Coherence-based characterization of macroscopic quantumness

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One of the most elusive problems in quantum mechanics is the transition between classical and quantum physics. This problem can be traced back to Schrödinger's cat thought experiment. A key element that lies at the center of this problem is the lack of a clear understanding and characterization of macroscopic quantum states. Our understanding of macroscopic quantumness relies on states such as the Greenberger-Horne-Zeilinger (GHZ) or the NOON state. Here we take a first-principle approach to this problem. We start from coherence as the key quantity that captures the notion of quantumness and require the quantumness to be collective and macroscopic. To this end, we introduce macroscopic coherence which is the coherence between macroscopically distinct quantum states. We construct a measure that quantifies how global and collective the coherence of the state is. Our work also provides a first-principle way to derive well-established states like the GHZ and the NOON state as the states that maximize our measure. Our approach paves the way towards a better understanding of the quantum-to-classical transition.

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For more than a century, quantum mechanics has successfully explained a wide range of phenomena in physics. There is, however, one simple yet challenging question that has puzzled some of the greatest minds in physics and still remains unsolved. Namely, it is still unclear why the macroscopic world around us is classical and what the nature of the transition from the quantum physics at the microscopic level to the classical one at the macroscopic level is. This problem was manifested by Schrödinger in the famous thought experiment of Schrödinger's cat [1]. Yet, after about a century, this problem is still the subject of active research and especially in the past two decades has attracted a lot of attention [2–8].

Different approaches have been taken to explain the discrepancy between the microscopic and macroscopic worlds. On the one hand, there are the collapse models which suggest that the theory of quantum mechanics needs to be modified to comply with our classical observations [9]. On the other hand, there are approaches that search for the solution within quantum mechanics [10–19]. For instance, in many cases, decoherence can explain the emergence of classical states from quantum ones. Or similarly, it has been shown that the lack of precision could make quantum states look like classical states [5,6,20].

One of the key challenges of finding a resolution to the quantum-to-classical transition is the ambiguity of the problem, i.e., the lack of a clear and cohesive picture of what macroscopic quantum states and effects are.

This problem has been intensively investigated for the past two decades and a variety of measures and definitions of macroscopic quantumness have been suggested [2,3,21–41]. These measures vary in approaches, formulations, and applicability. Some measures are based on comparison to well-established states such as the Greenberger-Horne-Zeilinger (GHZ) state [42] or the coherent cat states [1,43]. Some other measures quantify the macroscopic quantumness of a state by the oscillations in the probability distribution with respect to some measurement. For example, Lee and Jeong characterized the macroscopic quantumness of photonic states based on the intensity of oscillation frequencies of its Wigner function [37]. Following this idea, Fröwis and Dür proposed to use quantum Fisher information (QFI) for characterization of macroscopic quantumness [32,36,44].

Lack of cohesion and diversity of definitions and measures indicate that, although we have a better understanding of the problem, we still do not have a clear notion of what macroscopic quantumness is.

Here we present a simple approach to the characterization of macroscopic quantumness. We start with coherence [45] which is widely believed to be the underlying feature that distinguishes quantum and classical physics [45]. We construct a measure of macroscopic quantumness which is a monotone for quantum coherence that incentivizes the coherence between macroscopically distinguishable states. This measure is closely connected to the work by Yadin and Vedral [46]. They introduced a general framework for macroscopic quantumness in connection to quantum coherence. This framework establishes a class of coherence measures based on four conditions. The first three conditions are identical to the ones for a coherence monotone [45]. The last one ensures that the coherence measure would identify coherence at macroscopic level. Yadin and Vedral reviewed several examples for measures that satisfy these four conditions, including some of the well-established measures such as the Fisher information [32] and Lee and Jeong measure [37]. Our work is similar to the ones that fall in this family of coherence monotones. We emphasize that we are not reinventing the framework. Arguably, our measure can be seen as a simplified version

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of measures in this framework. For more information, see Appendix G.

This is a first-principle approach to define and quantify macroscopic quantumness. More specifically, our measure does not rely on specific example states such as the GHZ and instead it is constructed based on two principles: quantumness, which is represented by coherence, and macroscopisity, which is captured by the macroscopic distinguishability of the the states involved in the quantum coherence.

Naturally, macroscopic quantum states are expected to have relatively large amounts of coherence. However, for a state to be recognized as a macroscopic quantum state, not only it should have large measurable coherence, but the coherence should also be distributed macroscopically. To clarify this, consider the following two spin states:

$$|\psi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle^{\otimes(N-1)}, \quad |\psi_2\rangle = \frac{|0\rangle^{\otimes N} + |1\rangle^{\otimes N}}{\sqrt{2}}, \tag{1}$$

where $|0\rangle$ and $|1\rangle$ correspond to up and down spins respectively. Most coherence measures would assign the same amount of coherence to these two states since their density matrices both have similar off-diagonal elements, both in value and number. However, the off-diagonal elements of $|\psi_1\rangle$ are between $|00\cdots0\rangle$ and $|10\cdots0\rangle$, whereas for $|\psi_2\rangle$ they are between $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$. The difference between the two states is that, for the former, the states differ in only one spin and are not macroscopically distinguishable, whereas, for the latter, they could be distinguished for large enough N and with the right measurement. For instance, for a magnetization measurement in the z direction, $|0\rangle^{\otimes N}$ gives $N(\frac{\hbar}{2})$ whereas $|1\rangle^{\otimes N}$ gives $-N(\frac{\hbar}{2})$. This means that, for large enough N, the states $|0\rangle^{\otimes N}$ and $|1\rangle^{\otimes N}$ can be distinguished with a macroscopic magnetization measurement. In this sense, it can be argued that, although both states have the same amount of coherence (quantumness), $|\psi_2\rangle$ has the additional property that its quantumness is distributed macroscopically, i.e., coherence is between states that are macroscopically distinguishable. Here we present a characterization of macroscopic quantumness based on this notion. Namely, we start with a notion of quantumness, i.e., the coherence, and add the extra requirement that it should be macroscopic. The advantage of this approach is that it does not rely on well-established states or a phenomenological behavior of them. Instead, to some extent, it gives a first-principle approach to the characterization of macroscopic quantumness. We will show that this first-principle approach is consistent and can characterize the well-established macroscopic quantum states properly.

We start with our notation and terminology. For a density matrix $\rho = \sum_{i,j} \rho_{i,j} |i\rangle\langle j|$, the coherence is characterized by the off-diagonal elements $\rho_{i\neq j}$. We refer to $\rho_{i,j}$ as coherence elements between states $|i\rangle$ and $|j\rangle$.

Typical coherence monotones would treat all the coherence elements uniformly. However, as illustrated in the example in Eq. (1), this approach would not be suitable for characterization of macroscopic quantum states. For a coherence monotone to captures macroscopic quantumness, it has to incentivize coherence elements between states that are more macroscopically distinct; i.e., for a coherence element $\rho_{i,j}$, the more macroscopically distinct the two states $|i\rangle$ and $|j\rangle$, the more that element should contribute to the monotone. To this end, we introduce "macroscopic coherence," which refers to the coherence terms $\rho_{i,j}$ such that the states involved, i.e., $|i\rangle$ and $|j\rangle$, can be macroscopically distinguished with some measurement. For a schematic picture, see Fig. 1.

