Dark matter-wave gap solitons in dense ultracold atoms trapped by a one-dimensional optical lattice

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Optical lattices have been used as a versatile toolbox to control Bose-Einstein condensates (BECs) in recent years, and a wealth of emergent nonlinear phenomena have been found, including bright gap solitons and dark ones, among which the former has been realized in experiments. The latter, however, has only theoretical results and its fundamental properties are still not well understood. Here we theoretically and numerically explore an open issue of creating stable matter-wave dark gap solitons in a one-dimensional optical lattice, onto which the BECs with self-defocusing quintic nonlinearity are loaded. Using linear-stability analysis and direct simulations, the formation, structures, and properties of dark gap solitons in quintic nonlinearity have been compared to those upheld by cubic Kerr nonlinearity. In particular, we uncover that the dark gap solitons and soliton clusters are robustly stable in the first finite band gap of the underlying linear spectrum, and are hard to be stabilized in the second gap. The predicted dark gap solitons are observable in current experiments on dense ultracold atoms, using an optical lattice technique, and in the optics domain for nonlinear light propagation in periodic optical media with quintic nonlinearity.

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I. INTRODUCTION

Bose-Einstein condensates (BECs), created in ultracold bosonic atoms and degenerate quantum gases, are a macroscopic quantum phenomena and thus can be considered as a single particle in mean-field theory, whose conceptual framework is commonly referred to as Gross-Pitaevskii equations [1]. Recent decades have witnessed the emergence, prediction, and confirmation of a great number of interesting nonlinear phenomena in BECs, including profile, dynamical expansion and nonlinear excitations of trapped condensates [2], dark [3,4] and bright solitons [5–9], dark ring-shaped waves [10], Faraday [11] and shock waves [12,13], and four-wave mixing [14], to name just a few. The radical advance in laser technology has made the controlling and steering of BECs in an optical way possible, called optical lattices or standing optical waves, which can be easily created by using pairs of counterpropagating laser beams [15]. It is already known that by preparing the BECs or ultracold atomic gases onto optical lattices, we can study the existence of nonlinear matter waves and their dynamics [16–18], and simulate condensed-matter physics, including quantum many-body physics and beyond [19–21].

Localized waves or localized modes supported by periodic physical systems, such as BECs in optical lattices [15–18], and optical waves in photonic crystals and lattices [22–25], have become the focus of research in the frontier of nonlinear physics. Gap solitons are a novel kind of localized modes in periodic physical systems, whose chemical potential values are located inside the finite band gaps of the underlying linear spectrum. The gap solitons have been predicted theoretically [26–38] and observed in experiments [39–45], including the matter-wave gap solitons [39] (and exciton-polariton gap solitons [40–42]) in optical lattices, and the optical-wave gap solitons in Bragg gratings [43–45] and in negative index materials [46]. In addition, an extended version of gap solitons, which is being viewed as truncated nonlinear Bloch waves or gap waves [47,48], supported by periodic potentials as well, has been observed in BECs [49] and optics experiments [50,51] in the last decade. The periodic potentials are so fascinating in the respective optical and ultracold atomic media; the paramount reason is that they provide unique opportunities to manipulate the formation, propagation, and property of nonlinear optical and matter waves, in a very similar way to that for controlling the electrons in semiconductors [15–17,22–24].

Note that the aforementioned gap solitons represent a class of bright localized gap modes with a central peak, and recall that both bright and dark fundamental matter-wave solitons have been well understood and implemented in BECs (as mentioned above) [3–9]; in comparison, their dark counterparts, i.e., dark localized gap modes with a central dip, have not been well understood on the theoretical side (previous theoretical studies have found that the dark solitons can be controlled by optical superlattices [52]). In particular, the formation and property of dark gap solitons and how to stabilize them in dense BECs with quintic nonlinearity are still unclear; by contrast, their presence and dynamics in dilute BECs in optical lattices have been predicted and

