

Evolution of the Stokes parameters, polarization singularities, and optical skyrmionHiroshi Kuratsuji^{1,*} and Satoshi Tsuchida²¹*Office of Professor Emeritus, Ritsumeikan University-BKC, Kusatsu City, Shiga 525-8577, Japan*²*Department of Physics, Osaka City University, Osaka, Osaka 558-8585, Japan*

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A physical theory is presented for polarized light from an aspect of polarization singularity. This is carried out by analyzing the evolution equation of the Stokes parameters that is derived from the nonlinear Schrödinger type equation. The problem is explored from two aspects: The first is concerning the single-mode Stokes parameter that is described by the propagation distance z . The trajectory of the polarization singularities is simply given by the points satisfying a rule, $S_3(z) = \pm S_0$ and 0 (S_3 represents the third component of the Stokes parameters), which form a tubelike surface centered with the C line surrounded by L surfaces in the presence of the linear as well as the nonlinear birefringence. It is pointed out that the optical activity (or chirality) plays an “obstacle” to reveal the polarization singularity. As the other aspect, we consider the field theoretic aspect of the Stokes parameters, for which a special solution, called an optical skyrmion, is constructed to explicate the polarization singularity. The skyrmion forms a *tube*, the dynamical content of which is briefly discussed. On the basis of the work presented here, further analysis would be expected to reveal hidden aspects of polarization optics.

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The zero point of a complex wave function plays a peculiar role in physics. Among others, the striking consequence is an emergence of a nodal singularity accompanying the quantized monopole [1]. Now what happens if this is extended to a complex wave consisting of two components? A typical problem is the light polarization [2,3], which is described by a two-component wave or spinor. Hence one expects that a concept of the singularity is hidden in the light polarization. Indeed the polarization singularity was recognized in Refs. [4,5] and thoroughly discussed in the articles [6,7]. The essence is that the singularities are given by two kind of objects; one is the circular polarization point (or line) which is named the C point or line, and the other is the linear polarization line (or surface) called the L line or surface. The C point is given by the north (or south) pole on the Poincaré sphere representing the Stokes parameters. As is well known, if the point on a sphere is coordinated by the angular parameters (θ, ϕ) , we have difficulty defining the angle ϕ at both poles. Namely, the axis of rotation loses its meaning at the poles, which corresponds to circular polarization. As for the linear polarization, the situation is a little bit simpler: An equatorial circle divides the Poincaré sphere into two hemispheres and by this division the chirality of polarization suddenly changes.

In the works of Refs. [6,7] great detail of the behavior of polarization singularities has been given with special emphasis on the topological classification of the singularities. More specifically, in Ref. [6], the polarization singularities were analyzed in paraxial fields with applications to morphology and statistics, and in Ref. [7], the optical singularities of

birefringent dichroic chiral crystals were analyzed. Furthermore, the structure of lines of circular and linear polarization in three-dimensional ellipse fields was given in Ref. [8].

However, apart from such morphological analysis, it is intriguing to address the problem of how the singularity evolves during the propagation through anisotropic media. To explore this topic is the aim of the present article. This is carried out by analyzing the evolution equation of the Stokes parameters, which is based on the Maxwell equation that is expressed in terms of the paraxial scheme assisted by the envelope approximation.

We treat the problem from two aspects: one is the single-mode polarized wave and the other is the field theory aspect as is explained below. In Secs. II and III, we consider the single-mode Schrödinger type equation, which is described by the single coordinate z , i.e., the propagation distance, while the Hamiltonian depends on the transversal coordinates (x, y) (see Sec. II). This is transcribed to the equation of motion for the single Stokes parameters [9–11]. Here the crucial point is that the coordinates (x, y) should be treated only as parameters which control the polarization singularity. Subsequently, in Sec. III an analysis of the Stokes parameters is given in the presence of the linear and nonlinear birefringence which are induced by the external Kerr effect and/or nonlinear coupling of polarization components [9,12]. The trajectory of the C point, which is just the C line, is simply given by the point set satisfying $S_3(z) = \pm S_0$. Similarly, the L surface, which is traced by the L line, corresponds to the point set satisfying $S_3(z) = 0$. Here S_3 means the third component of the Stokes vector. It is important to mention here that the optical activity gives rise to difficulty in controlling the polarization singularity.

In Sec. IV, as a complementary aspect to the single mode, we consider the field equation for the Stokes parameters,

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which is arranged so as to include the distribution over the transversal plane (x, y) [13–15]. Within this framework the polarization singularity is treated as a special solution, similar to the “skyrmion” [16]. This object implies a common viewpoint that could be shared with the trajectory of polarization singularity for the single-mode Stokes parameters. A dynamical content of the optical skyrmion is briefly discussed.

II. PRELIMINARY

A. The Stokes parameters in paraxial scheme

Here we adopt the well-known procedure of the “paraxial approximation.” Let \mathbf{E} be the light field (electric field) which is traveling through the anisotropic medium in the z direction. We suppose the modified plane wave [2]: $\mathbf{E}(x, y, z) = \mathbf{f}(x, y, z) \exp[ikn_0z]$, where $k = \frac{\omega}{c}$ and $n_0 (\equiv \sqrt{\epsilon_0})$ is the refractive index of the isotropic medium that is just used for reference. We choose the z direction as one of three principal axes of the dielectric tensor $\hat{\epsilon}$ (see below). The amplitude $\mathbf{f}(x, y, z)$ is written in terms of the linear polarization basis: $\mathbf{f} = {}^t(f_1, f_2) = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$, with the mutual orthogonal basis $\mathbf{e}_{1,2}$. Alternatively, we transform the basis vectors to the circular polarization basis: $\mathbf{f} = \psi_1\mathbf{e}_+ + \psi_2\mathbf{e}_-$, with $\mathbf{e}_\pm = (1/\sqrt{2})(\mathbf{e}_1 \pm i\mathbf{e}_2)$. The corresponding two-component wave is given by

