Spin exchange in scattering with polarized orthopositronium beams

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We study electron and positron exchange without a spin flip in a single collision between two polarized orthopositronium beams. We use angular-momentum coupling and density-matrix techniques to calculate the probabilities of obtaining only parapositronium, both para- and orthopositronium or only orthopositronium, after the collision. The probabilities are functions of the angle between the polarization vectors of the beams and scattering amplitudes labeled with total electron spin and total positron spin. The real parts of the scattering amplitudes and products of the polarization tensors for the beams are given in terms of the probabilities for certain angles and initial spin orientations. We show that quenching probabilities for different transitions and the spin-exchange quenching cross section depend on the angle, but the ratio of the total quenching probability to the total probability for positronium scattering does not.

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I. INTRODUCTION

In this work, we study the effect of spin exchange in a single collision between two orthopositronium beams. Para- and orthopositronium are electron-positron bound states. Parapositronium has a spin of 0 and a lifetime of 0.125 16 ns [1]. Orthopositronium has a spin of 1 and a lifetime of 142.05 ns [2]. When an electron or positron from one beam is exchanged with an electron or positron from the other beam, without a spin flip, final states with only parapositronium, only orthopositronium, or a combination of both can be produced. The conversion of the longer-lived orthopositronium to parapositronium is referred to as quenching. The quenching of orthopositronium after electron exchange with either of the unpaired electrons of an oxygen molecule has been studied [3]. As noted [4,5], such an exchange *without* a spin flip is labeled Majorana exchange.

Scattering experiments with positronium (Ps) beams as well as theoretical studies of positronium formation and quenching are extensively reviewed in Refs. [6,7]. Schrader formulates the spin analysis required in the construction of the positronium-molecule wave function [8]. The production of a positronium molecule and spin-exchange quenching to para Ps in experiments with polarized ortho-Ps beams are described in Refs. [9–12]. Measurements of a spin-exchange cross section for nonthermalized positronium in porous silica [10] and the production in vacuum of polarized ortho Ps with spin magnetic quantum numbers of ± 1 [11] have been reported. Numerous theoretical methods have been used for calculating scattering lengths and cross sections in Ps-Ps collisions [13–21]. Shumway and Ceperley use a quantum Monte Carlo technique and give cross sections for triplet-triplet exciton scattering as well as triplet to singlet exciton conversion [13]. *S*-wave scattering lengths for Ps-Ps interactions are obtained semiempirically through the solution of the Schrödinger equation by Oda *et al.* [14]. The stochastic variational method is used to calculate the *s*-wave scattering length for spin-aligned ortho Ps–ortho Ps scattering [15] and scattering lengths as well as cross sections for Ps-Ps scattering [16]. A comparison of our probabilities to the expressions in Ref. [16] is presented in Sec. III. Other studies include the coupled-channel approach of Ref. [17], the close coupling model of Ref. [18], and the first Born approximation and Born-Oppenheimer model of Ref. [19]. More recently, a static exchange model was applied to ortho Ps–ortho Ps scattering and *s*-, *p*-, *d*-wave and quenching cross sections have been calculated [20]. In 2019, cross sections for transitions between specific positronium spin states were obtained from a four-body hyperspherical coordinate calculation of Ps-Ps scattering [21].

We use angular-momentum coupling to calculate the scattering matrix which contains amplitudes for transitions from the spin states of the initial two-orthopositronium system to those of the final two-positronium system. The total spin of the electrons, the total spin of the positrons, the total spin of the two-orthopositronium system, and the total spin magnetic quantum number are conserved. All the spin dependence in the scattering matrix elements is in the recoupling coefficients and the contribution of spin-dependent terms in the Hamiltonian is assumed to be negligible. The probabilities of producing only parapositronium, either type of positronium, or only orthopositronium in the final system are obtained from the diagonal elements of a spin-density matrix. We have used the density-matrix techniques described in Refs. [22–25] to study positron and orthopositronium collisions in Refs. [26–28].

II. METHOD

In our collision process, the initial orthopositronium atoms as well as the final para- or orthopositronium products are in the 1*s* state. We begin by constructing the scattering matrix.

The collision between the beams is described by $A + B \to C + D$, where the states A and B represent the incoming orthopositronium while the outgoing states C and D represent either type of positronium. The initial states A and B interact to form intermediate states in which electron and/or positron exchange can occur without a spin flip. After the exchange, the intermediate states develop into final states of either type of positronium C and D. During the evolution of the system from A and B to C and D, the total electron spin S_e, the total positron spin S_p, the total spin S_t, as well as the total spin magnetic quantum number M_t, are all conserved. The intermediate states are labeled $|[S_eS_p]S_tM_t\rangle$ and complex scattering amplitudes $f^{S_eS_p}$ which are independent of S_t and M_t, are defined as follows:

$$f^{S_eS_p} = \langle [S_eS_p]S_tM_t|M|[S_eS_p]S_tM_t\rangle, \tag{1}$$

$$(f^{S_eS_p})^* = \langle [S_eS_p]S_tM_t\left|M^\dagger\right|[S_eS_p]S_tM_t\rangle. \tag{2}$$

The spin assignments are (i) $S_t = 0$, $M_t = 0$ with $S_e = S_p = 0$, (ii) $S_t = 1$, $M_t = 0, \pm 1$ with $S_e = 0$ and $S_p = 1$ or $S_e = 1$ and $S_p = 0$, and (iii) $S_t = 0, 1, 2$ and $M_t = 0, \pm 1, \pm 2$ with $S_e = S_p = 1$. The elements of M contain the amplitudes for transitions from particular states of A and B to particular states of C and D and are given as

$$\begin{aligned}\langle S_CM_C|\langle S_DM_D|M|S_AM_A\rangle|S_BM_B\rangle \\ = \langle S_CM_C|\langle S_DM_D|\sum_{S_tM_t}|[S_eS_p]S_tM_t\rangle f^{S_eS_p} \\ \times \langle [S_eS_p]S_tM_t||S_AM_A\rangle|S_BM_B\rangle.\end{aligned} \tag{3}$$

