Qubit-environment-entanglement generation and the spin echo

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We analyze the relationship between qubit-environment entanglement that can be created during the pure dephasing of the qubit and the effectiveness of the spin-echo protocol. We focus here on mixed states of the environment. We show that whereas the echo protocol can obviously counteract classical environmental noise, it can also undo dephasing associated with qubit-environment entanglement, and there is no obvious difference in its efficiency in these two cases. Additionally, we show that qubit-environment entanglement can be generated at the end of the echo protocol even when it is absent at the time of application of the local operation on the qubit (the π pulse). We prove that this can occur only at isolated points in time, after fine-tuning of the echo protocol duration. Finally, we discuss the conditions under which the observation of specific features of the echo signal can serve as a witness of the entangling nature of the joint qubit-environment evolution.

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I. INTRODUCTION

Environmentally induced dephasing of superpositions of pointer states of a controlled quantum system is commonly associated with creation of system-environment entanglement, or, at least, the presence of the latter is deemed to be necessary in order to call this process quantum decoherence [1–3]. However, as has been pointed out in literature, this association holds only when the initial states of both the qubit *and the environment* are pure [1–4]. In the more general, and much more realistic, case of mixed environmental states, dephasing of the system *does not* have to be accompanied by establishment of system-environment entanglement, and intuitions concerning distinguishing between "quantum decoherence" and "dephasing due to classical environmental noise" (understood here strictly as leading to no system-environment entanglement) that are built in works focusing on pure-state vs "classical" environments become unreliable [5–13].

We shed light on this general problem by focusing on the relationship between the effectiveness of qubit coherence recovery in a spin-echo experiment [14–16], which is well known to lead to such a recovery when the environment is a source of external noise of mostly low-frequency character [17,18]. We show that the echo procedure can (but does not have to) lead to coherence recovery when the dephasing is *not* associated with qubit-environment entanglement (QEE), but it can also undo QEE, whereas using only local operations on the qubit. Interestingly, there is no obvious correlation between the efficiency of coherence recovery and presence or absence of QEE generated during the evolution of the qubit and its environment.

In fact, we show that it is possible for QEE to appear at the end of the echo protocol with no entanglement present at the time of application of the unitary operation to the qubit. This should not be surprising as the evolutions that are most interesting in the context of echo protocol typically have non-Markovian character, and at the time of application of the local unitary operation the state of the qubit and the environment is typically correlated. This effect can, however, only occur at isolated points in time, and this is the only feature of the echo experiment that conforms to the commonly encountered (but generally incorrect) intuitions that echo protocol should undo the generation of QEE as it typically undoes qubit dephasing.

Although most of our results underline the lack of strong correlation between efficacy of coherence recovery in the spin-echo protocol and the presence of QEE during the evolution, we show that there is, at least, one situation in which the appearance of a phase shift between the initial and the echoed coherence of the qubit signifies that the evolution is of QEE-generating character.

The paper is organized as follows. In Sec. II we introduce the echo protocol for the qubit undergoing pure dephasing due to an interaction with its environment and recapitulate the basic criterion for the appearance of QEE during pure-dephasing evolution. In Sec. III we discuss the conditions for the echo to work prefectly, i.e., to lead to the recovery of the initial pure state of the qubit. As the perfect echo necessarily leads to the removal of any entanglement (if any was in fact present during the evolution), in Sec. IV we focus on the imperfect echo and its relation to generation of entanglement during the evolution. There is no simple relation, and we show there that the echo can, in fact, lead to the creation of entanglement in the final state even if there was none at the time of application of the local operation to the qubit. However, as we show in Sec. V, it can happen only at certain points in time, and the π pulse applied to the qubit cannot transform a joint system evolution which is essentially nonentangling into an entangling one. Finally, in Sec. VI we describe the conditions for the initial environmental state and qubit-environment coupling that allows to use the echo signal as a witness of the entangling nature of the evolution of the qubit and its environment. Sec. VII concludes the paper.

II. PURE DEPHASING, ENTANGLEMENT, AND ECHO

A. Pure dephasing

In the following, we study the spin echo performed on a qubit in an arbitrary pure-dephasing scenario, meaning that the only constraint on the qubit-environment interaction is that it does not disturb the occupations of the qubit [9,19,20]. The most general form of the Hamiltonian which describes the pure-dephasing case is

$$\hat{H} = \hat{H}_\mathrm{Q} + \hat{H}_\mathrm{E} + |0\rangle\langle 0| \otimes \hat{V}_0 + |1\rangle\langle 1| \otimes \hat{V}_1. \tag{1}$$

The first term of the Hamiltonian describes the qubit and is given by $\hat{H}_\mathrm{Q} = \sum_{i=0,1} \varepsilon_i |i\rangle\langle i|$, the second describes the environment, whereas the remaining terms describe the qubit-environment interaction with the qubit states written on the left side of each term (the environment operators $\hat{V}_0$ and $\hat{V}_1$ are arbitrary as is the free Hamiltonian of the environment $\hat{H}_\mathrm{E}$). Hence, the only constraint on the Hamiltonian, which restricts the qubit evolution to pure dephasing, is that the interaction term is diagonal with respect to the qubit eigenstates.