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \equiv T \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \quad (1)$$

Now we introduce the Stokes parameters, for which it is efficient to use the wave (1) written in terms of the circular polarization basis [6,11] in contrast to the linear polarization basis (see, e.g., Ref. [17]). The former is particularly suitable to discuss the polarization singularity as is seen below. Using the Pauli-spin $\hat{\sigma}_i$, the Stokes parameters are written as

$$S_i = \psi^\dagger \hat{\sigma}_i \psi, \quad S_0 = \psi^\dagger \mathbf{1} \psi, \quad \text{with } (i = 1, 2, 3), \quad (2)$$

or, explicitly,

$$\begin{aligned} S_1 &= \psi_1^* \psi_2 + \psi_2^* \psi_1, \quad S_2 = i(\psi_2^* \psi_1 - \psi_1^* \psi_2), \\ S_3 &= \psi_1^* \psi_1 - \psi_2^* \psi_2. \end{aligned} \quad (3)$$

The vector $\mathbf{S} = [S_1, S_2, S_3]$ satisfies the relation $S_0^2 = S_1^2 + S_2^2 + S_3^2$, namely, S_0 represents the field strength, $S_0 \equiv |\mathbf{E}|^2$. Using the complex polar representation [18]

$$\psi_1 = \sqrt{S_0} \cos \frac{\theta}{2}, \quad \psi_2 = \sqrt{S_0} \sin \frac{\theta}{2} \exp[i\phi], \quad (4)$$

we obtain the polar form

$$\mathbf{S} = (S_0 \sin \theta \cos \phi, S_0 \sin \theta \sin \phi, S_0 \cos \theta). \quad (5)$$

This parametrization is used for the pictorial representation of the polarization on the Poincaré sphere, which is shown in Fig. 1.

Singular feature of the Stokes parameters. The expressions (4) and (5) bear the polarization singularity in an apparent way. This is shown by introducing the stereographic coordinate [20]

$$w = \frac{S_1 + iS_2}{S_0 + S_3} = \tan \frac{\theta}{2} \exp[i\phi], \quad (6)$$

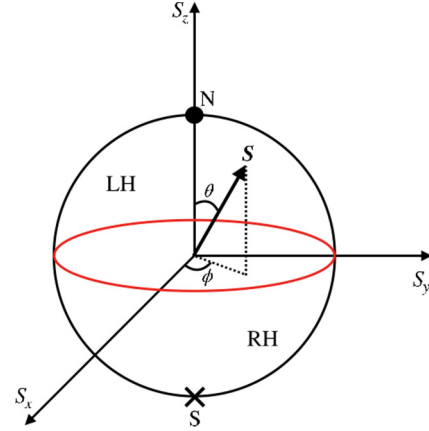


FIG. 1. Schematic image of the Poincaré sphere. N and S denote left-handed and right-handed circular polarization, respectively. “LH” and “RH” mean left-handed and right-handed polarization, respectively. The red line (equator) is the linear polarization.

which is the projection from the south pole. From this expression one sees that the phase ϕ cannot be defined at the point $w = 0$ or $\theta = 0$, which corresponds to the Stokes vector $\mathbf{S}_3 = +S_0$. We have another form of the stereographic coordinate that means the projection from the north pole:

$$w = \frac{S_1 - iS_2}{S_0 - S_3} = \cot \frac{\theta}{2} \exp[-i\phi], \quad (7)$$

for which the phase ϕ is also indefinite at $w = 0$ or $\theta = \pi$, which corresponds to the Stokes vector $\mathbf{S}_3 = -S_0$.

The above observation is reminiscent of the fact that the phase *loses* the meaning at the zero of wave function. This implies that the two circular polarizations reveal the singularities of the polarized light, called the C point. On the other hand, at the equatorial circle $\theta = \frac{\pi}{2}$, there do not reveal any singularities, and this circle divides the Poincaré sphere into two hemispheres corresponding to left-handed and right-handed polarizations. So we call the equatorial circle the L line.

B. Evolution equation

In the paraxial scheme, the Maxwell (Helmholtz) equation is written for the electric field \mathbf{E} :

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} + \nabla^2 \mathbf{E} + \left(\frac{\omega}{c}\right)^2 \hat{\epsilon} \mathbf{E} = 0, \quad (8)$$

where ∇^2 means the Laplacian in the transverse plane, $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The dielectric tensor $\hat{\epsilon}$ stands for the 2×2 Hermitian matrix.

First we consider the special case that the Laplacian can be discarded, which means that the modulation of the light field \mathbf{E} in the transverse plane (x, y) changes slowly compared with the change in the longitudinal direction z . This procedure may be called the single-mode approximation, for which the modulated amplitude depends only on the propagation distance z , namely, $\mathbf{E}(z)$. Hence we write

$$\frac{\partial^2 \mathbf{E}}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 \hat{\epsilon} \mathbf{E} = 0. \quad (9)$$

By applying the approximation of slowly varying amplitude (alias the envelope approximation) [21], namely, $|\frac{\partial^2 f}{\partial z^2}| \ll k|\frac{\partial f}{\partial z}|$, we keep only the first derivative term $\frac{\partial f}{\partial z}$. Writing it in terms of the circular polarization basis, we arrive at the two-component nonlinear Schrödinger equation for ψ :

$$i\lambda \frac{\partial \psi}{\partial z} = \hat{H} \psi. \quad (10)$$

Here, $\lambda (= \frac{2\pi}{k})$ is the wave length. The transformed Hamiltonian is given by $\hat{H} = -\frac{\pi}{n_0} T (\hat{\epsilon} - n_0^2) T^{-1}$. In terms of the Pauli spin, we write

$$\hat{H} = \alpha \hat{\sigma}_1 + \beta \hat{\sigma}_2 + \gamma \hat{\sigma}_3, \quad (11)$$

which may depend in general on the field (ψ^\dagger, ψ) as is shown below. In what follows the wave equation (10) is transcribed to the evolutionary equation for the Stokes parameters (see, e.g., Refs. [9–11]). The concise way to carry this out is given by using the action principle [12]. That is, we introduce the action function $I = \int L dz$ with the ‘‘Lagrangian’’

$$\begin{aligned} L &= \psi^\dagger \left(i\lambda \frac{\partial}{\partial z} - \hat{H} \right) \psi \\ &= -\frac{S_0 \lambda}{2} (1 - \cos \theta) \dot{\phi} - H(\theta, \phi). \end{aligned} \quad (12)$$

