Impossibility of cloning of quantum coherence

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It is well known that it is impossible to clone an arbitrary quantum state. However, this inability does not lead directly to no cloning of quantum coherence. Here, in this article, we show that it is impossible to clone the coherence of an arbitrary quantum state. In particular, with an ancillary system as machine state, we show that it is impossible to clone the coherence of states whose coherence is greater than the coherence of the known states on which the transformations are defined. Also, we characterize the class of states for which coherence cloning will be possible for a given choice of machine. Furthermore, we find the maximum range of states whose coherence can be cloned perfectly. The impossibility proof also holds when we do not include machine states. Lastly, we generalize the impossibility of cloning of coherence in terms of dimension of the quantum state and coherence measure taken into consideration.

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I. INTRODUCTION

The phenomenon of quantum superposition and entanglement lies at the heart of quantum mechanics acting as resources which we can harness to perform practical and important information-theoretic tasks [1]. Motivated by the increasing importance of quantum entanglement [2] in quantum information processing and communication schemes, a general study of the theory of resources within the paradigm of quantum mechanics and beyond is being formulated. We have several entanglement measures to quantify entanglement, however, until recently there was no standard way to quantify the coherence present in a quantum state. Quantum coherence can be viewed as a fundamental signature of nonclassicality in physical systems. Coherence can also be used as a resource for certain tasks like better cooling [3] or work extraction processes in nanoscale thermodynamics, in many quantum algorithms [4,5], in quantifying wave-particle duality [6–8], and in biological processes [9,10]. The resource theory of quantum coherence [11–22] along with other resource theories of entanglement and thermodynamics [23–26] has also been established. Once we have the measure based on a given set of axioms to quantify the coherence [27–35] we can build the resource theory of coherence. This seeks to quantify and study the amount of linear superposition a quantum state possesses with respect to a given basis. Given a state ρ, with its matrix elements as ρ_{ij}, the amount of coherence present in the state in the basis $\{|i\rangle\}$ is given by the quantity $C_l(\rho) = \sum_{i\neq j} |\langle i|\rho|j\rangle|$ which is known as the l_1 norm of coherence. Note that coherence is a basis-dependent quantity as the amount of coherence will be different in different bases. Since the l_1 norm is a function of the off-diagonal elements of the given density matrix representation, clearly the value of coherence will be zero in the eigenbasis of the density matrix, where there are no off-diagonal elements.

Quantum superposition and entanglement play a pivotal role in achieving information processing tasks that are otherwise not possible by any other classical resource. The same properties also forbid us to do certain tasks that are otherwise achievable classically. It started with the no-cloning theorem, which states that there does not exist a quantum operation which can perfectly duplicate any pure state [36].

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In particular, the no-cloning theorem states that if we have a cloning machine which can copy two orthogonal quantum states, then with the same cloning machine it is impossible to create an identical copy of an arbitrary quantum state. Pati and Braunstein later showed that we cannot delete either of the two quantum states perfectly [37]. In addition to these two famous no-cloning and no-deletion theorems there are many other no-go theorems like no-flipping (impossibility to flip an arbitrary quantum state) no-self-replication (cannot have a universal quantum constructor) [38]. A two-dimensional quantum system can always be represented as points on the Bloch sphere parametrized by azimuthal angle θ and the phase angle ϕ. It is interesting to note that there are no-go theorems like no partial erasure [39], no splitting [40], and no partial swapping [41], which together tells us the indivisibility of the information content present in a quantum system.

At this point it is interesting to ask the question *whether it is possible to clone coherence of arbitrary quantum states.* We know that cloning of arbitrary quantum states implies signaling. Therefore, the no signaling implies the no cloning but the no cloning does not imply the no signaling. Unlike in the case of cloning of states where each term of the density matrix has to be perfectly replicated, cloning of coherence does not limit us similarly. If we are able to clone the state perfectly, the coherence of the state also gets cloned. Hence, the cloning of quantum states implies the cloning of coherence present in the state. Its contrapositive tells us that no cloning of quantum coherence of a state will directly imply that the cloning of the state is not possible. So, in that sense no-cloning quantum coherence mathematically implies no cloning of quantum states. We have given a couple of examples of state cloners in the Appendix to illustrate that cloning of quantum coherence is not the same as the cloning of quantum state.

Impossibility of cloning coherence provides additional insights compared to that of the no-cloning theorem. Since coherence is a resource for creation of entanglement by incoherent operation, if we could clone coherence exactly, then we will be able to build up a large amount of entanglement starting from a minimal amount of coherence and repeated use of incoherent operations. Our no-go result suggests that this is not possible. Other insights can be gained in the context of thermal machines. If one aims to generate coherence in the energy eigenbasis from thermal resources, then our no-go result suggests that coherence cannot be created in initially incoherent systems while maintaining the machine in a fixed state. Thus, the impossibility of cloning coherence implies that various quantum resources cannot be generated arbitrarily starting from a given state having a fixed amount of nonzero coherence. In this paper, we show that indeed it is so and these two cloners are different. It is interesting to see that we cannot clone the coherence of arbitrary superposition of orthogonal states as long as the coherence of the state is more than the coherence of the orthogonal states in the given basis. This result holds when we define the cloning transformation with the machine states. However, we cannot say directly anything specific when the coherence of the input state is less than or equal to the coherence of these orthogonal states, but nevertheless in this zone we are able to characterize the states whose coherence can be cloned. We considered the action of an approximate quantum coherence cloner defined for two orthogonal states from the equatorial plane. We demonstrate its action on an arbitrary quantum state expressed in the basis of these two orthogonal states. Interestingly, we found that the higher the difference between the two angles of the initial states and the final state, the lower is the ratio of the initial and final coherence after cloning. We also find the maximum range of states whose coherence can be cloned perfectly. Further, we show that there does not exist any universal unitary operator as a coherence cloner even when we are not considering the ancillary states. In addition to these, we show that it is impossible to clone coherence of a qudit. Also, we show that this impossibility is valid for coherence measures like relative entropy measure, l_1 norm, quantum skew divergence, etc., where maximally coherent states are only pure states [42] (result of Theorem 1). We find that the impossibility of universal coherence cloning fundamentally depends upon the choice of the known states and is very much different from the cloning of the quantum state.

