Conductor's elastic response to the vacuum-field radiation pressure

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The momentum transfer from an electromagnetic field to reflecting and absorbing surfaces was asserted by the Maxwell equations. This phenomenon, with important implications for the development of the cosmos, has also been investigated in the discretized energy scale using optomechanical devices. With the quantization of the electromagnetic field, it was discovered that the vacuum field may influence the dynamics of some physical systems, such as in the Casimir effect and the radiative decay of an atom. Here, the effect of the radiation pressure by the electromagnetic vacuum on the surface of a compressible conductor is analyzed, and the model based on the Born-Markov master equation predicts a harmonic strain and momentum analog to the classical counterpart. A fundamental difference observed is the oscillating purity of the deformation state. In addition, it was demonstrated that the time-averaged force originated from the elastic reflection of the vacuum-field modes is comparable to the Casimir force for two ideal metallic plates separated by a distance proportional to the reflectivity cutoff wavelength.

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I. INTRODUCTION

With the advent of quantum optics, and the experimental realization of optomechanical devices and lasers, radiation pressure, normally associated with the optical tweezers [1,2] and with phenomena in the astronomical scale [3,4], has also been investigated at the level of the quantized energy. For example, systems such as suspended mirrors [5–8], membranes [9–11], and optical microresonators [12,13] have their trajectories affected upon the reflection of photons from a coherent field. Along with these advances, the quantization of the electromagnetic field revealed the influence of the vacuum field on the dynamics of some physical systems, such as the Casimir effect [14] and the radiative decay of atoms [15].

It is known that the reflection of the quantized vacuum-field modes originates the radiation pressure causing the Casimir force [16–18], which is attractive between two ideal metal plates due to the different boundary conditions inside and outside the cavity [16,17,19]. In this sense, it has already been demonstrated that the radiation pressure due to the vacuum field on metallic plates results in fluctuations on the position of the mirror [20], on the mass [21], and for a moving mirror, it is also responsible for the dissipation of energy and subsequent stabilization of the system [22,23]. In the experiments, the mirrors are obviously composed of materials with finite bulk modulus, and therefore are susceptible to deformations. Nevertheless, a theoretical description of the elastic behavior of conductors as a response to the force exerted by reflection of the vacuum-field modes is still not exhausted.

The purpose of this article is to investigate the time-dependent normal strain caused by the radiation pressure due to the vacuum-field fluctuations on a half cavity. Here, a Born-Markov master equation is derived and analytically solved. As a result, it is obtained that the trajectories of the expectation values for the position—which coincides with the deformation of the mirror—and momentum operators are analog to the ones calculated from its classical counterpart. It is also demonstrated that the same trajectories are attained from the coherent evolution of the system, by just ignoring the Lindblad superoperator. However, in this case, a correction on the energy of the system must be performed.

II. MODEL

The system considered here is composed of a conductor plate embedded in a photon reservoir at a temperature $T \rightarrow 0$. It is assumed that the bottom surface of the mirror is fixed in a substrate, therefore not significantly influenced by the radiation pressure. In addition, it is assumed that the area of the top surface is much greater than the area of the side face of the conductor slab, therefore, any dynamics of the side surface is ignored. The strain caused by the radiation pressure on the top surface of the conductor can be accounted through the Young's modulus Y of the mirror in the linear elasticity regime of uniaxial deformation at zero temperature. In this case, the equation of motion describing the dynamics of the mirror top surface is

$$M\ddot{z} = -\frac{2P}{c} - M\omega^2 z, \tag{1}$$

where M is the mass of the mirror, P is the total power of the field reflected by the surface of area A, and z is the displacement of the top surface of the mirror, which is a function of time and $z = 0$ when $P = 0$. In terms of the natural height of the mirror slab z_0, its mass density μ, and Y, the natural frequency can be written as $\omega = z_0^{-1}\sqrt{Y/\mu}$. The solution of Eq. (1), which is

$$z(t) = -\frac{4P}{M\omega^2 c}\sin^2\left(\frac{\omega t}{2}\right) \tag{2}$$

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for $z(0) = 0$ and $\dot{z}(0) = 0$, describes the classical dynamics of the mirror surface.

