Lower bounds on the failure probability of unambiguous discrimination

Xin Lü
School of Physics and Electrical Engineering, Liupanshui Normal University, Minghu Lu, Liupanshui 553004, Guizhou, China

(Received 22 March 2020; revised 3 November 2020; accepted 8 February 2021; published 19 February 2021)

We present a systematic approach for obtaining lower bounds on the failure probability of unambiguous state discrimination by introducing an ancillary degree of freedom. Specifically, the lower bounds are obtained by introducing a suitable homogeneous function of the off-diagonal entries of the reduced matrix in the ancillary degree of freedom. A series of lower bounds on the failure probability of unambiguous discrimination of both pure states and mixed states are derived using the l_r-norm of the off-diagonal entries. Our method adds insights into the problem of unambiguous state discrimination. In particular, since we have related the lower bound with off-diagonal entries of density matrices, e.g., the l_1-norm coherence, our work suggests that unambiguous state discrimination may provide further applications in quantifying coherence, and vise versa.

DOI: 10.1103/PhysRevA.103.022216

I. INTRODUCTION

It is fundamental in quantum mechanics that nonorthogonal states cannot be perfectly discriminated [1–11]. For example, nonorthogonal states are used in quantum cryptographic protocols [12–14], and state discrimination is intimately related to the no-signaling principle [15–17], quantum no-clone theory [18], sequential quantum measurements [19], quantum change point problem [20] and some fundamental quantum concepts, such as contextuality [21], entanglement [22,23], coherence [24–26], etc. Therefore state discrimination is not only important in understanding the nature of quantum mechanics, but also useful in many quantum information processing tasks. The objective of state discrimination is to find the optimal measurement scheme that produces the best discrimination results. In practice, there exist different ways to quantify discrimination results, e.g., sometimes one would like to minimize the average discrimination error, while sometimes one would like to minimize the error of inconclusive discrimination. These two figures of merit correspond to the two most popular discrimination strategies, i.e., ambiguous discrimination (also called minimum-error discrimination) [1,2] and unambiguous discrimination [3–6], respectively. These discrimination problems turn out to be difficult to solve, except in a few simple cases [1,3–6,27–30], people fail to find the optimal measurement scheme analytically, instead, some upper bounds of the success probability are derived without giving the corresponding measurement operators explicitly [31–37]. In addition, computing techniques, such as semidefinite programming, are applied to solve this problem numerically [38–41].

In the following discussion, we focus on unambiguous state discrimination exclusively. Given an ensemble of quantum states $\{\rho_j, w_j\}_{j=1}^n$ such that each positive number w_j is the weight of its corresponding state ρ_j and all the weights sum up to one, unambiguous discrimination of this ensemble is to find the optimal measurement operators $\{M_i\}_{i=0}^n$ such that the failure probability

$$p_0 = \sum_{j=1}^{n} w_j \mathrm{Tr}(M_0 \rho_j M_0^\dagger) \tag{1}$$

is minimized under the unambiguous condition

$$M_i \rho_j M_i^\dagger = \delta_{ij} M_j \rho_j M_j^\dagger \quad \text{with} \quad i, j = 1, \dots, n, \tag{2}$$

or equivalently, in terms of the probabilities of the measurement outcomes

$$\mathrm{Tr}(M_i \rho_j M_i^\dagger) = \delta_{ij} \mathrm{Tr}(M_j \rho_j M_j^\dagger) \quad \text{with} \quad i, j = 1, \dots, n. \tag{3}$$

Consequently, as long as $i \neq 0$, the occurrence of the ith measurement outcome implies that the state is ρ_i; on the other hand, when the zeroth outcome happens, the discrimination produces no conclusive results, and the probability p_0 in (1) of the zeroth measurement outcome is hence called the failure probability of unambiguous discrimination. This kind of discrimination problem is first considered by Ivanovic [3] to show the power of generalized measurements and manifests the fact that nonorthogonal states cannot be perfectly discriminated. Now it is known that unambiguous discrimination is possible only when the states are linearly independent [23,42], and several lower bounds are known for the failure probability p_0 [31–34]. In particular, Zhang *et al.* obtain the lower bound for ensembles of pure states in Ref. [31] and Feng *et al.* generalize this result to mixed states [34].