II. NO CLONING OF QUANTUM COHERENCE WITH MACHINE STATES

In the case of the no-cloning theorem for quantum states, we start with an assumption that we can clone two known orthogonal quantum states. Here, we start with an assumption that we can copy the coherence of two known orthogonal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ and then prove that it is impossible to clone the coherence of an unknown quantum state universally. At this point, one may ask the question that how do we know that we can clone coherence of two orthogonal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$. We can argue that since $|\psi_1\rangle$ and $|\psi_2\rangle$ are two known orthogonal states, we can make copies of these quantum states. Now, cloning of quantum states always implies that the cloning of coherence is true (although the reverse is not true). This is because when we can clone the entire state we can definitely clone the coherence content of the state. Therefore, it is natural to assume that we can clone coherence of two known orthogonal states.

Let U_{cc} be the unitary transformation that produces two copies of coherence starting from two orthogonal quantum states. The cloning transformation for coherence is given by

$$\begin{aligned} |\psi_1\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\Psi_1\rangle_{AB}|X_1\rangle_C, \\ |\psi_2\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\Psi_2\rangle_{AB}|X_2\rangle_C, \end{aligned} \tag{1}$$

where $|\psi_1\rangle$, $|\psi_2\rangle$ are input states, $|0\rangle$ is the blank state, and $|X_0\rangle$ is the initial machine state. Also, $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are states whose subsystems A and B have coherence the same as that of the input states, and $|X_1\rangle$ and $|X_2\rangle$ are the corresponding final machine states. The machine states satisfy $\langle X_2|X_1\rangle = 0$ due to unitarity of the transformation. Let us represent the two orthogonal states $|\psi_1\rangle_A$ and $|\psi_2\rangle_A$ in the $\{|0\rangle, |1\rangle\}$ basis as $|\psi_1\rangle_A = a|0\rangle_A + b|1\rangle_A$ and $|\psi_2\rangle_A = b^*|0\rangle_A - a^*|1\rangle_A$.

As the transformation demands coherence to be perfectly copied, we must have $C_l(|\psi_1\rangle_A) = C_l(\rho_A{}') = C_l(\rho_B{}')$ and $C_l(|\psi_2\rangle_A) = C_l(\rho_A{}'') = C_l(\rho_B{}'')$, where

$$\begin{aligned} \rho_A{}' = Tr_B|\Psi_1\rangle_{ABAB}\langle\Psi_1|,\ \rho_B{}' = Tr_A|\Psi_1\rangle_{ABAB}\langle\Psi_1|, \\ \rho_A{}'' = Tr_B|\Psi_2\rangle_{ABAB}\langle\Psi_2|,\ \rho_B{}'' = Tr_A|\Psi_2\rangle_{ABAB}\langle\Psi_2|. \end{aligned} \tag{2}$$

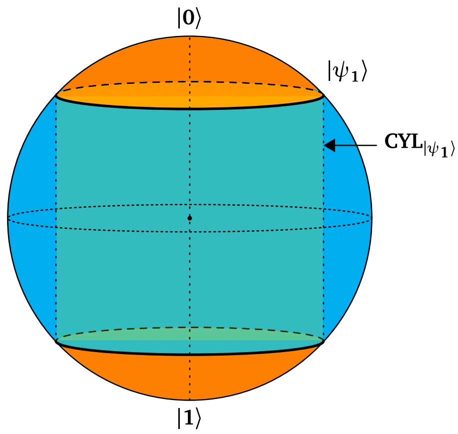

FIG. 1. Classification of states for a given cloner. The zone on the surface of the Bloch sphere that is between the two horizontal circles drawn with the dashed line denotes the pure states whose coherence can not be cloned. $CYL_{|\psi_1\rangle}$ represents all the states (pure as well as mixed) which have the same coherence as $|\psi_1\rangle$

Since the coherence of the orthogonal states is the same, we have $C_l(|\psi_1\rangle_A) = C_l(|\psi_2\rangle_A) = 2|a||b|$, where $C_l(\rho)$ is the l_1 norm for quantifying quantum coherence. It may be noted that in the case of cloning of quantum states we require two identical copies of the input state at the output port. However, for cloning of coherence, this is not the case as there can be two nonidentical states with the same coherence. Since any state can be represented on the Bloch sphere as $\rho = \frac{I+\vec{m}.\vec{\sigma}}{2}$ with $\vec{m} = (m_x, m_y, m_z)$ as the Bloch vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. The coherence in $\{|0\rangle, |1\rangle\}$ basis is given by $C_l(\rho) = \sqrt{m_x^2 + m_y^2}$. Hence, coherence only depends on m_x and m_y values. As shown in Fig. 1, we can say that all the states that lie on the curved surface of the cylinder with radius $\sqrt{m_x^2 + m_y^2}$ will have the same coherence. At this point, it is important to ask the following question: *Does quantum mechanics allow existence of a universal cloner for cloning the coherence of an arbitrary input state* $\alpha|\psi_1\rangle + \beta|\psi_2\rangle$*?* The answer to the question is *"no."*