We use angular-momentum coupling to calculate the matrix elements in four steps: (i) couple the initial orthopositronium states [Eq. (4)], (ii) transform the coupled initial states into intermediate states [Eq. (5)], (iii) transform the intermediate states into coupled final states [Eq. (6)], and (iv) uncouple the coupled final states [Eq. (7)]. Clebsch-Gordan coefficients are used in steps (i) and (iv) and 9-j coefficients are used in steps (ii) and (iii). The 9-j symbol used in coupling the four angular momenta, j_1, j_2, j_3, and j_4, is defined with curly brackets in Eq. (6.4.2) on p. 101 of Ref. [29]. The 9-j coefficients with the square brackets used in Eqs. (5) and (6) below include the factors $\sqrt{(2j_{12}+1)}$, $\sqrt{(2j_{34}+1)}$, $\sqrt{(2j_{13}+1)}$ and $\sqrt{(2j_{24}+1)}$, where j_{12}, j_{34}, j_{13}, and j_{24} are obtained by coupling j_1 and j_2, j_3 and j_4, j_1 and j_3, and j_2 and j_4, respectively.

$$|S_AM_A\rangle|S_BM_B\rangle = \sum_{S_tM_t} \langle S_AM_AS_BM_B|S_tM_t\rangle|[S_AS_B]S_tM_t\rangle, \tag{4}$$

$$|[S_AS_B]S_tM_t\rangle = \sum_{S_eS_p} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & S_e \\ \frac{1}{2} & \frac{1}{2} & S_p \\ S_A & S_B & S_t \end{bmatrix} |[S_eS_p]S_tM_t\rangle, \tag{5}$$

$$|[S_eS_p]S_tM_t\rangle = \sum_{S_CS_D} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & S_C \\ \frac{1}{2} & \frac{1}{2} & S_D \\ S_e & S_p & S_t \end{bmatrix} |[S_CS_D]S_tM_t\rangle, \tag{6}$$

$$|[S_CS_D]S_tM_t\rangle = \sum_{M_CM_D} \langle S_CM_CS_DM_D|S_tM_t\rangle|S_CM_C\rangle|S_DM_D\rangle. \tag{7}$$

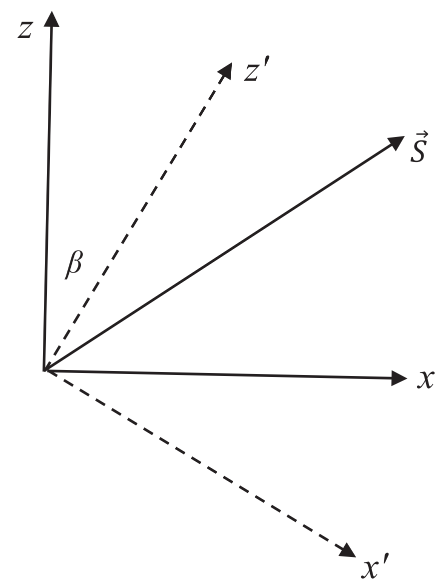


FIG. 1. The spin quantization vector $\vec{S} = S_{x'}\,\hat{i}' + S_{y'}\,\hat{j}' + S_{z'}\,\hat{k}'$ and the coordinate axes. The y and y' axes point into the plane of the page. The primed coordinate system is rotated by an angle β about the y (y') axis. The components of the spin quantization vector in the unprimed coordinate system are obtained with $S_x = \vec{S}\cdot\hat{i}$, $S_y = \vec{S}\cdot\hat{j}$, and $S_z = \vec{S}\cdot\hat{k}$.

The incoming spins are $S_A = S_B = 1$ and $M_A = M_B = 0, \pm 1$. The outgoing system can consist of only parapositronium ($S_C = S_D = 0$ and $M_C = M_D = 0$), both para- and orthopositronium ($S_C = 0$, $S_D = 1$ and $M_C = 0$, $M_D = 0, \pm 1$ or $S_C = 1$, $S_D = 0$ and $M_C = 0, \pm 1$, $M_D = 0$), or only orthopositronium ($S_C = S_D = 1$ and $M_C = M_D = 0, \pm 1$). The scattering matrix with 9 columns and 16 rows is shown in Table I.

The spin-density matrix for the incoming system ρ_{in}, is constructed as in Refs. [23,27,28], with unit matrix E, spin matrices S_i and S_{ij} for i, $j = x, y, z$, and $S_{ij} = \frac{3}{2}(S_iS_j + S_jS_i) - 2\delta_{ij}E$. The polarization vectors P_i and tensors P_{ij} are defined by $P_i = \mathrm{Tr}(\rho_{\text{in}}S_i)$ and $P_{ij} = \mathrm{Tr}(\rho_{\text{in}}S_{ij})$. The polarization vector of beam I is along the positive z axis (see Fig. 1) and the polarization tensor has only a pure z component. The density matrix for beam I is

$$\rho^{\text{I}}_{\text{in}} = \tfrac{1}{3}\left(E^{\text{I}} + \tfrac{3}{2}P^{\text{I}}_zS_z + \tfrac{1}{2}P^{\text{I}}_{zz}S_{zz}\right), \tag{8}$$

with $P^{\text{I}}_z = \pm 1$, $P^{\text{I}}_{zz} = 1$ or $P^{\text{I}}_z = 0$, $P^{\text{I}}_{zz} = -2$. The polarization vector of beam II is along positive z' which is obtained by a rotation about the y axis by angle β (see Fig. 1). The density matrix for beam II in the unprimed coordinate system with i, $j = x, y, z$, is given following Ref. [23] as

$$\rho^{\text{II}}_{\text{in}} = \frac{1}{3}\left(E^{\text{II}} + \frac{3}{2}\sum_i P^{\text{II}}_iS_i + \frac{1}{3}\sum_{ij} P^{\text{II}}_{ij}S_{ij}\right). \tag{9}$$

To express the matrix in terms of $P^{\text{II}}_{z'}$ and $P^{\text{II}}_{z'z'}$, we first relate the components of the spin quantization vector for beam II in

TABLE I. Spin scattering matrix for the process $A + B \rightarrow C + D$ with matrix elements $\langle S_C M_C | \langle S_D M_D | M | S_A M_A \rangle | S_B M_B \rangle$. The column labels I to IX represent the initial-state basis vectors $|S_A M_A\rangle |S_B M_B\rangle$, where A and B represent orthopositronium with $S_A = S_B = 1$. Specifically, I = $|11\rangle|11\rangle$, II = $|11\rangle|10\rangle$, III = $|11\rangle|1{-1}\rangle$, IV = $|10\rangle|11\rangle$, V = $|10\rangle|10\rangle$, VI = $|10\rangle|1{-1}\rangle$, VII = $|1{-1}\rangle|11\rangle$, VIII = $|1{-1}\rangle|10\rangle$, and IX = $|1{-1}\rangle|1{-1}\rangle$. The rows are labeled with final-state basis vectors $\langle S_C M_C | \langle S_D M_D |$ where C and D represent either parapositronium with $S_C = S_D = 0$ or orthopositronium with $S_C = S_D = 1$. The total electron spin $S_e = 0, 1$. The total positron spin $S_p = 0, 1$. We define combinations of the scattering amplitudes $f^{S_e S_p}$ as follows: $f^{1100} = \frac{1}{4} f^{11} + \frac{1}{4} f^{00}$, $g^{1100} = \frac{1}{4} f^{11} - \frac{1}{4} f^{00}$, $f^{0110} = \frac{1}{4} f^{01} + \frac{1}{4} f^{10}$, and $g^{0110} = \frac{1}{4} f^{01} - \frac{1}{4} f^{10}$.