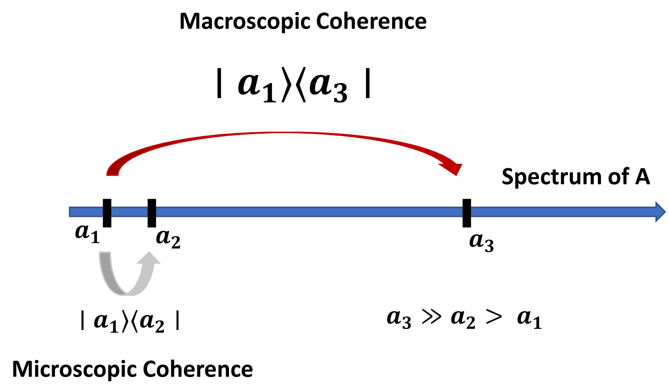

FIG. 1. Schematic picture for macroscopic coherence. The spectrum of an operator A, a_1, a_2, a_3, is depicted on the horizontal axis. Two coherence elements are shown, one between the first and the second eigenvalues of A and one between the first and the third. The idea is that the states involved in a coherence elements may or may not be macroscopically distinct, according to the measurement of some operator A. If they are macroscopically distinct, that makes the coherence macroscopic and these macroscopic coherence elements can characterize macroscopic quantumness.

Initially, there are two ambiguities in this approach. First, it is not clear what characterizes the macroscopic distinction between the two states $|i\rangle$ and $|j\rangle$, and second, the coherence elements depend on the basis. The former is due to the unclear border between macro and micro and for this, we can rely on what is considered a macroscopic distinction in an experimental setting. The latter is because coherence is a basis-dependent quantity. But both of these ambiguities are expected in the characterization of macroscopic quantumness. For instance, for a GHZ state with N spins, it is not clear for how large of a number N the state would qualify as a macroscopic state. Similarly, for identifying quantumness, the basis of the measured observable is important. This would mean that our measure for macroscopic quantumness should depend on the measurement.

To quantify the macroscopic coherence, we first need a monotone for coherence and next we need to quantify the macroscopisity of the coherence. For both of these, we need to specify the measured observable.

Assume that the observable of interest is $A = \sum_i a_i |i\rangle\langle i|$. The eigenbasis of A sets the basis for the coherence. For quantification of coherence we start with

$$\sum_{i\neq j}^{D^2-D} |\rho_{i,j}|, \tag{2}$$

where D is the dimension of the Hilbert space [45,47].

Next we need to quantify the macroscopic distinction between the states. Note that the elements of an orthonormal basis are mutually orthogonal and therefore the inner product does not capture the difference between say $|0\rangle\langle 1|$ and $|0\rangle\langle N|$. One natural choice for the macroscopic distinction between the two states $|i\rangle$ and $|j\rangle$ is $|a_i - a_j|$, i.e., the difference between the eigenvalues associated with $|i\rangle$ and $|j\rangle$. If the difference is large enough to be resolved with a macroscopic measurement, the states $|i\rangle$ and $|j\rangle$ are macroscopically distinct. For example, for a position measurement of a macroscopic object, one can argue that the states $|-1(\text{meter})\rangle$ and $|1(\text{meter})\rangle$ would be macroscopically distinct as they can be distinguished with naked eye. Mathematically we introduce the distance

$$d_A(i, j) = |a_i - a_j|. \tag{3}$$

For a measure of macroscopic quantumness, instead of uniformly considering all of the coherence elements, we weigh them based on their corresponding distances. This penalizes contribution of coherence elements with small $d_A(i, j)$ and incentivizes the contribution from elements with large $d_A(i, j)$.

To turn the coherence monotone in Eq. (2) into a monotone for macroscopic coherence, we add the distance to the measure, which gives

$$\sum_{i,j} d_A(i, j)\,|\rho_{i,j}|. \tag{4}$$

This incentivizes macroscopic coherence and suppresses the microscopic coherence. Note that we even included the diagonal elements that have no coherence in the sum but they are automatically suppressed by $d_A(i, i) = 0$ and the sum remains unchanged.

This, however, has a flaw, namely, there are two ways that the measure can increase: one is by increasing the coherence (not necessarily the macroscopic elements) and the other is by increasing the macroscopicity of the coherence elements. For instance, consider the state

$$|\psi_3\rangle = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)^{\otimes N}. \tag{5}$$

For large enough N, the quantity in Eq. (4) would be significantly affected by the large number of off-diagonal elements in the density matrix of $|\psi_3\rangle$ or equivalently, large amounts of coherence, although most of them are not macroscopic. To fix this issue, we can normalize the coherence elements. This means that instead of $|\rho_{i,j}|$, we use $\frac{|\rho_{i,j}|}{\sum_{i,j}|\rho_{i,j}|}$ which indicates the fraction of all of the elements in the density matrix corresponding to the coherence element $\rho_{i,j}$. This gives

$$M(\rho) = \frac{\sum_{i,j} d_A(i, j)\,|\rho_{i,j}|}{\sum_{i,j} |\rho_{i,j}|}. \tag{6}$$

This measure can be interpreted as the average of the distance $d_A(i, j)$ over all of the different elements of the density matrix. To see this more clearly, we can partition the elements of the density matrix into classes with different values for d_A, i.e.,

$$C_\delta = \{\rho_{i,j} | d_A(i, j) = \delta\}. \tag{7}$$

Based on this, we can define the following probability distribution:

$$P(\delta) = \frac{\sum_{\rho_{i,j}\in C_\delta} |\rho_{i,j}|}{\sum_{i,j} |\rho_{i,j}|}. \tag{8}$$

This is the probability of getting a coherence element with $d_A(i, j) = \delta$. This probability distribution translates the measure in Eq. (6) to

$$M(\rho) = \bar{d} = \sum_{\delta} P(\delta)\delta. \tag{9}$$

This, which is in fact the average distance between the states corresponding to the coherence terms $\rho_{i,j}$, i.e., $\bar{d}_A(i, j)$, gives a quantification for the macroscopic quantumness of the state.

Here there is an ambiguity associated with the reference that determines the macroscopic amount of M. For instance, one may change the value of M by altering the unit in which the observable is expressed. As we emphasized before, we define our measure with reference to what is considered macroscopic for the specific experiment. To decide whether M is macroscopic or not, we can compare the amount of M with the amount that is considered macroscopic for the specific experimental setup which is used to measure and investigate the state.

Alternatively, we can modify the definition of the measure to a dimensionless quantity. To this end, we can normalize the distance $d_{A(i,j)}$ in Eq. (3) to

$$\bar{d}_{A(i,j)} = \frac{|a_i - a_j|}{d_{A,\text{thresh}}}, \tag{10}$$

where $d_{A,\text{thresh}}$ is the threshold that identifies a macroscopic value. This value is set based on the specific experimental equipment, setup, and the measurement setting. This definition for the distance gives a dimensionless quantity for M in Eq. (6) that can be used to compare between different quantum states in different experimental settings. For instance, it can be used to compare between photonic and spin states.

For a state with its coherence elements focused between states that are not macroscopically distant according to the observable A or states with small coherence, the measure gives a small value. On the other hand, if the state has a large amount of coherence and the coherence elements are mostly focused between states that can be macroscopically distinguished, the measure assigns a large amount of macroscopic quantumness to the state.

As an example, consider the states in Eq. (1) under the measurement of the total magnetization in the z direction. Both states have two diagonal and two off-diagonal elements, all with the value of $1/2$. For ψ_1, the distance corresponding to the off-diagonal element, i.e., $d(|00\cdots 0\rangle, |10\cdots 0\rangle)$, is 1 and this gives $M(\psi_1) = 1/2$. For the GHZ state, the distance corresponding to the off-diagonal element is N, which gives $M(\psi_{\text{GHZ}}) = N/2$. This shows that the measure scales and grows with the system size for the GHZ state, but as expected, for $|\psi_1\rangle$, it stays constant. This gives a natural effective size for the system that describes the scale at which the coherence is distributed.