Recently the link between quantitative wave-particle duality and state discrimination is discovered [24–26,43], e.g., Bera *et al.* have established the duality relation of path distinguishability and coherence via unambiguous discrimination [24]. Such a relation provides insights into coherence from unambiguous discrimination. It is reasonable to wonder whether the converse is also possible, i.e., can coherence be used to obtain information about unambiguous discrimination. This is the motivation for us to consider the off-diagonal entries of density matrices here. In this paper, we present a unified approach for obtaining lower bounds on the failure probability p_0 of unambiguous discrimination by introducing an ancillary degree of freedom. We consider discrimination of both pure states and mixed states. As a result, a series of lower bounds is generated. In particular, the three known lower bounds [31,32,34] are recovered from this perspective. Our method is straightforward and involves no tedious computation, it returns lower bounds once a proper homogeneous function of the off-diagonal entries is constructed. Specifically, the l_r norm is used as an example, and by comparing the lower bounds thus derived, we give the best lower bound that can be derived from this consideration.

The problem of discrimination of pure states is considered in Sec. II, where two known lower bounds [31,32] of the failure probability p_0 are derived via the l_1 norm and l_2 norm, respectively. We also discuss the possibility of considering the general l_r norm and hence obtain a series of bounds in this section. In Sec. III, the consideration is generalized to mixed states, and a series of lower bounds of p_0 are derived similarly as in Sec. II. The paper is then closed with a summary.

II. DISCRIMINATION OF PURE STATES

In this section, we consider the problem of unambiguous discrimination of n pure states $\{|\phi_j\rangle, w_j\}$ with unequal *a priori* weights w_j by introducing an ancillary degree of freedom and therefore link the lower bound of the failure probability with upper bounds of the l_1 norm:

$$c_1(\rho) = \sum_j \sum_{k\neq j} |\rho_{jk}|, \tag{4}$$

i.e., the sum of the all the absolute values of the off-diagonal entries in the density matrix ρ and the l_2 norm:

$$c_2(\rho) = \left(\sum_j \sum_{k\neq j} |\rho_{jk}|^2 \right)^{1/2}. \tag{5}$$

Generally, the l_r norm is defined as

$$c_r(\rho) = \left(\sum_j \sum_{k\neq j} |\rho_{jk}|^r \right)^{1/r}. \tag{6}$$

We show that our method is still valid for the l_r norm with r bounded by a specific number r_n when there are n states to be discriminated.

Introduce the pure bipartite state

$$|\Phi\rangle_{\rm AB} = \sum_j \sqrt{w_j} |j\rangle_{\rm A} |\phi_j\rangle_{\rm B}, \tag{7}$$

where $\{|j\rangle_{\rm A}\}_{j=1}^n$ is an orthonormal basis of an ancillary n-dimension space A, and the states to be discriminated live in space B. In terms of density matrices, it is the following composite state:

$$\rho_{\rm AB} = \sum_{j,k} \sqrt{w_j w_k} |j\rangle\langle k| \otimes |\phi_j\rangle\langle\phi_k|, \tag{8}$$

where the subscript A or B is omitted for brevity. The corresponding state in each degree of freedom is obtained by taking a partial trace. Explicitly, the state in space A is

$$\rho_{\rm A} = \sum_{j,k} \sqrt{w_j w_k} \langle\phi_k|\phi_j\rangle |j\rangle\langle k|, \tag{9}$$