The variational equation $\delta I = 0$ leads to

$$\dot{\theta} = \frac{2}{S_0 \lambda} \frac{1}{\sin \theta} \frac{\partial H}{\partial \phi}, \quad \dot{\phi} = -\frac{2}{S_0 \lambda} \frac{1}{\sin \theta} \frac{\partial H}{\partial \theta}. \quad (13)$$

Using the Stokes vector, we have

$$\frac{d\mathbf{S}}{dz} = -\frac{2}{\lambda} \left(\mathbf{S} \times \frac{\partial H}{\partial \mathbf{S}} \right), \quad (14)$$

with the *expectation value* of the Hamiltonian $H = \psi^\dagger \hat{H} \psi$. The evolution equation (14) determines the orbit on the Poincaré sphere once the parameters (α, β, γ) are provided.

Following the conventional terminology, the meaning for the parameters appearing in the Hamiltonian is explained as follows: α and β represent the linear birefringence, whereas γ represents the optical activity or chirality. The crucial point here is that these parameters depend on the spatial coordinate, generally such that $\alpha(x, y, z)$, etc. In what follows, we restrict the argument to the special case in which these parameters have dependence only on the transversal coordinates, such that $\alpha(x, y)$.

III. TRAJECTORY OF THE POLARIZATION SINGULARITIES

Now we come to the first topic: to trace out the singularity of polarization in the linear birefringent medium. The birefringence concerned here is not the one that is inherent in the nature of media, e.g., the crystal structures, but it is caused by an external electromagnetic field or self-induced birefringence, which are discussed separately in what follows.

A. Linear birefringence caused by an external Kerr effect

This case is realized in such a way that the electric field is applied in the transverse plane: $\mathcal{E}(x, y) = (\mathcal{E}_x, \mathcal{E}_y)$, which

is arranged to depend on the transversal coordinates (x, y). In this way, the coordinates (x, y) could be treated as if they were constant during manipulation of the evolution of the Stokes vectors.

As the dielectric tensor, we adopt the one coming from an external Kerr effect [2], namely,

$$\epsilon_{ij} = n_0^2 \delta_{ij} + G \mathcal{E}_i \mathcal{E}_j. \quad (15)$$

Hence the wave equation for the amplitude f (namely, the linear polarization basis) is written as

$$i\lambda \frac{\partial}{\partial z} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad (16)$$

with $a = \frac{\pi}{n_0} G [\frac{\mathcal{E}_x^2 - \mathcal{E}_y^2}{2}]$ and $b = \frac{\pi}{n_0} G \mathcal{E}_x \mathcal{E}_y$. We have here omitted the term arising from the conventional Kerr effect, which is proportional to $G|\mathcal{E}|^2$, with $|\mathcal{E}|^2 = \mathcal{E}_x^2 + \mathcal{E}_y^2$. In terms of the circular polarization basis, this results in the following Hamiltonian:

$$\hat{H}_I = a \hat{\sigma}_1 + b \hat{\sigma}_2 \equiv \begin{pmatrix} 0 & a - ib \\ a + ib & 0 \end{pmatrix}. \quad (17)$$

Now it is instructive to examine the wave equation (16), for which there are two independent solutions: $f_\pm(z) = \exp[\pm i \frac{G}{2n_0} |\mathcal{E}|^2 kz]$. By combining these with the reference plane wave, one gets the total polarization wave:

$$f_{\parallel}(z) = \exp[in_{\parallel} kz], \quad f_{\perp}(z) = \exp[in_{\perp} kz], \quad (18)$$

which correspond to two refractive indices, namely: $n_{\parallel} = n_0 + \frac{G}{2n_0} |\mathcal{E}|^2$ and $n_{\perp} = n_0 - \frac{G}{2n_0} |\mathcal{E}|^2$, where the symbol \parallel means the polarization parallel to the applied field, while \perp means the polarization perpendicular to the applied field [22]. As such the emergence of two refractive indices represents the literal meaning of birefringence, which is in contrast to the optical activity.

We now examine the evolution of the Stokes parameters. The corresponding Hamiltonian reads $H = aS_1 + bS_2$, which is transformed to

$$H = cS'_1 \quad (19)$$

by using the orthogonal transform:

$$\begin{aligned} S'_1 &= \cos \Theta S_1 + \sin \Theta S_2, \\ S'_2 &= -\sin \Theta S_1 + \cos \Theta S_2, \quad S'_3 = S_3, \end{aligned} \quad (20)$$

with $\tan \Theta = \frac{b}{a}$. Here, for the sake of simplicity of notation, we write $c = \sqrt{a^2 + b^2}$. Then the evolution equation becomes

$$\lambda \frac{dS'_1}{dz} = 0, \quad \lambda \frac{dS'_2}{dz} = -2cS_3, \quad \lambda \frac{dS_3}{dz} = 2cS'_2, \quad (21)$$

leading to

$$S_3(z) = S_0 \cos \left[\frac{2cz}{\lambda} \right], \quad S'_2 = -S_0 \sin \left[\frac{2cz}{\lambda} \right], \quad (22)$$

with the initial phase being chosen to be zero.

