begin{pmatrix} \omega & 0 & 0 & 0 & 0\\ 0 & -\delta/2 & 0 & 0 & 0\\ 0 & 0 & \delta/2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\Gamma_i^{(0)} = \kappa, \quad L_i^{(0)} = |i\rangle\langle 0| \quad (i = 1, 2). \tag{10.5}
$$

The first-order term $K^{(1)}$ provides an approximation usually discussed in terms of adiabatic elimination, which in the present case is given by

$$
H^{(1)} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & -|g_1|^2 & -g_1^* g_2 & 0 & 0\\ 0 & -g_1 g_2^* & -|g_2|^2 & 0 & 0\\ 0 & 0 & 0 & |g_1|^2 + |g_2|^2 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\Gamma_\pm^{(1)} = \pm\frac{1}{4}|g_1 g_2|, \quad L_\pm^{(1)} = \frac{e^{-i\phi_1}|1\rangle \mp i e^{-i\phi_2}|2\rangle}{\sqrt{2}}\langle 4|, \tag{10.6}
$$

where $g_{1,2} = |g_{1,2}|e^{i\phi_{1,2}}$. Notice here that these approximations are valid only for limited time ranges. See Fig. 2. The Zeno generator $K_{\text{eff}}^{(0)} = K^{(0)}$ is a good approximation only for times up to $t = O(\gamma)$, while the evolution with $K_{\text{eff}}^{(1)} = K^{(0)} + K^{(1)}/\gamma$ by adiabatic elimination starts to deviate from the true evolution for $t = O(\gamma^2)$. The second- and third-order approximations $K^{(2)}$ and $K^{(3)}$ are given by

$$
H^{(2)} = \frac{1}{8}\delta\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & |g_1|^2 & 0 & 0 & 0\\ 0 & 0 & -|g_2|^2 & 0 & 0\\ 0 & 0 & 0 & -|g_1|^2 + |g_2|^2 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\begin{aligned}
&\Gamma_\pm^{(2)} = \pm\frac{1}{4}\kappa(|g_1|^2 + |g_2|^2),\\
&L_+^{(2)} = |3\rangle\langle 0|, \quad L_-^{(2)} = \frac{g_1^*|1\rangle + g_2^*|2\rangle}{\sqrt{|g_1|^2 + |g_2|^2}}\langle 0|,
\end{aligned}
\tag{10.7}
$$

and

$$
\begin{aligned}
H^{(3)} &= \frac{1}{16}(\delta^2 - |g_1|^2 - |g_2|^2)\\
&\quad \times \begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & -|g_1|^2 & -g_1^* g_2 & 0 & 0\\ 0 & -g_1 g_2^* & -|g_2|^2 & 0 & 0\\ 0 & 0 & 0 & |g_1|^2 + |g_2|^2 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\Gamma_1^{(3)} = +\frac{1}{4}\kappa\delta|g_1|^2, && L_1^{(3)} = |1\rangle\langle 0|,\\
&L_2^{(3)} = -\frac{1}{4}\kappa\delta|g_2|^2, && L_2^{(3)} = |2\rangle\langle 0|,\\
&L_3^{(3)} = -\frac{1}{4}\kappa\delta(|g_1|^2 - |g_2|^2), && L_3^{(3)} = |3\rangle\langle 0|,\\
&\Gamma_\pm^{(3)} = \pm\frac{1}{16}|g_1 g_2|(\delta^2 - |g_1|^2 - |g_2|^2),\\
&L_\pm^{(3)} = \frac{e^{-i\phi_1}|1\rangle \mp i e^{-i\phi_2}|2\rangle}{\sqrt{2}}\langle 4|.
\end{aligned}
\tag{10.8}
$$

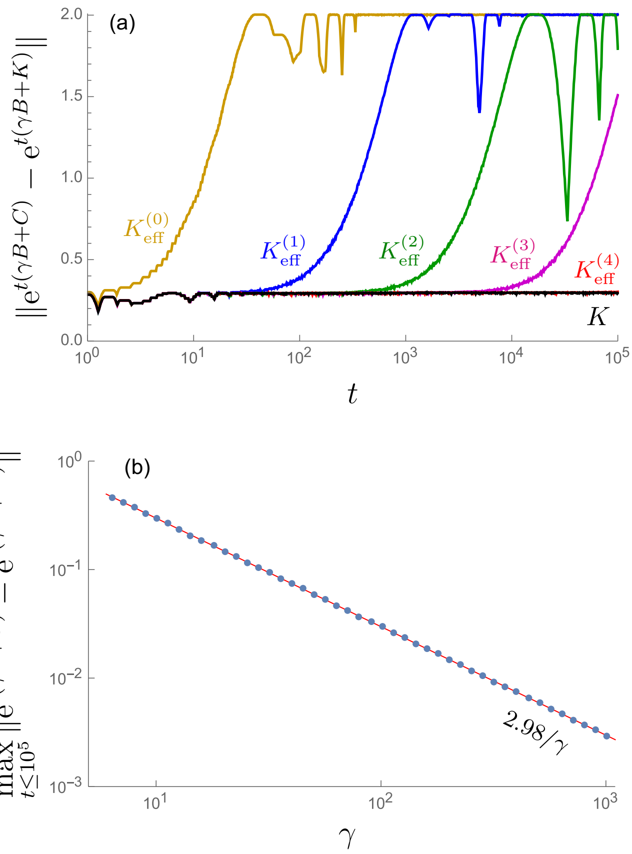

FIG. 2. (a) Norm distances as functions of time t between the full evolution $e^{t(\gamma B+C)}$ and the kth-order adiabatic approximations of the form $e^{t(\gamma B+K_{\rm eff}^{(k)})}$ with $K_{\rm eff}^{(k)} = \sum_{j=0}^{k} K^{(j)}/\gamma^j$ ($k = 0, 1, 2, 3, 4, \infty$), for the dissipative five-level system (10.1) with a Λ structure (see Fig. 1). The parameters are set at $\tilde{\delta} = \tilde{g}_1 = \tilde{g}_2 = 1$, $\tilde{\kappa} = 0.001$, $\tilde{\kappa}_0 = 1$, and $\gamma = 10$. We have chosen the spectral norm (the maximum of the singular values) of a matrix representation of the map to estimate the distance. The distances actually oscillate radically as quasiperiodic functions of time: their upper envelopes are plotted here. It is clearly observed that the kth-order approximation $K_{\rm eff}^{(k)}$ works well for times up to $t = O(\gamma^{k+1})$, while the nonperturbative adiabatic generator $K = K_{\rm eff}^{(\infty)}$ works eternally with the error remaining $O(1/\gamma)$ for long times. (b) Maximum distance $\max_{t\leqslant 10^5} \|e^{t(\gamma B+C)} - e^{t(\gamma B+K)}\|$ as a function of γ. The model and the parameters other than γ are the same as in (a). The error approximately decreases as $2.98/\gamma$ as γ is increased.

These extend the valid time range up to $t = O(\gamma^3)$ and $t = O(\gamma^4)$, respectively. In general, the kth-order adiabatic approximation $K_{\rm eff}^{(k)} = \sum_{j=0}^{k} K^{(j)}/\gamma^j$ works well for times up to $t = O(\gamma^{k+1})$, and the nonperturbative adiabatic generator $K = K_{\rm eff}^{(\infty)}$ works eternally, keeping the error $O(1/\gamma)$, as is clearly observed in Fig. 2.

For a nonvanishing δ, it is generally impossible to get an analytical expression for the nonperturbative adiabatic generator K, but it can be estimated numerically. For instance, for $\omega = \delta = g_1 = g_2 = \kappa = \kappa_0 = 1$, and $\gamma = 10$, we get $K = K_{\rm eff}^{(\infty)}$ in the GKLS form

$$K = -i[H,\, \bullet\,] - \frac{1}{2}\sum_i \Gamma_i (L_i^\dagger L_i \bullet + \bullet L_i^\dagger L_i - 2L_i \bullet L_i^\dagger), \qquad (10.9)$$

with

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -0.524 & -0.025 & 0 & 0 \\ 0 & -0.025 & 0.474 & 0 & 0 \\ 0 & 0 & 0 & 0.050 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} &\Gamma_1 = 1.000, \quad L_1 = (\cos\theta\,|1\rangle - e^{i\phi}\sin\theta\,|2\rangle)\langle 0|,\\ &\Gamma_2 = 0.995, \quad L_2 = (e^{-i\phi}\sin\theta\,|1\rangle + \cos\theta\,|2\rangle)\langle 0|,\\ &\Gamma_3 = 0.005, \quad L_3 = |3\rangle\langle 0|,\\ &\Gamma_\pm = \pm 0.025, \quad L_\pm = \frac{|1\rangle \mp i|2\rangle}{\sqrt{2}}\langle 4|, \end{aligned} \qquad (10.10)$$

where $\tan\theta = 0.909$, $\tan\phi = 0.029$. This provides an effective generator for the relevant subspaces, which closely (and eternally) approximates the evolution of the system. To get this nonperturbative generator K numerically, we used the adiabatic Bloch equation (3.2) as

$$\Omega_\ell = CP_\ell + S_\ell\Omega_\ell N_\ell - \frac{1}{\gamma}CS_\ell\Omega_\ell + \frac{1}{\gamma}S_\ell\Omega_\ell^2 \equiv f(\Omega_\ell), \quad (10.11)$$

and performed naive iterations over the function f, which for $\gamma = 10$ converged quickly with the initial guess $\Omega_\ell^{(0)} = \langle C\rangle P_\ell = \sum_{n=0}^{n_\ell-1} S_\ell^n C N_\ell^n P_\ell$, that is the zeroth-order solution of Ω_ℓ (there is no nilpotent N_ℓ in the present model and the

TABLE I. The spectra of B and $\gamma B + C$ of the dissipative Λ system (10.2) and (10.3) with $\delta = 0$. Here $g = \sqrt{|g_1|^2 + |g_2|^2}$.