The evolution operator corresponding to the Hamiltonian (1) may, in general, be written in the form

$$\hat{U}(t) = |0\rangle\langle 0| \otimes \hat{w}_0(t) + |1\rangle\langle 1| \otimes \hat{w}_1(t), \tag{2}$$

where $\hat{w}_i(t) = \exp(-\frac{i}{\hbar}\varepsilon_i t)\exp(-\frac{i}{\hbar}\hat{H}_i t)$ with $\hat{H}_i = \hat{H}_\mathrm{E} + \hat{V}_i$ (the first exponential term is responsible for the phase evolution which comes from the free Hamiltonian of the qubit). Note that, although $\hat{H}_\mathrm{Q}$ commutes with all the other terms in $\hat{H}$, this is not necessarily the case with $\hat{H}_\mathrm{E}$. We assume that the initial state has no correlations between the qubit and the environment,

$$\hat{\sigma}(0) = |\psi\rangle\langle\psi| \otimes \hat{R}(0), \tag{3}$$

with the initial qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ and $\hat{R}(0)$ being the initial state of the environment. The qubit-environment density matrix at a later time can be written as

$$\hat{\sigma}(t) = \begin{pmatrix} |a|^2 \hat{w}_0(t)\hat{R}(0)\hat{w}_0^\dagger(t) & ab^* \hat{w}_0(t)\hat{R}(0)\hat{w}_1^\dagger(t) \\ a^* b \hat{w}_1(t)\hat{R}(0)\hat{w}_0^\dagger(t) & |b|^2 \hat{w}_1(t)\hat{R}(0)\hat{w}_1^\dagger(t) \end{pmatrix}. \tag{4}$$

Here the matrix form only pertains to the qubit subspace and is written in terms of qubit pointer states. If only the state of the qubit is of interest, then the reduced density matrix of the qubit is obtained by tracing out the environment from the matrix (4), and we get

$$\hat{\rho}(t) = \mathrm{Tr}_E \hat{\sigma}(t) = \begin{pmatrix} |a|^2 & ab^* W(t) \\ a^* b W^*(t) & |b|^2 \end{pmatrix}, \tag{5}$$

with normalized coherence,

$$W(t) = \mathrm{Tr}[\hat{R}(0)\hat{w}_1^\dagger(t)\hat{w}_0(t)]. \tag{6}$$

B. Spin echo during pure dephasing

The procedure which is known as the spin echo [14–16] can be described as follows. After the initialization of the qubit state, the qubit and environment evolve for a certain time τ after which a π pulse about the x or y axis is applied to the qubit (for concreteness we focus here on pulses about the x axis). Such a pulse interchanges the amplitudes of $|0\rangle$ and $|1\rangle$ states. Then the system is allowed to evolve for the same time period τ, and another π pulse is applied. In the ideal case, this leads to the qubit regaining its initial state at time 2τ (after the second π pulse), but even in nonideal scenarios the decoherence which is observed after the echo sequence can be much smaller compared to the evolution without the echo when the environment is a source of external noise of mostly low-frequency character [17] (see Sec. III B below for a concise formal explanation of this fact).

The evolution in the echo experiment with the final time 2τ is described by the operator,

$$\hat{U}_\mathrm{echo}(2\tau) = \hat{\sigma}_x \hat{U}(\tau)\hat{\sigma}_x \hat{U}(\tau), \tag{7}$$

where $\hat{\sigma}_x$ is the appropriate Pauli matrix which describes the action of the π pulse on the qubit and $\hat{U}(\tau)$ is a joint system-environment evolution operator, which for pure dephasing is given by Eq. (2). The second π pulse at time 2τ interchanges the two complex-conjugate coherences in the final reduced state of the qubit. and it is added for convenience, to make the final coherence equal to the original one, not to its complex conjugate in the case of perfect echo.

We assume that the initial state of the qubit-environment system is given by Eq. (3). Then the joint system-environment state at time τ before the first π pulse is given by the density matrix (4). Modeling the whole procedure with the evolution operator (7) we get the qubit-environment state directly after the echo sequence is performed, which is given by

$$\hat{\sigma}(2\tau) = \begin{pmatrix} |a|^2 \hat{w}_1(\tau)\hat{w}_0(\tau)\hat{R}(0)\hat{w}_0^\dagger(\tau)\hat{w}_1^\dagger(\tau) & ab^* \hat{w}_1(\tau)\hat{w}_0(\tau)\hat{R}(0)\hat{w}_1^\dagger(\tau)\hat{w}_0^\dagger(\tau) \\ a^* b \hat{w}_0(\tau)\hat{w}_1(\tau)\hat{R}(0)\hat{w}_0^\dagger(\tau)\hat{w}_1^\dagger(\tau) & |b|^2 \hat{w}_0(\tau)\hat{w}_1(\tau)\hat{R}(0)\hat{w}_1^\dagger(\tau)\hat{w}_0^\dagger(\tau) \end{pmatrix}. \tag{8}$$

The echoed qubit state is obtained as in the case of simple decoherence (5) by tracing out the environment from Eq. (8), which yields $\hat{\rho}(2\tau) = \mathrm{Tr}_E \hat{\sigma}(2\tau)$, which has the same structure as Eq. (5) but with normalized coherence,

$$W(2\tau) = \mathrm{Tr}[\hat{R}(0)\hat{w}_1^\dagger(\tau)\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau)\hat{w}_0(\tau)]. \tag{9}$$

C. QEE condition for pure dephasing with and without echo

For any bipartite density matrix which can be written in the form (4), the if and only if condition of qubit-environment separability is

$$[\hat{w}_0^\dagger(t)\hat{w}_1(t), \hat{R}(0)] = 0\ , \tag{10}$$

as has been proven in Ref. [9]. The original derivation involves the positive-partial-transpose criterion [21,22] in one direction and the definition of mixed bipartite separable states in the other. Since the qubit-environment state at time τ before the π pulse is applied is given precisely by Eq. (4), the condition can be explicitly used to check for QEE present just before the application of the pulse (the prepulse entanglement).