Theorem 1. It is impossible to clone the coherence of an arbitrary quantum state $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$, with the cloning transformations given by Eq. (1) when the coherence of the state $|\psi\rangle$ is more than the coherence of the states $|\psi_i\rangle$ $(i = 1, 2)$ for a fixed choice of basis $\{|0\rangle, |1\rangle\}$.

Proof. Without loss of generality, let us use the l_1 norm as a measure of quantum coherence and assume a fixed basis as the computational basis. Any arbitrary state in $|\psi_1\rangle_A$, $|\psi_2\rangle_A$ basis can be written as $|\psi\rangle_A = \alpha|\psi_1\rangle_A + \beta|\psi_2\rangle_A$. The $l1$ norm of coherence of the state $|\psi\rangle_A$ in the $\{|0\rangle, |1\rangle\}$ basis is $2|(\alpha a + \beta b^*)(\alpha b - \beta a^*)|$. After the application of cloning transformation U_{cc}, the arbitrary state along with the blank and machine states becomes $(\alpha|\Psi_1\rangle_{AB}|X_1\rangle_C + \beta|\Psi_2\rangle_{AB}|X_2\rangle_C)$. Tracing out the subsystems B and C, we get $\rho_A^{\text{final}} = |\alpha|^2\rho_A{}' + |\beta|^2|\rho_A{}''$. From the convexity property of coherence measure we have $C_l(\rho_A^{\text{final}}) \leqslant [|\alpha|^2 C_l(\rho_A{}') + |\beta|^2 C_l(\rho_A{}'')] = 2(|\alpha|^2|a||b| + |\beta|^2|a||b|) = 2|a||b| = C(|\psi_i\rangle)$. Therefore, the final coherence of the subsystem A is at most $2|a||b|$. Therefore, it is evident that all the input states $|\psi\rangle$ whose initial coherence $C_l(|\psi\rangle)$ is greater than $2|a||b|$, which is the coherence of the known orthogonal states, it is impossible to clone the coherence perfectly. ■

Note 1. Theorem 1 holds for all coherence measures and is not only restricted to the l_1 norm of coherence. The convexity of any coherence measure ensures that the final coherence $C(\rho_A^{\text{final}})$ is bounded above by $C(|\psi_i\rangle_A)$, where $i = 1, 2$.

This tells that if the coherence of an arbitrary input state is greater than the coherence of the orthogonal states, then we cannot copy the coherence of the state into a blank state. Geometrically, if we consider the Bloch sphere as the state space, the orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$ represent two symmetric points on the surface of each hemisphere of the Bloch sphere. Taking the shortest distance of each of these points from the central axis as radius, these circles will represent all the states with same coherence value. We will have exactly two similar circles, one in each hemisphere representing orthogonal states. All the pure states with greater coherence value will be the points on the surface which are lying between these two circles. This theorem geometrically tells us that we cannot copy the coherence of the intermediate surface points (see Fig. 1). However, the theorem does not tell anything about the points on the surface which lies on the circles (except $|\psi_1\rangle$ and $|\psi_2\rangle$) and other points lying between those circles and poles. It may be possible to clone some of these states. The theorem only tells us that given a choice of known orthogonal states, there does not exist any universal cloner which will clone all pure states on the surface of the Bloch sphere. However, the theorem is only true as long as the orthogonal states are not from the equatorial circle of the Bloch sphere ($|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$ lying on the equator of the Bloch sphere can be one such example). In that scenario, we do not have any input state with a coherence greater than the coherence of these equatorial orthogonal states (which is 1). If we view this, circles from each hemisphere coincide with each other and there is no intermediate point. The important question is whether for such choice of cloner it is possible to clone all the states on the surface of Bloch sphere. The answer to this is once again "no" and indeed there does not exist a universal cloner for whatever choice of machine.

Corollary 1. For a cloning transformation given by Eq. (1), with the choice of known orthogonal states $|\psi_{1(E)}\rangle$ and $|\psi_{2(E)}\rangle$, taken from the equator, it is impossible to clone the coherence of an arbitrary quantum state $|\psi\rangle = \alpha|\psi_{1(E)}\rangle + \beta|\psi_{2(E)}\rangle$, for a fixed choice of basis $\{|0\rangle, |1\rangle\}$.