	I	II	III	IV	V	VI	VII	VIII	IX
$\langle 00 \| \langle 00 \|$			$-g^{1100}$		g^{1100}		$-g^{1100}$		
$\langle 00 \| \langle 11 \|$		g^{0110}		$-g^{0110}$					
$\langle 00 \| \langle 10 \|$			g^{0110}				$-g^{0110}$		
$\langle 00 \| \langle 1{-1} \|$						g^{0110}		$-g^{0110}$	
$\langle 11 \| \langle 00 \|$		$-g^{0110}$		g^{0110}					
$\langle 11 \| \langle 11 \|$	f^{11}								
$\langle 11 \| \langle 10 \|$		$0.5 f^{11} + f^{0110}$		$0.5 f^{11} - f^{0110}$					
$\langle 11 \| \langle 1{-1} \|$			$f^{1100} + f^{0110}$		g^{1100}		$f^{1100} - f^{0110}$		
$\langle 10 \| \langle 00 \|$			$-g^{0110}$				g^{0110}		
$\langle 10 \| \langle 11 \|$		$0.5 f^{11} - f^{0110}$		$0.5 f^{11} + f^{0110}$					
$\langle 10 \| \langle 10 \|$			g^{1100}		$0.5 f^{11} + f^{1100}$		g^{1100}		
$\langle 10 \| \langle 1{-1} \|$						$0.5 f^{11} + f^{0110}$		$0.5 f^{11} - f^{0110}$	
$\langle 1{-1} \| \langle 00 \|$						$-g^{0110}$		g^{0110}	
$\langle 1{-1} \| \langle 11 \|$			$f^{1100} - f^{0110}$		g^{1100}		$f^{1100} + f^{0110}$		
$\langle 1{-1} \| \langle 10 \|$						$0.5 f^{11} - f^{0110}$		$0.5 f^{11} + f^{0110}$	
$\langle 1{-1} \| \langle 1{-1} \|$									f^{11}

the unprimed and primed systems (see Fig. 1) as

$$S_x = S_{x'} \cos\beta + S_{z'} \sin\beta, \quad S_y = S_{y'},$$
$$S_z = -S_{x'} \sin\beta + S_{z'} \cos\beta. \tag{10}$$

Then we calculate the polarization vectors as

$$P_x^{\mathrm{II}} = P_{z'}^{\mathrm{II}} \sin\beta, \quad P_y^{\mathrm{II}} = 0, \quad P_z^{\mathrm{II}} = P_{z'}^{\mathrm{II}} \cos\beta \tag{11a}$$

and the polarizations tensors as

$$P_{xx}^{\mathrm{II}} = P_{z'z'}^{\mathrm{II}} \left(\tfrac{3}{2} \sin^2\beta - \tfrac{1}{2}\right), \quad P_{yy}^{\mathrm{II}} = -\tfrac{1}{2} P_{z'z'}^{\mathrm{II}},$$
$$P_{zz}^{\mathrm{II}} = P_{z'z'}^{\mathrm{II}} \left(\tfrac{3}{2} \cos^2\beta - \tfrac{1}{2}\right),$$
$$P_{xz}^{\mathrm{II}} = P_{zx}^{\mathrm{II}} = P_{z'z'}^{\mathrm{II}} \left(\tfrac{3}{2} \sin\beta \cos\beta\right),$$
$$P_{xy}^{\mathrm{II}} = P_{yx}^{\mathrm{II}} = P_{yz}^{\mathrm{II}} = P_{zy}^{\mathrm{II}} = 0. \tag{11b}$$

The formulas in Eq. (11) are consistent with Eqs. (4.21) and (4.22) on p. 741 of Ref. [22] which relate polarization vectors and tensors in two coordinate systems. Finally, substituting Eq. (11) in Eq. (9) and using $S_{xx} + S_{yy} + S_{zz} = 0$, we get

$$\rho_{\mathrm{in}}^{\mathrm{II}} = \tfrac{1}{3}\left(E^{\mathrm{II}} + \tfrac{3}{2} P_{z'}^{\mathrm{II}} [S_x \sin\beta + S_z \cos\beta]\right)$$
$$+ \tfrac{1}{3}\left(\tfrac{1}{2} P_{z'z'}^{\mathrm{II}} [S_{xx} \sin^2\beta + S_{zz} \cos^2\beta + S_{xz}(2 \sin\beta \cos\beta)]\right), \tag{12}$$

with $P_{z'}^{\mathrm{II}} = \pm 1$, $P_{z'z'}^{\mathrm{II}} = 1$ or $P_{z'}^{\mathrm{II}} = 0$, $P_{z'z'}^{\mathrm{II}} = -2$. The density matrix for the incoming system of the two beams obtained by taking the direct product of the matrices in Eqs. (8) and (12) is

$$\begin{aligned}
\rho_{\mathrm{in}}^{\mathrm{I+II}} = {} & \tfrac{1}{9}(E^{\mathrm{I}} \times E^{\mathrm{II}}) + \tfrac{1}{6} P_{z'}^{\mathrm{II}} [(E^{\mathrm{I}} \times S_x) \sin\beta + (E^{\mathrm{I}} \times S_z) \cos\beta] \\
& + \tfrac{1}{18} P_{z'z'}^{\mathrm{II}} [(E^{\mathrm{I}} \times S_{xx}) \sin^2\beta + (E^{\mathrm{I}} \times S_{zz}) \cos^2\beta + (E^{\mathrm{I}} \times S_{xz})(2 \sin\beta \cos\beta)] \\
& + \tfrac{1}{6} P_z^{\mathrm{I}} (S_z \times E^{\mathrm{II}}) + \tfrac{1}{4} P_z^{\mathrm{I}} P_{z'}^{\mathrm{II}} [(S_z \times S_x) \sin\beta + (S_z \times S_z) \cos\beta] \\
& + \tfrac{1}{12} P_z^{\mathrm{I}} P_{z'z'}^{\mathrm{II}} [(S_z \times S_{xx}) \sin^2\beta + (S_z \times S_{zz}) \cos^2\beta + (S_z \times S_{xz})(2 \sin\beta \cos\beta)] \\
& + \tfrac{1}{18} P_{zz}^{\mathrm{I}} (S_{zz} \times E^{\mathrm{II}}) + \tfrac{1}{12} P_{zz}^{\mathrm{I}} P_{z'}^{\mathrm{II}} [(S_{zz} \times S_x) \sin\beta + (S_{zz} \times S_z) \cos\beta] \\
& + \tfrac{1}{36} P_{zz}^{\mathrm{I}} P_{z'z'}^{\mathrm{II}} [(S_{zz} \times S_{xx}) \sin^2\beta + (S_{zz} \times S_{zz}) \cos^2\beta + (S_{zz} \times S_{xz})(2 \sin\beta \cos\beta)].
\end{aligned} \tag{13}$$