A. Optomechanical Hamiltonian and the Born-Markov master equation

The Lagrangian resulting in the equation of motion (1) may be written as

$$\mathcal{L} = \frac{M\dot{z}^2}{2} - \frac{M\omega^2 z^2}{2} - \frac{2Pz}{c}. \tag{3}$$

Using the Legendre transformation, promoting the coordinate z and its conjugate momentum p_z to operators as for a free harmonic oscillator, we obtain the Hamiltonian

$$\hat{H} = \hbar\omega\hat{b}^\dagger\hat{b} + \frac{\hat{P}}{\omega}\sqrt{\frac{2\hbar\omega}{Mc^2}}(\hat{b}^\dagger + \hat{b}) + \sum_{\boldsymbol{k},s}\hbar\omega_{\boldsymbol{k}s}\hat{a}^\dagger_{\boldsymbol{k}s}\hat{a}_{\boldsymbol{k}s}, \tag{4}$$

where $\hat{b}$ ($\hat{b}^\dagger$) is the annihilation (creation) operator for the mechanical harmonic oscillator, $\omega_{\boldsymbol{k},s}$ is the frequency of the mode with wave vector $\boldsymbol{k}$ and polarization s, and $\hat{a}_{\boldsymbol{k}s}$ ($\hat{a}^\dagger_{\boldsymbol{k}s}$) is the annihilation (creation) operator associated with the photon reservoir. The power has also been promoted to the operator because of its dependence on the quantized field of the reservoir, such as

$$\hat{P} = \frac{c\epsilon A}{2}\lim_{z\to 0^+}\hat{\boldsymbol{E}}\cdot\hat{\boldsymbol{E}}^\dagger, \tag{5}$$

with

$$\hat{\boldsymbol{E}} = \sum_{\boldsymbol{k},s}\left(\boldsymbol{u}_{\boldsymbol{k},s}\hat{a}_{\boldsymbol{k}s} + \boldsymbol{u}^*_{\boldsymbol{k},s}\hat{a}^\dagger_{\boldsymbol{k}s}\right), \tag{6}$$

and

$$\boldsymbol{u}_{\boldsymbol{k},s} = i\sqrt{\frac{\hbar\omega_{\boldsymbol{k}s}}{2\epsilon\mathcal{V}}}\boldsymbol{e}_{\boldsymbol{k}s}\exp\left(i\boldsymbol{k}\cdot\boldsymbol{r}\right), \tag{7}$$

where $\mathcal{V}$ is the normalization volume and $\boldsymbol{e}_{\boldsymbol{k}s}$ is a unit vector [24]. The dynamical Casimir effect, describing the photon emission as a response to the movement of the reflecting surface, is not included in this model.

The first approximation considered in the derivation of the master equation is the Born approximation, in which it is assumed that the dynamics of the system does not affect the vacuum state of the photon reservoir. Moreover, the rotating-wave approximation is applied and the terms oscillating at high frequencies are ignored. The Markov approximation is also applied, in which the predictability of the system evolution depends only on the present, not on the past. This is valid when the bandwidth of the vacuum field is much larger than the spectral diffusion of the system [24]. Finally, nonlinear terms involving the mechanical harmonic oscillator operators are also ignored because their expectation values are relatively very small. Therefore, after performing the partial trace over the reservoir operators, the Lindblad master equation achieved is

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{i}{\hbar}[\hat{H}_{\mathrm{m}}, \hat{\rho}] + \Gamma\sin(\omega t)(\hat{\mathcal{K}}[\hat{b}^\dagger] + \hat{\mathcal{K}}[\hat{b}]), \tag{8}$$

where

$$\hat{H}_{\mathrm{m}} = \hbar\omega\hat{b}^\dagger\hat{b} + \hbar\sqrt{\frac{\omega\Gamma}{2}}(\hat{b}^\dagger + \hat{b}), \tag{9}$$

and $\hat{\mathcal{K}}$ is the Lindblad superoperator, given by

$$\hat{\mathcal{K}}[\hat{\mathcal{O}}] = \hat{\mathcal{O}}\hat{\rho}\hat{\mathcal{O}}^\dagger - \frac{1}{2}\{\hat{\mathcal{O}}^\dagger\hat{\mathcal{O}}, \hat{\rho}\}. \tag{10}$$