We summarize a type of polarization, a ‘‘quantization rule,’’ and a trajectory for each typical value of $S_3(z)$ in Table I.

Thus, noting that $c (\equiv \sqrt{a^2 + b^2}) \propto G(\mathcal{E}_x^2 + \mathcal{E}_y^2)$, the above rule forms a surface in (x, y, z) space. From Table I, for the particular case $n = 0$, we have $\mathcal{E}_x = 0$ and $\mathcal{E}_y = 0$. These

TABLE I. The type of polarization, quantization rule, and trajectory for each typical value of $S_3(z)$ with integer n .

The value of $S_3(z)$	Polarization type	Quantization rule	Trajectory
$+S_0$	Circular (left-handed)	$\frac{2cz}{\lambda} = 2n\pi$	C point
$-S_0$	Circular (right-handed)	$\frac{2cz}{\lambda} = (2n-1)\pi$	C point
0	Linear	$\frac{2cz}{\lambda} = (n + \frac{1}{2})\pi$	L line

two equations determine the point in the transversal plane (x, y) . In other words, the zero point of the external electric field forms a C line in three-dimensional space. For nonzero n , the corresponding trajectory forms the objects; the C points become the “ C surface,” whereas the L line the “ L volume.” If cutting these objects by the plane $z = \text{const.}$, we have a cocentric pattern according to the value n . These represent the L surface and the C circle, so to speak, which are arranged according to the numbering of n .

At this point it is natural to consider the case such that $\mathcal{E} = |\mathcal{E}_0|\sqrt{f(r)}\hat{r}$ ($r = \sqrt{x^2 + y^2}$, $\hat{r} = \frac{r}{r}$), namely, the radial vector in the transverse plane, which is axially symmetric with respect to the z axis. In what follows we demonstrate the simplest choice, $f(r) = r$. The corresponding profile is given in Fig. 2, which represents the rotational body around the z axis. It is apparent that the z axis (that is $r = 0$) forms a C line.

Furthermore the following point must be mentioned: The configurational structure of polarization singularity can be controlled from an external condition through modulation of an electric field, for example, by allowing the time variation of the amplitude $\mathcal{E}_0(t)$.

Remarks on the role of optical activity. In the above argument, it is essential that the Hamiltonian does not include the term coming optical activity or chirality governed by S_3 , which causes S_3 to be invariant under Eq. (20). So one may wonder how about the effect of chirality or optical activity. It is easy to examine this effect: Instead of Eq. (19) let us consider $H = aS_1 + \gamma S_3 = \Gamma S'_1$, with $\gamma = v\mathcal{B}$ (\mathcal{B} is the magnetic field and v is the Verdet constant), $\Gamma = \sqrt{a^2 + \gamma^2}$. One has the orthogonal transform corresponding to Eq. (20):

$$\begin{aligned} S'_1 &= \cos \Phi S_1 + \sin \Phi S_3, \\ S'_3 &= -\sin \Phi S_1 + \cos \Phi S_3, \quad S'_2 = S_2, \end{aligned} \quad (23)$$

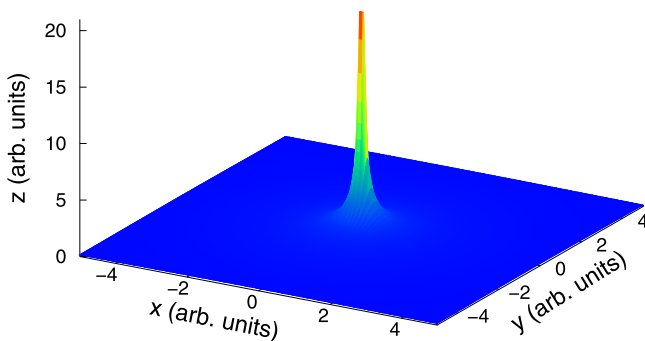


FIG. 2. Outline of the polarization singularity: $rz = \text{const.}$ The z axis, namely $r = 0$, indicates the C line.

with $\tan \Phi = \frac{\gamma}{a}$. Then the evolution equation becomes $\lambda \frac{dS'_1}{dz} = 0$, $\lambda \frac{dS'_2}{dz} = -2\Gamma S'_3$, $\lambda \frac{dS'_3}{dz} = 2\Gamma S_2$, from which one gets $S_3(z) = C \sin \Phi + S_0 \cos \Phi \cos [\frac{2}{\lambda}\Gamma z]$, where C is the constant of motion $S'_1 = C$. The trajectory of the C line (or L surface) is thus derived from

$$\begin{aligned} \cos \left[\frac{2}{\lambda}\Gamma z \right] &= \pm \frac{1}{\cos \Phi} - \frac{C \tan \Phi}{S_0} \quad (\text{Circular}), \\ &= -\frac{C \tan \Phi}{S_0} \quad (\text{Linear}). \end{aligned} \quad (24)$$

From this expression one sees that the additional term, which depends on the angle Φ as well as arbitrarily chosen constant of motion C , violates a simple quantization rule leading to the polarization singularities. This fact suggests that the optical activity plays the role of an *obstacle* for forming the polarization singularities. The exceptional case is that $\Phi = 0$ (which means γ vanishes), for which the problem is reduced to the case of the pure birefringence.

As an extreme case, we consider the pure optical activity, for which the evolution equation reads $\dot{S}_1 = -\frac{2}{\lambda}\gamma S_2$, $\dot{S}_2 = \frac{2}{\lambda}\gamma S_1$, $\dot{S}_3 = 0$. From this we have the constant of motion $S_3 = C$. Here the points $C = \pm S_0$ and 0; the circular and linear polarization should be careful. Namely, the circular polarization consists only of “isolated points” on a Poincaré sphere arbitrarily chosen in three-dimensional space, whereas the linear polarization describes the equatorial circle on the same sphere.