The QEE present in the system after the echo procedure is performed is similarly straightforward to study because the qubit-environment density matrix (8) is of the same form as the one that is obtained by a simple pure-dephasing interaction (4). The two can be reduced to one another by the transformation,

$$\hat{w}_0'(2\tau) = \hat{w}_1(\tau)\hat{w}_0(\tau), \tag{11a}$$

$$\hat{w}_1'(2\tau) = \hat{w}_0(\tau)\hat{w}_1(\tau). \tag{11b}$$

Then the condition for separability of the echoed state is

$$[\hat{w}_0'^{\dagger}(2\tau)\hat{w}_1'(2\tau), \hat{R}(0)] = [\hat{w}_0^{\dagger}(\tau)\hat{w}_1^{\dagger}(\tau)\hat{w}_0(\tau)\hat{w}_1(\tau), \hat{R}(0)] = 0. \tag{12}$$

III. CONDITIONS FOR PERFECT ECHO

A. General considerations

For the echo to be perfect, meaning that the qubit state which is obtained after performing the echo is equal to the initial qubit state $\mathrm{Tr}_E\hat{\sigma}(2\tau) = |\psi\rangle\langle\psi|$, the following condition needs to be met

$$[\hat{w}_0^{\dagger}(\tau), \hat{w}_1(\tau)] = 0. \tag{13}$$

The complementary condition $[\hat{w}_0(\tau), \hat{w}_1(\tau)] = 0$ follows from the above equation since commutation of two operators implies that there exists a basis in which both operators are diagonal and the Hermitian conjugate of any operator is always diagonal in the same basis as the operator itself.

In the situation when the echo reinstates the initial qubit state, it also severs any entanglement which may have been generated between the qubit and the environment during their joint evolution. However, the condition for perfect echo is not related in any way to the condition for absence of QEE at time τ, which is given by Eq. (10). The latter depends on the initial state of the density matrix of the environment and can be fulfilled both when the conditional evolution operators of the environment commute, and when they do not.

It is fairly straightforward to find an evolution which leads to a perfect echo for a given τ, or even for any τ, but does not lead to any QEE generation, and one that does lead to entanglement generation.

For example, if $[\hat{V}_i, \hat{H}_E]=0$ for $i = 0$, 1, and $\hat{R}(0) \propto \exp(-\beta\hat{H}_E)$, i.e., the environment is in a thermal equilibrium state achieved in absence of the qubit, then there is no entanglement generated at time τ as Eq. (10) is fulfilled. However, the echo is perfect only if additionally $[\hat{V}_0, \hat{V}_1]=0$.

On the other hand, if we assume all the commutation relations from the previous example to be fulfilled, but take $\hat{R}(0)$ such that $[\hat{R}(0), \hat{V}_0 - \hat{V}_1]\neq 0$, we have perfect echo at time 2τ, but the qubit-environment state is entangled at time τ. These examples already show that the behavior of echoed coherence reflects the general feature of dephasing caused by an environment in a mixed state: There is no direct correspondence between the generation of QEE and the amount of dephasing. The echo procedure can undo dephasing (even perfectly) not only in the "classical dephasing" case (using the terminology from Ref. [1]) in which no entanglement is established, but also in the "true quantum decoherence" case in which entanglement is created during the evolution.

B. Small decoherence limit

If the echoed coherence $W(2\tau)$ is close to unity as happens when 2τ is close to the time at which the echo is perfect, one can approximate it by an expression valid to second order in qubit-environment coupling. For simplicity, let us focus on a less general form of the $\hat{V}_i$ operators, namely,

$$\begin{aligned}\hat{V}_0 &= \tfrac{1}{2}\lambda(\eta + 1)\hat{V},\\ \hat{V}_1 &= \tfrac{1}{2}\lambda(\eta - 1)\hat{V},\end{aligned} \tag{14}$$

so that the qubit-environment coupling takes the form $\frac{1}{2}\lambda(\eta\hat{\mathbb{1}} - \hat{\sigma}_z) \otimes \hat{V}$. In the formulas above, λ is a dimensionless parameter controlling the strength of the coupling, whereas η controls the "bias" of the coupling. A commonly used "unbiased" coupling $\propto\hat{\sigma}_z \otimes \hat{V}$, which occurs, for example, for qubits based on spin-1/2 entities coupled to an environment via the magnetic dipole interaction [23,24], corresponds to $\eta = 0$, whereas the biased case of $\eta = -1$ applies, for example, to excitonic qubits [25–28] or to qubits based on $m = 0$ and $m = \pm 1$ levels of a qubit based on a spin-1 entity, such as a nitrogen-vacancy center in diamond [29,30]. A calculation of coherence up to λ^2 order gives [31,32]

$$W(2\tau) \approx 1 - \lambda^2\chi(2\tau) - i\eta\lambda^2\Phi(2\tau), \tag{15}$$

where the attenuation function $\chi(t)$ and the phase shift $\Phi(t)$ are real functions given by

$$\chi(2\tau) = \frac{1}{2}\int_0^{2\tau} dt_1 \int_0^{t_1} dt_2 f(t_1)f(t_2)C(t_1, t_2), \tag{16}$$

$$\Phi(2\tau) = \frac{1}{2}\int_0^{2\tau} dt_1 \int_0^{t_1} dt_2 f(t_2)K(t_1, t_2), \tag{17}$$

where

$$C(t_1, t_2) = \mathrm{Tr}_E[\hat{R}(0)\{\hat{V}(t_1), \hat{V}(t_2)\}] \tag{18}$$

is the autocorrelation function of the operator $\hat{V}(t) = \exp(i\hat{H}_E t)\hat{V}\exp(-i\hat{H}_E t)$, whereas

$$K(t_1, t_2) = -i\theta(t_1 - t_2)\mathrm{Tr}_E(\hat{R}(0)[\hat{V}(t_1), \hat{V}(t_2)]) \tag{19}$$

is the linear-response function [33,34] associated with this operator, and the temporal filter function [17,35] for the echo experiment is given by $f(t) = \Theta(t)\Theta(\tau - t) - \Theta(t - \tau)\Theta(2\tau - t)$, i.e., $|f(t)| = 1$ for $t \in [0, 2\tau]$ is zero otherwise, and changes sign at $t = \tau$. For the derivation of the expression for $\chi(2\tau)$ see Ref. [18], whereas the derivations of the formula for phase $\Phi(2\tau)$ can be found in Refs. [31,32].