and in space B

$$\rho_{\rm B} = \sum_j w_j |\phi_j\rangle\langle\phi_j|, \tag{10}$$

thus $\rho_{\rm B}$ is exactly the ensemble to be discriminated unambiguously. Performing measurement $\{M_i\}_{i=0}^n$ on system B yields

$$\rho_{\rm AB}^{(i)} = \sum_{j,k} \frac{\sqrt{w_j w_k}}{p_i} |j\rangle\langle k| \otimes M_i|\phi_j\rangle\langle\phi_k|M_i^\dagger, \tag{11}$$

with $p_i = {\rm Tr} M_i \rho_{\rm B} M_i^\dagger = \sum_j w_j \langle\phi_j|M_i^\dagger M_i|\phi_j\rangle$, so that p_0 is consistent with the earlier definition of the failure probability in (1). The unambiguous condition (2) requires that

$$M_i|\phi_j\rangle = \delta_{ij} M_j|\phi_j\rangle \text{ with } i \neq 0, \tag{12}$$

so that $p_i = w_i \langle\phi_i|M_i^\dagger M_i|\phi_i\rangle$ and consequently

$$\rho_{\rm A}^{(i)} = {\rm Tr}_{\rm B} \rho^{(i)}{}_{\rm AB} = \sum_{j,k} \frac{\sqrt{w_j w_k}}{p_i} \langle\phi_k|M_i^\dagger M_i|\phi_j\rangle |j\rangle\langle k|$$
$$= \frac{w_i}{p_i} \langle\phi_i|M_i^\dagger M_i|\phi_i\rangle |i\rangle\langle i| = |i\rangle\langle i| \text{ for } i \neq 0.$$

As a consequence, except $\rho^{(0)}{}_{\rm A}$, each $\rho^{(j)}{}_{\rm A}$ has no off-diagonal entries. The normalization requirement

$$\sum_{i=0}^n M_i^\dagger M_i = \mathbb{1} \tag{13}$$

of the measurement operators $\{M_i\}_{i=0}^n$ implies that

$$\rho_{\rm A} = {\rm Tr}_{\rm B} \left(\mathbb{1} \otimes \sum_i M_i^\dagger M_i \, \rho_{\rm AB} \right) = \sum_{i=0}^n p_i \rho_{\rm A}^{(i)}. \tag{14}$$

For that reason, concentrated on the off-diagonal entries of $\rho_{\rm A}$, we have

$$\rho_{\rm A,offdiag} = p_0 \rho_{\rm A,offdiag}^{(0)}, \tag{15}$$

so that, for any degree-d homogeneous function f of the off-diagonal entries, the following equation is satisfied:

$$f(\rho_{\rm A}) = f\big(p_0 \rho_{\rm A}^{(0)}\big) = p_0^d f\big(\rho_{\rm A}^{(0)}\big). \tag{16}$$

In particular, the l_1 norm c_1 (4) and l_2 norm c_2 (5) are such functions of degree one. Consequently, from (9) we establish that

$$c_1(\rho_{\rm A}) = \sum_{j=1}^n \sum_{k\neq j} \sqrt{w_j w_k} |\langle\phi_k|\phi_j\rangle| = p_0 c_1\big(\rho_{\rm A}^{(0)}\big), \tag{17}$$

$$c_2(\rho_{\rm A}) = \sqrt{\sum_{j=1}^n \sum_{k\neq j} w_j w_k |\langle\phi_k|\phi_j\rangle|^2} = p_0 c_2\big(\rho_{\rm A}^{(0)}\big). \tag{18}$$

In other words, the problem of calculating the failure probability p_0 is transferred to the off-diagonal entries of the reduced state in the ancillary system A. The positivity of any n-dimensional density matrix ρ or, more precisely, the non-negativeness of every 2×2 principal minor of ρ, and the normalization property of ρ imply that [where the maximum is achieved by letting each of the $n(n-1)$ off-diagonal entries equal $1/n$]

$$c_1\big(\rho_{\rm A}^{(0)}\big) \leqslant n-1, \quad \text{and} \tag{19}$$

$$c_2\big(\rho_{\rm A}^{(0)}\big) \leqslant \left(\frac{n-1}{n}\right)^{1/2}. \tag{20}$$