The matrix for the outgoing system is

$$\rho_{\mathrm{out}} = M \, \rho_{\mathrm{in}}^{\mathrm{I+II}} \, M^{\dagger}. \tag{14}$$

The projection operators $P^{(Ps)}$ which are used to select outgoing positronium states with specific spin are defined such that only certain matrix elements in Eq. (15) nonzero, namely, those with (i) $S_C = M_C = S_D = M_D = 0$ and operator $P^{(\mathrm{para})}$, (ii) $S_C = M_C = 0$, $S_D = 1$, $M_D = 0, \pm 1$ and operator $P^{(\mathrm{para-ortho})}$, (iii) $S_C = 1$, $M_C = 0, \pm 1$, $S_D = M_D = 0$ and operator $P^{(\mathrm{ortho-para})}$, and (iv) $S_C = S_D = 1$, $M_C = M_D = 0, \pm 1$

and operator $P^{(\mathrm{ortho})}$.

$$\begin{aligned}\langle S_C M_C|\langle S_D M_D|\rho_{\mathrm{out}} P^{(Ps)}|S_C M_C\rangle|S_D M_D\rangle\\ =\sum_{M_A M_B}\sum_{M_{A'}M_{B'}}\langle S_C M_C|\langle S_D M_D|M|1M_A\rangle|1M_B\rangle\\ \times\langle 1M_A|\langle 1M_B|\rho_{\mathrm{in}}^{\mathrm{I+II}}|1M_{A'}\rangle|1M_{B'}\rangle\\ \times\langle 1M_{A'}|\langle 1M_{B'}|M^{\dagger}|S_C M_C\rangle|S_D M_D\rangle.\end{aligned}\tag{15}$$

The probabilities for positronium states with specific spins are calculated from the diagonal elements of $\rho_{\mathrm{out}}P^{(\mathrm{para})}$, $\rho_{\mathrm{out}}P^{(\mathrm{para-ortho})}$, $\rho_{\mathrm{out}}P^{(\mathrm{ortho-para})}$, and $\rho_{\mathrm{out}}P^{(\mathrm{ortho})}$. From the traces, Tr $(\rho_{\mathrm{out}}P^{(\mathrm{para})})$, Tr $(\rho_{\mathrm{out}}P^{(\mathrm{ortho})})$, Tr $(\rho_{\mathrm{out}}P^{(\mathrm{para-ortho})}+\rho_{\mathrm{out}}P^{(\mathrm{ortho-para})})$, and Tr $(\rho_{\mathrm{out}}P^{(\mathrm{para})}+\rho_{\mathrm{out}}P^{(\mathrm{para-ortho})}+\rho_{\mathrm{out}}P^{(\mathrm{ortho-para})}+\rho_{\mathrm{out}}P^{(\mathrm{ortho})})$, we get the probabilities for final states with only parapositronium, only orthopositronium, a combination of para- and orthopositronium and the total probability, respectively.

III. RESULTS

The probabilities for specific outgoing positronium spin states are given in Table II. All the expressions are functions of β, the angle between the polarization vectors of the two beams and the real parts of the complex scattering amplitudes, $f^{S_eS_p}$, which have units of length. The complex scattering amplitudes $f^{S_eS_p}$ correspond to specific geometries of the scattered and initial beam directions. Differential scattering cross sections could be obtained from the trace Tr $((f^{S_eS_p})(f^{S_eS_p})^*\rho_{\mathrm{out}})$ and integrated to obtain total scattering cross sections. We calculate the total probability for positronium scattering from the trace Tr (ρ_{out}). As noted earlier S_e and S_p can be 0 or 1. With $f^{S_eS_p}=r^{S_eS_p}e^{i\theta_{S_eS_p}}$, the probabilities are functions of $(r^{00})^2$, $(r^{11})^2$, and $(r^{01})^2+(r^{10})^2$. The angle-integrated total scattering cross sections reported in the literature are proportional to the probabilities and can be used to estimate the values of $(r^{00})^2$, $(r^{11})^2$ and $(r^{01})^2+(r^{10})^2$. We define $a^2=\frac{1}{64}(r^{00})^2$, $d^2=\frac{1}{64}(r^{11})^2$, and $b^2+c^2=\frac{1}{64}(r^{01})^2+\frac{1}{64}(r^{10})^2$.

A. Total probabilities

The total probability for positronium scattering is

$$\begin{aligned}P_t=&\tfrac{4}{3}P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}(3\cos^2\beta-1)[a^2+d^2-(b^2+c^2)]\\ &-8P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}\cos\beta[a^2-3d^2+(b^2+c^2)]\\ &+\tfrac{16}{3}[a^2+7d^2+2(b^2+c^2)].\end{aligned}\tag{16}$$

The probability for producing final states with only parapositronium (P_p), a combination of para- and orthopositronium (P_b) or only orthopositronium (P_o) are given below in Eq. (17). Note that $P_p+P_b+P_o=P_t$.

$$P_p=\left[\tfrac{4}{3}-2P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}\cos\beta+P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}\left(\cos^2\beta-\tfrac{1}{3}\right)\right](a^2+d^2),\tag{17a}$$

$$P_b=\left[\tfrac{16}{3}-4P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}\cos\beta-2P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}\left(\cos^2\beta-\tfrac{1}{3}\right)\right](b^2+c^2),\tag{17b}$$

$$P_o=3P_p+P_b+\left(P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}\cos\beta+1\right)(32\,d^2).\tag{17c}$$