Only the photons traveling toward the top surface of the mirror were considered for the derivation of (8), so the characteristic rate is

$$\Gamma = \frac{\hbar\pi^4 c^2 V}{Y\lambda_c^8}, \tag{11}$$

with $V = Az_0$. The interaction between the mirror surface and the photon reservoir at a temperature $T \to 0$ is due to the vacuum fluctuations, and the lack of an upper bound on its spectral bandwidth leads to infinite energy exchange if the reflectance spectrum of the conductor is nonzero towards $\omega_{\boldsymbol{k}} \to \infty$. Fortunately, this does not correspond to the physical reality since the metals become transparent at frequencies much greater than the plasma frequency. In the calculation of the coupling rate Γ, for the sake of simplicity, the reflectance spectrum was defined as a Heaviside step function with discontinuity at a cutoff wavelength λ_c, necessary to avoid an unrealistic description of the mirror dynamics [25]. Thus, the integral over the modes was performed for $\omega_{\boldsymbol{k}}$ between 0 and ω_c. However, for a specific material, the reflectance spectrum should be taken into account in the integration over the $\boldsymbol{k}$, s modes.

III. RESULTS

In order to analytically solve the master equation (8), it is convenient to perform a further simplification, which can be achieved by performing the unitary transformation $\hat{\rho}_\alpha = \hat{D}(\alpha)\hat{\rho}\hat{D}^\dagger(\alpha)$, where $\hat{D}(\alpha)$ is the displacement operator with

$$\alpha = \sqrt{\frac{\Gamma}{2\omega}}(1 - e^{-i\omega t}). \tag{12}$$

In this case, the master equation in the interaction picture (superscript i) reduces to

$$\frac{\partial\hat{\rho}_\alpha^{(i)}}{\partial t} = \Gamma\sin(\omega t)(\hat{\mathcal{K}}[\hat{b}^\dagger] + \hat{\mathcal{K}}[\hat{b}]), \tag{13}$$

which can be solved through the vectorization of the Liouvillian and consequent disentanglement of the exponential operator [26]. Taking as the initial state $\hat{\rho}_0 = |0\rangle\langle 0|$ and returning to the original basis, the solution can be expressed in terms of displaced Fock states as

$$\hat{\rho} = \frac{1}{1+|\alpha|^2}\sum_{n=0}^{\infty}\left[\frac{|\alpha|^2}{1+|\alpha|^2}\right]^n\hat{D}^\dagger(\alpha)|n\rangle\langle n|\hat{D}(\alpha). \tag{14}$$

To describe the effects of the radiation pressure exerted by the electromagnetic vacuum on the mirror of the half cavity, it is necessary to calculate the expectation values of the operators $\hat{b}$, $\hat{b}^\dagger$, and $\hat{b}^\dagger\hat{b}$. Using the density operator from Eq. (14), they are

$$\langle\hat{b}^\dagger\hat{b}\rangle = 2|\alpha|^2, \tag{15}$$

$$\langle\hat{b}\rangle = -\alpha, \tag{16}$$

and $\langle \hat{b}^\dagger \rangle = -\alpha^*$. As a result, the deformation of the metal does not achieve the thermal equilibrium through the weak radiation pressure by the electromagnetic vacuum, as for a classical forced harmonic oscillator.