B	$\gamma B + C$
0	0 (threefold degenerated) $\pm\frac{i}{2}(\sqrt{\gamma^2+g^2}-\gamma)$ $-\kappa \pm i\omega$ $-\kappa \pm i[\omega + \frac{1}{2}(\sqrt{\gamma^2+g^2}-\gamma)]$ -2κ
$\pm i$	$\pm\frac{i}{2}(\gamma+\sqrt{\gamma^2+g^2})$ $\pm i\sqrt{\gamma^2+g^2}$ $-\kappa \pm i[\frac{1}{2}(\gamma+\sqrt{\gamma^2+g^2})-\omega]$
$-\frac{1}{2}\kappa_0 \pm i$	$-\frac{1}{2}\gamma\kappa_0 \pm \frac{i}{2}(3\gamma - \sqrt{\gamma^2+g^2})$
$-\frac{1}{2}\kappa_0 \pm 2i$	$-\frac{1}{2}\gamma\kappa_0 \pm 2i\gamma$ $-\frac{1}{2}\gamma\kappa_0 \pm \frac{i}{2}(3\gamma + \sqrt{\gamma^2+g^2})$ $-\frac{1}{2}\gamma\kappa_0 - \kappa \pm i(2\gamma - \omega)$
$-\kappa_0$	$-\gamma\kappa_0$

initial guess we used was simply CP_ℓ). A more sophisticated algorithm with advanced convergence speed and guaranteed solution using Newton iteration is provided in Ref. [63]. See Appendix D for the conditions for the existence and the uniqueness of the solution to the adiabatic Bloch equation (3.2) based on the Newton-Kantorovich theorem for the Newton iteration [43]. After obtaining $D_\ell = P_\ell \Omega_\ell P_\ell$ from Ω_ℓ, we also solved the conjugate adiabatic Bloch equation (7.2) numerically, constructed U_ℓ and $\tilde{U}_\ell$ through (5.3) and (7.8), respectively, and applied the similarity transformation $(\tilde{U}_\ell U_\ell)^{1/2}$ to get K_ℓ from D_ℓ according to (8.14). We can also solve the Bloch equations (5.11) and (7.12) in the same way to obtain U_ℓ and $\tilde{U}_\ell$ directly, instead of solving (3.2) and (7.2) for Ω_ℓ and $\tilde{\Omega}_\ell$. Then, we can construct K_ℓ according to (8.14).

One might have noticed that the perturbative terms presented above are all HP and TP, but not CP, except for the Zeno generator $K^{(0)}$, because of the non-positive-semidefinite Kossakowski matrices in the dissipators. In the nonperturbative adiabatic generator K in (10.10), summing up all the perturbative contributions, there remains one negative eigenvalue $\Gamma_- = -0.025$ in the Kossakowski matrix. It is associated with the strong decay from $|4\rangle$ to the Λ subspace. This negativity is not canceled by the dissipative part of the strong generator γB: the total adiabatic generator $\gamma B + K$ has a negative eigenvalue $\tilde{\Gamma}_- = -6.22\times 10^{-5}$ in its Kossakowski matrix with a Lindblad operator $\tilde{L}_- = (\cos\tilde{\theta}\,|1\rangle + i\sin\tilde{\theta}\,|1\rangle)\langle 4|$, where $\tan\tilde{\theta} = 0.0025$.

If one computes D for the present model, it is not CP even in the absence of the decays (i.e., even for $\kappa_0 = \kappa = 0$). It is turned into K by the Schrieffer-Wolff transformation and becomes skew-Hermitian and CP. The Schrieffer-Wolff transformation, however, does not amend CP in the presence of the decays. The unitary part, on the other hand, is properly amended by the Schrieffer-Wolff transformation, even in the presence of the decays. The decaying components anyway decay out, and the adiabatic evolution at long times within the decoherence-free subspaces $\{|1\rangle, |2\rangle\}$ and $\{|3\rangle\}$ are well described by the Hamiltonian part H of the resummed perturbative series. In any case, the error remains $O(1/\gamma)$ eternally. Within this approximation, the analysis is fully consistent and the violation of the CP condition of the effective evolution yields effects that are within the error $O(1/\gamma)$ *at all times*.

For $\delta = 0$, analytical expressions are available. The spectrum of $\gamma B + C$ is listed in Table I, and the nonperturbative adiabatic generator K is given in the GKLS form (10.9) with

$$H = \omega|0\rangle\langle 0| + \frac{1}{2}(\sqrt{\gamma^2+g^2}-\gamma)\begin{pmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & -|g_1|^2/g^2 & -g_1^* g_2/g^2 & 0 & 0\\ 0 & -g_1 g_2^*/g^2 & -|g_2|^2/g^2 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} &\Gamma_1 = \kappa, && L_1 = \frac{1}{g}(g_2|1\rangle - g_1|2\rangle)\langle 0|,\\ &\Gamma_2 = \kappa\frac{\gamma^2+\gamma\sqrt{\gamma^2+g^2}+g^2+8\kappa^2}{2(\gamma^2+g^2+4\kappa^2)}, && L_2 = \frac{1}{g}(g_1^*|1\rangle + g_2^*|2\rangle)\langle 0|,\\ &\Gamma_3 = \kappa\frac{\gamma^2-\gamma\sqrt{\gamma^2+g^2}+g^2}{2(\gamma^2+g^2+4\kappa^2)}, && L_3 = |3\rangle\langle 0|,\\ &\Gamma_\pm = \pm\frac{1}{2}(\sqrt{\gamma^2+g^2}-\gamma)\frac{|g_1 g_2|}{g^2}, && L_\pm = \frac{1}{\sqrt{2}}(e^{-i\phi_1}|1\rangle \mp i e^{-i\phi_2}|2\rangle)\langle 4|, \end{aligned} \tag{10.12}$$

where $g = \sqrt{|g_1|^2+|g_2|^2}$. Combined with the strong generator γB, the Kossakowski matrix of the total adiabatic generator $\gamma B + K$ has the same spectrum $\{\Gamma_i\}$ as (10.12) except for the last two terms with $\Gamma_\pm$ and $L_\pm$, which are replaced by

$$\tilde{\Gamma}_\pm = \frac{1}{2}\gamma\kappa_0\left(1 \pm \sqrt{1+4\tan^2\phi\,\frac{|g_1 g_2|^2}{g^4}}\right),\quad \tilde{L}_+ = (c_1 e^{-i\phi_1}|1\rangle - c_2 e^{-i\phi_2}|2\rangle)\langle 4|,\quad \tilde{L}_- = (c_2^* e^{-i\phi_1}|1\rangle + c_1^* e^{-i\phi_2}|2\rangle)\langle 4|, \tag{10.13}$$

where

$$\begin{cases} c_1 = (u_+|g_2| - u_- e^{i\phi}|g_1|)/g,\\ c_2 = (u_+|g_1| + u_- e^{i\phi}|g_2|)/g, \end{cases}\quad \tan\phi = \frac{\sqrt{\gamma^2+g^2}-\gamma}{2\gamma\kappa_0},\quad u_\pm = \sqrt{\frac{1}{2}\left(1 \pm \frac{1}{\sqrt{1+4\tan^2\phi\,|g_1 g_2|^2/g^4}}\right)}. \tag{10.14}$$

The eigenvalue $\tilde{\Gamma}_-$ is strictly negative, which is

$$\tilde{\Gamma}_- = -\frac{|g_1 g_2|^2}{16\gamma^3\kappa_0} + O(1/\gamma^5) \tag{10.15}$$

for large γ.

B. Single qubit with nilpotent

We can apply our formalism to open systems, even for a generator B that admits a nilpotent. Let us look at a simple qubit example,

$$B = -\frac{i}{2}[X,\ \bullet\,] - (1 - Z \bullet Z), \tag{10.16}$$

$$C = -i[X + Y,\ \bullet\,], \tag{10.17}$$

where X, Y, and Z are Pauli operators. In a matrix representation, the generator B is put in the Jordan normal form

$$B = R\begin{pmatrix} -2 & & & \\ & -1 & 1 & \\ & 0 & -1 & \\ & & & 0 \end{pmatrix}R^{-1}, \tag{10.18}$$

via a similarity transformation R. The eigenvalue -1 is degenerate and accompanies a nilpotent in its eigenspace. In this basis, the weak part C of the generator is represented by

$$C = R\begin{pmatrix} 0 & -2 & 0 & 0 \\ 2 & -2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}R^{-1}. \tag{10.19}$$

This simple model is tractable analytically. For instance, the spectrum of $\gamma B + C$ reads

$$\{0, -\gamma \pm 2i\sqrt{\gamma + 2}, -2\gamma\}. \tag{10.20}$$

Moreover, we can solve the adiabatic Bloch equation and get the nonperturbative adiabatic generator

$$\begin{aligned} K &= (\sqrt{\gamma^2 + 4\gamma + 8} - \gamma)R\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}R^{-1} \\ &= -\frac{i}{2}(\sqrt{\gamma^2 + 4\gamma + 8} - \gamma)[X,\ \bullet\,]. \end{aligned} \tag{10.21}$$

Note that even though K is endowed with the same block structure as B they do not commute, $[B, K] \neq 0$. Observe also that K is physical, i.e., HP, TP, and CP, in this example. The adiabatic generator $\gamma B + K$ is similar to the original generator $\gamma B + C$ as

$$\gamma B + K = W^{-1}(\gamma B + C)W \tag{10.22}$$

with

$$W = R\begin{pmatrix} 1 & -\frac{2}{\sqrt{\gamma^2+4\gamma+8}} & \frac{2}{\sqrt{\gamma^2+4\gamma+8}} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{\gamma+2} & 1 - \frac{\gamma+2}{\sqrt{\gamma^2+4\gamma+8}} & \frac{\gamma+2}{\sqrt{\gamma^2+4\gamma+8}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}R^{-1}, \tag{10.23}$$

and they share the same spectrum (10.20).