Here we assumed that the observable A is a discrete operator; however, the measure can be extended to continuous operators by discretizing the spectrum and defining bins. The discretization, i.e., the bin size, can be set based on the precision of the measurements.

This measure provides a way to define ideal states, i.e., states with maximum macroscopic coherence. A maximum macroscopic quantum state (MMQS) can be defined as a state which maximizes the measure in Eq. (9). For instance, it is easy to show that the GHZ state is an MMQS for spin-type systems. In general it is possible to prove the following theorem for our measure.

Theorem 1. For a bounded observable A with nondegenerate maximum and minimum eigenvalues a_0 and a_N, the following state maximizes the measure in Eq. (9):

$$|\psi_{\rm MMQS}\rangle = \frac{|a_0\rangle + e^{i\phi}|a_N\rangle}{\sqrt{2}}, \tag{11}$$

with $|a_0\rangle$ and $|a_N\rangle$ the eigenvectors of A corresponding to the minimum and maximum eigenvalues a_0 and a_N respectively and ϕ some phase.

For more details and the proof, see Appendix A.

This theorem indicates that states like the GHZ and NOON states maximize our measure for macroscopic quantumness in their corresponding Hilbert spaces.

This characterization, as mentioned before, depends on the measured observable. But it is also possible to make it measurement independent by maximizing over all possible measurements. However, it is often impractical and sometimes impossible to carry out the maximization [8]. For practical purposes, it it is more convenient to specify a measurement or set of measurements and investigate the states with respect to those measurements.

This measure can also be used to define an effective size for the macroscopic quantumness of a state. This is similar to [21,29,32,38,48]. To this end, we compare the value of the measure with the corresponding MMQS. More precisely, consider a system that consists of N entities with state ρ and assume that the measure returns a value $M(\rho)$ for the macroscopic quantumness of the state. We define the effective size $N_{\rm eff}$ as the size of the smallest MMQS that has the same amount of macroscopic quantumness, $M(\rho)$. Mathematically, that is

$$N_{\rm eff}(\rho) = \min\{n \mid M(\rho) \leqslant M(\text{MMQS(n)})\}, \tag{12}$$

where MMQS(n) is the MMQS with n particles. For a spin system like the examples we considered, $N_{\rm eff}(\rho) = 2M(\rho)$.

I. EXAMPLES

Next we calculate our measure for some well-known states. We consider two systems: first spin ensembles and then photonic quantum states. In the examples, the threshold distance $d_{\rm thresh}$ is considered the unit; however, it is important to notice that its amount must be determined with respect to the specific experiment.

A. Spin ensemble systems

We start with an ensemble of spin 1/2 particles. Here we consider the total magnetization, which is a natural and practical measurement for spin systems.

Without loss of generality, we take this to be the measurement of magnetization in z direction. The corresponding observable is $A = \sum_i \sigma_z^{(i)}$ with $\sigma_z^{(i)} = |0\rangle\langle 0| - |1\rangle\langle 1|$ on the ith spin of the ensemble.

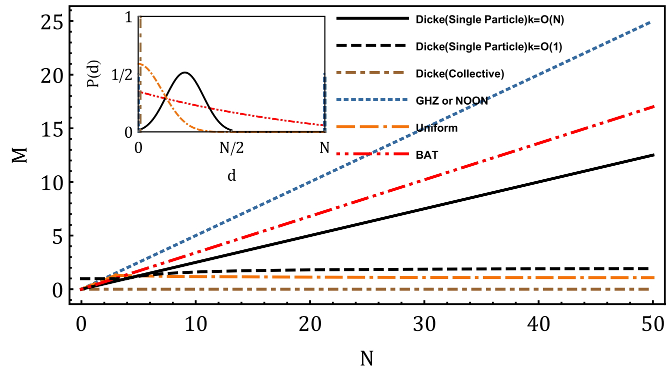

FIG. 2. This plots shows how our measure for macroscopic quantumness varies with N for the GHZ, uniform, Dicke, NOON, and BAT states (see the Examples Sec. I for the details of calculations). N is the number of elements (spins or photons) in the system. The inset gives a schematic plot of the probability distribution function $P_{(d)}$ states.

We start with the GHZ, which is the state $|\psi_2\rangle$ in Eq. (1). As was explained before, the measure gives

$$M_{\rm GHZ} = \frac{N}{2}. \tag{13}$$

The probability distribution $P(\delta)$ is plotted in the inset of the Fig. 2 and it is clear that the mean distance is $N/2$.

It is interesting to compare the GHZ state with $|\psi_3\rangle$. We refer to this state as the uniform state. Similarlu to the GHZ state, the uniform state is macroscopic and has nonzero coherence elements. The difference is that, in contrast to the GHZ state, the coherence is not collective and each spin has its independent coherence. The probability distribution corresponding to this state is also plotted in the inset of Fig. 2.

For large number of spins N, using Stirling's approximation $\ln N! = N\ln N - N$, the measure asymptotically converges to

$$M_{\rm uni} \approx e^{N\ln(\frac{(N+\frac{1}{2})^2}{(N-1)(N+2)})}. \tag{14}$$

For more details, see Appendix B.

This gives

$$\lim_{N\to\infty} M_{\rm uni} = 1 = O(N^0), \tag{15}$$

i.e., it converges to the constant value 1. This is consistent with the fact the the coherence in this state is the collection of the individual coherences.

Another interesting state is the Dicke state [49,50]. For a system with N elements with binary states, $\{|0\rangle, |1\rangle\}$, the Dicke state is defined as the uniform superposition of all the states with k elements in the 0 state and $N-k$ in the 1 state. If the elements are spin 1/2 particles, the Dicke state would be an eigenstate of total magnetization in the z direction. So for the collective magnetization measurement (the total magnetization in the z direction), the distances corresponding to all of the coherence elements are zero, which leads to $M_{\rm Dicke} = 0$. But, if we consider the single-particle measurements of magnetization in z direction, the value of the measure for this state is $M_{\rm Dicke} = \frac{k(N-k)}{N}$. For more details see Appendix C. Thus,

TABLE I. The value of the measure for photonic states. N is the number of photons, α is the annihilation eigenvalue of the coherent state $|\alpha\rangle$, and ε is the squeezing parameter.

State	NOON	BAT	SCS	Two-mode SCS	Two-mode squeezed state
Measure	$\frac{N}{2}$	$O(N)$	$\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	$\frac{4\tanh\varepsilon}{1-(\tanh\varepsilon)^2}$

for $k = O(N/2)$ and $N \gg 1$, the state is highly macroscopic quantum.

The example of the Dicke state also shows how our measure depends on the basis and measurement.

B. Photonic systems

Next we investigate photonic states with our measure. For the measured observable, we consider energy or, equivalently, the photon number. The state that we consider is the NOON state, which is defined as

$$|\text{NOON}\rangle = \frac{|N/2\rangle|0\rangle + |0\rangle|N/2\rangle}{\sqrt{2}}. \tag{16}$$

This state consists of two modes. These could be the vertical and horizontal polarization that can be separated with a polarizing beam splitter.

The calculation of the measure is similar to the one for the GHZ state and gives

$$M_{\text{NOON}} = \frac{N}{2}. \tag{17}$$

Another state of interest is the superposition of coherent states (SCS), also known as the cat state [43]. Mathematically that is

$$|\text{SCS}\rangle = (|\alpha\rangle + |-\alpha\rangle)/z, \tag{18}$$

where $|\alpha\rangle$ is a coherent state and $z = \sqrt{2 + 2\,\text{Re}(\langle\alpha| - \alpha\rangle)}$ is the normalization factor.