The expectation values for the position and momentum can be easily calculated from (16), giving

$$\langle \hat{z} \rangle = -\sqrt{\frac{4\hbar\Gamma}{M\omega^2}} \sin^2\left(\frac{\omega t}{2}\right), \tag{17}$$

$$\langle \hat{p}_z \rangle = -\sqrt{M\hbar\Gamma} \sin(\omega t). \tag{18}$$

So, the conductor slab is compressed by the field of the electromagnetic vacuum and the time-averaged normal strain is given by

$$\left\langle \frac{\langle \hat{z} \rangle}{z_0} \right\rangle_t = -\frac{\hbar\pi^2 c}{Y\lambda_c^4}. \tag{19}$$

The time-averaged compression value is very small and its dependence on only intensive properties of the material makes it relatively difficult to manipulate. As an example, if the mirror is composed of a material with $Y = 80$ GPa and $\lambda_c \sim 100$ nm, the strain is $\langle\langle \hat{z} \rangle / z_0\rangle_t \sim 3.9 \times 10^{-8}$, meaning that for every millimeter of the slab height, the average displacement of the top surface by the electromagnetic vacuum is on the order of the Bohr radius. The compression caused by the vacuum-field radiation pressure is, indeed, expected to be much smaller than one. This imposes a lower bound to the cutoff wavelength, which is

$$\lambda_c \gg \left(\frac{\hbar c \pi^2}{Y}\right)^{1/4}. \tag{20}$$

For most of the solid state conductors, the Young's modulus is on the order of 1–100 GPa, so a rough estimate gives $\lambda_c \gg 3$ nm, which is the case for most of the real metals. The time-averaged force exerted by the vacuum field can be obtained from the compression (19), giving

$$\langle F \rangle_t = \frac{\hbar c A \pi^2}{\lambda_c^4}. \tag{21}$$

For an insight about the magnitude of this force, it can be directly compared to the Casimir force between two perfectly reflecting mirrors distant by $L \approx \lambda_c / 2\sqrt[4]{15}$ [17]. Differing from the expectation value of the position (17), the time average of the expectation value of the momentum is zero, however, its amplitude may be significant since it depends on the slab volume as

$$\sqrt{M\hbar\Gamma} = \frac{\hbar\pi^2 c V}{\lambda_c^4}\sqrt{\frac{\mu}{Y}}. \tag{22}$$

Using the numbers from the previous example, the amplitude of the expectation value of the momentum per volume unit is $\approx$1.1 kg/m^{2}s for $\mu = 10$ g/cm^3 and $Y = 80$ GPa.

Regarding the energy transfer from the vacuum field to the metal slab, a straight comparison with the solution for the classical harmonic oscillator (2) leads to

$$P = \frac{\omega c}{2}\sqrt{M\hbar\Gamma}. \tag{23}$$

In term of the time-averaged strain, the intensity $I = P/A$ can be written as

$$I = -\frac{Yc}{2}\left\langle \frac{\langle \hat{z} \rangle}{z_0} \right\rangle_t. \tag{24}$$

Hence, the corresponding classical intensity depends only on the cutoff wavelength λ_c. If the mirror is a disk with 1 μm of radius, then $P \approx 1.5$ W for $\lambda_c = 100$ nm, which is comparable to the power of lasers. Nevertheless, from the expectation value of the Hamiltonian operator, it is obtained that only the time-dependent amount of power $P_\mathrm{m} = \hbar\omega\Gamma \sin(\omega t)$ is transferred to the mechanical harmonic oscillator, leading to

$$\frac{P_\mathrm{m}}{P} = \sqrt{\frac{4\hbar\Gamma}{Mc^2}} \sin(\omega t). \tag{25}$$

In order to stay in the regime of the Born approximation, the energy exchange between the system and the reservoir must be small, thus $|P_\mathrm{m}| \ll P$. This condition is easily fulfilled, since M is a macroscopic mass. As a consequence, the condition propagated to the cutoff wavelength is

$$\lambda_c \gg \left(\frac{4\hbar^2\pi^4}{Y\mu}\right)^{1/8}. \tag{26}$$

The mass densities of solid state conductors are found within 3–20 g/cm^3. Then, a rough estimate suggests $\lambda_c \gg 1$ Å, which is within the range obtained from (20).