A method for measuring the tensor polarizations is presented in Ref. [30]. We use the β dependence to obtain expressions for the products of the polarization vectors and tensors of the incoming beams. We set $\beta=0^\circ$ and 90° in Eqs. (16) and (17) and manipulate the results to get

$$P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}=\frac{2\left(P_o^{\beta=0}-P_p^{\beta=0}\right)-P_t^{\beta=0}}{2\left(P_o^{\beta=90}-P_p^{\beta=90}\right)-P_t^{\beta=90}}-1.\tag{18}$$

We set $\beta=90^\circ$ and $54.7^\circ(\cos^{-1}\sqrt{\frac{1}{3}}\approx 54.7^\circ)$ in Eq. (17a), use Eq. (18), and rearrange to get

$$P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}=4-\frac{P_p^{\beta=90}}{P_p^{\beta=54.7}}\left[4-\sqrt{12}\left(\frac{2\left(P_o^{\beta=0}-P_p^{\beta=0}\right)-P_t^{\beta=0}}{2\left(P_o^{\beta=90}-P_p^{\beta=90}\right)-P_t^{\beta=90}}-1\right)\right].\tag{19}$$

The four possible choices for the polarization vector and tensor products are as follows: (i) the spin magnetic quantum number M of each beam is 1 or -1 (spins aligned), $P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}=1$, $P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}=1$, (ii) the spins are oppositely aligned, $P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}=-1$, $P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}=1$, (iii) the M value for one of the beams equals zero, $P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}=0$, $P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}=-2$, and (iv) the M values for both beams equal zero, $P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}=0$, $P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}=4$. The tensor-product values for the four choices add as follows: $(1)+(1)-(-2)-(4)=0$. The probabilities given in Eqs. (16) and (17) also add in the same way, for example,

$$P_t^{(\mathrm{i})}+P_t^{(\mathrm{ii})}-P_t^{(\mathrm{iii})}-P_t^{(\mathrm{iv})}=0,\tag{20a}$$

$$P_p^{(\mathrm{i})}+P_p^{(\mathrm{ii})}-P_p^{(\mathrm{iii})}-P_p^{(\mathrm{iv})}=0.\tag{20b}$$

$P_t^{(\mathrm{i})}$ is the value of the total probability for choice (i) with $P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}=1$, $P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}=1$ and $P_p^{(\mathrm{i})}$ is the value of the probability for producing final states with only parapositronium for choice (i).

It is difficult to calculate the scattering amplitudes directly. Instead, we set $\beta=0^\circ$ in Eq. (16), substitute the values of $P^{\mathrm{I}}_zP^{\mathrm{II}}_{z'}$ and $P^{\mathrm{I}}_{zz}P^{\mathrm{II}}_{z'z'}$ for choices (i), (ii), and (iii) and derive

$$(r^{11})^2=P_t^{(\mathrm{i})},\tag{21a}$$

$$(r^{01})^2+(r^{10})^2=4P_t^{(\mathrm{iii})}-2P_t^{(\mathrm{i})},\tag{21b}$$

$$(r^{00})^2=P_t^{(\mathrm{i})}+4P_t^{(\mathrm{ii})}-4P_t^{(\mathrm{iii})}.\tag{21c}$$

B. Probabilities for specific spins

The probabilities for all possible final states are given in Table II. Here, we focus on the cases in which the spins of the initial beams are aligned, represented by the basis vectors $|11\rangle|11\rangle$ and $|1-1\rangle|1-1\rangle$, or oppositely aligned, with vectors

TABLE II. Probabilities for all possible final states just after a single collision between two orthopositronium beams. Initial states with spins $S_A = S_B = 1$ and M values equal to 1, 0, or -1 are given in column 2. Final states with only parapositronium ($S_C = S_D = 0$ and M values equal to 0), a combination of para- and orthopositronium ($S_C = 0$, $S_D = 1$ or $S_C = 1$, $S_D = 0$ and M values equal to 1, 0, or -1) or only orthopositronium ($S_C = 1$, $S_D = 1$, and M values equal to 1, 0, or -1), are listed in column 3. The values of $|\Delta M| = |(M_C + M_D) - (M_A + M_B)|$ are in column 4. The probabilities are functions of β, the angle between the polarization vectors of the two beams and scattering amplitudes defined as follows: $a^2 = \frac{1}{64}(r^{00})^2$, $d^2 = \frac{1}{64}(r^{11})^2$ and $b^2 + c^2 = \frac{1}{64}(r^{01})^2 + \frac{1}{64}(r^{10})^2$.