We assume that the environment is initially in a stationary state of its free Hamiltonian $[\hat{R}(0), \hat{H}_E]=0$, which implies that $C(t_1, t_2)$ is a function of a single variable $\Delta t = t_1 - t_2$. We can then introduce the power spectral density (PSD) of the noise, defined by

$$S(\omega) = \int_{-\infty}^{\infty} e^{i\omega\,\Delta t} C(\Delta t) d\,\Delta t\ , \tag{20}$$

and express the attenuation function and the phase shift as

$$\chi(2\tau) = \int_{-\infty}^{\infty} \frac{8\,\sin^4\frac{\omega\tau}{2}}{\omega^2} S(\omega)\frac{d\omega}{2\pi}, \tag{21}$$

$$\Phi(2\tau) = \int_{-\infty}^{\infty} \frac{8\,\sin^4\frac{\omega\tau}{2}}{\omega^2} \text{cotan}\frac{\omega\tau}{2}\tanh\frac{\beta\omega}{2} S(\omega)\frac{d\omega}{2\pi}. \tag{22}$$

Here, in order to derive the second of these expressions, we have assumed that the environment is actually in a thermal state, i.e., $\hat{R}(0) = e^{-\beta\hat{H}_E}/\text{Tr}\, e^{-\beta\hat{H}_E}$.

Vanishing $\chi(2\tau)$ is necessary for the perfect echo. Here we see that, taking into account the fact that $S(\omega)$ is positive definite, this can happen at $\tau \neq 0$ only when PSD consists of a series of delta peaks at frequencies $\omega_k = 2\pi k/\tau$ for integer k. The most commonly encountered case is of PSD concentrated only at very low frequencies (only the $k=0$ peak is present), i.e., $S(\omega) \propto \delta(\omega)$. This corresponds to a time-independent symmetric correlator of $\hat{V}(t)$, i.e., $C(\Delta t)$, which requires $[\hat{H}_E, \hat{V}] = 0$. This situation is, thus, equivalent to the previously discussed case of the perfect echo. The situation of $S(\omega)$ with periodically positioned narrow peaks in frequency is more interesting as it corresponds to $\hat{V}(t)$ that has nontrivial dynamics. It is also not particularly artificial: It corresponds to situations in which the second-order correlation function of the environmental operator $\hat{V}$ has a well-defined periodicity. A perfect echo can occur at isolated points in time in this case.

Let us note that, although the response function $K(\Delta t)$ vanishes when the environment is completely mixed, the symmetric correlation function $C(\Delta t)$ has no reason to vanish in this situation. The presence of a finite attenuation function χ and, thus, of finite decay of qubit coherence, obviously does not require the presence of QEE: Note that the condition (10) for QE separability is fulfilled for a completely mixed initial environmental state.

IV. IMPERFECT ECHO AND QEE

A. Echo-induced entanglement

Let us consider the situation when at time τ at which we apply a *local* operation to one part (the qubit) of our bipartite system, the condition of qubit-environment separability is fulfilled (10), but the perfect-echo condition (13) is not. Since the perfect echo kills any QEE that was generated during the evolution, one could expect that a nonperfect echo, still leading to a partial recovery of coherence, should diminish its amount compared to values attained during the evolution, for example, at the time of application of the pulse. In particular, if the evolution does not entangle the qubit with is environment at the time the first π pulse is applied, it should not lead to QEE after the whole echo procedure is performed. In the following, we will show that this is, in fact, not necessarily the case. This is nothing else, but another result of the general fact that the magnitude of system dephasing is rather weakly affected by presence or absence of system-environment entanglement *when the environmental state is far from being pure*.

The condition of separability (10) is equivalent to the statement that there exists a basis in which both the operator $\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau)$ and the initial density matrix of the environment $\hat{R}(0)$ are diagonal. Although diagonality in this basis is obviously preserved for the conjugate of $\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau)$, there is no reason why the operators $\hat{w}_1^\dagger(\tau)$ and $\hat{w}_0(\tau)$ should also be diagonal in this basis. In other words, for any two evolution operators $\hat{w}_0^\dagger(\tau)$ and $\hat{w}_1(\tau)$ which do not commute at a given time τ [which means that $\hat{w}_0^\dagger(\tau)$ is diagonal in a different basis than $\hat{w}_1(\tau)$], there exists a set of initial environmental states for which $[\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau), \hat{R}(0)] = 0$. If the initial state of the environment is described by one of these density matrices then at time τ (both before and after the first π pulse), the qubit-environment density matrix obtained by using the evolution operator (2) is separable but is no longer a product state. The state (after the π pulse) can be written as

$$\sigma(\tau) = \begin{pmatrix} |b|^2\hat{R}_{00}(\tau) & a^*b\hat{w}_1(\tau)\hat{w}_0^\dagger(\tau)\hat{R}_{00}(\tau) \\ ab^*\hat{R}_{00}(\tau)\hat{w}_0(\tau)\hat{w}_1^\dagger(\tau) & |a|^2\hat{R}_{00}(\tau) \end{pmatrix}, \tag{23}$$

where $\hat{R}_{00}(\tau) = \hat{w}_0(\tau)\hat{R}(0)\hat{w}_0^\dagger(\tau)$ and the fact that