B. Including the nonlinear birefringence

Turning to the nonlinear Kerr media, it is known that the dielectric tensor is modified to be expressed in terms of the complex components of the spinor [2,23]. Namely this is given as

$$\epsilon_{ij} = n_0^2 \delta_{ij} + g(E_i^* E_j + E_j^* E_i). \quad (25)$$

The corresponding Hamiltonian operator becomes

$$\hat{V}_{nl} = -g \begin{pmatrix} -\frac{|\psi_1|^2 - |\psi_2|^2}{2} & \psi_2^* \psi_1 \\ \psi_1^* \psi_2 & \frac{|\psi_1|^2 - |\psi_2|^2}{2} \end{pmatrix}, \quad (26)$$

which results in the expectation value $\psi^\dagger \hat{V}_{nl} \psi = -\frac{g}{2}(S_1^2 + S_2^2 - S_3^2) \equiv gS_3^2$ up to additional constants. By taking account of this term, the Hamiltonian (19) is modified as

$$H = cS'_1 + gS_3^2. \quad (27)$$

By this modified Hamiltonian, the equation of motion turns out to be [9,12]

$$\lambda \frac{dS'}{dz} = \begin{pmatrix} -4gS'_2 S_3 \\ -2cS_3 + 4gS'_1 S_3 \\ 2cS'_2 \end{pmatrix}, \quad (28)$$

for which we have the two constants of motion:

$$S_1^2 + S_2^2 + S_3^2 = S_0^2, \quad cS'_1 + gS_3^2 = E. \quad (29)$$

One sees that by choosing the energy constant as $E = gS_0^2$, the equation of motion is derived for the third component S_3 . According to the ratio between the linear birefringence and

TABLE II. The type of polarization, quantization rule, and trajectory for each typical value of $S_3(z)$ with integer n .

The value of $S_3(z)$	Polarization type	Quantization rule	Trajectory
$+S_0$	Circular (left-handed)	$\frac{2cz}{\lambda} = 4nK(\kappa)$	C point
$-S_0$	Circular (right-handed)	$\frac{2cz}{\lambda} = 2nK(\kappa) (\neq 4nK)$	C point
0	Linear	$\frac{2cz}{\lambda} = (2n - 1)K(\kappa)$	L line

nonlinear coupling, namely, $\kappa = \frac{gS_0^2}{c}$ (called the *moduli parameter*), we have two cases of equations for S_3 : (A) $c > gS_0^2$ and (B) $c < gS_0^2$ [24]. By putting $S_3 = S_0X$ and $\tau = \frac{2cz}{\lambda}$, these are written as

$$\left(\frac{dX}{d\tau}\right)^2 = (1 - X^2)(1 - \kappa^2 + \kappa^2 X^2) \text{ for case (A),} \quad (30)$$

$$\left(\frac{dX}{d\tau}\right)^2 = \kappa^2(1 - X^2)(X^2 - \bar{\kappa}^2) \text{ for case (B),} \quad (31)$$

with $\bar{\kappa} = \sqrt{1 - \frac{1}{\kappa^2}}$. Hence the corresponding solutions of S_3 are given by

$$S_3(\tau) = S_0 \text{cn}(\tau, \kappa) \text{ for case (A),} \quad (32)$$

$$S_3(\tau) = S_0 \text{dn}(\kappa\tau, \bar{\kappa}) \text{ for case (B),} \quad (33)$$

where cn and dn represent the Jacobi elliptic functions.

The former solution indicates S_3 oscillates between $-S_0$ and S_0 , while the latter exhibits the small oscillation in the vicinity of the north (or south) poles; this feature is implied from $\bar{\kappa}S_0 \leq |S_3| \leq S_0$. In what follows we give analysis for the trajectory of the C points and the L lines, $S_3(\tau) = \pm S_0, 0$.

For case (A), $\kappa < 1$, we summarize a type of polarization, quantization rule, and trajectory for each typical value of $S_3(z)$ in Table II.

In Table II, $K(\kappa)$ denotes the period of the cn function, which is given by the complete elliptic integral:

$$K(\kappa) = \int_0^1 \frac{d\tau}{\sqrt{(1 - \tau^2)(1 - \kappa^2\tau^2)}}. \quad (34)$$

The contents of Table II correspond to the contents obtained for the case of pure linear birefringence.

The trajectory revealing these singularities is rather complicated to display, because of the highly sophisticated nature of the elliptic functions. Here we note that $n = 0$ is not allowed, because it yields $c \propto G(\mathcal{E}_x^2 + \mathcal{E}_y^2) = 0$, but this contradicts the constraint $c > gS_0^2 > 0$. In other words, there does not appear the case that $\mathcal{E} = (\mathcal{E}_x, \mathcal{E}_y)$ vanishes leading to the C line. This feature is in contrast to the case of the linear birefringence.

We discuss the trajectory of left-handed circular polarization by adopting the same profile as the linear birefringence for the applied external field, namely, $|\mathcal{E}|^2 = |\mathcal{E}_0|^2 r$. We take up the case $n = 1$, that is, the period $4K(\kappa)$. Then, we write

$$\bar{r} \left(\equiv \frac{\pi}{2n_0} \frac{G|\mathcal{E}_0|^2 r}{gS_0^2} \right) = \frac{1}{\kappa}. \quad (35)$$

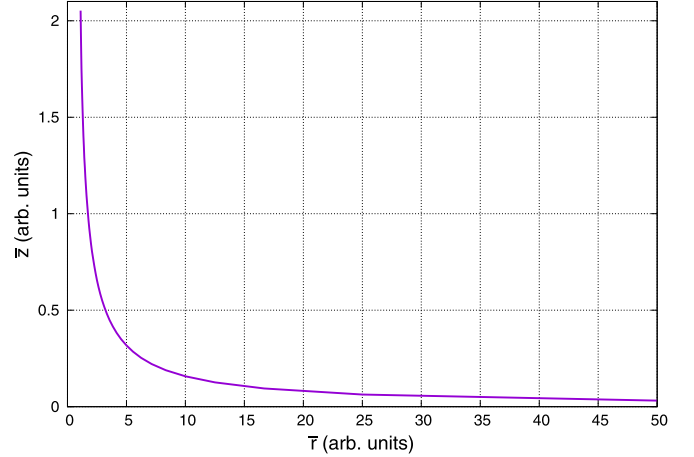


FIG. 3. The trajectory in the (\bar{r}, \bar{z}) plane.