Proof. As per the assumption that the unitary cloning transformation (U_{cc}) perfectly clones the coherence of $|\psi_{1(E)}\rangle_A$, $|\psi_{2(E)}\rangle_A$ into subsystem of $|\Psi_1\rangle_{AB}$, $|\Psi_2\rangle_{AB}$ we have $2|x| = 1$ and $2|y| = 1$, where, $x = \rho'_{A01}$ and $y = \rho''_{A01}$ are the off-diagonal terms of the subsystem $\rho_A{}'$ and $\rho_A{}''$, respectively. Here we will only look at system A and prove that there exist some states for which cloning is not possible. With the constraint $2|a||b| = 1$ and the normalization condition we have $|a| = |b| = \frac{1}{\sqrt{2}}$. Now, the initial coherence of the input state becomes $C_l(|\psi\rangle_A) = 2|\alpha^2 ab - \beta^2 a^* b^*| = 2\sqrt{\frac{1}{4} - 2\,\text{Re}[\alpha^2\beta^{*2}(ab)^2]}$. However, the final coherence of the subsystem A is given by $C_l(\rho_A^{\text{final}}) = 2|(|\alpha|^2 x + |\beta|^2 y)|$.

Let us assume that there is a universal machine that clones coherence of any arbitrary state $|\psi\rangle$. Now, we clearly see that the initial coherence depends on the values of α and β but the final coherence depends only on $|\alpha|$ and $|\beta|$. A requirement to perfect cloning of coherence is that $C_l(|\psi\rangle)$ should be equal to $C_l(\rho_A^{\text{final}})$ for every α and β value. To show contradiction, we give examples of two states such that their final coherence is the same while their initial coherence is different.

There exist α_1 and β_1 such that $|\alpha| = |\alpha_1|$ and $|\beta| = |\beta_1|$ but $\alpha \neq \alpha_1$ or/and $\beta \neq \beta_1$. In that case, though the final coherences will be equal, the initial coherences are not. That clearly means that for at least one of the states the cloning coherence is not happening perfectly, hence proving Corollary 1. ■

Example. Let $|\chi_1\rangle = \alpha_1|\psi_1\rangle + \beta_1|\psi_2\rangle$ and $|\chi_2\rangle = \alpha_2|\psi_1\rangle + \beta_2|\psi_2\rangle$, where $\alpha_1 = \frac{1}{\sqrt{2}}$, $\beta_1 = \frac{1}{\sqrt{2}}$, $\alpha_2 = \frac{i}{\sqrt{2}}$, $\beta_2 = \frac{1}{\sqrt{2}}$. Here $|\psi_1\rangle$ and $|\psi_2\rangle$ are the states defined in Eq. (1). Then, initial coherence of $|\chi_1\rangle$ is given by $C_l(|\chi_1\rangle) = |ab - a^*b^*|$ and that of $|\chi_2\rangle$ is $C_l(|\chi_2\rangle) = |ab + a^*b^*|$ given that $|\mathrm{a}| = |\mathrm{b}| = \frac{1}{\sqrt{2}}$. Final coherence $C_l(\rho_1^{\text{final}}) = C_l(\rho_2^{\text{final}}) = |\rho'_{A01} + \rho''_{A01}|$, where ρ'_{A01} and ρ''_{A01} are the off-diagonal terms of the subsystem ρ'_A and ρ''_A, respectively.

Clearly, we see that there is a mismatch of the initial coherence of states $|\chi_1\rangle$ and $|\chi_2\rangle$, but their final coherence is the same. Therefore, at least for one of the states the coherence is not getting perfectly copied.

III. CLASSIFICATION OF STATES GIVEN A COHERENCE CLONER

In this section, we try to characterize the states whose coherence can be perfectly cloned given a machine defined over $|\psi_1\rangle$ and $|\psi_2\rangle$. Geometrically, we attempt to find out points on the surface of the sphere for which the cloning of coherence is possible. The entire Bloch sphere can be divided in two zones, namely, $C_l(|\psi\rangle_A) \leqslant 2|a||b|$ and $C_l(|\psi\rangle_A) > 2|a||b|$. In Theorem 1, we have already shown that cloning of coherence is not possible when $C_l(|\psi\rangle_A) > 2|a||b|$, however, it is not clear when $C_l(|\psi\rangle_A) \leqslant 2|a||b|$.

Let us take an arbitrary state $|\psi\rangle$ from the top or the bottom-most zones, which is the orange zone, as shown in Fig. 1. All the states both pure and mixed that have the same coherence value as $|\psi\rangle$ lie on $CYL_{|\psi\rangle}$ as shown in Fig. 4. For the coherence of $|\psi\rangle$ to be perfectly cloned, the output states ρ_A^{final} and ρ_B^{final} should lie on $CYL_{|\psi\rangle}$. As we have seen earlier $\rho_A^{\text{final}} = |\alpha|^2\rho_A{}' + |\beta|^2|\rho_A{}''$ and $\rho_B^{\text{final}} = |\alpha|^2\rho_B{}' + |\beta|^2|\rho_B{}''$ are convex combination of $\rho_A{}'$, $\rho_A{}''$ and $\rho_B{}'$, $\rho_B{}''$, respectively. Here, $\rho_A{}'$, $\rho_A{}''$ and $\rho_B{}'$, $\rho_B{}''$ are the mixed output states for the known orthogonal states $|\psi_i\rangle$ and should lie on the wider cylinder $CYL_{|\psi_1\rangle}$. This would mean that ρ_A^{final} is an intersection of the line segment joining $\rho_A{}'$, $\rho_A{}''$, and $CYL_{|\psi\rangle}$, a similar condition must hold for ρ_B^{final}.

For perfect cloning to happen, the line segment joining ρ'_A and ρ''_A and the line segment joining ρ'_B and ρ''_B should intersect $CYL_{|\psi\rangle}$ in equal proportions, as it is evident from the expressions of ρ_A^{final} and ρ_B^{final}. Let us imagine that $\rho_{A(B)}^{\text{final 1 (final 2)}}$ are four intersection points. Figure 2 shows some of the possible orientations of these four points.