$$\hat{w}_0(\tau)\hat{R}(0)\hat{w}_0^\dagger(\tau) = \hat{w}_1(\tau)\hat{R}(0)\hat{w}_1^\dagger(\tau) \tag{24}$$

is a straightforward consequence of the separability criterion (10) being fulfilled at time τ. Applying the other half of the echo procedure [unitary evolution $U(\tau)$ followed by the σ_x operator] yields

$$\sigma(2\tau) = \begin{pmatrix} |a|^2\hat{w}_1(\tau)\hat{R}_{00}(\tau)\hat{w}_1^\dagger(\tau) & ab^*\hat{w}_1(\tau)\hat{R}_{00}(\tau)\hat{w}_0(\tau)\hat{w}_1^\dagger(\tau)\hat{w}_0^\dagger(\tau) \\ a^*b\hat{w}_0(\tau)\hat{w}_1(\tau)\hat{w}_0^\dagger(\tau)\hat{R}_{00}(\tau)\hat{w}_1^\dagger(\tau) & |b|^2\hat{w}_0(\tau)\hat{R}_{00}(\tau)\hat{w}_0^\dagger(\tau) \end{pmatrix}. \tag{25}$$

This qubit-environment density matrix is separable if and only if the condition,

$$[\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau), \hat{R}_{00}(\tau)] = 0 \tag{26}$$

is fulfilled. The condition is equivalent to the separability criterion for a product initial state of the qubit, and the environment initially in state $\hat{R}_{00}(\tau)$ when the evolution is governed by the operators $\hat{w}_0(\tau)$ and $\hat{w}_1(\tau)$, Eq. (10). Interestingly, the resulting state (25) is different than the state which would be obtained at time τ from an initial environmental state $\hat{R}(0) = \hat{R}_{00}(\tau)$. This becomes obvious when the elements of the density matrix proportional to ab^* are compared in both cases since $\hat{w}_0(\tau)\hat{w}_1^\dagger(\tau)\hat{w}_0^\dagger(\tau) \neq \hat{w}_1^\dagger(\tau)$ [because we assumed that $\hat{w}_0(\tau)$ and $\hat{w}_0^\dagger(\tau)$ do not commute with $\hat{w}_1^\dagger(\tau)$].

B. Example of qubit-environment entanglement generated via the spin echo at time 2τ for a separable state at time τ

As an example let us study a qubit interacting with an environment of dimension $N = 2$. We will study a pair of interaction operators $\hat{w}_0(\tau)$ and $\hat{w}_1(\tau)$ that do not lead to entanglement in the density matrix (23) but lead to entanglement in the echoed density-matrix (25) for a set of initial environmental states.

Our exemplary operators $\hat{w}_0(\tau)$ and $\hat{w}_1(\tau)$ written in the eigenbasis of the initial environment density-matrix $\hat{R}(0) = c_0|0\rangle\langle 0| + c_1|1\rangle\langle 1|$ are

$$\hat{w}_0^\dagger(\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{27a}$$

$$\hat{w}_1(\tau) = \hat{w}_1^\dagger(\tau) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{27b}$$

The operators do not commute, and we find that

$$\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau) = \hat{w}_1^\dagger(\tau)\hat{w}_0(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{28}$$

are diagonal in the eigenbasis of $\hat{R}(0)$ meaning that the evolution (without the echo) does not yield entanglement at time τ for any c_0 since $[\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau), \hat{R}(0)] = 0$. On the other hand, this does not mean that there is no qubit decoherence since the off-diagonal elements of the qubit density matrix are proportional to

$$\mathrm{Tr}[\hat{w}_1^\dagger(\tau)\hat{w}_0(\tau)\hat{R}(0)] = c_0 - c_1. \tag{29}$$

Hence, the qubit state remains pure only for an initial pure state of the environment, $c_0 = 0$ or 1, with the purity reaching its minimal possible value in the type of evolutions described for a completely mixed environment $c_0 = c_1 = 1/2$.

It is now straightforward to find the operators which govern QEE in the case of the quantum echo,

$$\hat{w}_0^\dagger(\tau)\hat{w}_1^\dagger(\tau)\hat{w}_0(\tau)\hat{w}_1(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{30}$$

This operator is obviously not diagonal in the eigenbasis of the initial environment density matrix. Furthermore,

$$[\hat{w}_0^\dagger(\tau)\hat{w}_1^\dagger(\tau)\hat{w}_0(\tau)\hat{w}_1(\tau), \hat{R}(0)] = (c_1 - c_0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{31}$$

and the condition for separability (12) is fulfilled only for $c_0 = c_1 = \frac{1}{2}$, another words, only when the initial density matrix of the environment is proportional to unity, $\hat{R}(0) \sim \mathbb{I}$.

When it comes to qubit decoherence, we always have

$$\mathrm{Tr}[\hat{w}_1^\dagger(\tau)\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau)\hat{w}_0(\tau)\hat{R}(0)] = 0, \tag{32}$$

which means that the qubit at time 2τ is always fully decohered, regardless of the initial state of the environment. In this extreme case, the spin echo can do no damage in the best scenario, whereas for most states of the environments, the procedure strongly enhances decoherence. This should not be surprising in light of discussion from Sec. III B as for such a small (two-dimensional) environment the correlation function of any environmental operator has to be periodic.