Comparison with the coherent evolution

Supposing that the Lindblad superoperator term in Eq. (8) can be neglected under the argument that its coefficient averages to zero, the Schrödinger equation is retrieved as

$$\left[\hbar\omega\left(\hat{b}^\dagger\hat{b} + \frac{1}{2}\right) + \hbar\sqrt{\frac{\Gamma}{2\omega}}(\hat{b}^\dagger + \hat{b})\right]|\psi\rangle = i\hbar\frac{\partial}{\partial t}|\psi\rangle, \tag{27}$$

and has as a solution the coherent state $|\psi\rangle = |-\alpha\rangle$, with the initial state $|\psi_0\rangle = |0\rangle$. Thus, the expectation value of the operator $\hat{b}$ is identical to the one obtained from the master equation (16). As a result, the trajectory of the mirror surface, which is completely described by its position and momentum, is determined by the coherent evolution in this case of an oscillating Lindblad coefficient. However, the number of excitations is now $\langle \hat{b}^\dagger \hat{b} \rangle = |\alpha|^2$, which is only half of the value obtained in Eq. (15). Therefore, although the oscillating Lindblad term does not lead to the thermal equilibrium, it does correct the number of expected excitations of the mechanical harmonic oscillator, and consequently the energy.

A fundamental discrepancy between the two approaches relies on the purity of the state in Eq. (14), which decreases with the time-dependent expectation value of the number of excitations as

$$\mathrm{Tr}[\hat{\rho}^2] = \frac{1}{1 + 2|\alpha|^2}. \tag{28}$$

Hence, for low values of Γ/ω, the purity oscillates with small amplitude with a crest equal to one. The amplitude of the purity oscillations increase with the ratio Γ/ω and the trough lasts much longer than the crest for $\Gamma/\omega \gg 1$, as shown in Fig. 1.

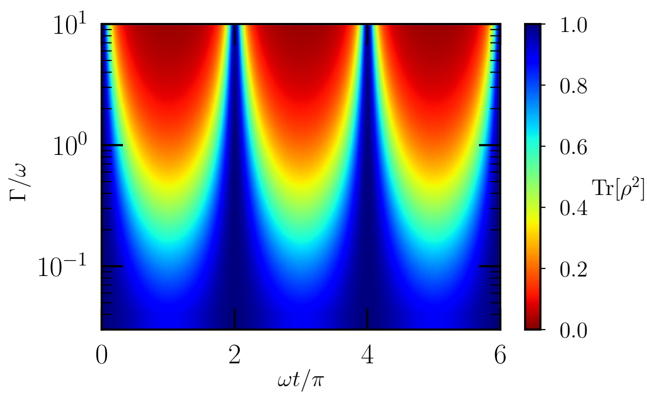

FIG. 1. Purity of the displaced Fock state describing the mirror surface dynamics as a function of ωt and the ratio Γ/ω.

IV. CONCLUSION

In this article, the evolution of the mirror compression due to the vacuum-field fluctuations was investigated from the Born-Markov master equation perspective. With the analytical solution of the master equation resulting in a density operator composed of displaced Fock states in its diagonal, the trajectories of the compression and momentum operators were obtained. The same trajectories were obtained from the coherent evolution of the system, however, in this case, a correction to the expectation value of the number operator had to be performed. These trajectories are analog to the ones derived from the classical mechanics upon replacing the vacuum field by a classical field, and a straight comparison resulted in an estimate of the power exchange between the vacuum field and the mirror surface. In addition, it was demonstrated that, as a consequence of the energy exchange between the system and the reservoir, the purity of the density operator describing the system oscillates with an amplitude strongly dependent on the ratio between the incoherent coupling rate Γ and the natural frequency of the mechanical harmonic oscillator ω.

The expectation value of the compression of the conductor is generally a small number, with a dependence on the range of reflected wavelength. On the other hand, the momentum transferred from the electromagnetic vacuum to the metal surface is proportional to the area where the interaction happens, which could lead the way towards experimental realizations. A possible effect of the mirror surface vibration is the photon emission through the dynamical Casimir effect, which is not included in this model.

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