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very recently surveyed theoretically (although an experimental observation is still lacking) [53,54]. In highly dense BECs, where the atom-atom collision is dominated by three-body interactions, the nonlinear term of the underlying theoretical framework (Gross-Pitaevskii equation) is the self-focusing or self-defocusing quintic nonlinearity [55–60]. In such a regime, the condensate of rubidium atoms with hard-core interactions, also known as the Tonks-Girardeau gases, governed by three-body inelastic collisions has been realized experimentally [61]. For very dense BECs, the three-body inelastic collisions induce unavoidable losses which can affect the robustness of solitons. On the other hand, for the BECs with two-body collision but without any three-body interaction, and trapped by a strong confinement in the transverse plane, a universal quintic nonlinear term is present in the underlying theoretical model when the dimension is reduced from three to one [62–64]. In the context of nonlinear optics, theoretical predictions [65,66] and experimental observations [67–69] have confirmed the existence of quintic nonlinearity in nonlinear optical media.

The objective of this work is to survey an open issue outlined above of the existence and dynamics of dark gap solitons in a dense BEC supported by the reciprocity between a 1D external periodic potential (optical lattice) and quintic repulsive nonlinearity. We report systematic results, which combine theory and numerical calculations, for dark matter-wave gap solitons and soliton clusters in shallow and medium shallow optical lattices, as well as in a moderately deep one. We uncover that for the former, the soliton solutions are partially stable in the first band gap, which resemble our recent predictions in the dilute BEC with cubic nonlinearity [53]; the corresponding stability region is greatly expanded for the latter two (medium shallow and moderately deep lattices). Our recent related article on a dilute BEC demonstrated that the gap-type dark localized modes are stable objects in most of the second band-gap region; on the contrary, here we find that these dark gap modes, constructed in the second gap too, can hardly be stabilized in the dense BEC. The stability property and evolution process of the dark gap modes in both band gaps are provided by linear-stability analysis and direct perturbed numerical simulations. Our results provide further insight into the unobserved dark gap modes in the fifth state of matter—the BEC, driving the development of soliton studies in the theoretical and experimental communities in BEC and beyond (e.g., in nonlinear optics). With the preceding descriptions in mind, we foresee that in the coming years, the predicted dark gap modes reported here and elsewhere will be tested experimentally in the corresponding ultracold atoms laboratories and in the context of optics territory with nonlinear optical media possessing cubic or quintic nonlinearity.

II. THEORETICAL MODEL AND NUMERICAL METHODS

In mean-field theory, the dense BEC trapped by an optical lattice is described by the conventional Gross-Pitaevskii equation with nonlinear terms, which is expressed by the order parameter (wave function) U [1,2,29,30],

$$i\hbar\frac{\partial U}{\partial \tau} = -\frac{\hbar^2}{2m}\nabla^2 U + V_{trap}U + \beta|U|^\alpha U, \tag{1}$$

where the parameters $\hbar$, m, and τ represent, respectively, Planck's constant, the mass of the atom, and evolution time; and where Laplacian $\nabla^2 = \partial_X^2$ in spatial dimension $D = 1$ for low-dimensional condensed quantum gases (usually referred to as the cigar-shaped BEC), and V_{trap} is the periodic trapping potential called the optical lattice induced by the interference of counterpropagating laser beams.

The order of the nonlinearity is given by $\alpha = 2$ and $\alpha = 4$ for a dilute BEC and a dense one, respectively, with nonlinear strength β being $\beta = 2\hbar^2 a_s/(ma_\perp^2)$ and $\beta = g_2/(3\pi^2 a_\perp^4)$, where a_s is the s-wave scattering length and $a_\perp$ is the transverse linear oscillator length, and the coefficient g_2 depends on the three-body interactions [55]. Note that the above equation is in the same form of the groundbreaking nonlinear Schrödinger equation; such a similarity confirms the fact that the last term of Eq. (1) stands for cubic (Kerr) nonlinearity as $\alpha = 2$, and the nonlinear term should be called quintic nonlinearity in the case of $\alpha = 4$. Since our focus is restricted to the dense BECs with quintic nonlinearity, we shall set $\alpha = 4$ in the following, unless otherwise stated. Without loss of generality, the governing Eq. (1) can be rescaled to a dimensionless form by rewriting the variables $t = \tau/\tau_0$, $\psi = (mg_0/3\pi^2\hbar^2 a_\perp^4)^{1/4}U$, $V = m/\hbar^2 V_{trap}$, and $\tau_0 = \hbar/m$, leading to the following reduced system for scaled wave function ψ:

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{1}{2}\partial_x^2\psi + V(x)\psi + g|\psi|^4\psi, \tag{2}$$

where $g = g_2/g_0$ is the dimensionless nonlinear strength, with g_0 the nonlinear coefficient, and the positive $g > 0$ represents the local strength of defocusing (self-repulsive) nonlinearity arising from repulsive interatomic interactions. The external periodic potential $V(x)$ is an optical lattice expressed as $V(x) = V_0\sin^2(x)$, with constant parameter V_0 being the amplitude of the periodic trapping (modulation depth). The quintic nonlinearity can be realized as Tonks-Girardeau gases in dense BECs [61]. In the context of nonlinear optics, the materials with quintic nonlinearity have also been demonstrated experimentally in recent years [65,66].

The wave function at certain chemical potential μ can be expressed as $\psi = \phi\exp(-i\mu t)$. Substituting it into Eq. (2) leads to a stationary equation for the wave function ϕ:

$$\mu\phi = -\tfrac{1}{2}\partial_x^2\phi + V(x)\phi + g\phi^5, \tag{3}$$

which can be obtained from the corresponding Lagrangian,

$$2L = \int_{-\infty}^{+\infty}\left[\mu\phi^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(x)\phi^2 - \frac{g}{3}\phi^6\right]dx. \tag{4}$$

We seek stationary solutions in the form of dark gap solitons, as mentioned above, for Eq. (2) under the condition of a defocusing sign of the quintic nonlinearity (i.e., $g > 0$). This can be done by solving the corresponding stationary equation (3) in the way of Newton's iteration. Then a crucially paramount issue that deserves to be carefully weighed is the stability of the stationary solutions thus found. We address such issue in a conventional way, called linear-stability analysis, which, principally, deals with the stability of the underlying stationary states against small perturbations (i.e., the Bogoliubov–de Gennes equations). This is achieved by taking the perturbed solution as

$$\psi(x,z) = e^{-i\mu t}[\phi(x) + p(x)e^{\lambda t} + q^*(x)e^{\lambda^* t}], \tag{5}$$

where ϕ is the undisturbed wave function (dark gap soliton) with chemical potential μ, taken as per stationary equation (3), and $p(r)$ and $q^*(r)$ represent small perturbed eigenmodes with eigenvalue (instability growth rate) λ. The substitution of Eq. (5) into Eq. (2) and the linearization results in the following eigenvalue problem for λ:

$$i\lambda p = -\tfrac{1}{2}\partial_x^2 p + [V(x) - \mu]p + g|\phi|^4(3p + 2q), \tag{6}$$

$$i\lambda q = +\tfrac{1}{2}\partial_x^2 q - [V(x) - \mu]q - g|\phi|^4(3q + 2p). \tag{7}$$

According to the principle of linear-stability analysis, it is seen from Eqs. (6) and (7), which were solved by means of the finite-difference method, that the stability of the so-found dark gap soliton solution requires only pure imaginary eigenvalues λ. In other words, the corresponding real part always obeys $\mathrm{Re}(\lambda) = 0$ [70]. We stress that the stability property of the dark gap solitons is also further checked in direct numerical simulation of the perturbed solutions, performed in the framework of dynamical equation (2) by also adopting the commonly used finite-difference method. The Dirichlet boundary condition is adopted in all our simulations. Without loss of generality, we set the nonlinear strength $g = 1.5$ in the following.