For large enough $|\alpha|$, the two states $|\alpha\rangle$ and $|-\alpha\rangle$ are approximately orthogonal to each other, i.e., $\langle\alpha|{-\alpha}\rangle \approx 0$ and therefore, in the basis of the quadrature operator $X(\theta) = X\cos\theta + P\sin\theta$ with $\tan\theta = \frac{\text{Im}(\alpha)}{\text{Re}(\alpha)}$, the density matrix can be approximated as

$$\rho_{\text{SCS}} = \tfrac{1}{2}(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha| + |\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha|). \tag{19}$$

From Eq. (19) it is easy to see that $P(d)$ is $\frac{1}{2}$ for both $d = 0$ and $d = |\alpha - (-\alpha))| = 2|\alpha|$. So for the measurement of $X(\theta)$ our measure gives

$$M_{\text{SCS}} \approx \tfrac{1}{2} \times 0 + \tfrac{1}{2} \times 2|\alpha| = |\alpha|. \tag{20}$$

Since $|\alpha|^2$ is the mean number of photons in the system, the measure increases by increasing the number of photons.

A similar state is the two-mode SCS state which is defined as

$$|\text{two-mode SCS}\rangle = \frac{|\alpha\rangle|\alpha\rangle + |-\alpha\rangle|-\alpha\rangle}{z'}, \tag{21}$$

where $z' = \sqrt{2 + 2Re(\langle\alpha| - \alpha\rangle^2)}$. For the value of measure in the basis of the quadrature $X_1(\theta) \otimes I_2 + I_1 \otimes X_2(\theta)$, similar calculations give

$$M_{\text{two-mode SCS}} = 2|\alpha|. \tag{22}$$

We also consider the two-mode squeezed state and BAT state. The results are shown in Table I. Also we plotted the amount of our measure for some of these states in Fig. 2. For more examples and further details of the calculations, see Appendix E.

II. COMPARISON WITH OTHER MEASURES

In this section we compare our measure with other measures. The results are summarized in Table II. The comparison is done based on how different measures classify specific states of interest.

We do not consider all the existing measures for macroscopic quantumness. We focus on some of the more well-known measures that can also be applied as generally and as widely as our measure. For instance, some measures are limited to specific classes of states, e.g., spin states, or states of the form $|A\rangle + |D\rangle$ [2,3,21,23,29,34,35,48]. Also, for the ones we consider, if they are not applicable for a specific state that is used for the comparison, the results is indicated as not applicable (N.A.) in the table. A more through comparison is included in Appendix F.

III. CONCLUSION

In conclusion, we presented an approach for the characterization of macroscopic quantumness which is in fact a coherence measure. But in addition to the coherence, it also quantifies how global and collective the coherence is. Our

TABLE II. Comparison of our measure with other measures for well-established states. The green color means that the amount of the correspondent measure is in agreement with the amount obtained by our measure. The yellow color means that the correspondent measure is in agreement with the amount obtained by our measure but it has different scaling order either in N or $|\alpha|$. N.A. stands for "not applicable." Also, $1 \leqslant p \leqslant 2$ denotes the measure of Shimizu and Miyadera.

	Our measure	Bjork and Mana (J. Opt. B 2004 [23])	Shimizu and Miyadera (PRL 2002 [22])	Lee and Jeong (PRL 2011 [31])	Fröwis and Dür (NJP 2012 [32])	Yadin and Vedral (PRA 2015 [38])
GHZ	$N/2$	N.A.	$p = 2$	$O(N)$	$O(N)$	$O(N)$
NOON	$N/2$	$\sqrt{N}$	$p = 2$	$O(N)$	$O(N)$	N.A.
Dicke [$k = O(N/2)$]	$O(N)$	N.A.	$p = 2$	$O(N)$	$O(N)$	$O(N)$
Dicke [$k = O(1)$]	$O(1)$	N.A.	$p = 1$	$O(1)$	$O(1)$	$O(1)$
SCS	$\lvert\alpha\rvert$	$2\lvert\alpha\rvert$	N.A.	$\lvert\alpha\rvert^2$	$\lvert\alpha\rvert^2$	N.A.

approach can be seen as a more axiomatic alternative to established measures of macroscopic quantumness.

It also provides a first-principle approach to derive maximum macroscopic quantum states (MMQS) such as the GHZ state. We showed that the maximization of our measure over all the states would lead to MMQS. This provides a way to arrive at states such as the GHZ state in the context of macroscopic quantumness without making any assumption about their macroscopic quantumness.

This approach opens up an avenue for understanding macroscopic quantumness and paves the way towards a cohesive and unified characterization of macroscopic quantumness.

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APPENDIX A: MMQS

Theorem. In a system and in the basis of eigenvectors of the bounded observable $\hat{A}$, the state

$$|\psi_{\text{MMQS}}\rangle = \frac{|a_0\rangle + e^{i\phi}|a_N\rangle}{\sqrt{2}} \tag{A1}$$

maximizes the measure M. Here, $|a_0\rangle$ and $|a_N\rangle$ are the eigenvectors of $\hat{A}$ with minimum and maximum eigenvalues respectively and ϕ is a phase. Irrespective of ϕ and if $\hat{A}$ is not degenerate, $|\psi_{\text{MMQS}}\rangle$ is unique.

Proof. First of all we prove the state $|\psi_{\text{MMQS}}\rangle$ maximizes M among all pure states in the range of spectrum of $\hat{A}$. Consider an arbitrary pure state $|\psi\rangle$ in the spectrum of $\hat{A}$ as below:

$$|\psi\rangle = \sum_{i=0}^{N} c_i|a_i\rangle \equiv \rho = \sum_{ij=0}^{N} c_i c_j^* |a_i\rangle\langle a_j|; \tag{A2}$$

the $|a_k\rangle$'s, $k \in \{0, \dots, N\}$, are the eigenvectors of $\hat{A}$ corresponding to the eigenvalues a_k. If $i \leqslant j$, $i, j \in \{0, \dots, N\}$, then $a_i \leqslant a_j$. For $|\psi\rangle$ the measure is:

$$M = \frac{\sum_{ij} |c_i||c_j|d_{ij}}{\sum_{ij} |c_i||c_j|}. \tag{A3}$$

We know that $\sum_{i=0}^{N} |c_i|^2 = 1$. Maximizing M, we neglect this constraint but later we will turn back to it.

Differentiating M in $|c_k|$ and equating it to zero, we find the following set of equations:

$$\begin{aligned} &\forall k \in \{0, \dots, N\}, \\ &\frac{\partial M}{\partial |c_k|} = \frac{\partial}{\partial |c_k|}\left(\frac{\sum_{ij} |c_i||c_j|d_{ij}}{\sum_{ij} |c_i||c_j|}\right) \\ &= \frac{\left(2\sum_i |c_i|d_{ik}\right)\left(\sum_{ij} |c_i||c_j|\right)}{\left(\sum_{ij} |c_i||c_j|\right)^2} \\ &\quad - \frac{\left(2\sum_i |c_i|\right)\left(\sum_{ij} |c_i||c_j|d_{ij}\right)}{\left(\sum_{ij} |c_i||c_j|\right)^2} = 0. \end{aligned} \tag{A4}$$

As $M = \frac{\sum_{ij} |c_i||c_j|d_{ij}}{\sum_{ij} |c_i||c_j|}$, we substitute M in the second fraction of the relation (A4), thus it is simplified:

$$\begin{aligned} &\forall k \in \{0, \dots, N\}, \\ &\frac{\sum_i |c_i|d_{ik} - \sum_i |c_i|M}{\sum_{ij} |c_i||c_j|} = 0 \\ &\Leftrightarrow M\sum_i |c_i| = \sum_i |c_i|d_{ik} \\ &\Leftrightarrow M = \frac{\sum_i |c_i|d_{ik}}{\sum_i |c_i|}. \end{aligned} \tag{A5}$$

The c_k,s maximizing M satisfy the Eq. (A5).