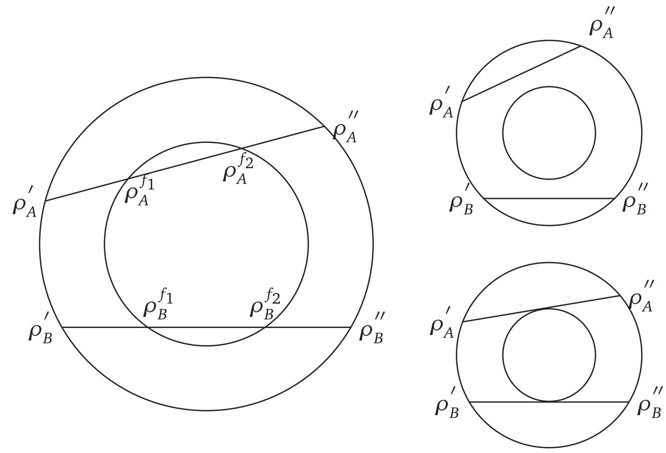

FIG. 2. Top view to depict existence of solutions.

Without loss of generality, let us just look at the subsystem A. Let $|\alpha| = k$. To find all the pure states that have the same coherence as $|\psi\rangle$, whose coherence can be perfectly cloned, we only need to see what are the points of intersection of rims of $CYL_{|\psi\rangle}$ and the circle CIRC_k, where CIRC_k contains all the points $\alpha|\psi_1\rangle + \beta|\psi_2\rangle$ whose $|\alpha| = k$, as shown in Fig. 3. Depending on the $CYL_{|\psi\rangle}$, $CYL_{|\psi_1\rangle}$, and the values of k, the number of points of intersections will vary from 0 to 4, as shown in Fig. 3. Similarly, when $|\alpha| = 1 - k$ we get the same number of solutions. Therefore, the total number of solutions vary as 0, 2, 4, 6, or 8.

IV. MAXIMIZATION OF COHERENCE CLONERS

In an earlier section, we have seen classification of states given a particular coherence cloner. It is clear that the range of states whose coherence can be cloned perfectly depends on the cloner U_{cc}. In this section, we discuss the techniques to maximize this cloner so as to have more range of states whose coherence can be cloned perfectly.

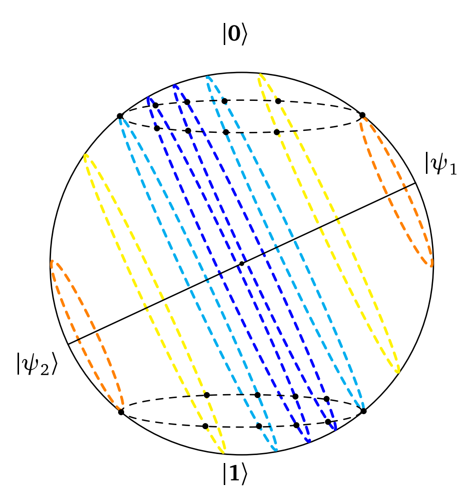

FIG. 3. Different cases of possible solutions: The figure shows depending on the choice of k different CIRC_k (the oblique circles) intersect the rims of the $CYL_{|\psi\rangle}$ (the two horizontal circles), the number of points of intersections will vary from 0 to 4. Total number of possible solutions will vary as 0, 2, 4, 6, or 8

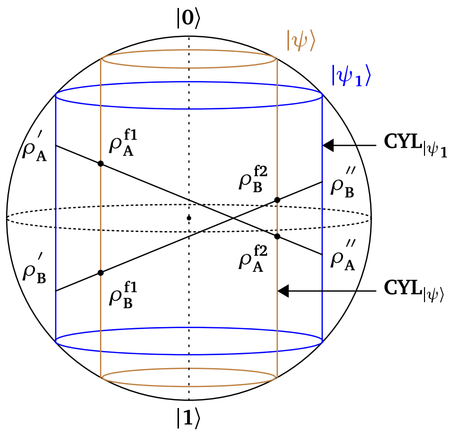

FIG. 4. Solutions with convex combinations: The cylinder with the bigger diameter represents the states having the same coherence as the known orthogonal states. The cylinder with the smaller diameter represents the states having the same coherence as the input states. Here, $\rho^{f1(f2)}_{A(B)}$ are nothing but $\rho^{\text{final 1 (final 2)}}_{A(B)}$.

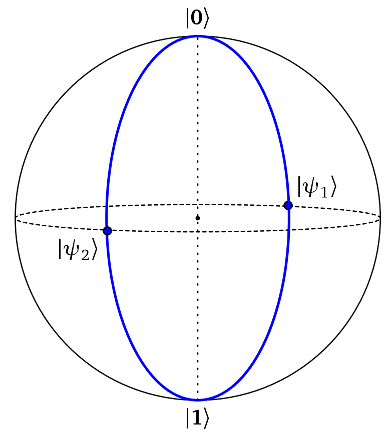

FIG. 5. The vertical circle going through the states $|\psi_1\rangle$ and $|\psi_2\rangle$ indicates the states whose coherence can be perfectly cloned.