	Initial	Final	$\lvert\Delta M\rvert$	Probabilities
1	$\lvert 11\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 00\rvert$	2	$(a^2+d^2)(1-\cos\beta)^2$
2	$\lvert 1{-}1\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 00\rvert$	2	$(a^2+d^2)(1-\cos\beta)^2$
3	$\lvert 11\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 00\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
4	$\lvert 1{-}1\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 00\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
5	$\lvert 10\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 00\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
6	$\lvert 10\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 00\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
7	$\lvert 11\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 00\rvert$	0	$(a^2+d^2)(1+\cos\beta)^2$
8	$\lvert 1{-}1\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 00\rvert$	0	$(a^2+d^2)(1+\cos\beta)^2$
9	$\lvert 10\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 00\rvert$	0	$4(a^2+d^2)\cos^2\beta$
10	$\lvert 11\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 10\rvert$, $\langle 10\rvert\langle 00\rvert$	2	$(b^2+c^2)(1-\cos\beta)^2$
11	$\lvert 1{-}1\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 10\rvert$, $\langle 10\rvert\langle 00\rvert$	2	$(b^2+c^2)(1-\cos\beta)^2$
12	$\lvert 10\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 00\rvert$	2	$(b^2+c^2)(1-\cos\beta)^2$
13	$\lvert 10\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 11\rvert$, $\langle 11\rvert\langle 00\rvert$	2	$(b^2+c^2)(1-\cos\beta)^2$
14	$\lvert 11\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 11\rvert$, $\langle 11\rvert\langle 00\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
15	$\lvert 1{-}1\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 00\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
16	$\lvert 11\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 11\rvert$, $\langle 11\rvert\langle 00\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
17	$\lvert 1{-}1\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 00\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
18	$\lvert 11\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 10\rvert$, $\langle 10\rvert\langle 00\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
19	$\lvert 1{-}1\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 10\rvert$, $\langle 10\rvert\langle 00\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
20	$\lvert 10\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 11\rvert$, $\langle 00\rvert\langle 1{-}1\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
21	$\lvert 10\rangle\lvert 10\rangle$	$\langle 11\rvert\langle 00\rvert$, $\langle 1{-}1\rvert\langle 00\rvert$	1	$2(b^2+c^2)(1-\cos^2\beta)$
22	$\lvert 11\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 10\rvert$, $\langle 10\rvert\langle 00\rvert$	0	$(b^2+c^2)(1+\cos\beta)^2$
23	$\lvert 1{-}1\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 10\rvert$, $\langle 10\rvert\langle 00\rvert$	0	$(b^2+c^2)(1+\cos\beta)^2$
24	$\lvert 10\rangle\lvert 11\rangle$	$\langle 00\rvert\langle 11\rvert$, $\langle 11\rvert\langle 00\rvert$	0	$(b^2+c^2)(1+\cos\beta)^2$
25	$\lvert 10\rangle\lvert 1{-}1\rangle$	$\langle 00\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 00\rvert$	0	$(b^2+c^2)(1+\cos\beta)^2$
26	$\lvert 11\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 11\rvert$, $\langle 11\rvert\langle 00\rvert$	0	$4(b^2+c^2)\cos^2\beta$
27	$\lvert 1{-}1\rangle\lvert 10\rangle$	$\langle 00\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 00\rvert$	0	$4(b^2+c^2)\cos^2\beta$
28	$\lvert 11\rangle\lvert 11\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	2	$[a^2+d^2+(b^2+c^2)](1-\cos\beta)^2$
29	$\lvert 1{-}1\rangle\lvert 1{-}1\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	2	$[a^2+d^2+(b^2+c^2)](1-\cos\beta)^2$
30	$\lvert 11\rangle\lvert 11\rangle$	$\langle 10\rvert\langle 10\rvert$	2	$(a^2+d^2)(1-\cos\beta)^2$
31	$\lvert 1{-}1\rangle\lvert 1{-}1\rangle$	$\langle 10\rvert\langle 10\rvert$	2	$(a^2+d^2)(1-\cos\beta)^2$
32	$\lvert 11\rangle\lvert 1{-}1\rangle$	$\langle 11\rvert\langle 11\rvert$	2	$16d^2(1-\cos\beta)^2$
33	$\lvert 1{-}1\rangle\lvert 11\rangle$	$\langle 1{-}1\rvert\langle 1{-}1\rvert$	2	$16d^2(1-\cos\beta)^2$
34	$\lvert 10\rangle\lvert 11\rangle$	$\langle 10\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 10\rvert$	2	$(4d^2+b^2+c^2)(1-\cos\beta)^2$
35	$\lvert 10\rangle\lvert 1{-}1\rangle$	$\langle 10\rvert\langle 11\rvert$, $\langle 11\rvert\langle 10\rvert$	2	$(4d^2+b^2+c^2)(1-\cos\beta)^2$
36	$\lvert 11\rangle\lvert 11\rangle$	$\langle 11\rvert\langle 10\rvert$, $\langle 10\rvert\langle 11\rvert$	1	$2[4d^2+(b^2+c^2)](1-\cos^2\beta)$
37	$\lvert 1{-}1\rangle\lvert 1{-}1\rangle$	$\langle 1{-}1\rvert\langle 10\rvert$, $\langle 10\rvert\langle 1{-}1\rvert$	1	$2[4d^2+(b^2+c^2)](1-\cos^2\beta)$
38	$\lvert 11\rangle\lvert 1{-}1\rangle$	$\langle 11\rvert\langle 10\rvert$, $\langle 10\rvert\langle 11\rvert$	1	$2[4d^2+(b^2+c^2)](1-\cos^2\beta)$
39	$\lvert 1{-}1\rangle\lvert 11\rangle$	$\langle 1{-}1\rvert\langle 10\rvert$, $\langle 10\rvert\langle 1{-}1\rvert$	1	$2[4d^2+(b^2+c^2)](1-\cos^2\beta)$
40	$\lvert 10\rangle\lvert 10\rangle$	$\langle 10\rvert\langle 11\rvert$, $\langle 10\rvert\langle 1{-}1\rvert$	1	$2[4d^2+(b^2+c^2)](1-\cos^2\beta)$
41	$\lvert 10\rangle\lvert 10\rangle$	$\langle 11\rvert\langle 10\rvert$, $\langle 1{-}1\rvert\langle 10\rvert$	1	$2[4d^2+(b^2+c^2)](1-\cos^2\beta)$
42	$\lvert 11\rangle\lvert 10\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	1	$2(a^2+d^2+b^2+c^2)(1-\cos^2\beta)$
43	$\lvert 1{-}1\rangle\lvert 10\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	1	$2(a^2+d^2+b^2+c^2)(1-\cos^2\beta)$
44	$\lvert 11\rangle\lvert 10\rangle$	$\langle 10\rvert\langle 10\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
45	$\lvert 1{-}1\rangle\lvert 10\rangle$	$\langle 10\rvert\langle 10\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
46	$\lvert 10\rangle\lvert 11\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
47	$\lvert 10\rangle\lvert 1{-}1\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	1	$2(a^2+d^2)(1-\cos^2\beta)$
48	$\lvert 11\rangle\lvert 10\rangle$	$\langle 11\rvert\langle 11\rvert$	1	$32d^2(1-\cos^2\beta)$
49	$\lvert 1{-}1\rangle\lvert 10\rangle$	$\langle 1{-}1\rvert\langle 1{-}1\rvert$	1	$32d^2(1-\cos^2\beta)$
50	$\lvert 10\rangle\lvert 11\rangle$	$\langle 10\rvert\langle 10\rvert$	1	$2(a^2+9d^2)(1-\cos^2\beta)$
51	$\lvert 10\rangle\lvert 1{-}1\rangle$	$\langle 10\rvert\langle 10\rvert$	1	$2(a^2+9d^2)(1-\cos^2\beta)$
52	$\lvert 11\rangle\lvert 11\rangle$	$\langle 11\rvert\langle 11\rvert$	0	$16d^2(1+\cos\beta)^2$
53	$\lvert 1{-}1\rangle\lvert 1{-}1\rangle$	$\langle 1{-}1\rvert\langle 1{-}1\rvert$	0	$16d^2(1+\cos\beta)^2$