Now consider the equations associated with $k = 0$ and $k = 1$:

$$\begin{aligned} k = 0, \quad M &= \frac{\sum_i |c_i|d_{i0}}{\sum_i |c_i|}, \\ k = 1, \quad M &= \frac{\sum_i |c_i|d_{i1}}{\sum_i |c_i|}. \end{aligned} \tag{A6}$$

By cross multiplication, we can write

$$\begin{aligned} k = 1, \quad M = \frac{\sum_i |c_i|d_{i1}}{\sum_i |c_i|} &\Leftrightarrow M\sum_i |c_i| = \sum_i |c_i|d_{i1} \\ &\Leftrightarrow M\sum_i |c_i| = |c_0|d_{01} + \sum_{i\neq 0} |c_i|d_{i1}. \end{aligned} \tag{A7}$$

Knowing $d_{i1} = d_{i0} - d_{10}$ for $i > 1$ and replacing it in (A7),

$$\begin{aligned} M\sum_i |c_i| &= \sum_i |c_i|d_{i1} = |c_0|d_{10} + \sum_{i\neq 0} |c_i|(d_{i0} - d_{10}) \\ &= \left(|c_0| - \sum_{i\neq 0} |c_i|\right)d_{10} + \sum_{i\neq 0} |c_i|d_{i0}. \end{aligned} \tag{A8}$$

Regarding the relations (A6), the last term in (A8) is $M\sum_i |c_i|$, so they cancel each other and we have

$$|c_0| = |c_N| + \sum_{i\neq 0,N} |c_i|. \tag{A9}$$

Now we do the same procedure for $k = N$ and $k = N - 1$:

$$\begin{aligned} k = N - 1, \quad M &= \frac{\sum_i |c_i|d_{i,N-1}}{\sum_i |c_i|}, \\ k = N, \quad M &= \frac{\sum_i |c_i|d_{i,N}}{\sum_i |c_i|}. \end{aligned} \tag{A10}$$

By cross multiplication, we can write

$$\begin{aligned} &k = N - 1, \\ &M = \frac{\sum_i |c_i|d_{i,N-1}}{\sum_i |c_i|} \\ &\quad \Leftrightarrow M\sum_i |c_i| = \sum_i |c_i|d_{i,N-1} \\ &\quad \Leftrightarrow M\sum_i |c_i| = |c_N|d_{N,N-1} + \sum_{i\neq N} |c_i|d_{i,N-1}. \end{aligned} \tag{A11}$$

Knowing $d_{i,N-1} = d_{i,N} - d_{N,N-1}$ for $i < N - 1$ and replacing it in (A11),

$$M \sum_i |c_i| = \sum_i |c_i| d_{i,N-1} = |c_N| d_{N,N-1} + \sum_{i\neq N} |c_i|(d_{i,N} - d_{N,N-1}) = \left(|c_N| - \sum_{i\neq N} |c_i|\right) d_{N,N-1} + \sum_{i\neq N} |c_i| d_{i,N}. \tag{A12}$$

Regarding the relations (A10), the last term in (A12) is $M\sum_i |c_i|$, so they cancel each other and we have

$$|c_0| = |c_N| - \sum_{i\neq 0,N} |c_i|. \tag{A13}$$

Equations (A9) and (A13) imply that

$$k \neq 0, \quad N \to c_i = 0, \quad |c_0| = |c_N|. \tag{A14}$$

Hence, when $|c_0| = |c_N|$ and the other c_i's are zero, M is an extremum. If $|c_0| = |c_N|$ and also is nonzero, the extremum is maximum too, because for all nonzero values of $|c_0| = |c_N|$, regardless of any constraints, the amount of the extremum is $\frac{d_{\max}}{2}$:

$$M_{\max} = \frac{2|c_0|^2 \times 0 + 2|c_0|^2 \times d_{\max}}{2|c_0|^2 + 2|c_0|^2} = \frac{d_{\max}}{2}. \tag{A15}$$

The state $|\psi_{\text{MMQS}}\rangle$ is the only pure state in the range of the spectrum of $\hat{A}$ that $|c_0| = |c_N| \neq 0$, therefore it maximizes M.

Note. Generally, by doing the exact same procedure for each k and $k+1$, the set of equations in (A4) turns to the below set of equations below, which are equivalent to (A4):

$$\forall k \in \{0,\dots,N\}, \quad \sum_{i=0}^{k} |c_i|^2 = \sum_{i=k+1}^{N} |c_i|^2. \tag{A16}$$

These equations only have answers when either all c_i's are zero (in this case $M = 0$ and is minimum) or just $|c_0|$ and $|c_N|$ are nonzero and equal. The latter obtains the maximum for M.

Now, we prove the state (A1) also maximizes M among mixed states in the spectrum of $\hat{A}$.

Consider the mixed state ρ; we can decompose it in N ensembles:

$$\rho = \sum_i^N P_i |\psi_i\rangle\langle\psi_i|, \tag{A17}$$

where $i \in \{0,\dots,N\}$ and $|\psi_i\rangle$'s are orthogonal. The $|\psi_i\rangle$'s and their corresponding density matrices can be written as follows:

$$|\psi_i\rangle = \sum_{x=0}^{N} c_x^i |a_x\rangle \equiv |\psi_i\rangle\langle\psi_i| = \sum_{x,y=0}^{N} a_{xy}^i |a_x\rangle\langle a_y|, \tag{A18}$$

where $x, y \in \{0,\dots,N\}$. We know the relations between a_{xy}^i and c_x^i:

$$a_{xy}^i = c_x^i c_y^{i*}, \quad a_{xx}^i = c_x^i c_x^{i*} = |c_x|^2. \tag{A19}$$

Also,

$$|c_y|^2 = \frac{|c_x^i|^2 |c_y^i|^2}{|c_x^i|^2} = \frac{|a_{xy}^i|^2}{a_{xx}}. \tag{A20}$$

Regarding $\sum_y |c_y|^2 = 1$ and with respect to Eq. (A20), we have the following constraints for a_{xy}^i:

$$\sum_y |c_y|^2 = 1 \Rightarrow \sum_y \frac{|a_{xy}^i|^2}{a_{xx}} = 1, \quad f_x^i = \sum_y |a_{xy}^i|^2 - a_{xx}^i = 0 \tag{A21}$$

and

$$\sum_x |c_x|^2 = 1 \Rightarrow \sum_x a_{xx} = 1, \quad f_0^i = \sum_x a_{xx} = 1. \tag{A22}$$

The measure M for ρ is

$$M = \frac{\sum_{x,y} |\sum_i P_i a_{xy}^i| d_{xy}}{\sum_{x,y} |\sum_i P_i a_{xy}^i|} = \frac{\sum_{x,y} \sqrt{\left(\sum_i P_i a_{xyR}^i\right)^2 + \left(\sum_i P_i a_{xyI}^i\right)^2} d_{xy}}{\sum_{x,y} \sqrt{\left(\sum_i P_i a_{xyR}^i big)^2 + \left(\sum_i P_i a_{xyI}^i\right)^2}}. \tag{A23}$$

a_{xyR}^i and a_{xyI}^i are the real and imaginary parts of a_{xy}^i respectively. Besides, we denote the denominator of M on the right side of (A23) with D.