III. NUMERICAL RESULTS AND DISCUSSIONS

This section is devoted to present the numerical results of the dark gap localized modes (matter-wave solitons and the soliton clusters) of a dense BEC loaded into periodic potentials (three types of optical lattices as mentioned above), with an emphasis on how the modulation depth (V_0) degree of freedom would affect the formation of stable localized modes, and explain how their structure feature and stability are achieved. It starts by providing the band-gap structure of the underlying linear periodic physical model; then the structures of dark gap modes within the band-gap regions (first and second band gaps) are produced numerically and compared to their counterparts in dilute BEC with cubic nonlinearity. Two numerical approaches, which include the above-mentioned linear-stability analysis and direct numerical simulations, are thereafter used to examine the stability of the dark gap modes, discussing their physical intension and evolution dynamics.

In order to investigate the formation of matter-wave dark gap solitons, we first need to know what the relevant band-gap diagram looks like. From the physical point of view, the periodic potential induces a band-gap structure for Bloch waves in the linear regime [for the noninteracting BEC in Eq. (3) by discarding the last (nonlinear) term], based upon the energy band theory of the crystalline solids [71]. The numerical results of the spectral bands computed for the optical lattice, $V(x) = V_0 \sin^2(x)$, are depicted in Fig. 1. The spectral gaps, which situate the place between the two adjacent spectral bands, open and widen with an increase of lattice depth V_0 [Fig. 1(a)]. We choose three types of optical lattices for supporting dark gap solitons, which we name shallow, medium shallow, and moderate deep optical lattices respectively, at the specific lattice depths $V_0 = 3$, 6, and 9, and show the corresponding band-gap structures in Figs. 1(b)–1(d). For now, we are interested in the first and second band gaps, within which the stable dark gap solitons may be localized in the spectral gaps of the induced band-gap structure due to the interplay between the optical lattice (which induces the band-gap structure) and nonlinearity.

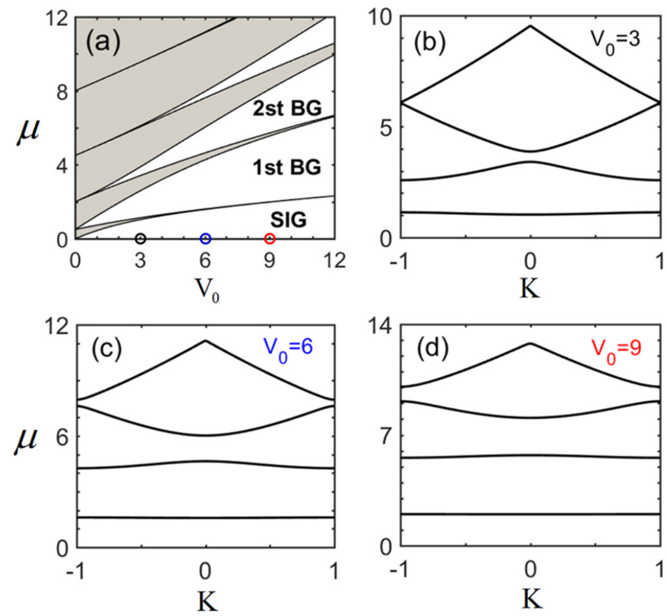

FIG. 1. Numerical results showing the emergence of the band-gap structure for 1D optical lattice $V(x) = V_0 \sin^2(x)$. (a) Linear Bloch spectrum (bands are shaded) as a function of modulation depth of the lattice V_0. (b)–(d) The relevant Bloch spectra at $V_0 = 3$, 6, and 9. SIG: semi-infinite gap; 1st (2nd) BG: first (second) band gap. K represents the momentum.