TABLE II. The spectra of B and $\gamma B + C$ for the three-level system (10.24) and (10.25).

B	$\gamma B + C$
0	0 (twofold degenerated) -2
$\pm\frac{i}{3}$	$-\frac{1}{2} \pm \frac{i}{3}\gamma$
$\pm\frac{2i}{3}$	$-\frac{1}{2} \pm \frac{2i}{3}\gamma$
$\pm i$	$-1 \pm i\sqrt{\gamma^2 - 1}$

C. Impossibility of physical generator

In the previous qubit example, K is physical (HP, TP, and CP), but it is just a lucky case. Indeed, in the first example (dissipative Λ system), the adiabatic generator K is not of proper physical structure. We are sure about HP and TP of K, as proved in Sec. IX, but CP is not guaranteed in general. One might think that CP can be amended via an additional small similarity transformation on $\gamma B + K$ keeping the block structure of B. However, it is generally impossible, as we prove here.

We provide a counterexample,

$$B = -i[H_0,\ \bullet\,], \quad C = -(1 - L_0 \bullet L_0^\dagger), \tag{10.24}$$

with

$$H_0 = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{10.25}$$

The strong generator B has seven spectral blocks,

$$B = R\begin{pmatrix} 0 &&&&&&&& \\ & 0 &&&&&&& \\ && 0 &&&&&& \\ &&& -i/3 &&&&& \\ &&&& i/3 &&&& \\ &&&&& -2i/3 &&& \\ &&&&&& 2i/3 && \\ &&&&&&& -i & \\ &&&&&&&& i \end{pmatrix}R^{-1}. \tag{10.26}$$

All the sectors are nondecaying. The spectrum of the total generator $\gamma B + C$ is given in Table II, and decays are induced by the perturbation C in the nondecaying eigenspaces of B. For this model, the adiabatic generator K is obtained via the generalized

Schrieffer-Wolff transformation in the GKLS form (10.9) with

$$H = \frac{1}{3}(\gamma - \sqrt{\gamma^2 - 1})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\Gamma_1 = \Gamma_2 = \frac{1}{2}, \quad L_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\Gamma_\pm = \pm\frac{1}{3\sqrt{3}}(\gamma - \sqrt{\gamma^2 - 1}), \quad L_\pm = \begin{pmatrix} e^{\pm\pi i/3} & 0 & 0 \\ 0 & e^{\mp\pi i/3} & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{10.27}$$

This K is HP and TP, but not CP.

Now we try to find an adiabatic generator $\tilde{K}$ that is endowed with the same block structure as B, shares the same spectrum with $\gamma B + C$, and is physical (HP, TP, and CP), via an additional similarity transformation on $\gamma B + K$. Let us first impose HP, TP, and the block structure of B on $\gamma B + \tilde{K}$. Then a possible adiabatic generator $\gamma B + \tilde{K}$ is constrained to

$$\gamma B + \tilde{K} = R\begin{pmatrix} r_1 & r_2 & r_3 & & & & & & \\ r_4 & r_5 & r_6 & & & & & & \\ -r_1 - r_4 & -r_2 - r_5 & -r_3 - r_6 & & & & & & \\ & & & r_7 + ir_8 & & & & & \\ & & & & r_7 - ir_8 & & & & \\ & & & & & r_9 + ir_{10} & & & \\ & & & & & & r_9 - ir_{10} & & \\ & & & & & & & r_{11} + ir_{12} & \\ & & & & & & & & r_{11} - ir_{12} \end{pmatrix} R^{-1}, \tag{10.28}$$

parametrized by 12 real parameters $(r_1, \ldots, r_{12})$. By further requiring that $\gamma B + \tilde{K}$ should have the same spectrum as $\gamma B + C$ listed in Table II, we realize that the parameters should satisfy the conditions

$$r_7 = r_9 = -\frac{1}{2}, \quad r_{11} = -1, \tag{10.29}$$

$$r_8 = -\frac{1}{3}\gamma, \quad r_{10} = -\frac{2}{3}\gamma, \quad r_{12} = -\sqrt{\gamma^2 - 1}, \tag{10.30}$$

and

$$(r_1 + r_5) - (r_3 + r_6) = -2, \tag{10.31}$$

$$(r_1 - r_3)(r_5 - r_6) - (r_2 - r_3)(r_4 - r_6) = 0. \tag{10.32}$$

The last two constraints are for the top-left 3×3 block to admit the eigenvalues 0 and -2. In this way we are left with four free parameters. By tuning the remaining four parameters, we try to make $\gamma B + \tilde{K}$ physical. Since it is already required to be HP and TP, we try to achieve CP. In terms of the remaining parameters, the spectrum of the Kossakowski matrix of $\gamma B + \tilde{K}$ is given by

$$\left\{\frac{1}{2}r_2, \frac{1}{2}r_3, \frac{1}{2}r_4, \frac{1}{2}r_6, -\frac{1}{2}(r_1 + r_4), -\frac{1}{2}(r_2 + r_5), \pm\frac{1}{12}\sqrt{9(r_1 + r_5 + 1)^2 + 3(r_1 - r_5 + 1)^2 + 12(\gamma - \sqrt{\gamma^2 - 1})^2}\right\}. \tag{10.33}$$

All these eigenvalues should be nonnegative for $\gamma B + \tilde{K}$ to be CP. However, the last eigenvalue is strictly negative, and it is impossible to achieve the goal by tuning the parameters and to make $\gamma B + \tilde{K}$ physical.

This counterexample leads us to the following conclusion. If we wish to find an adiabatic generator endowed with the physical structure (HP, TP, and CP), we have to sacrifice some of the axioms listed in the Introduction. We emphasize again that the breakdown of the CP structure does not imply the failure of the approximation and working assumptions. The distance of the effective evolution from the true evolution is guaranteed to be $O(1/\gamma)$ for arbitrarily long times, and the violation of CP is small.

XI. CONCLUSIONS

We have developed a general perturbation theory based on an iterated adiabatic theorem for arbitrary finite-dimensional quantum systems. Special cases previously known are given by Zeno dynamics, adiabatic elimination, Bloch generators, des Cloizeaux generators, and by the Schrieffer-Wolff approach. Although we showed that an ideal effective generator cannot always be provided in open quantum systems, our generalization provides a good approach to highlight the eternal adiabatic resilience of quantum systems to perturbations. We were able to provide concise bounds for this. We note that many of our theorems can be generalized easily to bounded operators on infinite-dimensional Hilbert spaces, provided that appropriate bounds on the spectral gap appearing in the reduced resolvent are assumed.

In this work we focused on static systems, with time-independent generators and time-independent perturbations. From a quantum control perspective, the eternal adiabaticity would also have important applications in driven quantum systems. See for instance Refs. [64–66]. In the unitary case, the generalization of Bloch's perturbation theory to the time-dependent case is studied in Ref. [67], and it would be interesting to see how our current framework can be extended towards the study of driven systems.