Maximizing M, we differentiate M in a_{xyR}^i and a_{xyI}^i and with respect to the constraints f_x^i, we use the Lagrange multipliers method. We can directly apply the constraints (A22) in D:

$$D = \sum_{x,y} \sqrt{\left(\sum_i P_i a_{xyR}^i\right)^2 + \left(\sum_i P_i a_{xyI}^i\right)^2} = \sum_x \sqrt{\left(\sum_i P_i a_{xxR}^i\right)^2 + \left(\sum_i P_i a_{xxI}^i\right)^2} + \sum_{x,y,x\neq y} \sqrt{\left(\sum_i P_i a_{xyR}^i\right)^2 + \left(\sum_i P_i a_{xyI}^i\right)^2}. \tag{A24}$$

Because $a_{xx}^i \geqslant 0$ are real, we can write

$$D = \sum_i P_i \sum_x a_{xx}^i + \sum_{x,y,x\neq y} \sqrt{\left(\sum_i P_i a_{xyR}^i\right)^2 + \left(\sum_i P_i a_{xyI}^i\right)^2}. \tag{A25}$$

By the constraints (A22), the first term in the right side of (A25) is $\sum_i P_i \sum_x a^i_{xx} = 1$, thus

$$D = 1 + \sum_{x,y,x\neq y} \sqrt{\left(\sum_i P_i a^i_{xyR}\right)^2 + \left(\sum_i P_i a^i_{xyI}\right)^2}. \quad \text{(A26)}$$

Note. By applying the constraints (A22), and with respect to $d_{xx} = 0$, M is no longer a function of a^i_{xy}.

Carrying the calculations for maximizing M, we reach the equations below:

$$x \neq y, \quad \frac{\rho_{xyR}(d_{xy} - M)}{D|\rho_{xy}|} = \frac{\lambda^i_x}{2P_i} a^i_{xyR}, \quad \text{(A27)}$$

$$x \neq y, \quad \frac{\rho_{xyI}(d_{xy} - M)}{D|\rho_{xy}|} = \frac{\lambda^i_x}{2P_i} a^i_{xyI}, \quad \text{(A28)}$$

$$\lambda^i_x (2a^i_{xx} - 1) = 0. \quad \text{(A29)}$$

$\rho_{xyR} = \mathrm{Re}(\rho_{xy})$, $\rho_{xyI} = \mathrm{Im}(\rho_{xy})$, and λ^i_x's are the Lagrange multipliers associated with f^i_x.

We show the calculations for deriving Eqs. (A27); the other equations are derived in the same way. By differentiating M in a^i_{xyR},

$$\frac{\partial M}{\partial a^i_{xyR}} = \frac{\frac{4P_i\left(\sum_i P_i a^i_{xyR}\right) d_{xy}}{\sqrt{\left(\sum_i P_i a^i_{xyR}\right)^2 + \left(\sum_i P_i a^i_{xyI}\right)^2}} D}{D^2} - \frac{\frac{4P_i\left(\sum_i P_i a^i_{xyR}\right)}{\sqrt{\left(\sum_i P_i a^i_{xyR}\right)^2 + \left(\sum_i P_i a^i_{xyI}\right)^2}}}{D^2} \times \sum_{x,y} \sqrt{\left(\sum_i P_i a^i_{xyR}\right)^2 + \left(\sum_i P_i a^i_{xyI}\right)^2}\, d_{xy}. \quad \text{(A30)}$$

In the above relation we can substitute the following terms:

$$\rho_{xy} = \sqrt{\left(\sum_i P_i a^i_{xyR}\right)^2 + \left(\sum_i P_i a^i_{xyI}\right)^2}, \quad \text{(A31)}$$

$$\rho_{xyR} = \sum_i P_i a^i_{xyR}, \quad \text{(A32)}$$

$$M = \frac{\sum_{x,y} \sqrt{\left(\sum_i P_i a^i_{xyR}\right)^2 + \left(\sum_i P_i a^i_{xyI}\right)^2}\, d_{xy}}{D}. \quad \text{(A33)}$$

With these substitutions, $\frac{\partial M}{\partial a^i_{xyR}}$ in (A30) is simplified as

$$\frac{\partial M}{\partial a^i_{xyR}} = \frac{4P_i \rho_{xyR}(d_{xy} - M)}{D|\rho_{xy}|}. \quad \text{(A34)}$$

Applying the constraints (A21) is done by subtracting $\lambda^i_x \frac{\partial f^i_x}{\partial a^i_{xyR}}$ from $\frac{\partial M}{\partial a^i_{xyR}}$. Because

$$\frac{\partial f^i_x}{\partial a^i_{xyR}} = \frac{\partial}{\partial a^i_{xyR}} \left(\sum_y |a^i_{xy}|^2 - a^i_{xx}\right) = 2a^i_{xyR}, \quad \text{(A35)}$$

at last we end up the following equation:

$$\frac{\rho_{xyR}(d_{xy} - M)}{D|\rho_{xy}|} = \frac{\lambda^i_x}{2P_i} a^i_{xyR},$$

which is the same as (A27).

Replacing a^i_{xyR} and a^i_{xyI} from Eqs. (A27) and (A28) in (A21),

$$\sum_{y\neq x} \frac{[(\rho_{xyR})^2 + (\rho_{xyI})^2](d_{xy} - M)^2}{D^2|\rho_{xy}|^2} = \frac{\lambda^{i2}_x}{4P^2_i}\left(a^{i2}_{xx} - a^i_{xx}\right). \quad \text{(A36)}$$

Because $\rho^2_{xyR} + \rho^2_{xyI} = |\rho_{xy}|^2$,

$$\sum_{y\neq x} \frac{(d_{xy} - M)^2}{D^2} = \frac{\lambda^{i2}_x}{4P^2_i}\left(a^{i2}_{xx} - a^i_{xx}\right). \quad \text{(A37)}$$

Since d_{xy}'s have different amounts and $M \geqslant \frac{d_{\max}}{2}$ [if $M \leqslant \frac{d_{\max}}{2}$ the theorem is proved because $M(|\psi_{\mathrm{MMQS}}\rangle) = \frac{d_{\max}}{2}$], the left side of (A37) is positive, therefore λ^i_x's must be nonzero.

From Eqs. (A29), a^i_{xx}'s are $\frac{1}{2}$, consequently from the constraints (A22) we find that $|\psi_i\rangle$'s are pure states with 2×2 density matrices in which the diagonal elements are $\frac{1}{2}$, therefore $|\psi_i\rangle$'s must be of the form $(e^{i\phi}|a_i\rangle + |a_j\rangle)/\sqrt{2}$. Thus if only all $|\psi_i\rangle = |\psi_{\mathrm{MMQS}}\rangle$, M is maximized and ρ is the density matrix corresponding to $|\psi_{\mathrm{MMQS}}\rangle$ and the theorem is proved. ■

APPENDIX B: CALCULATION OF THE MEASURE FOR THE UNIFORM STATE

Here we calculate the measure for the uniform state in the basis of total spin z. Total spin z in a spin ensemble system in which the particles take the values 0 or 1 for the spin-z observable is equal to the number of particles having the value of spin z equal to 1, so in the basis of total spin z we can represent the density matrix of the uniform state as

$$\frac{1}{2^N} \sum_{\{i\},\{j\}} |i_1 i_2 \cdots i_N\rangle\langle j_1 j_2 \cdots j_N|, \quad \text{(B1)}$$

in which i_k and j_k indicate the spins of the kth particles in the z direction and take the values 0 or 1.