TABLE II. *(Continued.)*

	Initial	Final	$\lvert\Delta M\rvert$	Probabilities
54	$\lvert 11\rangle\lvert 1{-}1\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	0	$[a^2+d^2+(b^2+c^2)](1+\cos\beta)^2$
55	$\lvert 1{-}1\rangle\lvert 11\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	0	$[a^2+d^2+(b^2+c^2)](1+\cos\beta)^2$
56	$\lvert 11\rangle\lvert 1{-}1\rangle$	$\langle 10\rvert\langle 10\rvert$	0	$(a^2+d^2)(1+\cos\beta)^2$
57	$\lvert 1{-}1\rangle\lvert 11\rangle$	$\langle 10\rvert\langle 10\rvert$	0	$(a^2+d^2)(1+\cos\beta)^2$
58	$\lvert 10\rangle\lvert 11\rangle$	$\langle 11\rvert\langle 10\rvert$, $\langle 10\rvert\langle 11\rvert$	0	$(4d^2+b^2+c^2)(1+\cos\beta)^2$
59	$\lvert 10\rangle\lvert 1{-}1\rangle$	$\langle 1{-}1\rvert\langle 10\rvert$, $\langle 10\rvert\langle 1{-}1\rvert$	0	$(4d^2+b^2+c^2)(1+\cos\beta)^2$
60	$\lvert 11\rangle\lvert 10\rangle$	$\langle 11\rvert\langle 10\rvert$, $\langle 10\rvert\langle 11\rvert$	0	$4(4d^2+b^2+c^2)\cos^2\beta$
61	$\lvert 1{-}1\rangle\lvert 10\rangle$	$\langle 1{-}1\rvert\langle 10\rvert$, $\langle 10\rvert\langle 1{-}1\rvert$	0	$4(4d^2+b^2+c^2)\cos^2\beta$
62	$\lvert 10\rangle\lvert 10\rangle$	$\langle 11\rvert\langle 1{-}1\rvert$, $\langle 1{-}1\rvert\langle 11\rvert$	0	$4(a^2+d^2)\cos^2\beta$
63	$\lvert 10\rangle\lvert 10\rangle$	$\langle 10\rvert\langle 10\rvert$	0	$4(a^2+9d^2)\cos^2\beta$

$|11\rangle|1{-}1\rangle$ and $|1{-}1\rangle|11\rangle$. For the collision $A+B\rightarrow C+D$, $|\Delta M| = |(M_C+M_D)-(M_A+M_B)|$.

Consider first the final states with only parapositronium. When the incoming spins are aligned (rows 1 and 2 of Table II), the probabilities equal $(a^2+d^2)(1-\cos\beta)^2$ and $|\Delta M| = 2$. When the spins are oppositely aligned (rows 7 and 8), the probabilities equal $(a^2+d^2)(1+\cos\beta)^2$ and $|\Delta M| = 0$. For the probabilities in rows 7, 8, and 9 to be equal, $\cos\beta = -\frac{1}{3}$. This value of $\cos\beta$ gives probabilities of $\frac{16}{9}(a^2+d^2)$ for $|\Delta M| = 1$ or 2, which are higher by a factor of 4 than the values for $|\Delta M| = 0$.

Next, consider the final states with a combination of para- and orthopositronium. When the incoming spins are aligned and $|\Delta M| = 2$ (rows 10 and 11), the probabilities equal $(b^2+c^2)(1-\cos\beta)^2$. With $|\Delta M| = 1$, and the spins either aligned (rows 14 and 15), or oppositely aligned (rows 16 and 17), the probabilities equal $2(b^2+c^2)(1-\cos^2\beta)$, and for $|\Delta M| = 0$, with oppositely aligned spins (rows 22 and 23), the probabilities equal $(b^2+c^2)(1+\cos\beta)^2$. For the probabilities in rows 24 and 25 to equal those in 26 and 27, respectively, $\cos\beta = -\frac{1}{3}$. This value of $\cos\beta$ gives probabilities of $\frac{16}{9}(b^2+c^2)$ for $|\Delta M| = 1$ or 2, which are higher by a factor of 4 than the values for $|\Delta M| = 0$.

Finally, consider the final states with only orthopositronium. When the incoming spins are aligned, the probabilities are $[a^2+d^2+(b^2+c^2)](1-\cos\beta)^2$ (rows 28 and 29) or $(a^2+d^2)(1-\cos\beta)^2$ (rows 30 and 31) for $|\Delta M| = 2$, $2[4d^2+(b^2+c^2)](1-\cos^2\beta)$ (rows 36 and 37) for $|\Delta M| = 1$ and $16d^2(1+\cos\beta)^2$ (rows 52 and 53) for $|\Delta M| = 0$. On the other hand, with oppositely aligned spins, the probabilities are $16d^2(1-\cos\beta)^2$ (rows 32 and 33) for $|\Delta M| = 2$, $2[4d^2+(b^2+c^2)](1-\cos^2\beta)$ (rows 38 and 39) for $|\Delta M| = 1$ and $[a^2+d^2+(b^2+c^2)](1+\cos\beta)^2$ (rows 54 and 55) or $(a^2+d^2)(1+\cos\beta)^2$ (rows 56 and 57) for $|\Delta M| = 0$. For the probabilities in rows 58 and 59 to equal those in 60 and 61, respectively, $\cos\beta = -\frac{1}{3}$. With this value of $\cos\beta$, for either aligned or oppositely aligned spins the probabilities when $|\Delta M| = 0$ are lower than the probabilities for $|\Delta M| = 2$ by a factor of 4, but for $|\Delta M| = 1$, the probabilities are the same.

C. Quenching probabilities

The quenching probability is the probability of producing parapositronium from the conversion of the incoming orthopositronium, just after the collision. We consider two possibilities, final states with only para Ps or with a combination of para- and ortho Ps. Final states with only para Ps were discussed earlier. The probability of finding both types of Ps in the final state equals $(b^2+c^2)(6-4\cos\beta-2\cos^2\beta)$ (add rows 10, 11, 14, and 15 of Table II), when the initial ortho-Ps spins are aligned and $(b^2+c^2)(6+4\cos\beta-2\cos^2\beta)$ (add rows 16, 17, 22, and 23 of Table II) when the initial spins are oppositely aligned.

The total probability for final states with only para Ps, both para and ortho Ps, and only ortho Ps, equals $12(a^2+d^2)$, $48(b^2+c^2)$ and $36\ (a^2+9d^2)+48\ (b^2+c^2)$, respectively, that is, there is no β dependence. Using these expressions, we get the ratio of the quenching probability to the total probability for positronium scattering as

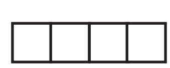
$$R = \frac{a^2+d^2+4(b^2+c^2)}{4[a^2+7d^2+2(b^2+c^2)]}. \tag{22}$$

D. Comparisons with published work

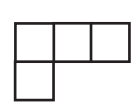
As noted in the Introduction, cross sections for specific transitions in positronium-positronium scattering have been reported in the literature. The differential cross section for a polarized beam is equal to the trace of the outgoing density matrix (see p. 749, Ref. [22] or p. 523, Ref. [25]). The probabilities for the production of specific spin states, listed in Table II, were obtained from the diagonal elements of the outgoing density matrix [Eqs. (14) and (15)]. In our comparisons, we assume that the reported cross sections are proportional to the probabilities by a factor Ω. We focus on the work of Ivanov *et al.* [16].