Substituting (19) into (17), we immediately have

$$p_0 \geqslant \frac{1}{n-1}\sum_{j=1}^{n}\sum_{k\neq j}\sqrt{w_j w_k}|\langle\phi_k|\phi_j\rangle|, \tag{21}$$

which is exactly the lower bound found in Ref. [31], and similarly (18) and (20) imply

$$p_0 \geqslant \left(\frac{n}{n-1}\sum_{j=1}^{n}\sum_{k\neq j} w_j w_k|\langle\phi_k|\phi_j\rangle|^2\right)^{1/2}, \tag{22}$$

the lower bound in Ref. [32]. Straightforward application of Hölder's inequality shows that, for $r \geqslant s \geqslant 1$,

$$\left(\frac{1}{n(n-1)}\right)^{1/r} c_r(\rho) \geqslant \left(\frac{1}{n(n-1)}\right)^{1/s} c_s(\rho), \tag{23}$$

where $c_r(\rho)$ denotes the l_r norm of the off-diagonal entries of ρ defined in (6). In particular, the lower bound in (22) is greater than or equal to the bound in (21).

As a result, the two known lower bounds in Refs. [31,32] are obtained by a unified approach. One may suggest from (23) that the l_r norm with $r > 2$ will produce an even better lower bound, and intuitively, the l_∞ norm will offer the best lower bound which depends only on the greatest off-diagonal entry of the density matrix. Unfortunately, as observed in Ref. [32], this is not the case. The problem is that, for the l_r norm with sufficiently large r, its maximum is no longer reached with equal off-diagonal entries, and therefore one may get a larger upper bound than expected from (19) and (20). To be explicit, it is easy to see that the tight upper bound of $c_\infty(\rho)$ is $1/2$ for any dimensional density matrix ρ, so that our consideration produces the simple bound

$$p_0 \geqslant 2\max_{k\neq j}\{\sqrt{w_j w_k}|\langle\phi_j|\phi_k\rangle|\}, \tag{24}$$

which is significant only when the maximum of the off-diagonal entries of $\rho_{\rm A}$ in (9) is large enough. An oversimplification. But, on the other hand, as long as the maximum of $c_r(\rho)$ is achieved with equal off-diagonal entries, inequality (23) suggests a better lower bound. Consequently, there must be a largest number r_n, which depends on the number n of the states to be discriminated, such that the maximum of $c_{r_n}(\rho)$ still happens when all the off-diagonal entries are equal, and it will provide us the best lower bound from this reasoning.

Actually, for any matrix $\mu_{\rm A}$, the following two conditions:

(C1) Tr $\mu_{\rm A} = 1$, $\mu_{\rm A} = \mu_{\rm A}^\dagger$ and every diagonal entries of $\mu_{\rm A}$ are non-negative;

(C2) every 2×2 principal minor of $\mu_{\rm A}$ is non-negative,

are sufficient to show the bounds

$$c_1\big(\mu_{\rm A}^{(0)}\big) \leqslant n-1, \quad \text{and} \tag{25}$$

$$c_2\big(\mu_{\rm A}^{(0)}\big) \leqslant \left(\frac{n-1}{n}\right)^{1/2}, \tag{26}$$

which is everything one needs in the consideration. By switching the notation from $\rho_{\rm A}$ to $\mu_{\rm A}$, we emphasize that the restriction of the reduced matrix in space A to be a proper quantum state is loosened. Of course, this relaxation does not matter much for pure states, but it helps to obtain better bounds for mixed states. To summarize, after introducing an ancillary degree of freedom A such that the reduced matrix $\mu_{\rm A}$ in A fulfills the above two conditions, all one needs next is a suitable homogeneous function of the off-diagonal entries whose upper bound due to conditions (C1, C2) is apparent. In the particular case of the l_r norm, with constraints (C1, C2), the maximum of $c_r(\mu_{\rm A}^{(0)})$ is