First we calculate $P_{(d)}$; we need to find the density matrix elements corresponding to the distances with the amount of d. These elements are those in which the discrepancy of numbers of 1 in $|i_1 i_2 \cdots i_N\rangle$ and $\langle j_1 j_2 \cdots j_N|$ is equal to d. If $|i_1 i_2 \cdots i_N\rangle$ has total z magnetization equal to m, then m number of i_k's must be 1 and the other $(N - m)$ are zero, so we have $\binom{N}{m}$ possible choices. In order to require $|i_1 i_2 \cdots i_N\rangle\langle j_1 j_2 \cdots j_N|$ to be associated with the distance d, the total z magnetization of $\langle j_1 j_2 \cdots j_N|$ must be $m + d$ or $m - d$, so for the first we have $\binom{N}{m+d}$ and for the last we have $\binom{N}{m-d}$ possible choices. Thus, based on the product rule, the number of elements associated with the distance d is

$$N_d = 2\sum_{m=0}^{N} \binom{N}{m}\binom{N}{m+d}. \quad \text{(B2)}$$

Because all of the elements in the density matrix of uniform state are equal to $\frac{1}{2^N}$, and the number of elements is $2^N \times 2^N$:

$$P_{(d)} = \frac{N_d}{2^N * 2^N} = \frac{2\sum_{m=0}^{N}\binom{N}{m}\binom{N}{m+d}}{2^{2N}}. \tag{B3}$$

Having $P_{(d)}$, we can calculate the measure directly for this state:

$$M = \frac{2\sum_{d=0}^{N}\sum_{m=0}^{N}\binom{N}{m}\binom{N}{m+d}d}{2^{2N}}. \tag{B4}$$

M can be simplified as

$$M = \frac{(N+1)!(2N+1)!}{N!(N+2)!2^{2N}} \tag{B5}$$

At last, in the limit $N \gg 1$ using Stirling's approximation, we have

$$M = e^{N\ln(\frac{(N+\frac{1}{2})^2}{(N-1)(N+2)})}.$$

APPENDIX C: EVALUATION OF THE MEASURE FOR DICKE THE STATE

A Dicke state is defined as

$$|\text{Dicke}\rangle = \frac{1}{\sqrt{\binom{N}{k}}}\sum_{\text{all permutations}}|N,k\rangle_i, \tag{C1}$$

in which the ket $|N,k\rangle_i$ is an eigenvector of z magnetization that belongs to a system having N number of spins, with eigenvalue of k and ith permutation of k out of N. The correspondent density matrix takes the form

$$\rho_{\text{Dicke}} = \frac{1}{\binom{N}{k}}\sum_{\text{all permutations } i,j}|N,k\rangle_{ij}\langle N,k|. \tag{C2}$$

All the components of ρ_{Dicke} have equal value of $1/\binom{N}{k}$, so to evaluate the measure, we only have to find the number of off-diagonal elements associated with the distance d, which we denote by N_d. In the case of single-particle spin-z measurement, the distance d is associated with the elements $|N,k\rangle_{ij}\langle N,k|$ in the density matrix in which d number of spins in the same position have different amounts of spin z. If we fix $|N,k\rangle_i$ there are $\binom{k}{d}\cdot\binom{N-k}{d}$ choices for $|N,k\rangle_j$ and overall we have $\binom{N}{k}$ choices for $|N,k\rangle_i$. Therefore, by the product rule we have

$$N_d = \binom{k}{d}\binom{N-k}{d}\binom{N}{k}. \tag{C3}$$

Having N_d and considering that all of the elements of the density matrix ρ_{Dicke} are equal, it will be straightforward to calculate M:

$$M_{\text{Dicke}} = \frac{\sum_{d=0}^{k}\binom{k}{d}\binom{N-k}{d}\binom{N}{k}d}{\binom{N}{k}^2}. \tag{C4}$$

M_{Dicke} in (C4) can be simplified as

$$M_{\text{Dicke}} = \frac{k(N-k)}{N}. \tag{C5}$$

APPENDIX D: GENERALIZED GHZ

Another interesting state is the generalized GHZ state considered in [8,21]. The generalized GHZ state is defined as

$$|\phi_\epsilon\rangle = \frac{|0\rangle^{\otimes N} + (\cos\epsilon|0\rangle + \sin\epsilon|1\rangle)^{\otimes N}}{2 + 2\cos^N\epsilon}. \tag{D1}$$

We calculate the measure for this state in the limits $N \gg 1$ and $\epsilon \ll 1$ and $N\epsilon < 1$, which gives

$$M_{\text{GHZ}_\epsilon} \approx \frac{N\epsilon}{2}. \tag{D2}$$

As we see, quantum macroscopicity of generalized GHZ, evaluated by our measure, is plausible compared to the amount that the Dür *et al.* measures obtain in the same limits, which is $N\epsilon^2$ [50].

APPENDIX E: SOME OTHER PHOTONIC STATES

1. Mixed SCS

Mixed SCS is defined as

$$\rho \;\propto\; |\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|. \tag{E1}$$

In the case of $|\alpha| \gg 1$, the two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ could be considered orthogonal to each other [51]. Hence in the basis of the quadrature $X\cos\theta + P\sin\theta$ with $\tan\theta = \frac{\text{Im}(\alpha)}{\text{Re}(\alpha)}$ and for large amounts of $|\alpha|$, the density matrix of mixed SCS turns to a diagonal one and the measure becomes zero for the state. This result has meaning when we compare the mixed SCS with SCS; Compared to SCS, a mixed SCS has lost its coherence terms in the aforementioned basis and it should not be macroscopic quantum.

2. Thermal state

The thermal state [51] is a thermal classical mix of photons with the density matrix

$$\rho_{\text{Thermal}} = \frac{\sum_N e^{-\beta N}|N\rangle\langle N|}{Z}, \tag{E2}$$

$Z = \sum_N e^{-\beta N}$ is the normalization factor (i.e., in terms of statistical mechanics it is the partition function). Because the density matrix has no coherence (off-diagonal) terms,

$$M_{\text{Thermal}} = 0. \tag{E3}$$

3. Two-mode squeezed state

We also consider the two-mode squeezed state which is defined as

$$|\text{2-mode squeezed}\rangle = \sqrt{1-(\tanh\varepsilon)^2}\sum_{n=0}^{\infty}(\tanh\varepsilon)^n|n,n\rangle. \tag{E4}$$

Calculation of the measure for this state gives

$$M = \frac{2\sum_{n>m}(\tanh\varepsilon)^{n+m}(n-m)}{\sum n,m(\tanh\varepsilon)^{n+m}}. \tag{E5}$$

The term in Eq. (E5) can be simplified as

$$M = \frac{4\tanh\varepsilon}{1-(\tanh\varepsilon)^2}. \tag{E6}$$

As we see in the relation (E6), if $\tanh\varepsilon \to 1$, M goes to infinity, so for $\varepsilon \gg 1$ the two-mode squeezed state is a macroscopic quantum state.