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APPENDIX A: KEY FORMULA FOR THE ADIABATIC THEOREM

Here we show the derivation of the key formula (2.10) for the iterative application of the adiabatic theorem. Recall first $(B-b_\ell)S_\ell = 1-P_\ell$ in (2.9), satisfied by the reduced resolvent S_ℓ defined in (2.7). Note also that

$$e^{(t-s)(\gamma B+C)}(B-b_\ell) = -\frac{1}{\gamma}\left(\frac{\partial}{\partial s}(e^{(t-s)(\gamma B+C)}e^{s(\gamma b_\ell+C)})\right)e^{-s(\gamma b_\ell+C)}. \tag{A1}$$

Combining these relations we have

$$e^{(t-s)(\gamma B+C)}(1-P_\ell) = e^{(t-s)(\gamma B+C)}(B-b_\ell)S_\ell = -\frac{1}{\gamma}\left(\frac{\partial}{\partial s}(e^{(t-s)(\gamma B+C)}e^{s(\gamma b_\ell+C)})\right)e^{-s(\gamma b_\ell+C)}S_\ell. \tag{A2}$$

Then, for an arbitrary operator A, we get

$$\begin{aligned}
&\int_0^t ds\, e^{(t-s)(\gamma B+C)}AP_\ell e^{s(\gamma B+D_\ell)}\\
&= \int_0^t ds\, e^{(t-s)(\gamma B+C)}P_\ell AP_\ell e^{s(\gamma B+D_\ell)} + \int_0^t ds\, e^{(t-s)(\gamma B+C)}(1-P_\ell)AP_\ell e^{s(\gamma B+D_\ell)}\\
&= \int_0^t ds\, e^{(t-s)(\gamma B+C)}P_\ell AP_\ell e^{s(\gamma B+D_\ell)} - \frac{1}{\gamma}\int_0^t ds\left(\frac{\partial}{\partial s}(e^{(t-s)(\gamma B+C)}e^{s(\gamma b_\ell+C)})\right)e^{-s(\gamma b_\ell+C)}S_\ell AP_\ell e^{s(\gamma B+D_\ell)}\\
&= \int_0^t ds\, e^{(t-s)(\gamma B+C)}P_\ell AP_\ell e^{s(\gamma B+D_\ell)} - \frac{1}{\gamma}[e^{(t-s)(\gamma B+C)}S_\ell AP_\ell e^{s(\gamma B+D_\ell)}]_{s=0}^{s=t}\\
&\quad + \frac{1}{\gamma}\int_0^t ds\, e^{(t-s)(\gamma B+C)}e^{s(\gamma b_\ell+C)}\frac{\partial}{\partial s}(e^{-s(\gamma b_\ell+C)}S_\ell AP_\ell e^{s(\gamma B+D_\ell)})\\
&= \int_0^t ds\, e^{(t-s)(\gamma B+C)}P_\ell AP_\ell e^{s(\gamma B+D_\ell)} + \frac{1}{\gamma}e^{t(\gamma B+C)}S_\ell AP_\ell - \frac{1}{\gamma}S_\ell AP_\ell e^{t(\gamma B+D_\ell)} - \frac{1}{\gamma}\int_0^t ds\, e^{(t-s)(\gamma B+C)}\mathcal{K}_\ell(A)P_\ell e^{s(\gamma B+D_\ell)},
\end{aligned} \tag{A3}$$

where $\mathcal{K}_\ell$ is defined in (2.11). The key formula (2.10) is thus obtained.

APPENDIX B: BOUNDING THE LAST TERM OF (2.13)

We show that the last term of (2.13) decays as $n\to+\infty$. To show this, let us bound $A_\ell^{(n)}/\gamma^n = \mathcal{K}_\ell^n(C-D_\ell)/\gamma^n$, where $\mathcal{K}$ is defined in (2.11). Recall that there exists an integer $n_\ell \geqslant 1$ such that $N_\ell^{n_\ell}=0$. This limits the highest possible power of γ in the expansion of $\mathcal{K}_\ell^n$ to $n-\lfloor n/n_\ell\rfloor$, where $\lfloor x\rfloor$ is the largest integer less than or equal to x. This is because, in the expansion of $\mathcal{K}_\ell^n$, the nilpotent N_ℓ can repeat only $n_\ell-1$ times sequentially and D_ℓ should interrupt the sequence. The highest-order terms look like $\gamma^{n-\lfloor n/n_\ell\rfloor}S_\ell^n \bullet N_\ell^p D_\ell (N_\ell^{n_\ell-1}D_\ell)^{\lfloor n/n_\ell\rfloor-1}N_\ell^q$ with integers p and q satisfying $p,q\leqslant n_\ell-1$ and $p+q=n-(\lfloor n/n_\ell\rfloor-1)n_\ell-1$.

Therefore, $A_\ell^{(n)}$ is bounded by

$$\|A_\ell^{(n)}\| \leqslant \sum_{r=0}^{n-\lfloor n/n_\ell\rfloor} \binom{n}{r} (\|C\|\|S_\ell\| + \|S_\ell\|\|D_\ell\|)^{n-r} (\gamma\|S_\ell\|\|N_\ell\|)^r \|C - D_\ell\|. \tag{B1}$$

It is a rough bound since it is overcounting also vanishing terms containing N_ℓ^m with $m > n_\ell - 1$, but this suffices for our purpose. For $\gamma > 1$, it is further bounded by

$$\leqslant \gamma^{n-\lfloor n/n_\ell\rfloor} \|S_\ell\|^n \sum_{r=0}^{n-\lfloor n/n_\ell\rfloor} \binom{n}{r} (\|C\| + \|D_\ell\|)^{n-r} \|N_\ell\|^r \|C - D_\ell\| \leqslant \gamma^{n-\lfloor n/n_\ell\rfloor} [\|S_\ell\|(\|C\| + \|D_\ell\| + \|N_\ell\|)]^n \|C - D_\ell\|. \tag{B2}$$

Since $(n+1)/n_\ell - 1 \leqslant \lfloor n/n_\ell\rfloor \leqslant n/n_\ell$,

$$\leqslant \gamma^{n-(n+1)/n_\ell+1} [\|S_\ell\|(\|C\| + \|D_\ell\| + \|N_\ell\|)]^n \|C - D_\ell\| = \gamma^{n-1/n_\ell+1} \left(\frac{[\|S_\ell\|(\|C\| + \|D_\ell\| + \|N_\ell\|)]^{n_\ell}}{\gamma}\right)^{n/n_\ell} \|C - D_\ell\|. \tag{B3}$$

Therefore, $\|A_\ell^{(n)}\|/\gamma^n \to 0$ as $n \to +\infty$, provided $\gamma > \max\{1, [\|S_\ell\|(\|C\| + \|D_\ell\| + \|N_\ell\|)]^{n_\ell}\}$.

APPENDIX C: LINK WITH BLOCH'S PERTURBATION THEORY

We want to translate our adiabatic Bloch equation (3.2) with (3.3) for Ω_ℓ into the equation for the similarity transformation U_ℓ defined in (5.3). This will show that our theory is equivalent to Bloch's perturbation theory in the unitary case [38] and generalizes it to the nonunitary case.

Let us first try to invert the relation (5.3) between U_ℓ and Ω_ℓ, i.e.,

$$U_\ell = P_\ell - \frac{1}{\gamma} S_\ell \Omega_\ell. \tag{C1}$$

It yields $S_\ell\Omega_\ell/\gamma = P_\ell - U_\ell$. We use it to replace Ω_ℓ with U_ℓ in our adiabatic Bloch equation (3.2),

$$\begin{aligned}\Omega_\ell &= \frac{1}{\gamma} S_\ell \Omega_\ell^2 - \frac{1}{\gamma} C S_\ell \Omega_\ell + S_\ell \Omega_\ell N_\ell + C P_\ell \\ &= (P_\ell - U_\ell)\Omega_\ell - C(P_\ell - U_\ell) + \gamma(P_\ell - U_\ell)N_\ell + CP_\ell \\ &= CU_\ell + (P_\ell - U_\ell)(\Omega_\ell + \gamma N_\ell) \\ &= CU_\ell - (1 - P_\ell)U_\ell(\Omega_\ell + \gamma N_\ell),\end{aligned} \tag{C2}$$

where we have used $P_\ell U_\ell = P_\ell$ from (5.6). This implies

$$P_\ell \Omega_\ell = P_\ell C U_\ell. \tag{C3}$$

Therefore, by inserting it back into the right-hand side of (C2) and by noting $U_\ell P_\ell = U_\ell$ from (5.6), we get

$$\Omega_\ell = CU_\ell - (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell). \tag{C4}$$

This is the inversion of the relation (C1).

By inserting this expression into the right-hand side of the relation (C1), we obtain the equation for U_ℓ as

$$U_\ell = P_\ell - \frac{1}{\gamma} S_\ell (CU_\ell - U_\ell C U_\ell) + S_\ell U_\ell N_\ell, \tag{C5}$$

with

$$U_\ell P_\ell = U_\ell. \tag{C6}$$

These equations are presented in (5.11) and (5.12) of the main text. Note that Eq. (C5) automatically reproduces one of the two properties of U_ℓ in (5.6), $P_\ell U_\ell = P_\ell$, while the other one $U_\ell P_\ell = U_\ell$ is independent of (C5). We need (C6) in addition to Eq. (C5) to characterize U_ℓ.

When B and C are Hamiltonians (multiplied by $-i$), there is no nilpotent N_ℓ in B, and Eq. (C5) for U_ℓ is nothing but the well-known Bloch equation [38]. Our Eq. (C5) generalizes Bloch's equation to the case where B and C are not skew-Hermitian and B might be even nondiagonalizable. In particular, our formalism can describe noisy quantum dynamics.