$$\sum_j\sum_{k\neq j}(x_j x_k)^{r/2}, \tag{27}$$

where x_j denotes the jth diagonal entry of $\mu_{\rm A}^{(0)}$ so that $\sum_j x_j = 1$ and $x_j \geqslant 0\ \forall\ j$ by (C1), and the norm of the jkth off-diagonal entry is not greater than $\sqrt{x_j x_k}$ by (C2). Applying the Lagrange multipliers method, one finds that the local maximums of (27) with the constraint (C1) happens when

$$\begin{aligned} x_{j_1} &= \cdots = x_{j_m} = 1/m, \\ x_{k_1} &= \cdots = x_{k_{n-m}} = 0, \end{aligned} \tag{28}$$

with $m \leqslant n$, i.e., m diagonal entries are nonzero and have the same value, while the other $n-m$ diagonal entries are zero. The corresponding local maximum of (27) is then

$$c_r^{(m)} = [m^{1-r}(m-1)]^{1/r}. \tag{29}$$

Let $m_1 < m_2$, direct calculation shows that $c_r^{(m_1)} \leqslant c_r^{(m_2)}$ if and only if

$$r \leqslant \frac{\ln(m_2-1)-\ln(m_1-1)}{\ln m_2 - \ln m_1} + 1. \tag{30}$$

We would like $c_r^{(n)}$ to be the largest among all the $\{c_r^{(m)}\}$, so necessarily $c_r^{(n-1)} \leqslant c_r^{(n)}$, i.e.,

$$r \leqslant \frac{\ln(n-1)-\ln(n-2)}{\ln n - \ln(n-1)} + 1. \tag{31}$$

On the other hand, fixing m_2, the right-hand side of (30) is a decreasing function of m_1, which guarantees that for such r satisfying (31), $c_r^{(n)}$ is indeed larger than any $c_r^{(m_1)}$ with $m_1 < n$. In conclusion, the maximum of the l_r-norm c_r happens with equal off-diagonal entries, i.e.,the situation of $m = n$ in (29), if and only if the exponent $r \leqslant r_n$, where

$$r_n = \frac{\ln(n-1)-\ln(n-2)}{\ln n - \ln(n-1)} + 1 > 2. \tag{32}$$

Consequently, in dimension n, the l_{r_n} norm will provide the best possible general lower bound in our consideration as

$$p_0 \geqslant \left(\frac{n^{r_n}}{n(n-1)}\sum_j\sum_{k\neq j}\big(\sqrt{w_j w_k}|\langle\phi_j|\phi_k\rangle|\big)^{r_n}\right)^{1/r_n}, \tag{33}$$

with r_n defined in (32). This r_n also answers negatively to the question left in Ref. [32]: The condition

$$r \leqslant \frac{\ln(n-1)}{\ln n - \ln 2} + 1 \tag{34}$$

is too coarse to guarantee a better bound.

III. DISCRIMINATION OF MIXED STATES

Now we move on to the general situation of unambiguous discrimination of the ensemble $\{\rho_j, w_j\}_{j=1}^n$. Similarly as in Sec. II, an ancillary degree of freedom is introduced, and we consider the following matrix:

$$\mu_{\rm AB} = \sum_{j,k} \sqrt{w_j w_k} |j\rangle\langle k| \otimes \sqrt{\rho_j} U_{jk} \sqrt{\rho_k}, \tag{35}$$

with the requirement

$$U_{jj} = \mathbb{1} \quad \text{and} \quad U_{jk} = U_{kj}^\dagger \; \forall \; j \neq k, \tag{36}$$