4. BAT state

Another photonic state which is of interest in this context is the BAT state [52]. The BAT state is defined as

$$|\mathrm{BAT}\rangle = \sum_{k=0}^{N/4} \frac{\sqrt{(N/2-k)!}\sqrt{(2k!)}}{k!(N/4-k)!\sqrt{2^(N/2)}} |N/2-2k\rangle_1 |2k\rangle_2. \tag{E7}$$

Directly calculating the measure for BAT state, we obtain

$$M = \frac{4\sum_{k,l}^{N/4} \frac{\sqrt{(N/2-k)!(2k)!(N/2-l)!(2l)!}}{k!l!(N/4-k)!(N/4-l)!}|k-l|}{\sum_{k,l}^{N/4} \frac{\sqrt{(N/2-k)!(2k)!(N/2-l)!(2l)!}}{k!l!(N/4-k)!(N/4-l)!}}. \tag{E8}$$

We have plotted the amount of M_{BAT} versus N in the interval $0 < N < 100$, as shown in Fig. 2. It is evident from the plot that the measure for the BAT state is $O(N)$. So, for large number of photons $N \gg 1$, the BAT state has high amount of macroscopic quantumness based on our measure.

APPENDIX F: COMPARISON WITH OTHER MEASURES

Dür *et al.* presented two measures for macroscopic quantumness of a generalized GHZ state based on comparison with maximum size of a GHZ state that can be obtained by the distillation process or has the same decoherence rate [21]. Both measures give $O(N\epsilon^2)$ as the value of quantum macroscopicity of a generalized GHZ in the limits $N \gg 1$ and $\epsilon \ll 1$. Our measure gives $N\epsilon/2$ for this setting, which is similar in the scaling with respect to N, although it has a different scaling in ϵ.

Bjork and Mana suggested a criterion of macroscopic quantumness for Schrödinger's cat states, i.e., $|A\rangle + |D\rangle$ based on how fast the state becomes orthogonal to itself when subjected to a specific unitary transformation [23]. In their paper, they calculate their measure for the NOON, generalized GHZ (in the limits $N \gg 1$ and $\epsilon \ll 1$), and SCS states and obtain the amounts of $\sqrt{N}$, $\sqrt{N}\epsilon$, and $2|\alpha|$ respectively. The measure of Bjork and Mana classifies these states similarly to ours, although, the scalings are slightly different. However, our measure is in better agreement with the scaling of other measures for these states.

Shimizu and Miyadera have introduced a measure of macroscopic quantumness based on the the spread of entanglement in a multipartite macroscopic system [22]. To characterize this insight, they chose the variance of an additive operator. They quantify macroscopic quantumness by an index p, where $1 \leqslant p \leqslant 2$, that shows the scaling order of the variance in the number of elements (e.g., particles, spins) [26]. The index p takes its maximum, i.e., $p = 2$, for the GHZ and NOON states. For a Dicke state, if $k = O(N/2)$ then $p = 2$ and if $k = O(1)$ then $p = 1$, thus in the latter case the state has no quantum macroscopicity. All these results are in agreement with our measure. We showed in Theorem 1 that our measure is maximized for NOON and GHZ. Also, for single-particle measurement of the magnetization in the z direction, for the Dicke state, our measure gives zero if $k = O(1)$ and has maximum scaling order in N if $k = O(N/2)$.

Lee and Jeong proposed that the macroscopic quantumness of a photonic state is related to the amplitude and intensity of frequencies in its Wigner function [37]. In their work, they calculated their measure for SCS, GHZ and NOON state and obtained the amounts of $|\alpha|^2$, N, and N respectively. Similarly to our measure, their measure also takes its maximum for these states. For SCS, our measure scales with $|\alpha|$ but that of Lee and Jeong scales with $|\alpha|^2$. Another state that Lee and Jeong investigated was the generalized GHZ, for which their measure takes $N\epsilon^2/2$ in the limits $N \gg 1$ and $\epsilon \ll 1$.

Fröwis and Dür proposed to use Fisher information for the characterization of macroscopic quantumness for a many-body state [32]. Their measure gives an effective size for the state, which for some of the states of interest takes the following forms:

(1) GHZ: $N_{\mathrm{eff}} = O(N)$.
(2) NOON: $N_{\mathrm{eff}} = O(N)$.
(3) Dicke state with $k = O(N/2)$: $N_{\mathrm{eff}} = O(N)$.
(4) Dicke state with $k = O(1)$: $N_{\mathrm{eff}} = O(1)$.
(5) SCS: $N_{\mathrm{eff}} = 4|\alpha|^2$.
(6) Generalized GHZ (in the limits $N \gg 1$ and $\epsilon \ll 1$): $N_{\mathrm{eff}} = O(N\epsilon^2)$.

For 1–4 our measure obtains the same scaling order in N and is in agreement with the measure of Fröwis and Dür. Also, $O(N)$ is the maximum scaling order in N for these states in both measures. For SCS, both measures suggest high macroscopic quantumness for large amounts of $|\alpha|$; however, the scaling order is different. For generalized GHZ the scaling order in N is equal in both proposals, although they scale differently in ϵ.

Yadin and Vedral, in addition to the general framework, also introduced a measure for quantification of the macroscopic quantumness in spin ensembles. It works based on the maximum size of a GHZ state that can be obtained by a distillation process from the state of interest [38]. Their measure gives the following values for the GHZ, Dicke, and generalized GHZ state

(1) GHZ: $N_f = N$.
(2) Dicke state: $N_f = 1 + \frac{2k(N-k)}{N}$.
(3) Generalized GHZ (in the limits $N \gg 1$ and $\epsilon \ll 1$): $N_f = N\epsilon^2/2$.

Here N_f denotes their measure. Our measure is in complete agreement with the results 1–3, except that for the generalized GHZ the scaling in ϵ is different. Both measures are maximized for the GHZ state and the scaling is of the order $O(N)$. Also for Dicke state, if $k = O(N)$, the proposals take their maximum order in N which is $O(N)$, and if $k = O(1)$ they are of the order $O(1)$.

APPENDIX G: RELATION WITH THE GENERAL FRAMEWORK OF YADIN AND VEDRAL [46]

In their general framework of macroscopic quantumness [46], Yadin and Vedral discuss that, for an observable A, a state exhibits macroscopic coherence if there exists superposition of eigenstates of A having macroscopically different eigenvalues [46]. They also argue that any measure for quantifying the concept of macroscopic coherence should meet the conditions below [46]:

M1: $M(\rho) \geqslant 0$ and $M(\rho) = 0 \iff \rho = 0$.

M2a: For a free operation ξ we have $M(\rho) \geqslant M(\xi(\rho))$.

M2b: For $\xi = \sum_\alpha \xi_\alpha$ in which ξ_α are free operations, $M(\rho) \geqslant \sum_\alpha P_\alpha M(\sigma_\alpha)$, in which $\sigma_\alpha = \xi_\alpha(\rho)/P_\alpha$ has the probability $P_\alpha = \mathrm{Tr}[\xi_\alpha(\rho)]$.

M3: Convexity: $M(\sum_i P_i \rho_i) \leqslant \sum_i P_i M(\rho_i)$.

M4: Consider $|\psi\rangle = \frac{|i\rangle + |j\rangle}{\sqrt{2}}$ and $|\phi\rangle = \frac{|m\rangle + |n\rangle}{\sqrt{2}}$. If $|a_i - a_j| \geqslant |a_m - a_n|$ then $M(|\psi\rangle\langle\psi|) \geqslant M(|\phi\rangle\langle\phi|)$.

The first three conditions are identical to the ones for a resource theory of coherence [45]. Condition M2 implies that a free operation in the context of coherence should not increase coherence of the system.

Condition M3 implies that mixing cannot increase coherence.

Condition M4 is the most important one in the context of macroscopic quantumness. This condition makes sure that the coherence is a macroscopic coherence.

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