Let us check the validity of the results just obtained. First, we assume that Ω_ℓ satisfies our adiabatic Bloch equation (3.2) with (3.3) and show that U_ℓ introduced through the relation (C1) solves the generalized Bloch equation (C5). Before starting to show it, note that our adiabatic Bloch equation (3.2) multiplied by P_ℓ from the left yields

$$-P_\ell\left(1 + \frac{1}{\gamma} C S_\ell\right)\Omega_\ell + P_\ell C P_\ell = 0. \tag{C7}$$

Now, by inserting the relation (C1) for U_ℓ,

$$\begin{aligned}&U_\ell - P_\ell + \frac{1}{\gamma} S_\ell (CU_\ell - U_\ell C U_\ell) - S_\ell U_\ell N_\ell \\ &= \left(P_\ell - \frac{1}{\gamma} S_\ell\Omega_\ell\right) - P_\ell + \frac{1}{\gamma} S_\ell \left[C\left(P_\ell - \frac{1}{\gamma} S_\ell\Omega_\ell\right) - \left(P_\ell - \frac{1}{\gamma} S_\ell\Omega_\ell\right)C\left(P_\ell - \frac{1}{\gamma} S_\ell\Omega_\ell\right)\right] - S_\ell\left(P_\ell - \frac{1}{\gamma} S_\ell\Omega_\ell\right)N_\ell \\ &= -\frac{1}{\gamma} S_\ell \left[\Omega_\ell - C\left(P_\ell - \frac{1}{\gamma} S_\ell\Omega_\ell\right) - \frac{1}{\gamma} S_\ell\Omega_\ell\left(P_\ell C P_\ell - \frac{1}{\gamma} P_\ell C S_\ell \Omega_\ell\right) - S_\ell \Omega_\ell N_\ell\right]\end{aligned}$$

$$= -\frac{1}{\gamma}S_\ell\left[\Omega_\ell - C\left(P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell\right) - \frac{1}{\gamma}S_\ell\Omega_\ell P_\ell\Omega_\ell - S_\ell\Omega_\ell N_\ell\right]$$

$$= \frac{1}{\gamma}S_\ell\left[\frac{1}{\gamma}S_\ell\Omega_\ell^2 - \left(1 + \frac{1}{\gamma}CS_\ell\right)\Omega_\ell + CP_\ell + S_\ell\Omega_\ell N_\ell\right] = 0. \tag{C8}$$

We have used $S_\ell P_\ell = 0$ and $\Omega_\ell = \Omega_\ell P_\ell$ from (3.3) for the second equality, used (C7) to get the third equality, and used our adiabatic Bloch equation (3.2) for the last equality. This proves that the generalized Bloch equation (C5) is satisfied. Equation (C6) also follows from the definition of U_ℓ in (C1) and $\Omega_\ell P_\ell = \Omega_\ell$ from (3.3).

The converse is also true. We now assume that U_ℓ satisfies the generalized Bloch equation (C5) with (C6) and show that Ω_ℓ introduced through the relation (C4) solves our Bloch equation (3.2). By inserting the relation (C4) for Ω_ℓ,

$$\frac{1}{\gamma}S_\ell\Omega_\ell^2 - \left(1 + \frac{1}{\gamma}CS_\ell\right)\Omega_\ell + CP_\ell + S_\ell\Omega_\ell N_\ell$$

$$= \frac{1}{\gamma}S_\ell[CU_\ell - (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell)]^2 - \left(1 + \frac{1}{\gamma}CS_\ell\right)[CU_\ell - (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell)]$$

$$+ CP_\ell + S_\ell[CU_\ell - (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell)]N_\ell$$

$$= \frac{1}{\gamma}S_\ell[CU_\ell - U_\ell(CU_\ell + \gamma N_\ell)]CU_\ell - CU_\ell + (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell) - C\frac{1}{\gamma}S_\ell[CU_\ell - U_\ell(CU_\ell + \gamma N_\ell)]$$

$$+ CP_\ell + \frac{1}{\gamma}S_\ell[CU_\ell - U_\ell(CU_\ell + \gamma N_\ell)]\gamma N_\ell$$

$$= (P_\ell - U_\ell)CU_\ell - CU_\ell + (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell) - C(P_\ell - U_\ell) + CP_\ell + \gamma(P_\ell - U_\ell)N_\ell = 0. \tag{C9}$$

We have used $S_\ell(1 - P_\ell) = S_\ell$ and $U_\ell(1 - P_\ell) = 0$ from (C6) for the second equality, used the generalized Bloch equation (C5) to get the third equality, and used $P_\ell U_\ell = P_\ell$, which follows from the generalized Bloch equation (C5), for the last equality. This proves that our adiabatic Bloch equation (3.2) is satisfied. Equation (3.3) also follows from the relation (C4) and $U_\ell P_\ell = U_\ell$ from (C6).

Finally, let us also check that (C1) and (C4) are indeed the inverses of each other, provided that both Bloch equations (3.2) with (3.3) and (C5) with (C6) hold: by inserting (C4) for Ω_ℓ into the right-hand side of (C1) we immediately get

$$P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell = P_\ell - \frac{1}{\gamma}S_\ell[CU_\ell - (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell)] = U_\ell, \tag{C10}$$

thanks to the generalized Bloch equation (C5), while by inserting (C1) for U_ℓ into the right-hand side of (C4) we get

$$\begin{aligned} CU_\ell - (1 - P_\ell)U_\ell(CU_\ell + \gamma N_\ell) &= C\left(P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell\right) - (1 - P_\ell)\left(P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell\right)\left[C\left(P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell\right) + \gamma N_\ell\right] \\ &= C\left(P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell\right) + \frac{1}{\gamma}S_\ell\Omega_\ell\left[P_\ell C\left(P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell\right) + \gamma N_\ell\right] \\ &= C\left(P_\ell - \frac{1}{\gamma}S_\ell\Omega_\ell\right) + \frac{1}{\gamma}S_\ell\Omega_\ell(P_\ell\Omega_\ell + \gamma N_\ell) \\ &= \frac{1}{\gamma}S_\ell\Omega_\ell^2 - \frac{1}{\gamma}CS_\ell\Omega_\ell + CP_\ell + S_\ell\Omega_\ell N_\ell \\ &= \Omega_\ell, \end{aligned} \tag{C11}$$

where we have used (C7), which follows from our Bloch equation (3.2). Everything is thus consistent.

APPENDIX D: SOLVABILITY OF THE ADIABATIC BLOCH EQUATIONS

For a given ℓ, the adiabatic Bloch equations (3.2) and (5.11) for Ω_ℓ and U_ℓ, respectively, are quadratic matrix equations. Lancaster and Rokne [63] studied the existence and the uniqueness problem of a similar quadratic equation using the Newton-Kantorovich theorem [43]. We can follow similar proofs for the adiabatic Bloch equations (3.2) and (5.11) using Ref. [43] directly. It shows the existence of a solution constructively by a converging Newton iteration finding a solution of the equation. Let us show here the solvability of the adiabatic Bloch equation (5.11) for the wave operator U_ℓ.

We can also analyze the other adiabatic Bloch equation (3.2) for Ω_ℓ in the same way. Strictly speaking the adiabatic Bloch equation is a set of coupled equations (5.11) and (5.12). We will see that the Newton iteration preserves the latter condition (5.12), so we can solve both equations simultaneously.

The adiabatic Bloch equation (5.11) for the wave operator U_ℓ is a quadratic matrix equation in $X = U_\ell$ of the form

$$\mathcal{F}(X) = X - S_\ell X N_\ell + \frac{1}{\gamma} S_\ell (CX - XCX) - P_\ell = 0. \tag{D1}$$

The (Fréchet) derivative of $\mathcal{F}(X)$ reads

$$\mathcal{F}'_X(A) = A - S_\ell A N_\ell + \frac{1}{\gamma} S_\ell (CA - XCA - ACX). \tag{D2}$$

The derivative $\mathcal{F}'_X$ is invertible for large γ,

$$(\mathcal{F}'_X)^{-1} = \left(\mathcal{I} + \frac{1}{\gamma}\mathcal{G}_X\right)^{-1} = \mathcal{I}^{-1}\left(1 + \frac{1}{\gamma}\mathcal{G}_X \mathcal{I}^{-1}\right)^{-1}, \tag{D3}$$

where

$$\mathcal{I}(A) = A - S_\ell A N_\ell, \quad \mathcal{I}^{-1}(A) = \sum_{n=0}^{n_\ell - 1} S_\ell^n A N_\ell^n, \tag{D4}$$

$$\mathcal{G}_X(A) = S_\ell (CA - XCA - ACX). \tag{D5}$$

The Newton iteration is then given by

$$X_{k+1} = X_k - (\mathcal{F}'_{X_k})^{-1}(\mathcal{F}(X_k)). \tag{D6}$$

It is reasonable to choose the zeroth-order solution of the perturbative equation as an initial guess. With

$$X_0 = U_\ell^{(0)} = \mathcal{I}^{-1}(P_\ell) = P_\ell, \tag{D7}$$

we have

$$\mathcal{F}(X_0) = \frac{1}{\gamma} S_\ell C P_\ell \tag{D8}$$

and

$$\mathcal{G}_{X_0}(A) = S_\ell (CA - ACP_\ell). \tag{D9}$$

Explicit bounds are readily obtained from geometric series:

$$\|\mathcal{I}^{-1}\| \leqslant \sum_{n=0}^{n_\ell - 1} (\|S_\ell\| \|N_\ell\|)^n = \frac{1 - (\|S_\ell\| \|N_\ell\|)^{n_\ell}}{1 - \|S_\ell\| \|N_\ell\|} \equiv \mu_\ell, \tag{D10}$$

$$\|\mathcal{F}(X_0)\| \leqslant \frac{1}{\gamma} \|S_\ell\| \|C\| \|P_\ell\|, \tag{D11}$$

$$\|\mathcal{G}_{X_0}\| \leqslant 2 \|S_\ell\| \|C\| \|P_\ell\|, \tag{D12}$$

where we have used $\|P_\ell\| \geqslant 1$. Therefore,

$$\|(\mathcal{F}'_{X_0})^{-1}\| \leqslant \frac{\|\mathcal{I}^{-1}\|}{1 - \frac{1}{\gamma}\|\mathcal{G}_{X_0}\| \|\mathcal{I}^{-1}\|} \leqslant \frac{\mu_\ell}{1 - \frac{2}{\gamma}\mu_\ell \|S_\ell\| \|C\| \|P_\ell\|} \equiv \beta_\ell, \tag{D13}$$

$$\|(\mathcal{F}'_{X_0})^{-1}(\mathcal{F}(X_0))\| \leqslant \frac{1}{\gamma} \frac{\mu_\ell \|S_\ell\| \|C\| \|P_\ell\|}{1 - \frac{2}{\gamma}\mu_\ell \|S_\ell\| \|C\| \|P_\ell\|} \equiv \nu_\ell. \tag{D14}$$