For a process with two positronium atoms, an antisymmetric four-particle total wave function is required. With two electrons and two positrons, each with a spin of $\frac{1}{2}$, this is a complicated process (see p. 723, Ref. [25]). The spin-wave function can be totally symmetric as shown in the Young tableaux (i), below, or have intermediate symmetry as shown in (ii) and (iii).

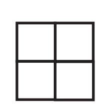
(i)



(ii)



(iii)



Orbital wave functions that match each of the three symmetries must be properly combined with the spin-wave functions to get the total wave function. We are only interested in the total spins, which using the sign conventions for Young tableaux, are (i) $S_t = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 2$, (ii) $S_t = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 1$, and (iii) $S_t = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} = 0$.

In our analysis, a total spin $S_t = 1$ can be obtained with total electron spin $S_e = 0$ and total positron spin $S_p = 1$ or with $S_e = 1$ and $S_p = 0$. In Refs. [13,14,16,21], *s*-wave scattering is considered and transitions with $S_t = 1$ are not allowed. We include $S_t = 1$ transitions; therefore, the scattering amplitudes for $S_e = 0$, $S_p = 1$ or $S_e = 1$, $S_p = 0$ are nonzero, and the probabilities for transitions containing these amplitudes are functions of $(b^2 + c^2)$.

We have studied ortho Ps–ortho Ps scattering represented by $A + B \to C + D$, not Ps–Ps scattering as in Ref. [16]; therefore, we do not have any transitions with para Ps in the initial states. However, the probabilities for transitions with ortho Ps in the initial states, specifically those in rows 7–9 and rows 52–63 of Table II, *do* correspond to specific cross sections in Eqs. (13), (16), and (18–21) of Ref. [16]. In making our comparisons, we set $\beta = 0^\circ$ and adopt the convention of Ref. [16] for the spin magnetic quantum numbers, namely, that $M_A \leqslant M_B$ and $M_C \leqslant M_D$. We find that the cross sections $\sigma_{11\to 11}$ in Eq. (20) and $\sigma_{11\to 00}$ in Eq. (21) of Ref. [16] equal $6(a^2 + 9d^2)\,\Omega$ and $2(a^2 + d^2)\,\Omega$, respectively. Using the values of $\sigma_{11\to 11} = 1.11 \times 10^{-14}$ cm^2 (see Fig. 3, Ref. [16]) and $\sigma_{11\to 00} = 6.51 \times 10^{-16}$ cm^2 (see text below Fig. 5, Ref. [16]), we get

$$(r^{11})^2 = 1.22 \times 10^{-14}\ \Omega^{-1}\ \text{cm}^2, \tag{23a}$$

$$(r^{00})^2 = 8.6 \times 10^{-15}\ \Omega^{-1}\ \text{cm}^2. \tag{23b}$$

We calculate $a^2 = \frac{1}{64}(r^{00})^2$ and $d^2 = \frac{1}{64}(r^{11})^2$, set $(b^2 + c^2) = 0$, and substitute the values in Eq. (22) to get $R = 0.055$. We note that the transitions not listed in Ref. [16] correspond to transitions in Table II with probabilities that equal zero for $\beta = 0^\circ$ or $(b^2 + c^2) = 0$.

The time taken for an initial ensemble of partially polarized orthopositronium atoms to be converted to a fully polarized one is important for experiments attempting to make a positronium Bose-Einstein condensate [31]. An estimate for the time is made using the cross section for mutual spin conversion of orthopositronium to parapositronium via electron exchange [32]. The spin-exchange quenching cross section σ_{SEQ} is defined by Cassidy and Mills and calculated to be 5.2×10^{-15}cm^2 [10] using values from Ref. [16]. In terms of the probabilities from Table II, with $\beta = 0^\circ$,

$$\sigma_{\text{SEQ}} = \sigma_{1-111\to 0000} + \sigma_{1-111\to 1010} = 16(a^2 + d^2)\Omega. \tag{24}$$

An increase in this cross section will result in a smaller conversion time [32]. If the possibility of quenching to a final state involving both para- and orthopositronium is included, then

$$\begin{aligned}\sigma_{\text{SEQ}} &= \sigma_{1-111\to 0000} + \sigma_{1-111\to 1010} + \sigma_{1-111\to 0010} \\ &\quad + \sigma_{1-111\to 1000} \\ &= 4(1 + \cos\beta)^2(a^2 + d^2 + b^2 + c^2)\Omega.\end{aligned} \tag{25}$$

Measurement of this cross section can be used to obtain the amplitudes $(r^{01})^2 + (r^{10})^2$.

Ps–Ps scattering was also studied in 2019 by Higgins *et al.* [21]. The transitions from initial ortho-Ps states, with nonzero cross sections, shown in Table III of Ref. [21], correspond to transitions in rows 7–9, 52–57, 60–61, and 63 of Table II of our work. The transitions in rows 58, 59, and 62 of Table II are missing in Ref. [21].

IV. SUMMARY

Angular-momentum coupling is used to determine all possible final spin states in a single collision between two orthopositronium beams in which both electron and positron exchange can occur without a spin flip. The scattering matrix for the transitions is given. The probabilities for the production of para- and orthopositronium are calculated from the diagonal elements of the density matrix for the outgoing system. The probabilities for all possible final states are given in terms of the scattering amplitudes and the angle β between the polarization vectors of the two beams. We show that for certain values of β, (i) the real parts of the scattering amplitudes can be obtained from measurements of total probabilities for specific initial orthopositronium spins, (ii) values for the polarization vector and tensor products can be obtained from measurements of probabilities, (iii) nonzero probabilities can be obtained for all possible final states, (iv) the probabilities are lower when the sum of the spin magnetic quantum numbers of the initial states equals that of the final states, and (v) the probability for the quenching of the incoming orthopositronium is larger when the beam spins are aligned than when they are opposite. We give an expression for the ratio of the quenching probability to the total probability which is independent of β. We compare our work to that of Ref. [16] and obtain numerical estimates for the real parts of the scattering amplitudes and the ratio of the quenching probability to the total probability. We expand the definition of spin-exchange quenching given in Ref. [10]. The techniques used in this work can be applied to positronium ion formation in the process $Ps + Ps \to e^+ + Ps-$.

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