There can be an infinite number of unitaries that are defined based on the transformation rules defined in Eq. (1). Every unitary depends on six states $|\psi_1\rangle_A$, $|\psi_2\rangle_A$, ρ'_A, ρ''_A, ρ'_B, ρ''_B. The cloner U_{cc} on system ABC transforms subsystem A which was an arbitrary quantum state $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$ and subsystem B which was a blank state $|0\rangle$ to $\rho^{\text{final}}_A = |\alpha|^2\rho'_A + |\beta|^2\rho''_A$ and $\rho^{\text{final}}_B = |\alpha|^2\rho'_B + |\beta|^2\rho''_B$, respectively. $\rho^{\text{final}}_{A(B)}$ is a convex combination of $\rho'_{A(B)}$ and $\rho''_{A(B)}$. Therefore, final state $\rho^{\text{final}}_{A(B)}$ should lie somewhere on the line segment joining the states $\rho'_{A(B)}$ and $\rho''_{A(B)}$ in the Bloch sphere, as we can see in Fig. 4. Therefore, we can say that the possibility of perfect coherence cloning for a state depends on whether the line segments intersect the cylinder with $C_l(\rho^{\text{final}}_{A(B)})$ or not. Given the fact that they do intersect, they have to intersect with the same ratio as each other, only then perfect cloning will be possible on both the subsystems, otherwise we can definitely say that the $C_l(|\psi\rangle)$ cannot be perfectly copied.

This brings us to our first level of maximization of our cloner U_{cc}. We can see in Fig. 2 that the cloners which have their line segments joining ρ'_A and ρ''_A and line segment joining ρ'_B and ρ''_B pass through the central axis allow for the possibility of perfect coherence cloning for a bigger range of states as they will intersect all the cylinders above them.

The second level of maximization can be done in the following way. We can see that if the starting states $|\psi_1\rangle_A$ and $|\psi_2\rangle_A$ of the assumed cloner lie on the equatorial plane, i.e., $C_l(|\psi_1\rangle) = C_l(|\psi_2\rangle) = 1$ and the output states will have equal coherence to that of the starting states, then this cloner will give maximum number of perfectly cloned copies as it will intersect all the cylinders on the sphere. Then, the cloning transformation is given by

$$\begin{aligned} |\psi_1\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\psi_1{}'\rangle_A|\psi_1{}''\rangle_B|X_1\rangle_C, \\ |\psi_2\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\psi_2{}'\rangle_A|\psi_2{}''\rangle_B|X_2\rangle_C. \end{aligned} \tag{3}$$

Here, $\langle\psi'_1|\psi'_2\rangle = 0$ and $\langle\psi''_1|\psi''_2\rangle = 0$ as this would ensure that the line segment joining ρ'_A and ρ''_A and the line segment joining ρ'_B and ρ''_B both pass through the central axis.

Interestingly, it is observed that a class of states whose coherence can be cloned perfectly given the transformations defined in Eq. (3) and the conditions $\langle\psi'_1|\psi'_2\rangle = 0$ and $\langle\psi''_1|\psi''_2\rangle = 0$ are the states that lie on the great circle passing through the states $|\psi_1\rangle$, $|\psi_2\rangle$, $|0\rangle$, and $|1\rangle$ on the Bloch sphere. The calculations for the same and the introduction of approximate quantum coherence cloner is given in the ensuing paragraph.

Approximate quantum coherence cloner for arbitrary quantum state. The states $|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{i\phi_1}|1\rangle$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{i(\phi_1+\pi)}|1\rangle$ represent a pair of orthogonal states on the equatorial circle of the Bloch sphere. Then, any arbitrary state $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$ can be written as $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi_2}|1\rangle$ in $\{|0\rangle, |1\rangle\}$ basis. Then, $C_l(|\psi\rangle) = |\sin\theta|$ in the $\{|0\rangle, |1\rangle\}$ basis. As $\alpha = \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e^{i(\phi_2-\phi_1)})$ and $\beta = \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}e^{i(\phi_2-\phi_1)})$, the final coherence which is given by $C_l(|\psi_{\text{final}}\rangle) = ||\alpha|^2 - |\beta|^2| = |2|\alpha|^2 - 1|$ becomes $|\sin\theta\cos(\phi_2-\phi_1)|$. Here, as we can see that the ratio of $\frac{C_l(|\psi_{\text{final}}\rangle)}{C_l(|\psi\rangle)} = \cos(\phi_2-\phi_1)$, which means the higher the difference between two angles, the lower is the ratio of the initial and final coherence. Similar to ratio, we can also see them as the difference between initial coherence and final coherence. Therefore, the approximate or perfect cloning of coherence depends on the difference between those two angles. We see that the only solution where initial coherence is equal to final coherence is when $\phi_1 = \phi_2$. Therefore, for all values of θ the final coherence $C_l(|\psi_{\text{final}}\rangle) = C_l(|\psi\rangle)$ if $\phi_1 = \phi_2$. This means that the cloner defined in Eq. (3) perfectly clones coherence for all the states on the great circle passing through $|\psi_1\rangle$, $|\psi_2\rangle$, $|0\rangle$, $|1\rangle$ as shown in Fig. 5.