On the other hand, the trajectory of left-handed circular polarization is also written as

$$\bar{z} \left(\equiv \frac{2gS_0^2}{\lambda} z \right) = 4\kappa K(\kappa). \quad (36)$$

With the aid of these two relations, one can eliminate the moduli parameter κ to get a trajectory in the (\bar{r}, \bar{z}) plane (see Fig. 3). Note that in the limit $\kappa \rightarrow 0$ ($g \rightarrow 0$), we recover the trajectory for the linear birefringence.

Now turning to the case (B), $\kappa > 1$, we see that there appear two orbits, for which we give a brief sketch. These are determined by $\text{dn} = \pm 1$, which turn out to be $\frac{2cz}{\lambda} = 4K(1/\kappa)$ for $S_3 = +S_0$ and $\frac{2cz}{\lambda} = 2K(1/\kappa)$ for $S_3 = -S_0$. There are no orbits for the linear polarization, $S_3 = 0$, because the dn function takes never zero.

Remarks. If the term coming from the optical activity is added to the Hamiltonian, the equation of motion is not reduced to a simple form that is described by the third component of the Stokes parameters. This implies that one has the same problem as in the linear birefringence case; that is, the optical activity destroys a simple quantization rule to extract the configurational structure revealing the polarization singularities.

IV. NONLINEAR FIELD AND OPTICAL SKYRMION

We now consider the case that the polarized field extends over the transversal plane; the field is described by $\psi(x, y, z)$; that is, the Laplacian is recovered [25]:

$$i\lambda \frac{\partial}{\partial z} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \left(-\frac{\lambda^2}{2n_0} \nabla^2 \mathbf{1} + \hat{V}_{nl} \right) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad (37)$$

with $\mathbf{1}$ being the 2×2 unit matrix. The corresponding Hamiltonian is written in terms of the angular field $[\theta(x, y), \phi(x, y)]$ [25]:

$$\tilde{H} = \int \left[\frac{S_0\lambda^2}{2n_0} \left\{ \frac{1}{4}(\nabla\theta)^2 + \sin^2 \frac{\theta}{2} (\nabla\phi)^2 \right\} + gS_0^2 \cos^2 \theta \right] d^2x, \quad (38)$$

where we have omitted an additional constant term. Then the problem is what is expected for the polarization singularity in the framework of this field equation. A plausible way to

answer this question may be given by utilizing the special solution that is inherent in the nonlinear field equation (37). If we consider the characteristics of the nonlinear birefringence, this can be achieved by the vortex type solution, which is called the optical skyrmion in terms of the current terminology [26]. By this specific solution the structure of the polarization singularity of the C line and the L surface could be naturally incorporated. Actually the same solution has been discussed in the previous work [27], but the argument from the polarization singularity remains untouched. Here we give a discussion from this renewal point of view.

To construct the special solution we need to settle its profile. First, taking into account the topological characteristics, the phase function is chosen such that $\phi = n \tan^{-1}(\frac{y}{x})$, with $n = 1, 2, \dots$, being the winding number, which follows the idea of the usual vortex [28]. Next the angular function θ , which determines the profile of the skyrmion, is chosen as a function of the radial variable $r (= \sqrt{x^2 + y^2})$. Keeping these characteristics in mind, we write the Hamiltonian (38) in terms of the field $\theta(r)$:

$$H = \frac{S_0 \lambda^2}{8n_0} \int \left[\left\{ \left(\frac{d\theta}{dr} \right)^2 + \frac{4n^2}{r^2} \sin^2 \frac{\theta}{2} \right\} + g' \cos^2 \theta \right] r dr. \quad (39)$$

Here, we put $g' = \frac{8gn_0S_0}{\lambda^2}$, which means that g' takes a positive value. The profile function $\theta(r)$ may be derived from the extremum of H ; namely, the Euler-Lagrange equation leads to

$$\frac{d^2\theta}{d\xi^2} + \frac{1}{\xi} \frac{d\theta}{d\xi} - \frac{n^2}{\xi^2} \sin \theta + \frac{1}{2} \sin 2\theta = 0, \quad (40)$$

where we adopt the scaling of the variable: $\xi = \sqrt{g'}r$. An efficient way to extract the polarization singularity is given by checking the behavior of $\theta(\xi)$ at specific boundary conditions at $\xi = 0$ and $\xi = \infty$.

The behavior near the origin $\xi = 0$ is controlled by the differential equation that serves as the Bessel equation; thus we obtain $\theta(\xi) \simeq J_n(\xi)$, which satisfies $\theta(0) = 0$. This means that the optical state is left-handed circular polarization at the origin. On the other hand, the behavior at $\xi = \infty$ is seen by checking the stationary feature of the solution; if putting $\theta(\xi) = \frac{\pi}{2} + \alpha$, with α being the infinitesimal deviation, then we have the linearized equation $\alpha'' - \alpha \simeq 0$ near $\xi = \infty$, which results in $\alpha \simeq \exp[-\xi]$. This means that the solution should approach $\theta(\infty) = \frac{\pi}{2}$. Thus, the state becomes linear polarization at infinity.