Moreover, since

$$\mathcal{F}'_X(A) - \mathcal{F}'_Y(A) = -\frac{1}{\gamma} S_\ell [(X - Y)CA + AC(X - Y)], \tag{D15}$$

we have

$$\|\mathcal{F}'_X - \mathcal{F}'_Y\| \leqslant \frac{2}{\gamma} \|S_\ell\| \|C\| \|X - Y\| \leqslant L_\ell \|X - Y\|, \tag{D16}$$

with

$$L_\ell = \frac{2}{\gamma} \|S_\ell\| \|C\| \|P_\ell\|. \tag{D17}$$

According to Ref. [43], if

$$h_\ell = \beta_\ell L_\ell \nu_\ell \leqslant \frac{1}{2}, \tag{D18}$$

there is a solution of $\mathcal{F}(X) = 0$ within

$$\|X - X_0\| \leqslant \Theta_\ell = \frac{1 - \sqrt{1 - 2h_\ell}}{\beta_\ell L_\ell}. \tag{D19}$$

Moreover, there is at most one solution within

$$\|X - X_0\| < \Xi_\ell = \frac{1 + \sqrt{1 - 2h_\ell}}{\beta_\ell L_\ell}. \tag{D20}$$

Finally, the convergence is at least quadratic if $h_\ell < 1/2$.

In the present case,

$$h_\ell = \beta_\ell L_\ell \nu_\ell = \frac{1}{\gamma^2} \frac{2\mu_\ell^2 \|S_\ell\|^2 \|C\|^2 \|P_\ell\|^2}{\left(1 - \frac{2}{\gamma}\mu_\ell \|S_\ell\| \|C\| \|P_\ell\|\right)^2} \tag{D21}$$

and

$$\Theta_\ell = \frac{1 - \sqrt{1 - \gamma_\ell/\gamma}}{1 + \sqrt{1 - \gamma_\ell/\gamma}} = \Xi_\ell^{-1}, \tag{D22}$$

with

$$\gamma_\ell = 4\mu_\ell \|S_\ell\| C\| \|P_\ell\|. \tag{D23}$$

The condition $h_\ell \leqslant 1/2$ for the solvability of the Bloch equation (5.11) requires

$$\gamma \geqslant \gamma_\ell. \tag{D24}$$

Under this condition, a solution U_ℓ exists within

$$\|U_\ell - P_\ell\| \leqslant \Theta_\ell = O(1/\gamma), \tag{D25}$$

and there is at most one solution within

$$\|U_\ell - P_\ell\| < \Xi_\ell = O(\gamma). \tag{D26}$$

We note that $X_0 = X_0 P_\ell$. Furthermore, since $\mathcal{F}$ contains right multiplication with only N_ℓ, it preserves $X = XP_\ell$, i.e., $\mathcal{F}(X) = \mathcal{F}(X)P_\ell$. The same holds for $F'_X(X)$ because it only contains right multiplication by N_ℓ and CX, i.e., $\mathcal{F}'_X(X) = \mathcal{F}'(X)P_\ell$. Therefore, the Newton iteration (D6) preserves this property, and the limit X_∞ fulfills both $\mathcal{F}(X_\infty) = 0$ and $X_\infty = X_\infty P_\ell$. The solution $U_\ell = X_\infty$ obtained by the Newton iteration satisfies (5.12). In addition, the small distance $O(1/\gamma)$ from the initial guess $X_0 = P_\ell$ justifies the perturbative approach taken in Sec. IV.

Finally, the bound on U_ℓ in (D25) allows us to estimate the size of the adiabatic generator D_ℓ. Recalling that $D_\ell = P_\ell C U_\ell$, its norm is bounded by

$$\|D_\ell\| = \|P_\ell C U_\ell\| \leqslant \|P_\ell\|\|C\|(1+\|U_\ell - P_\ell\|)\,\|P_\ell\| \leqslant \frac{2\|C\|\|P_\ell\|^2}{1+\sqrt{1-\gamma_\ell/\gamma}}. \tag{D27}$$

APPENDIX E: ETERNAL BOUNDS

We can also work on the conjugate Bloch equation (7.12) for $\tilde{U}_\ell$, and get

$$\|\tilde{U}_\ell - P_\ell\| \leqslant \Theta_\ell, \tag{E1}$$

with the same Θ_ℓ given in (D22). This and the bound on U_ℓ in (D25) allow us to explicitly bound the norm distance between the approximate adiabatic evolution $e^{t(\gamma B+K)}$ and the true evolution $e^{t(\gamma B+C)}$ eternally.

The similarity between the generators $\gamma B + C$ and $\gamma B + K$ in (8.21) implies the similarity between the evolutions $e^{t(\gamma B+C)}$ and $e^{t(\gamma B+K)}$. The difference between the two evolutions is then estimated to be

$$\begin{aligned} e^{t(\gamma B+C)} - e^{t(\gamma B+K)} &= e^{t(\gamma B+C)} - W^{-1}e^{t(\gamma B+C)}W \\ &= -e^{t(\gamma B+C)}(W-1) \\ &\quad -(W^{-1}-1)e^{t(\gamma B+C)}W \\ &= -\sum_\ell e^{t(\gamma B+C)}(W_\ell - P_\ell) \\ &\quad +\sum_\ell (W_\ell - P_\ell)W_\ell^{-1}e^{t(\gamma B+C)}W_\ell. \end{aligned} \tag{E2}$$

Note the intertwining relations

$$W_\ell = W_\ell P_\ell = \tilde{P}_\ell W_\ell, \tag{E3}$$

$$W_\ell^{-1} = P_\ell W_\ell^{-1} = W_\ell^{-1}\tilde{P}_\ell \tag{E4}$$

in (8.8) and (8.12). Recall here the definitions of W_ℓ and W_ℓ^{-1} in (8.7) and (8.11), and the pseudoinverse $(\tilde{U}_\ell U_\ell)^{-1}$ in (8.2). Since

$$U_\ell = U_\ell P_\ell, \quad P_\ell U_\ell = P_\ell, \tag{E5}$$

$$\tilde{U}_\ell = P_\ell \tilde{U}_\ell, \quad \tilde{U}_\ell P_\ell = P_\ell, \tag{E6}$$

as noted in (5.6) and (7.11), we have

$$W_\ell = [1+(U_\ell - P_\ell)][1+(\tilde{U}_\ell - P_\ell)(U_\ell - P_\ell)]^{-1/2}P_\ell, \tag{E7}$$

$$W_\ell^{-1} = P_\ell[1+(\tilde{U}_\ell - P_\ell)(U_\ell - P_\ell)]^{-1/2}[1+(\tilde{U}_\ell - P_\ell)], \tag{E8}$$

and

$$W_\ell - P_\ell = [1+(U_\ell - P_\ell)][1+(\tilde{U}_\ell - P_\ell)(U_\ell - P_\ell)]^{-1/2} - 1, \tag{E9}$$

$$W_\ell^{-1} - P_\ell = [1+(\tilde{U}_\ell - P_\ell)(U_\ell - P_\ell)]^{-1/2}[1+(\tilde{U}_\ell - P_\ell)] - 1. \tag{E10}$$

These are bounded by

$$\|W_\ell\|,\, \|W_\ell^{-1}\| \leqslant \frac{1+\Theta_\ell}{\sqrt{1-\Theta_\ell^2}}\|P_\ell\|, \tag{E11}$$

$$\|W_\ell - P_\ell\|,\, \|W_\ell^{-1} - P_\ell\| \leqslant \frac{1+\Theta_\ell}{\sqrt{1-\Theta_\ell^2}} - 1, \tag{E12}$$

using the bounds $\|U_\ell - P_\ell\| \leqslant \Theta_\ell$ and $\|\tilde{U}_\ell - P_\ell\| \leqslant \Theta_\ell$ in (D25) and (E1). We hence get

$$\begin{aligned} \|e^{t(\gamma B+C)} - e^{t(\gamma B+K)}\| &\leqslant \sum_\ell \big(\|W_\ell - P_\ell\| + \|(W_\ell - P_\ell)W_\ell^{-1}\|\|W_\ell\|\big)\|e^{t(\gamma B+C)}\| \\ &\leqslant \sum_\ell \frac{2}{1-\Theta_\ell}\left(\sqrt{\frac{1+\Theta_\ell}{1-\Theta_\ell}} - 1\right)\|P_\ell\|\|e^{t(\gamma B+C)}\| \\ &= \sum_\ell \left(\frac{1}{\sqrt{1-\gamma_\ell/\gamma}} + 1\right)\left(\frac{1}{\sqrt[4]{1-\gamma_\ell/\gamma}} - 1\right)\|P_\ell\|\|e^{t(\gamma B+C)}\|, \end{aligned} \tag{E13}$$

where

$$\gamma_\ell = 4\|S_\ell\|\|C\|\|P_\ell\|\frac{1-(\|S_\ell\|\|N_\ell\|)^{n_\ell}}{1-\|S_\ell\|\|N_\ell\|}. \tag{E14}$$

This can be loosely bounded as in (8.23) for $\gamma \geqslant 2\max_\ell \gamma_\ell$, in the norm induced by the operator trace norm.