This example shows that the echo may lead to the increase in entanglement with respect to the entanglement present in the system at the end of the free-evolution period in the echo procedure (since it can create such entanglement). This is contrary to intuition since it is natural to try to extend the notion that since a perfect echo procedure diminishes all QEE (while diminishing all decoherence), an imperfect echo should lead to lesser entanglement whereas it leads to lesser decoherence in the echoed state. As we see here, there exist situations when the echo not only increases entanglement, but also increases decoherence and can be counterproductive. Using the physical picture discussed for weak dephasing in Sec. III B (and taking it, strictly speaking, outside of domain of its quantitative applicability, unless we assume a Gaussian environment [18] for which $|W(2\tau)| = \exp[-\chi(2\tau)]$), we see that this can occur when the PSD of the environmental noise is periodic, but τ is such that it is the *maximum* of the filter $|\tilde{f}(\omega)|^2$ in Eq. (21) that overlaps with the peaks of $S(\omega)$.

C. Entangling evolution—pure environmental states

Let us study the special case of a pure initial state of the environment (we expect from the results of the previous subsection that this situation will enhance the differences between the prepulse entanglement and echoed entanglement). Then the joint state of the system and the environment is pure at any time, so it is pure at time τ (prepulse) and at echo time 2τ. In this situation, entanglement at any time can be evaluated in a straightforward manner using the von Neumann entropy of one of the entangled subsystems, which is a good entanglement measure for pure states. The measure is defined as

$$E[|\psi(t)\rangle] = -\frac{1}{\ln 2}\mathrm{Tr}[\rho(t)\ln\,\rho(t)], \tag{33}$$

where $|\psi(t)\rangle$ is the pure system-environment state so $\sigma(t) = |\psi(t)\rangle\langle\psi(t)|$, $\rho(t) = \mathrm{Tr}_E|\psi(t)\rangle\langle\psi(t)|$ is the density matrix of the qubit at time t (obtained by tracing out the environment), and the entanglement measure is normalized to yield unity for maximally entangled states. The same result would be obtained when tracing out the qubit degrees of freedom instead of the environmental degrees of freedom, but the small dimensionality of the qubit makes this way much more convenient.

Let us denote the pure initial state of the environment as $|R_0\rangle$. Then qubit-environment state at time τ (prepulse) is given by

$$|\psi(\tau)\rangle = a|0\rangle \otimes \hat{w}_0(\tau)|R_0\rangle + b|1\rangle \otimes \hat{w}_1(\tau)|R_0\rangle, \tag{34}$$

and the corresponding echoed state (at time 2τ) is

$$|\psi(2\tau)\rangle = a|0\rangle \otimes \hat{w}_1(\tau)\hat{w}_0(\tau)|R_0\rangle + b|1\rangle \otimes \hat{w}_0(\tau)\hat{w}_1(\tau)|R_0\rangle. \tag{35}$$

The qubit density matrices are then of the general form (5) with $W(\tau) = \langle R_0|\hat{w}_1^\dagger(\tau)\hat{w}_0(\tau)|R_0\rangle$ prepulse, and $W(2\tau) = \langle R_0|\hat{w}_1^\dagger(\tau)\hat{w}_0^\dagger(\tau)\hat{w}_1(\tau)\hat{w}_0(\tau)|R_0\rangle$ for the echoed state. Hence, the absolute values of functions $W(\tau)$ and $W(2\tau)$ constitute the degrees of coherence retained in the qubit system at the time of application of the pulse and at the echo time, respectively.

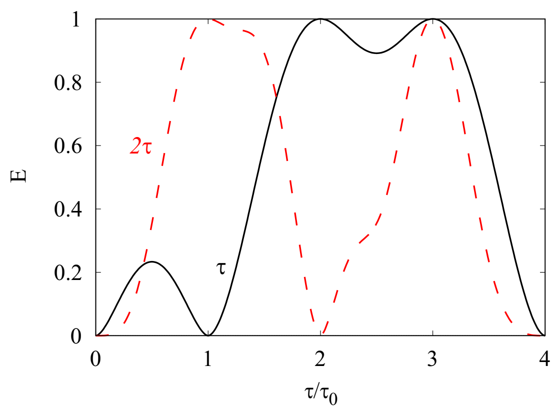

FIG. 1. Exemplary QEE evolution for a single qubit environment initially in a pure-state prepulse (at time τ, solid black line) and the corresponding echoed entanglement (at time 2τ, dashed red line).

The entanglement measure of Eq. (33) can be calculated using Eq. (5) which yields

$$E(|\psi(t)\rangle) = -\frac{1}{\ln 2}\left[\frac{1+\sqrt{\Delta(t)}}{2}\ln\frac{1+\sqrt{\Delta(t)}}{2} + \frac{1-\sqrt{\Delta(t)}}{2}\ln\frac{1-\sqrt{\Delta(t)}}{2}\right], \tag{36}$$

with $\Delta(t) = 1 - 4|a|^2|b|^2 + |a|^2|b|^2|W(t)|^2$. Note that $\Delta(t)$ is an increasing function of the degree of coherence $|W(t)|$, whereas entanglement measured by $E(|\psi(t)\rangle)$ is a decreasing function of $\Delta(t)$, so entanglement is a decreasing function of coherence $|W(t)|$, which means (as expected) that the higher the qubit coherence, the lower the QEE. Consequently, the situation described at the beginning of Sec. IV A, when the prepulse state $\sigma(\tau)$ has no QEE, but the echoed state $\sigma(2\tau)$ is entangled for a pure initial state of the environment translates to the prepulse qubit state being more coherent than the echoed qubit state, meaning that the echo can have an opposite effect on the qubit coherence than intended. This should be kept in mind when dealing with rather small environments that have a discrete spectrum, and which are close to being in pure state (e.g., their temperature is very low, or, in the case of spin environments, a large nonequilibrium polarization of the environmental spins was previously established, see Ref. [12] for discussion of QEE in this case).