From the above construction of the optical skyrmion, we see that this reveals a three-dimensional object, if it is extended in the propagation direction. The center line forms a C line, whereas the boundary forms an L surface. From the pictorial viewpoint (see Fig. 4), there is somehow similarity with Fig. 2 which profiles the polarization singularity for the linear birefringence. Specifically, to be noted is that the C line, the central axis $r = 0$, corresponds to the center line of the optical skyrmion. The C line may be deformed by an action coming from externally driving (pinning) force. In the presence of such an external force, the behavior of the C line becomes like that in Fig. 4(b), which is rather intricate in general to extract the analytical solution for the behavior of the C line.

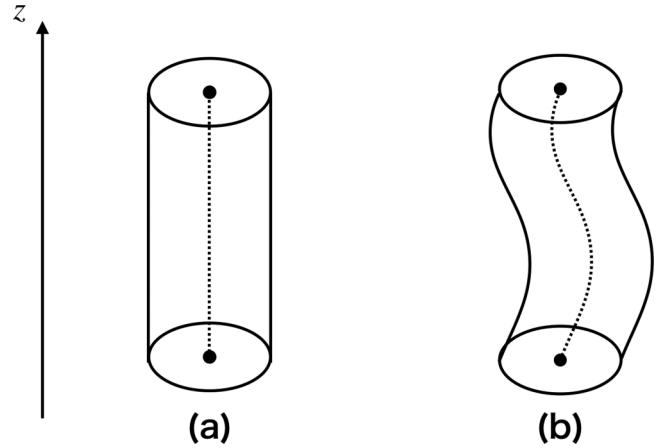


FIG. 4. Conceptual image of the skyrmion tube. The surface of the tube, namely, the L surface, corresponds to the equator of the Poincaré sphere. The center of the tube represents the C line. (a) No external force. (b) Including external force.

Apart from the configurational aspect of the optical skyrmion, which is quoted as ψ_{SK} , the more interesting aspect is its dynamical content, following the original idea of Skyrme [16]. Namely, the tube domain carries a *kinetic energy* along with the propagation direction z . This can be added to Eq. (38) as a form that is written in terms of the spinor wave ψ : $T \propto |\frac{\partial \psi}{\partial z}|^2$, which is the energy corresponding to the second derivative $\frac{\partial^2 E}{\partial z^2}$ of the starting Maxwell equation (8), which is discarded in the envelope approximation. [Note that in terms of the Stokes parameter, it is written as $T \propto (\frac{\partial S}{\partial z})^2$]. This term implies that the Stokes vector gives rise to the structure of the *spinning* (or rotational) degree together with the translational mode. Indeed the quadratic term of ψ may bring about the kinetic energy; that is, if we write the wave incorporating the *collective degree* Φ and \mathbf{R} (here, Φ means the global phase, and \mathbf{R} is the coordinate, which represents the trajectory of the C point),

$$\psi(x, y, z) = \exp[i\Phi(z)]\psi_{SK}[\mathbf{x} - \mathbf{R}(z)], \quad (41)$$

it follows that $T \propto A\dot{\Phi}^2 + B\dot{\mathbf{R}}^2$, the sum of the spinning and translational energy of the optical skyrmion. Here we note that there appears an essentially different term inherent in the spin degree, called the *gyration term*, that is proportional to $\dot{\mathbf{R}}$, which comes from the first-order derivative of the Stokes parameter [27].

Here a remark is given on the stability of the skyrmion against the various kinds of external perturbations. In order to examine this feature, let us consider the dynamical and topological aspects. The dynamical stability of the skyrmion will be carried out by the eigenmode analysis for the linearized equation, which describes the deviation from the solution (θ, ϕ) . It is crucial that, in the present context, without detailed analysis of this linearized equation, the feature of polarization singularity is kept even in the presence of external perturbation. That is, there is a stability from a topological aspect, which may be described by the topological charge. The main feature of the skyrmion can be topologically pro-

tected against the external disturbance [16]. The details of the eigenmode analysis will be left for future study.

We further mention the other aspect of skyrmions, e.g., the occurrence of multiple skyrmions in analogy with the multiple vortices that appear in the conventional single-component nonlinear Schrödinger equation. Such an aspect could not be expected in the dynamics of the single-mode Stokes parameters.

V. DISCUSSION AND SUMMARY

A brief remark on a possible experimental realization.

(i) For the single mode case, the first thing is to arrange the linear birefringence by using the electric field. This is provided in such a way that the electric charge is uniformly distributed on a “line” (or rod) so as to be in an axially symmetric way about the line; then an axially symmetric electric field may be produced according to the law of electrostatics, which is as given in Sec. III A. This can be extended to the case where there is nonlinear birefringent media; that is, the charged line is inserted in the media. Having given the setting, one can observe the C line (L line) *in principle*. Namely, appropriately arranged polarized light is transmitted through the birefringent apparatus, and then the output polarization is expected to be observed with the aid of an analyzer (e.g., Berry and Dennis [7]). Here particularly mentioned is the time variation of the birefringence, which is controlled by the time-varying amplitude of the electric field. According to the formula given in Tables I and II, the period c , which stands for the change of the C point, depends on time explicitly. The time variation would give rise to a direct modulation of the C point for fixed coordinates (r, z) . A similar argument can be applied to the case for L lines.

(ii) On the other hand, for the case of the optical skyrmion, the following setting will be supposed. According to the experimental setting of Ref. [14], let us consider the circular polarized beam passing through a nonlinear birefringent me-

dia. During the propagation through the media, the vortex (skymion) will be produced. The crucial point is that the vortex mode somehow forms the *waveguide (tube)*. If probe light that is circularly polarized is injected through this tube, then one could observe the output circularly polarized light by the analyzer at the end. So the actual occurrence of this event may be evidence of the optical skyrmion.