V. NO CLONING OF QUANTUM COHERENCE WITHOUT MACHINE STATES

In the previous section, we have shown that there does not exist universal cloning transformation which will be able to clone the coherence of any arbitrary state. In the previous proof, the cloning transformation includes the ancilla states representing the machine states. In this section, we investigate whether there exists any unitary in general which will act on the input state and blank state without invoking an ancillary state that will clone coherence for any arbitrary state. We find that there exists no such unitary. Like in the previous section, here also we assume that the perfect cloning is possible for two known orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$. The transformation is given by

$$\begin{aligned}|\psi_1\rangle_A|0\rangle_B &\longrightarrow |\Psi_1\rangle_{AB},\\ |\psi_2\rangle_A|0\rangle_B &\longrightarrow |\Psi_2\rangle_{AB},\end{aligned} \tag{4}$$

where $\langle\psi_1|\psi_2\rangle = 0$. Therefore, $\langle\Psi_1|\Psi_2\rangle = 0$.

Theorem 2. It is impossible to clone the coherence of any arbitrary quantum state $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$, with the cloning transformations given by Eq. (4).

Proof. Let us assume that there exists a unitary that clones coherence of any arbitrary quantum state. Then this unitary should clone coherence for the states $|+\rangle$ and $|-\rangle$ as well. As these states are maximally coherent states, and this machine can clone the coherence perfectly, then the output states should also be maximally coherent states. The transformation would be given by Eq. (5). The output states in this case are all pure because there are no mixed states whose coherence can be 1:

$$\begin{aligned}|+\rangle_A|0\rangle_B &\longrightarrow |\psi_1{}'\rangle_A|\psi_1{}''\rangle_B,\\ |-\rangle_A|0\rangle_B &\longrightarrow |\psi_2{}'\rangle_A|\psi_2{}''\rangle_B,\end{aligned} \tag{5}$$

where either $\langle\psi_1{}'|\psi_2{}'\rangle = 0$ or $\langle\psi_1{}''|\psi_2{}''\rangle = 0$. Let $|\phi\rangle = \gamma|+\rangle + \delta|-\rangle$ be an arbitrary quantum state on the equatorial circle of the Bloch sphere. The transformation as given in Eq. (5) results in the state of the system AB to $|\Phi_{\text{final}}\rangle = \gamma|\psi_1{}'\rangle_A|\psi_1{}''\rangle_B + \delta|\psi_2{}'\rangle_A|\psi_2{}''\rangle_B$. For the coherence to be cloned perfectly, $|\Phi_{\text{final}}\rangle$ needs to be a separable system of two maximally coherent states. This makes either $|\psi_1{}'\rangle_A = |\psi_2{}'\rangle_A$ or $|\psi_1{}''\rangle_A = |\psi_2{}''\rangle_A$ because one of this pair has to be orthogonal.

Without loss of generality, let us assume that $\langle\psi_1{}'|\psi_2{}'\rangle = 0$ and $|\psi_1{}''\rangle_A = |\psi_2{}''\rangle_A$, then any state $|\psi_1\rangle$ with $C_l(|\psi_1\rangle) < 1$ from Eq. (4) can be written as $\alpha|+\rangle + \beta|-\rangle$, but under this transformation rules the system will transform to $(\alpha|\psi_1{}'\rangle + \beta|\psi_2{}'\rangle_A|\psi_1{}''\rangle_B$. Although the coherence of subsystem A is preserved, the coherence of subsystem B is still 1. ■

VI. NO CLONING OF COHERENCE OF A QUDIT

In previous sections, we have shown the impossibility of cloning of coherence of a qubit based on a particular measure of coherence, i.e., l_1 norm. Here in this section we generalize the proofs in terms of the dimension of the quantum state and coherence measures for which maximally coherent states are pure states [42] (Result 1). Let us assume the dimension of the quantum state ρ as d_1 and denote the coherence measure as $C(\rho)$ in computational basis. Like in the previous sections, here also we assume the perfect cloning is possible for known orthogonal states $|\psi_i\rangle,\ i = 1, \ldots, d_1$ The cloning transformation U_{cc} for coherence is given as

$$\begin{aligned}|\psi_1\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow \quad |\Psi_1\rangle_{AB}|X_1\rangle_C,\\ |\psi_2\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow \quad |\Psi_2\rangle_{AB}|X_2\rangle_C,\\ &\ \ \vdots\\ |\psi_{d_1}\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow \quad |\Psi_{d_1}\rangle_{AB}|X_{d_1}\rangle_C.\end{aligned} \tag{6}$$

Here, $\langle X_j|X_i\rangle = \delta_{ij}$ based on the dependence of the above transformation on machine states. The dimensions of the input, blank, and machine states are d_1, d_2, and d_3, respectively. As we want to clone the coherence of the input state on the blank state, it is important to note that as some of the coherence measures are not normalized, the coherence value of maximally coherent state of dimension d_1 should be less than equal to that of maximally coherent state of dimension d_2.

Theorem 3. It is impossible to clone the coherence of any arbitrary quantum state $|\psi\rangle = \sum_i^{d_1} \alpha_i|\psi_i\rangle$ of dimension d_1, with the cloning transformations given in Eq. (6).