The distance between $e^{t(\gamma B+C)}$ and $e^{t(\gamma B+D)}$, which are similar to each other through U, can be bounded in a similar way. Note the intertwining relations

$$U_\ell = U_\ell P_\ell = \tilde{P}_\ell U_\ell, \tag{E15}$$

$$U_\ell^{-1} = P_\ell U_\ell^{-1} = U_\ell^{-1}\tilde{P}_\ell, \tag{E16}$$

where

$$U_\ell^{-1} = (\tilde{U}_\ell U_\ell)^{-1}\tilde{U}_\ell \tag{E17}$$

is a pseudoinverse satisfying

$$U_\ell^{-1}U_\ell = P_\ell, \quad U_\ell U_\ell^{-1} = \tilde{P}_\ell. \tag{E18}$$

It is bounded by

$$\|U_\ell^{-1}\| \leqslant \frac{1+\Theta_\ell}{1-\Theta_\ell^2}. \tag{E19}$$

Then the difference

$$e^{t(\gamma B+C)}-e^{t(\gamma B+D)}=-\sum_{\ell}e^{t(\gamma B+C)}(U_\ell-P_\ell)+\sum_{\ell}(U_\ell-P_\ell)U_\ell^{-1}e^{t(\gamma B+D)}U_\ell \tag{E20}$$

is bounded by

$$\begin{aligned}\|e^{t(\gamma B+C)}-e^{t(\gamma B+D)}\|&\leqslant\sum_{\ell}\big(\|U_\ell-P_\ell\|+\|(U_\ell-P_\ell)U_\ell^{-1}\|\|U_\ell\|\big)\|e^{t(\gamma B+C)}\|\\&\leqslant\sum_{\ell}\frac{2\Theta_\ell}{1-\Theta_\ell}\|P_\ell\|\|e^{t(\gamma B+C)}\|\\&=\sum_{\ell}\left(\frac{1}{\sqrt{1-\gamma_\ell/\gamma}}-1\right)\|P_\ell\|\|e^{t(\gamma B+C)}\|.\end{aligned} \tag{E21}$$

This bound is smaller than the bound on the distance $\|e^{t(\gamma B+C)}-e^{t(\gamma B+K)}\|$ in (E13). Since $1/\sqrt{1-x}-1<x$ for $0<x\leqslant 1/2$, this can be loosely bounded as in (6.6) for $\gamma\geqslant 2\max_\ell\gamma_\ell$, in the 1-1 norm induced by the operator trace norm.

Moreover, in the unitary case, by using the spectral norm, so that $\|A\|=\|A^\dagger A\|^{1/2}=\|AA^\dagger\|^{1/2}$, tighter bounds are available. For instance, by using the unitarity of W and $e^{t(\gamma B+C)}$, whose norms are $\|W\|=\|e^{t(\gamma B+C)}\|=1$, and the orthogonality $(W_k-P_k)(W_\ell-P_\ell)^\dagger=0$ for $k\neq\ell$, we can bound the distance as

$$\begin{aligned}\|e^{t(\gamma B+C)}-e^{t(\gamma B+K)}\|&=\|-e^{t(\gamma B+C)}(W-I)+(W-I)W^{-1}e^{t(\gamma B+C)}W\|\\&\leqslant 2\|W-I\|\\&=2\left\|\sum_{\ell}(W_\ell-P_\ell)\right\|\\&=2\left\|\sum_{k}(W_k-P_k)\sum_{\ell}(W_\ell-P_\ell)^\dagger\right\|^{1/2}\\&=2\left\|\sum_{\ell}(W_\ell-P_\ell)(W_\ell-P_\ell)^\dagger\right\|^{1/2}\\&\leqslant 2\left(\sum_{\ell}\|W_\ell-P_\ell\|^2\right)^{1/2}\\&\leqslant 2\sqrt{\sum_{\ell}\left(\sqrt{\frac{1+\Theta_\ell}{1-\Theta_\ell}}-1\right)^2}\\&\leqslant 2\sqrt{d}\max_{\ell}\left(\sqrt{\frac{1+\Theta_\ell}{1-\Theta_\ell}}-1\right)\\&=2\sqrt{d}\left(\frac{1}{\sqrt[4]{1-4\|C\|/(\gamma\eta)}}-1\right),\end{aligned} \tag{E22}$$

where d is the number of distinct eigenvalues of B, and

$$\eta=\min_{k\neq\ell}|b_k-b_\ell| \tag{E23}$$

is the spectral gap of B. Note that $\mu_\ell=1$, $\|P_\ell\|=1$, and hence $\gamma_\ell=4\|S_\ell\|\|C\|\leqslant 4\|C\|/\eta$ in the unitary case.

For the distance between $e^{t(\gamma B+C)}$ and $e^{t(\gamma B+D)}$, the similarity transformation U between them is not unitary even for unitary evolution, but anyway, we can bound it as

$$\begin{aligned}\|e^{t(\gamma B+C)}-e^{t(\gamma B+D)}\|&=\|-e^{t(\gamma B+C)}(U-I)+(U-I)U^{-1}e^{t(\gamma B+C)}U\|\\&\leqslant\|U-I\|+\|(U-I)U^{-1}e^{t(\gamma B+C)}U\|\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_\ell (U_\ell - P_\ell) \right\| + \left\| \sum_\ell (U_\ell - P_\ell) U_\ell^{-1} e^{t(\gamma B + C)} U_\ell \right\| \\
&\leqslant \left(\sum_\ell \|U_\ell - P_\ell\|^2 \right)^{1/2} + \left(\sum_\ell \left\| (U_\ell - P_\ell) U_\ell^{-1} e^{t(\gamma B + C)} U_\ell \right\|^2 \right)^{1/2} \\
&\leqslant \sqrt{\sum_\ell \Theta_\ell^2} + \sqrt{\sum_\ell \left(\Theta_\ell \frac{1+\Theta_\ell}{1-\Theta_\ell} \right)^2} \\
&\leqslant \sqrt{d} \max_\ell \left(\frac{2\Theta_\ell}{1-\Theta_\ell} \right) \\
&= \sqrt{d} \left(\frac{1}{\sqrt{1 - 4\|C\|/(\gamma\eta)}} - 1 \right),
\end{aligned} \tag{E24}
$$

where we have used the orthogonality $U_k U_\ell^\dagger = 0$ for $k \neq \ell$. This bound is larger than the bound on the distance $\|e^{t(\gamma B + C)} - e^{t(\gamma B + K)}\|$ in (E22).

[1] A. Messiah, *Quantum Mechanics* (Dover, New York, 2017).

[2] T. Kato, On the adiabatic theorem of quantum mechanics, J. Phys. Soc. Jpn. **5**, 435 (1950).

[3] P. Facchi, Quantum Zeno effect, adiabaticity and dynamical superselection rules, in *Fundamental Aspects of Quantum Physics*, edited by L. Accardi and S. Tasaki, Vol. 17 of QP-PQ: Quantum Probability and White Noise Analysis (World Scientific, Singapore, 2003), pp. 197–221.

[4] P. Facchi and S. Pascazio, Quantum Zeno Subspaces, Phys. Rev. Lett. **89**, 080401 (2002).

[5] P. Facchi, S. Tasaki, S. Pascazio, H. Nakazato, A. Tokuse, and D. A. Lidar, Control of decoherence: Analysis and comparison of three different strategies, Phys. Rev. A **71**, 022302 (2005).

[6] P. Facchi and S. Pascazio, Quantum Zeno dynamics: Mathematical and physical aspects, J. Phys. A: Math. Theor. **41**, 493001 (2008).

[7] P. Facchi and M. Ligabò, Quantum Zeno effect and dynamics, J. Math. Phys. **51**, 022103 (2010).

[8] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions: Basic Process and Appilcations* (Wiley, Weinheim, 1998).

[9] E. Brion, L. H. Pedersen, and K. Mølmer, Adiabatic elimination in a lambda system, J. Phys. A: Math. Theor. **40**, 1033 (2007).

[10] J. E. Avron, M. Fraas, and G. M. Graf, Adiabatic Response for Lindblad Dynamics, J. Stat. Phys. **148**, 800 (2012).

[11] M. V. Berry, Quantum phase corrections from adiabatic iteration, Proc. R. Soc. London Ser. A **414**, 31 (1987).

[12] M. V. Berry and R. Lim, Universal transition prefactors derived by superadiabatic renormalization, J. Phys. A: Math. Gen. **26**, 4737 (1993).

[13] M. V. Berry, Transitionless quantum driving, J. Phys. A: Math. Theor. **42**, 365303 (2009).

[14] S. Alipour, A. Chenu, A. T. Rezakhani, and A. del Campo, Shortcuts to adiabaticity in driven open quantum systems: Balanced gain and loss and non-Markovian evolution, Quantum **4**, 336 (2020).

[15] J. Schmid, Adiabatic theorems for general linear operators with time-independent domains, Rev. Math. Phys. **31**, 1950014 (2019).