so that $\mu_{\rm A} = {\rm Tr}_{\rm B}\mu_{\rm AB}$ satisfies the condition (C1). Viewing the matrix $\mu_{\rm AB}$ in (35) as a block matrix, it is straightforward to verify that the block matrix

$$\begin{pmatrix} \rho_j & \sqrt{\rho_j} U_{jk} \sqrt{\rho_k} \\ \sqrt{\rho_k} U_{jk}^\dagger \sqrt{\rho_j} & \rho_k \end{pmatrix} = X_{jk} X_{jk}^\dagger \geqslant 0 \tag{37}$$

for any pairs $j \neq k$, with

$$X_{jk} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\rho_j} & \sqrt{\rho_j} U_{jk} \\ \sqrt{\rho_k} U_{jk}^\dagger & \sqrt{\rho_k} \end{pmatrix}. \tag{38}$$

Similarly, any 2×2 block matrix of the form

$$\begin{pmatrix} M_i \rho_j M_i^\dagger & M_i \sqrt{\rho_j} U_{jk} \sqrt{\rho_k} M_i^\dagger \\ M_i \sqrt{\rho_k} U_{jk}^\dagger \sqrt{\rho_j} M_i^\dagger & M_i \rho_k M_i^\dagger \end{pmatrix} \tag{39}$$

is also positive semidefinite. In conclusion, the matrix $\mu_{\rm AB}$ defined in (35) and every matrix ${\mu^{(i)}}_{\rm AB}$ defined as

$$\mu_{\rm AB}^{(i)} = \frac{1}{p_i} \sum_{j,k} \sqrt{w_j w_k} |j\rangle\langle k| \otimes M_i \sqrt{\rho_j} U_{jk} \sqrt{\rho_k} M_i^\dagger, \tag{40}$$

where

$$p_i = \sum_j w_j {\rm Tr}(M_i \rho_j M_i^\dagger), \tag{41}$$

is the probability of the ith measurement outcome, and the set of unitaries $\{U_{jk}, j, k = 1, \dots, n\}$ fulfills condition (36), will provide us the reduced matrices

$$\mu_{\rm A} = \sum_{j,k} \sqrt{w_j w_k} {\rm Tr}(\sqrt{\rho_j} U_{jk} \sqrt{\rho_k}) |j\rangle\langle k| \quad \text{and}$$
$$\mu_{\rm A}^{(i)} = \sum_{j,k} \frac{\sqrt{w_j w_k}}{p_i} {\rm Tr}(M_i \sqrt{\rho_j} U_{jk} \sqrt{\rho_k} M_i^\dagger) |j\rangle\langle k|$$

satisfying the conditions (C1, C2) listed in Sec. II, so that similar analysis can be conducted.

The unambiguous condition (2) implies that

$$M_i \sqrt{\rho_j} = \delta_{ij} M_j \sqrt{\rho_j} \text{ with } i \neq 0, \tag{42}$$

therefore similarly as in Sec. II,

$$\mu_{\rm A}^{(i)} = |i\rangle\langle i| \text{ for } i \neq 0, \tag{43}$$

so that condition (15) is valid. Everything else then follows in the same way as in Sec. II, and we obtain the results

$$p_0 \geqslant \left(\frac{n^r}{n(n-1)} \sum_j \sum_{k\neq j} \left| \sqrt{w_j w_k} {\rm Tr}(\sqrt{\rho_j} U_{jk} \sqrt{\rho_k}) \right|^r \right)^{1/r},$$

with the index r bounded above by the number r_n defined in (32). Now fixing the unitaries $\{U_{jk}\}$ by the polar decomposition

$$U_{jk} \sqrt{\rho_k} \sqrt{\rho_j} = |\sqrt{\rho_k} \sqrt{\rho_j}| \text{ with } j < k, \tag{44}$$

and the requirement (36), we obtain the lower bound

$$p_0 \geqslant \left(\frac{n^r}{n(n-1)} \sum_j \sum_{k\neq j} [w_j w_k F(\rho_j, \rho_k)]^{r/2} \right)^{1/r}, \tag{45}$$