Summary. The circular and linear polarizations play a peculiar role in the light polarization, which is represented symbolically by the terminology of “polarization singularity.” In particular the circular polarization is characterized by the feature that the azimuthal angle ϕ is indefinite at the north and south poles. Under this standpoint we have explored the polarization singularity from two aspects: On the one hand, the configurational structure of polarization singularity, which is manifested by the single-mode Stokes vector, is given by the trajectory satisfying $S_3(z) = \pm S_0$ and 0 that is induced by the linear as well as the nonlinear birefringence. Here particularly mentioned is that the optical activity plays a role in preventing the formation of the polarization singularity. The polarization singularity may be manipulated by modulating the external electric field. On the other hand, as a supplement to the single Stokes parameters, we have discussed the nonlinear field by taking account of the transverse coordinate, which generates a spontaneous structure of the polarization singularities in the form of optical skyrmions. This provides a self-organized polarization singularity that is caused by the collaborated effect of nonlinear birefringence and the field effect extended over the transverse plane. A further analysis would be expected to reveal hidden aspects of polarization optics on the basis of the attempt presented here.

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- [1] P. A. M. Dirac, *Proc. R. Soc. London, Ser. A* **133**, 60 (1931).
 - [2] L. D. Landau, E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics in Continuous Media, Course of Theoretical Physics* Vol. 8 (Pergamon, Oxford, 1968), Chap. 11.
 - [3] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, London, 2019).
 - [4] J. F. Nye, *Proc. R. Soc. London A* **387**, 105 (1983).
 - [5] J. F. Nye and J. V. Hajnal, *Proc. R. Soc. London, Ser. A* **409**, 21 (1987).
 - [6] M. R. Dennis, *Opt. Commun.* **213**, 201 (2002).
 - [7] M. Berry and M. R. Dennis, *Proc. R. Soc. London, Ser. A* **459**, 1261 (2003).
 - [8] I. Freund, *Opt. Commun.* **283**, 1 (2010); **283**, 16 (2010).
 - [9] K. L. Sala, *Phys. Rev. A* **29**, 1944 (1984).
 - [10] M. V. Tratnik and J. E. Sipe, *Phys. Rev. A* **35**, 2965 (1987).
 - [11] H. Kuratsuji and S. Kakigi, *Phys. Rev. Lett.* **80**, 1888 (1998).
 - [12] H. Kuratsuji, R. Botet, and R. Seto, *Prog. Theor. Phys.* **117**, 195 (2007).
 - [13] A. W. Snyder, L. Poladian, and D. J. Mitchell, *Opt. Lett.* **17**, 789 (1992).
 - [14] G. A. Swartzlander Jr., and C. T. Law, *Phys. Rev. Lett.* **69**, 2503 (1992).
 - [15] D. Rozas, Z. S. Sacks, and G. A. Swartzlander Jr., *Phys. Rev. Lett.* **79**, 3399 (1997).
 - [16] T. H. R. Skyrme, *Proc. R. Soc. London, Ser. A* **260**, 127 (1961).
 - [17] C. Brosseau, *Fundamentals of Polarized Light: A Statistical Optics Approach* (Wiley & Sons, Hoboken, NJ, 1998), Chap. 3.
 - [18] The angular representation adopted here is different from that of Ref. [6]. We use the standard spinor parametrization [19], which leads to the natural definition of the spin vector (Stokes parameters) given below. See also Ref. [20].
 - [19] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (Non-Relativistic Theory)*, Course of Theoretical Physics Vol. 3, 3rd ed. (Butterworth-Heinemann, Oxford, 1981).

- [20] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, *Phys. Rev. A* **6**, 2211 (1972).
- [21] See S. A. Akhmanov and S. Yu. Nikitin, *Physical Optics* (Clarendon, Oxford, 1997), Chap. 14. We note that the envelope approximation used here is essentially equivalent to the eikonal approximation; see also Ref. [2].
- [22] An alternative expression for n_{\parallel} and n_{\perp} are $n_{\parallel} = n_0$ and $n_{\perp} = n_0 - \frac{G}{n_0} |\mathcal{E}|^2$, which are simply obtained by adding the contribution from the isotropic term $\frac{G}{2n_0} |\mathcal{E}|^2$. See Ref. [2].
- [23] P. D. Maker, R. W. Terhune, and C. M. Savage, *Phys. Rev. Lett.* **12**, 507 (1964); **16**, 832(E) (1966).
- [24] The third case $c = gS_0^2$ gives $S_3 = \text{sech}\tau$, which means a localized soliton mode. This case is apparently irrelevant.
- [25] H. Kuratsuji, *J. Phys. Soc. Jpn.* **77**, 044403 (2008).
- [26] There are numerous papers on the skyrmion to be cited here, so we only cite the early one that is relevant to the present work: K. Moon, H. Mori, K. Yang, S. M. Girvin, A. H. MacDonald, L. Zheng, D. Yoshioka, and S.-C. Zhang, *Phys. Rev. B* **51**, 5138 (1995).
- [27] H. Kuratsuji and S. Kakigi, in *Proceedings of the 1st International Symposium on TOP 2005* (World Scientific, Singapore, 2006), p. 295; see also [arXiv:cond-mat/9809257](https://arxiv.org/abs/cond-mat/9809257); S. Tsuchida and H. Kuratsuji, *Phys. Rev. E* **102**, 032204 (2020).
- [28] It is known that the conventional skyrmion is characterized by the topological invariant (charge). For the present optical skyrmion we can construct the correspondent quantity, which is given by $\sigma = \int (\nabla \times \mathbf{v}) d^2x$, where $\mathbf{v} = (1 - \cos\theta)\nabla\phi$ is the velocity field. It turns out to be $\int \sin\theta d\theta \wedge d\phi$, which means the mapping from the compactified plane $\{\mathbf{R} \cup \infty\}$ to the Poincaré sphere. See, e.g., Refs. [29,30].
- [29] D. Mermin and T. L. Ho, *Phys. Rev. Lett.* **36**, 594 (1976).
- [30] H. Kuratsuji and H. Yabu, *J. Phys. A* **29**, 6505 (1996).