[16] S. Pascazio, On noise-induced superselection rules, J. Mod. Opt. **51**, 925 (2004).

[17] P. Zanardi and L. Campos Venuti, Coherent Quantum Dynamics in Steady-State Manifolds of Strongly Dissipative Systems, Phys. Rev. Lett. **113**, 240406 (2014).

[18] P. Zanardi and L. Campos Venuti, Geometry, robustness, and emerging unitarity in dissipation-projected dynamics, Phys. Rev. A **91**, 052324 (2015).

[19] V. V. Albert, B. Bradlyn, M. Fraas, and L. Jiang, Geometry and Response of Lindbladians, Phys. Rev. X **6**, 041031 (2016).

[20] J. Marshall, L. Campos Venuti, and P. Zanardi, Noise suppression via generalized-Markovian processes, Phys. Rev. A **96**, 052113 (2017).

[21] D. Burgarth, P. Facchi, H. Nakazato, S. Pascazio, and K. Yuasa, Generalized adiabatic theorem and strong-coupling limits, Quantum **3**, 152 (2019).

[22] D. Burgarth, P. Facchi, H. Nakazato, S. Pascazio, and K. Yuasa, Quantum Zeno dynamics from general quantum operations, Quantum **4**, 289 (2020).

[23] N. Barankai and Z. Zimborás, Generalized quantum Zeno dynamics and ergodic means, arXiv:1811.02509.

[24] T. Möbus and M. M. Wolf, Quantum Zeno effect generalized, J. Math. Phys. **60**, 052201 (2019).

[25] S. Becker, N. Datta, and R. Salzmann, Quantum Zeno effect for open quantum systems, arXiv:2010.04121.

[26] J. I. Cirac, R. Blatt, P. Zoller, and W. D. Phillips, Laser cooling of trapped ions in a standing wave, Phys. Rev. A **46**, 2668 (1992).

[27] D. J. Atkins, H. M. Wiseman, and P. Warszawski, Approximate master equations for atom optics, Phys. Rev. A **67**, 023802 (2003).

[28] M. Mirrahimi and P. Rouchon, Singular perturbations and Lindblad-Kossakowski differential equations, IEEE Trans. Autom. Control **54**, 1325 (2009).

[29] F. Reiter and A. S. Sørensen, Effective operator formalism for open quantum systems, Phys. Rev. A **85**, 032111 (2012).

[30] D. Finkelstein-Shapiro, D. Viennot, I. Saideh, T. Hansen, T. Pullerits, and A. Keller, Adiabatic elimination and subspace

evolution of open quantum systems, Phys. Rev. A **101**, 042102 (2020).

[31] R. Azouit, A. Sarlette, and P. Rouchon, Adiabatic elimination for open quantum systems with effective Lindblad master equations, in *Proceedings of the 2016 IEEE 55th Conference on Decision and Control (CDC)* (IEEE, New York, 2016), pp. 4559–4565.

[32] R. Azouit, F. Chittaro, A. Sarlette, and P. Rouchon, Towards generic adiabatic elimination for bipartite open quantum systems, Quant. Sci. Tech. **2**, 044011 (2017).

[33] P. Forni, A. Sarlette, T. Capelle, E. Flurin, S. Deléglise, and P. Rouchon, Adiabatic elimination for multi-partite open quantum systems with non-trivial zero-order dynamics, in *Proceedings of the 2018 IEEE Conference on Decision and Control (CDC)* (IEEE, New York, 2018), pp. 6614–6619.

[34] A. Sarlette, P. Rouchon, A. Essig, Q. Ficheux, and B. Huard, Quantum adiabatic elimination at arbitrary order for photon number measurement, arXiv:2001.02550.

[35] Z. Gong, N. Yoshioka, N. Shibata, and R. Hamazaki, Universal Error Bound for Constrained Quantum Dynamics, Phys. Rev. Lett. **124**, 210606 (2020).

[36] Z. Gong, N. Yoshioka, N. Shibata, and R. Hamazaki, Error bounds for constrained dynamics in gapped quantum systems: Rigorous results and generalizations, Phys. Rev. A **101**, 052122 (2020).

[37] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed. (Springer, Berlin, 1976).

[38] C. Bloch, Sur la théorie des perturbations des états liés, Nucl. Phys. **6**, 329 (1958).

[39] I. Lindgren, The Rayleigh-Schrödinger perturbation and the linked-diagram theorem for a multi-configurational model space, J. Phys. B: At. Mol. Phys. **7**, 2441 (1974).

[40] J. des Cloizeaux, Extension d'une formule de Lagrange à des problèmes de valeurs propres, Nucl. Phys. **20**, 321 (1960).

[41] D. J. Klein, Degenerate perturbation theory, J. Chem. Phys. **61**, 786 (1974).

[42] J. P. Killingbeck and G. Jolicard, The Bloch wave operator: Generalizations and applications: Part I. The time-independent case, J. Phys. A: Math. Gen. **36**, R105 (2003).

[43] J. M. Ortega, The Newton-Kantorovich theorem, Am. Math. Month. **75**, 658 (1968).

[44] C. E. Soliverez, General theory of effective Hamiltonians, Phys. Rev. A **24**, 4 (1981).

[45] T. F. Havel, Robust procedures for converting among Lindblad, Kraus and matrix representations of quantum dynamical semigroups, J. Math. Phys. **44**, 534 (2003).

[46] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, 2nd ed. (Springer, Berlin, 2007).

[47] D. Chruściński and S. Pascazio, A brief history of the GKLS equation, Open Sys. Inf. Dyn. **24**, 1740001 (2017).

[48] I. Shavitt and L. T. Redmon, Quasidegenerate perturbation theories: A canonical Van Vleck formalism and its relationship to other approaches, J. Chem. Phys. **73**, 5711 (1980).

[49] S. Bravyi, D. P. DiVincenzo, and D. Loss, Schrieffer-Wolff transformation for quantum many-body systems, Ann. Phys. **326**, 2793 (2011).

[50] J. R. Schrieffer and P. A. Wolff, Relation between the Anderson and Kondo Hamiltonians, Phys. Rev. **149**, 491 (1966).

[51] M. Sanz, E. Solano, and Í. L. Egusquiza, Beyond adiabatic elimination: Effective Hamiltonians and singular perturbation, in *Applications+Practical Conceptualization+Mathematics= Fruitful Innovation*, edited by R. S. Anderssen, P. Broadbridge, Y. Fukumoto, K. Kajiwara, T. Takagi, E. Verbitskiy, and M. Wakayama, Mathematics for Industry Vol. 11 (Springer, Tokyo, 2016), pp. 127–142.

[52] R. Han, H. K. Ng, and B.-G. Englert, Raman transitions without adiabatic elimination: A simple and accurate treatment, J. Mod. Opt. **60**, 255 (2013).

[53] V. Paulisch, H. Rui, H. K. Ng, and B.-G. Englert, Beyond adiabatic elimination: A hierarchy of approximations for multi-photon processes, Eur. Phys. J. Plus **129**, 12 (2014).

[54] E. M. Kessler, Generalized Schrieffer-Wolff formalism for dissipative systems, Phys. Rev. A **86**, 012126 (2012).

[55] D. Burgarth, P. Facchi, H. Nakazato, S. Pascazio, and K. Yuasa, KAM-stability for conserved quantities in finite-dimensional quantum systems, arXiv:2011.04707.

[56] Note that in quantum mechanics the generators B and C are superoperators acting on density operators, and the operator norm of such superoperators depends on the choice of the norm on the density operators. Here we adopt the natural choice of the trace norm on density operators $\|\sigma\|_1 = \text{tr}\,|\sigma|$, which induces the norm (6.5) on the superoperators.

[57] D. Pérez-García, M. M. Wolf, D. Petz, and M. B. Ruskai, Contractivity of positive and trace-preserving maps under L_p norms, J. Math. Phys. **47**, 083506 (2006).

[58] N. J. Higham, *Functions of Matrices: Theory and Computation* (SIAM, Philadelphia, 2008).

[59] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, 1991).

[60] M. M. Wolf, Quantum Channels & Operations: Guided Tour, URL: https://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf.

[61] J. Watrous, *The Theory of Quantum Information* (Cambridge University Press, Cambridge, 2018).

[62] Strictly speaking, the following perturbative terms do not fit in the GKLS form, since some of their Kossakowski matrices are not positive semidefinite, and should not be called GKLS form. We will however call the form (10.4) GKLS form.

[63] P. Lancaster and J. G. Rokne, Solutions of nonlinear operator equations, SIAM J. Math. Anal. **8**, 448 (1977).

[64] L. S. Theis and F. K. Wilhelm, Nonadiabatic corrections to fast dispersive multiqubit gates involving Z control, Phys. Rev. A **95**, 022314 (2017).

[65] L. S. Theis, F. Motzoi, S. Machnes, and F. K. Wilhelm, Counteracting systems of diabaticities using DRAG controls: The status after 10 years, Europhys. Lett. **123**, 60001 (2018).

[66] G. S. Uhrig, Quantum coherence from commensurate driving with laser pulses and decay, SciPost Phys. **8**, 040 (2020).

[67] G. Jolicard and J. P. Killingbeck, The Bloch wave operator: Generalizations and applications: II. The time-dependent case, J. Phys. A: Math. Gen. **36**, R411 (2003).