Proof. Let us assume that the unitary transformation U_{cc} given in Eq. (6) clones the coherence of any arbitrary quantum state of dimension d_1 into the blank state of dimension d_2. Thus, this unitary should also clone the coherence of maximally coherent state $|\psi_m\rangle = \sum_i^{d_1} \beta_i|\psi_i\rangle$ which should be a pure state as mentioned before. Applying the unitary transformation U_{cc} on this maximally coherent state, we get

$$|\psi_m\rangle_A|0\rangle_B|X_0\rangle_C \longrightarrow \sum_i^{d_1} |\Psi_i\rangle_{AB}|X_i\rangle_C. \tag{7}$$

But, as we know that the maximally coherent states are pure states only and as we have assumed U_{cc} to be a universal cloner, the coherence of $|\psi_m\rangle$ should be cloned perfectly and the resultant state of subsystem A should be a pure state. It is only possible when $|\psi_i\rangle = |\psi_{m'}\rangle,\ \forall\ i$. Here, $|\psi_{m'}\rangle$ is another maximally coherent pure state. Therefore, the subsystem A should be separable from the subsystem BC and the coherence of subsystem A should be $C(|\psi_m\rangle)$. As the subsystem A always comes out to be separable from subsystem BC and its coherence is equal to $C(\psi_m)$, U_{cc} will not retain the coherence of any other arbitrary state $|\psi\rangle$ with $C(|\psi\rangle)$ on subsystem A. Using contradiction on our initial assumption, we can say that U_{cc} is not a universal coherence cloner irrespective of its dependence on machine states. ■

VII. CONCLUSION

To summarize, we have shown that it is impossible to clone the coherence of the states whose coherences are greater than the coherence of known states with which transformations are defined. This establishes the impossibility of cloning of coherence of any arbitrary quantum state. Since we know that the coherence acts as a resource for creation of entanglement by incoherent operation, if we could clone coherence exactly, then we will be able to build up a large amount of entanglement, starting from a minimal amount of coherence and repeated use of incoherent operations. However, from our result it is evident that we can not build up a large amount of entanglement from an arbitrary quantum state by cloning the coherence. Similarly, in the same spirit, if one aims to generate coherence in the energy eigenbasis from thermal resources, then our no-go result suggests that coherence cannot be created in initially incoherent systems while maintaining the machine in a fixed state. Aside from this, we characterize the class of states for which coherence cloning will be possible for a given choice of machine built on known orthogonal states and find the maximum range of states whose coherence can be cloned perfectly. Interestingly, we also show that the universal cloner does not exist even in the situation where we have no ancillary inputs. At last, we generalize the impossibility of cloning of coherence in terms of dimension of the quantum state and given all coherence measures.

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APPENDIX: HOW CLONING OF COHERENCE IS DIFFERENT FROM CLONING OF STATES

Here, we illustrate some instances where coherence cloning is not the same as the cloning of quantum states. As an example, consider the Wootter-Zurich (WZ) cloning machine [36] that performs the following operation:

$$\begin{aligned} |0\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |0\rangle_A|0\rangle_B|X_1\rangle_C, \\ |1\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |1\rangle_A|1\rangle_B|X_2\rangle_C. \end{aligned} \tag{A1}$$

Here, A, B, and C are the input, output, and machine qubits, respectively. When we apply the same WZ cloning machine on an arbitrary quantum state $\alpha|0\rangle + \beta|1\rangle$ ($|\alpha|^2 + |\beta|^2 = 1$) whose coherence is $2|\alpha||\beta|$ in the $\{|0\rangle, |1\rangle\}$ basis, it transforms to the state $\alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1|$ which has zero coherence in the $\{|0\rangle, |1\rangle\}$ basis. This shows that even if the state gets cloned approximately with the WZ cloning machine, there is no cloning of the coherence as the WZ cloning machine does not take into account the off-diagonal terms of the state.

Let us consider another example of the Buzek-Hillery (BH) cloning machine [43] which is proved to be an optimal state-independent quantum cloning machine [44]. The two-dimensional BH cloning transformation is given as

$$\begin{aligned} |\Psi_1\rangle_A|0\rangle_B|X_0\rangle_C \rightarrow\ & c|\Psi_1\rangle_A|\Psi_1\rangle_B|X_{11}\rangle_C \\ & + d(|\Psi_1\rangle_A|\Psi_2\rangle_B + |\Psi_2\rangle_A|\Psi_1\rangle_B)|Y_{12}\rangle_C, \\ |\Psi_2\rangle_A|0\rangle_B|X_0\rangle_C \rightarrow\ & c|\Psi_2\rangle_A|\Psi_2\rangle_B|X_{22}\rangle_C \\ & + d(|\Psi_2\rangle_A|\Psi_1\rangle_B + |\Psi_1\rangle_A|\Psi_2\rangle_B)|Y_{21}\rangle_C, \end{aligned}$$

where the coefficients c and d are real. The notations A, B, and C represent the input, output, and machine qubits, respectively. In case of cloning a single qubit, using the no-signaling constraint and the fidelity as a parameter of quantum cloning machine, Gisin proved that the BH state-independent quantum cloner is the optimal one with the fidelity $\frac{5}{6}$ [44]. But, if we consider the ratio of the final coherence to the initial coherence (l_1 norm), the BH cloner gives $\frac{2}{3}$. That is, in other words, two thirds of coherence is getting copied with the BH cloner. This example shows that even though information gets cloned up to $\frac{5}{6}$, coherence gets cloned only up to $\frac{2}{3}$. Possibly, this suggests us that when we clone quantum information we try to clone both the wave information and the particle information. As the BH machine only clones wave information up to $\frac{2}{3}$ it may be the case that the higher value of $\frac{5}{6}$ is due to particle nature getting cloned more compared to wave information.

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