Figure 1 shows an exemplary evolution of the QEE, measured by the normalized von Neumann entropy of Eq. (33) for an environment restricted to a single qubit which is initially in a pure state. The evolution operators (in the subspace of the environment) are given by

$$\hat{w}_i(t) = e^{i\omega_i t}|\psi_i\rangle\langle\psi_i| + e^{i\omega_i' t}|\psi_i'\rangle\langle\psi_i'|, \tag{37}$$

with $i = 0, 1$, $\omega_0 = \pi/(4\tau_0)$, $\omega_0' = -\pi/(4\tau_0)$, $\omega_1 = \pi/\tau_0$, $\omega_1' = 2\pi/\tau_0$, and

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|R_0\rangle - \frac{i}{\sqrt{2}}|R_1\rangle, \tag{38}$$

$$|\psi_0'\rangle = \frac{1}{\sqrt{2}}|R_0\rangle + \frac{i}{\sqrt{2}}|R_1\rangle, \tag{39}$$

$$|\psi_1\rangle = \frac{\sqrt{2-\sqrt{2}}}{2}|R_1\rangle - \frac{\sqrt{2+\sqrt{2}}}{2}|R_0\rangle, \tag{40}$$

$$|\psi_1'\rangle = \frac{\sqrt{2+\sqrt{2}}}{2}|R_0\rangle + \frac{\sqrt{2-\sqrt{2}}}{2}|R_1\rangle, \tag{41}$$

where $|R_1\rangle$ is the state perpendicular to the initial environmental state $|R_0\rangle$. Obviously, the evolution is periodic and repeats itself every $4\tau_0$, whereas at $t = \tau_0$ the evolution operators are equal to the operators introduced in Sec. IV B for which a nonentangled state before the pulse leads to an entangled echoed qubit-environment state.

The solid black line in Fig. 1 (denoted as τ) shows the amount of entanglement between the qubit and the environment as a function of time τ when no echo is performed. The dashed red line (denoted as 2τ), on the other hand, shows qubit-environment entanglement at time 2τ in the situation when a π pulse was applied to the qubit at time τ again as a function of τ. Hence, the two curves in Fig. 1 show prepulse entanglement and the corresponding echoed entanglement as a function of the same parameter τ. The evolution of echoed entanglement is much more involved, and the interplay of the two curves shows that apart from the previously predicted $\tau = \tau_0$ case (when no prepulse entanglement is observed, but there is echoed entanglement), there are many situations when applying the pulse enhances qubit-environment entanglement at a later time. Note, that for a pure initial state of the environment, there is a strict correspondence between QEE and qubit coherence, meaning that every time entanglement is enhanced by the echo, the coherence of the qubit is damped, and the effect of the echo is contrary to its purpose.

V. ECHO-INDUCED ENTANGLEMENT IS NOT POSSIBLE FOR PRINCIPALLY NONENTANGLING EVOLUTIONS

Although the examples discussed above show that the spin-echo procedure can lead to the appearance of QEE at echo time when the qubit-environment state was separable before the application of the pulse to the qubit, this occurs in rather special situations.

Let us show now that it is only possible at isolated points of time, and there are no finite time intervals $t \in [\tau_1, \tau_2]$ for which the prepulse state $\hat{\rho}(t)$ is separable, whereas the echoed state $\hat{\rho}(2t)$ is entangled. Since this is the case, we can extend the time interval to encompass the whole prepulse evolution $t \in [0, \infty]$, which yields the result that the echo procedure cannot be used to modify a nonentangling evolution into an entangling one.

The argument is as follows. Separable evolutions, which obviously must fulfill the criterion (24), can be divided into two categories: One encompasses all types of evolutions for which the environment does not evolve:

$$\mathrm{Tr}_Q\hat{\sigma}(t) = \hat{R}_{00}(t) = \hat{R}_{11}(t) = \hat{R}(0). \tag{42}$$

Here the trace is taken over the qubit degrees of freedom, so what is left is the evolution only in the subspace of the environment. Note that such evolutions also lead to pure dephasing of the qubit, it is only that this process cannot be witnessed by any measurements on the environment. The other encompasses all types of evolutions which do involve the evolution of the environment,

$$\mathrm{Tr}_Q\hat{\sigma}(t) = \hat{R}_{00}(t) = \hat{R}_{11}(t) = R(t) \neq \hat{R}(0). \tag{43}$$

The density matrix of the environment conditional on the qubit being in state $|1\rangle$ is defined as $\hat{R}_{11}(t) = \hat{w}_1(t)\hat{R}(0)\hat{w}_1^\dagger$ in analogy to $\hat{R}_{00}(t)$.

An evolution of the first category can never lead to echoed entanglement since if $\hat{w}_0(t)\hat{R}(0)\hat{w}_0^\dagger = \hat{w}_1(t)\hat{R}(0)\hat{w}_1^\dagger = \hat{R}(0)$, we have

$$\hat{R}(0) = \hat{w}_1(t)\hat{R}(0)\hat{w}_1^\dagger = \hat{w}_1(t)\hat{w}_0(t)\hat{R}(0)\hat{w}_0^\dagger\hat{w}_1^\dagger,$$
$$\hat{R}(0) = \hat{w}_0(t)\hat{R}(0)\hat{w}_0^\dagger = \hat{w}_0(t)\hat{w}_1(t)\hat{R}(0)\hat{w}_1^\dagger\hat{w}_0^\dagger,$$

so the separability criterion for the echoed state (26) is obviously fulfilled at all times without any additional assumption. Even isolated instances of time, which would lead to entanglement in the echoed state for a separable prepulse state are impossible.