where the trace ${\rm Tr}|\sqrt{\rho_k}\sqrt{\rho_j}|$ is expressed by the square root of the fidelity between ρ_j and ρ_k [44,45]:

$$F(\rho_k, \rho_j) = ({\rm Tr}|\sqrt{\rho_k}\sqrt{\rho_j}|)^2, \tag{46}$$

which is the square of the fidelity in Ref. [46]. The above result (45) is the best lower bound that one can get from matrix (35) by varying the unitaries, since it is well known that

$$\sqrt{F(\rho_j, \rho_k)} = \max_{\text{unitary } U} {\rm Tr}(\sqrt{\rho_k}\sqrt{\rho_j} U). \tag{47}$$

In particular, let $r = 2$, we obtain the known bound

$$p_0 \geqslant \left(\frac{n}{n-1} \sum_j \sum_{k\neq j} w_j w_k F(\rho_j, \rho_k) \right)^{1/2}, \tag{48}$$

which has been derived in Ref. [34].

Note that, different from the situation of pure states, the matrix $\mu_{\rm AB}$ in (35) is not positive in general, therefore it cannot be considered as a valid quantum state in the composite system. Nevertheless, since only the weaker positivity condition (C2) is necessary, mathematically we are still able to derive lower bounds for mixed states this way. As a direct application of these lower bounds, the bounds (21) and (48) suggest the quantitative wave-particle duality relation proposed in Refs. [24] and [26], respectively, which is not surprising, since both quantitative wave particle duality and unambiguous discrimination depend on the positivity of density matrices. To be precise, all known wave-particle duality relations follow from the conditions (C1, C2). It is interesting to wonder whether the bound (45) also suggests quantitative wave-particle duality relations. We remark that Dürr has already shown that the l_2 norm is the standard deviation of the interference pattern averaged over all the phases [47], so these lower bounds do have physical significance.

IV. CONCLUSION

We have systematically obtained a series of lower bounds [(21), (22), (33), (45)] on the failure probability p_0 of unambiguous discrimination. Notably, we have provided the largest index r_n in (32) such that the l_{r_n} norm of the off-diagonal entries will return a good lower bound of the failure probability p_0. By introducing an ancillary degree of freedom, the failure probability is directly linked with homogeneous functions of off-diagonal entries of any matrix satisfying conditions (C1, C2). As a result, there is no tedious mathematics in our approach as in earlier similar works. It would be interesting to generalize the consideration by using other homogeneous functions of the off-diagonal entries other than the l_r norm, and to apply the idea also in the problem of ambiguous discrimination to obtain upper bounds on its success probability. Besides, since we have related the lower bound with off-diagonal entries of density matrices, e.g., the l_1 norm coherence, our work also suggests that unambiguous state discrimination may provide further applications in quantifying coherence, and vise versa.

ACKNOWLEDGMENT

The author is funded by the Young Scientific Talents Growth Project of the Department of Education of Guizhou Province (QJHKYZ[2018]377), Research Fund for High-level Talents of Liupanshui Normal University (LPSSYKYJJ201813).