We now proceed to present the numerical results of the dark gap localized modes, which can be constructed through solving Eq. (3) using the Newton-iteration method, with the initial input being a hyperbolic tangent function. Typical examples of the fundamental dark gap solitons, residing within the first band gap, of a dense BEC (under the quintic repulsive nonlinearity) trapped by an optical lattice are shown in Fig. 2, where the comparison to their counterparts in cubic nonlinearity is also made. It is seen from the figure that the amplitude of the solitons increases with the increasing of the energy of the BEC (chemical potential μ), abiding by the

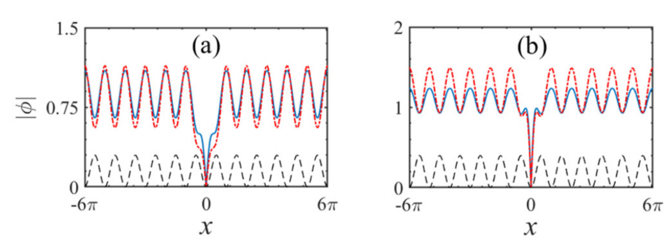

FIG. 2. Profiles of 1D fundamental dark gap solitons with cubic (blue solid line) and quintic (red dashed line) nonlinearity in the first band gap under different chemical potential μ at $V_0 = 3$. (a) $\mu = 2.5$; (b) $\mu = 3.7$. The black dashed line in both panels represents the shape of the scaled optical lattice.

common ground of stronger Bragg scatterings in spectral gaps when going deeper inside the gaps. Compared with the cases in cubic nonlinearity, we can see clearly that the dark gap solitons in the quintic nonlinearity have a wider waist and a larger amplitude. This feature is in strong agreement with their bright counterparts (gap solitons) supported by quintic or cubic repulsive nonlinearity; the bright gap solitons enhance their amplitudes and waists under the quintic case as compared with their counterparts in the cubic one (not shown here). Such a striking feature can be understood from the intrinsic properties of dilute and dense BECs, such as the definition of the number of atoms N,

$$N = \frac{1}{\mu}\int_{-L/2}^{+L/2}\left[\frac{1}{2}\left|\frac{\partial\phi}{\partial x}\right|^2 + V(x)|\phi|^2 + g|\phi|^{\alpha+2}\right]dx. \quad (8)$$

Here, L is the total calculation length. The above expression is originated from the corresponding stationary equation (3) by simply multiplying by ϕ^* and integrating with a simplification; meanwhile, such expression can also be derived from the Lagrangian (4). When keeping the same values of μ, $V(x)$, and g, an increase of α ($\alpha = 2$ for dilute BECs and $\alpha = 4$ for dense BECs, as described above) would lead to the increase of N, indicating the increasing of $|\phi|$ (and the relevant amplitudes), recalling the definition of atoms number $N = \int_{-L/2}^{+L/2} |\phi|^2 dx$. Further, it is also observed from both Figs. 2(a) and 2(b) that the central dip of the so-found dark gap solitons is situated at the location of the potential (optical lattice) minimum, analogous to what we reported above under cubic nonlinearity conditions. A physical principle worth pointing out is that the nonlinear Bloch waves, i.e., the periodic solutions of the underlying Eq. (3) at given μ, are the foundation of dark gap solitons—that is, the dark gap solitons are grounded in the corresponding nonlinear Bloch waves.

With the in-depth investigation, we obtain the relation between atom number N and chemical potential μ for the 1D dark gap solitons in which we are interested, and collect it at lattice depth $V_0 = 3$ in Fig. 3(a). Such $N(\mu)$ relation tells us that the dark gap solitons adhere to the previous found stability criterion for localized gap modes, the "anti-Vakhitov-Kolokolov" (anti-VK) criterion, $dN/d\mu > 0$ [31,32,36,53,72]. The corresponding dependence in forms of the maximal real segment of eigenvalues $\mathrm{Re}(\lambda)$ and μ is as well depicted in Fig. 3(a), which is double checked by our systematic simulations in directly solving the dynamical equation (2) with initial small perturbations. The stability results imply that the dark gap solitons trapped in a shallow optical lattice and under quintic nonlinearity are partially stable in the first band gap, and even in the midst of where the unstable region remains there; on the contrary, their counterparts in the context of cubic nonlinearity are unstable only near the band edge. Furthermore, we reveal that the dark gap solitons are almost not able to be stabilized in the second band gap—the stability region is extremely narrow; this is different from the results obtained in the cubic model, where they are proved to be robustly stable [53]. To expand the stability territory, we try to increase the modulation depth of the lattice V_0 and see what happens next in terms of medium shallow and moderate deep optical lattices, with the depth being, respectively, $V_0 = 6$ and 9. As might be expected, the stability region of the dark gap