In the other situation, we know that such instances of time exist due to the examples above. To check if there exist time intervals in the prepulse evolution for which the echo generates entanglement, let us study a time interval $t \in [\tau_1, \tau_2]$ such that for any time t within this interval we have $\hat{R}_{00}(t) = \hat{R}_{11}(t)$ (which guarantees prepulse separability). For there to be entanglement in the echoed state we need $\hat{w}_1(t)\hat{R}_{00}(t)\hat{w}_1^\dagger \neq \hat{w}_0(t)\hat{R}_{11}(t)\hat{w}_0^\dagger$, but because of the prepulse separability, we can exchange the conditional environmental states and get $\hat{w}_1(t)\hat{R}_{11}(t)\hat{w}_1^\dagger \neq \hat{w}_0(t)\hat{R}_{00}(t)\hat{w}_0^\dagger$, or, equivalently,

$$\hat{R}_{00}(2t) \neq \hat{R}_{11}(2t). \tag{44}$$

Hence, for there to exist time intervals for which the echo protocol leads to entanglement generation, the qubit-environment evolution without the echo procedure would have to fulfill a very specific requirement. Namely, there would have to exist time intervals in which the evolution is separable, followed by time intervals in which QEE is generated. In other words, sudden birth of entanglement [36,37] would have to be possible in the system.

The results of Ref. [11] show that for pure-dephasing evolutions, such as studied here, separability is equivalent to the lack of quantum discord [38–40] with respect to the environment. This means that the set of separable states has zero volume, and, therefore, sudden death of entanglement (which is a consequence of the geometry of separable states [41]) will not occur. Hence, also the transformation of separable evolutions to entangling ones via the quantum echo when the evolution remains separable for finite or infinite time intervals is not possible, and such occurrences are limited to isolated instances in time.

VI. THE ECHO SIGNAL AS THE QUBIT-ENTANGLEMENT ENVIRONMENT WITNESS

In the previous sections we have given examples showing that, in general, there is no correlation between the effectiveness of the echo protocol (measured by its capability to lead to coherence revival at time 2τ) and the generation of QEE. Although this conclusion stands, as it is simply a manifestation of the fact that for an environment in a mixed state, the correlation between amount of QEE and the strength of dephasing is rather weak, let us finish here with a more positive result for a specific case.

Let us use the separability condition for the prepulse evolution of the qubit-environment system lasting for time τ in the form given by Eq. (24). Let us then focus on a qubit that couples to the environment in biased way [31,32] so that $\hat{V}_0 = 0$ and only $\hat{V}_1 = \lambda\hat{V}$ is nontrivial. This means that $\hat{R}_{00}(\tau) = \hat{R}(0)$, and QEE is generated if and only if $\hat{R}_{11}(\tau) \neq \hat{R}(0)$. A necessary condition for the latter is $[\hat{H}_1, \hat{R}(0)] \neq 0$. It is also a sufficient condition for QEE to appear at all τ but a subset of isolated points. This follows from an argument about impossibility of sudden death or birth of QEE from the previous section: for $[\hat{H}_1, \hat{R}(0)] \neq 0$, QEE appears at the beginning of the evolution, and it cannot then vanish and stay zero for a finite stretch of time.

We focus now on the system in which the initial state of the environment is stationary with respect to the free Hamiltonian of the environment $[\hat{R}(0), \hat{H}_E] = 0$. The if and only if (with the exception of isolated points in time) condition for nonzero QEE is then $[\hat{V}_1, \hat{R}(0)] \neq 0$. A simple calculation of the commutator in expression for imaginary contribution to dephasing, Eq. (17), shows that the function $\Phi(2\tau)$ vanishes if the commutator of $\hat{V}_1$ and $\hat{R}(0)$ is zero. This leads to the following statement: If the environment is such a state, and the qubit's coupling is biased, the appearance of nonzero $\Phi(t)$ contribution to the echo signal means that qubit and environment were entangled during the evolution (with a possible exception of isolated points in time). This means that if the qubit is initialized with its Bloch vector in some direction (say x), then at echo time 2τ the length of this vector is not only going to be diminished due to nonzero $\chi(2\tau)$, but due to nonzero $\Phi(t)$ the direction of the final vector is going to be rotated with respect to the original one. Under all the listed conditions, the appearance of such an environment-induced rotation of the echoed state of the qubit is equivalent to entangling nature of the evolution of the composite qubit-environment system.

VII. CONCLUSION

We have studied the spin echo performed on a qubit that interacts with an environment due to a type of Hamiltonian which leads to pure dephasing of the qubit. Our intent was to find a relation between the performance of the echo procedure to reduce decoherence, and the entanglement which can be generated between the qubit and its environment. Quite surprisingly, we have found that the effectiveness of the echo and entanglement generation are two distinct issues. The perfect echo for which full coherence is restored can occur both in cases of entangling and separable evolutions.

We have further analyzed the situation when the echo is not perfect and found that it is possible for a qubit-environment state to be separable prior to the application of the local operation on the qubit (the π pulse) whereas the final echoed state is entangled. It turns out that, although such a possibility does exist, it is limited to isolated instances of time. The important consequence here is that, although the spin echo can result in the generation of entanglement from a point of time when there is no prepulse entanglement, this is a special case in an evolution which leads to entanglement generation on average. It cannot result in the change in the nature of evolution from nonentangling to entangling, so it cannot lead to a robust creation of quantum correlations.

Finally, we have shown that there is, at least, one case in which one can use the echo signal as a witness of the entangling character of the evolution of a qubit and its environment. When the environment is initially in a stationary state with respect to the free Hamiltonian of the environment, and only one of two levels of the qubit is coupled to the environment (as happens for qubits for which only one of their levels has a finite dipole moment, e.g., excitonic qubits [25–28] or spin qubits based on $m = 0$ and $m = 1$ levels of the spin $S = 1$ system, such as the nitrogen-vacancy center [29,30]), the appearance of the phase shift of coherence [31,32] proves then the entangling nature of the evolution.

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