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1979).
[3] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).
[4] D. Dieks, Phys. Lett. A **126**, 303 (1988).
[5] A. Peres, Phys. Lett. A **128**, 19 (1988).
[6] G. Jaeger and A. Shimony, Phys. Lett. A **197**, 83 (1995).
[7] J. A. Bergou, U. Herzog, M. Hillery, and M. Hillery, *Discrimination of Quantum States* in Quantum State Estimation (Springer, Berlin, Heidelberg, 2004), pp. 417–465.
[8] J. A. Bergou, J. Phys: Conf. Ser. **84**, 012001 (2007).
[9] S. M. Barnett and S. Croke, Adv. Opt. Photon. **1**, 238 (2009).
[10] J. A. Bergou, J. Mod. Opt. **57**, 160 (2010).
[11] J. Bae and L. C. Kwek, J. Phys. A: Math. Theor. **48**, 083001 (2015).
[12] C. H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992).
[13] J. Bae and A. Acin, Phys. Rev. A **75**, 012334 (2007).
[14] J. A. Bergou and M. Orszag, J. Opt. Soc. Am. B **24**, 384 (2007).
[15] S. M. Barnett and E. Andersson, Phys. Rev. A **65**, 044307 (2002).
[16] W.-Y. Hwang, Phys. Rev. A **71**, 062315 (2005).
[17] J. Bae, J. W. Lee, J. Kim, and W.-Y. Hwang, Phys. Rev. A **78**, 022335 (2008).
[18] N. Gisin, Phys. Lett. A **242**, 1 (1998).
[19] J. Bergou, E. Feldman, and M. Hillery, Phys. Rev. Lett. **111**, 100501 (2013).
[20] G. Sentís, E. Bagan, J. Calsamiglia, G. Chiribella and R. Muñoz-Tapia, Phys. Rev. Lett. **117**, 150502 (2016).
[21] D. Schmid and R. W. Spekkens, Phys. Rev. X **8**, 011015 (2018).
[22] A. Chefles and S. M. Barnett, Phys. Lett. A **236**, 177 (1997).
[23] A. Chefles, Phys. Lett. A **239**, 339 (1998).
[24] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, Phys. Rev. A **92**, 012118 (2015).
[25] E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery, Phys. Rev. Lett. **116**, 160406 (2016).
[26] X. Lü, Phys. Rev. A **102**, 022201 (2020).
[27] S. M. Barnett, Phys. Rev. A **64**, 030303(R) (2001).
[28] E. Andersson, S. M. Barnett, C. R. Gilson, and K. Hunter, Phys. Rev. A **65**, 052308 (2002).
[29] C.-L. Chou and L. Y. Hsu, Phys. Rev. A **68**, 042305 (2003).
[30] Y. C. Eldar, A. Megretski, and G. C. Verghese, IEEE Trans. Inf. Theory **50**, 1198 (2004).
[31] S. Y. Zhang, Y. Feng, X. M. Sun, and M. S. Ying, Phys. Rev. A **64**, 062103 (2001).
[32] Y. Feng, S. Y. Zhang, R. Y. Duan, and M. S. Ying, Phys. Rev. A **66**, 062313 (2002).
[33] T. Rudolph, R. W. Spekkens, and P. S. Turner, Phys. Rev. A **68**, 010301(R) (2003).
[34] Y. Feng, R. Y. Duan, and M. S. Ying, Phys. Rev. A **70**, 012308 (2004).
[35] A. Montanaro, Commun. Math. Phys. **273**, 619 (2007).
[36] D. Qiu, Phys. Rev. A **77**, 012328 (2008).
[37] J. Tyson, J. Math. Phys. **50**, 032106 (2009).
[38] M. A. Jafarizadeh, M. Rezaei, N. Karimi, and A. R. Amiri, Phys. Rev. A **77**, 042314 (2008).
[39] Y. C. Eldar, A. Mergretski, and G. C. Verghese, IEEE Trans. Inf. Theory **49**, 1007 (2003).
[40] Y. C. Eldar, IEEE Trans. Inf. Theory **49**, 446 (2003).
[41] S. Pang and S. Wu, Phys. Rev. A **80**, 052320 (2009).
[42] L. M. Duan and G. C. Guo, Phys. Rev. Lett. **80**, 4999 (1998).
[43] X. Lü, Phys. Lett. A **384**, 126538 (2020).
[44] A. Uhlmann, Rep. Math. Phys. **9**, 273 (1976).
[45] R. Jozsa, J. Mod. Opt. **41**, 2315 (1994).
[46] M. Nielsen and I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[47] S. Dürr, Phys. Rev. A **64**, 042113 (2001).