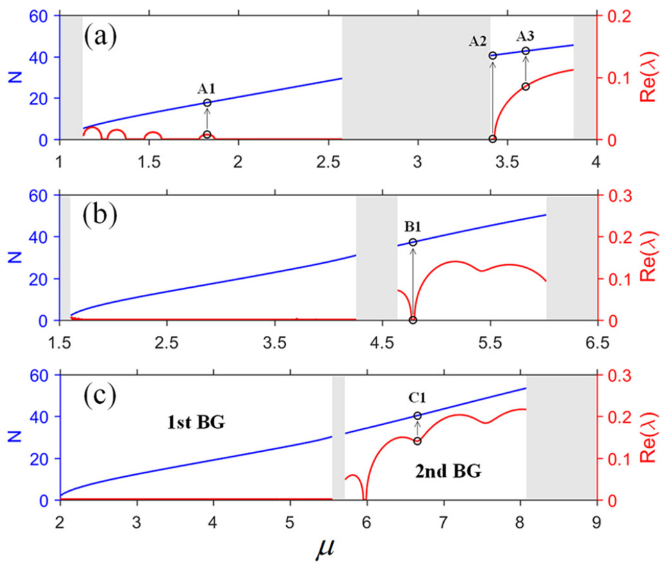

FIG. 3. Number of atoms N (left, blue y axis) and maximal real part of eigenvalues $\mathrm{Re}(\lambda)$ (right, red y axis) vs chemical potential μ for 1D matter-wave dark gap solitons supported by quintic nonlinearity and optical lattice potential. The lattice's modulation depth (a) $V_0 = 3$, (b) $V_0 = 6$, and (c) $V_0 = 9$. The linear Bloch bands in each panel are shaded gray. The structure feature and dynamics of the dark gap solitons marked by circles (A1, A2, A3, B1, C1) are shown in Fig. 4.

solitons in the first band gap has been greatly expanded at a bit higher depth (V_0). However, once again, the stability of the dark solitons in the second band gap cannot be enlarged at all, and the stable dark solitons can exist only within a limit region, according to Figs. 3(a) and 3(b).

To show how the dynamic evolution process of the dark gap solitons develops in perturbed evolution, we show five characteristic examples of dark gap solitons supported by the above-mentioned three types of lattices (shallow, medium shallow, and moderate deep ones) in the first band gap and the second one in Fig. 4. We also show their relevant shapes and linear-stability spectra calculated from the eigenvalue problem given by Eqs. (6) and (7). Direct perturbed evolutions of these dark gap solitons, both stable examples and unstable ones, are portrayed in the bottom panels of Fig. 4, where both the three-dimensional and top-down (projection) views are shown. We can observe the dynamic process from the panels and conclude that the stable dark gap solitons maintain their forms unchanged during the evolutions, while the instability for the unstable solitons always begins from the inside (the central dip); then they transform themselves into oscillating waves and, finally, immerse into the background with the nonlinear Bloch waves mentioned above. We stress that these stability and instability properties are analogous to their counterparts found in the similar physical model, but under cubic repulsive nonlinearity conditions [53].

Apart from the fundamental dark gap modes, it is interesting to see if our theoretical model can support higher-order modes (the higher-order modes are usually characterized by complex structures, which are different from the simple and isotropic fundamental modes), e.g., the dark gap soliton

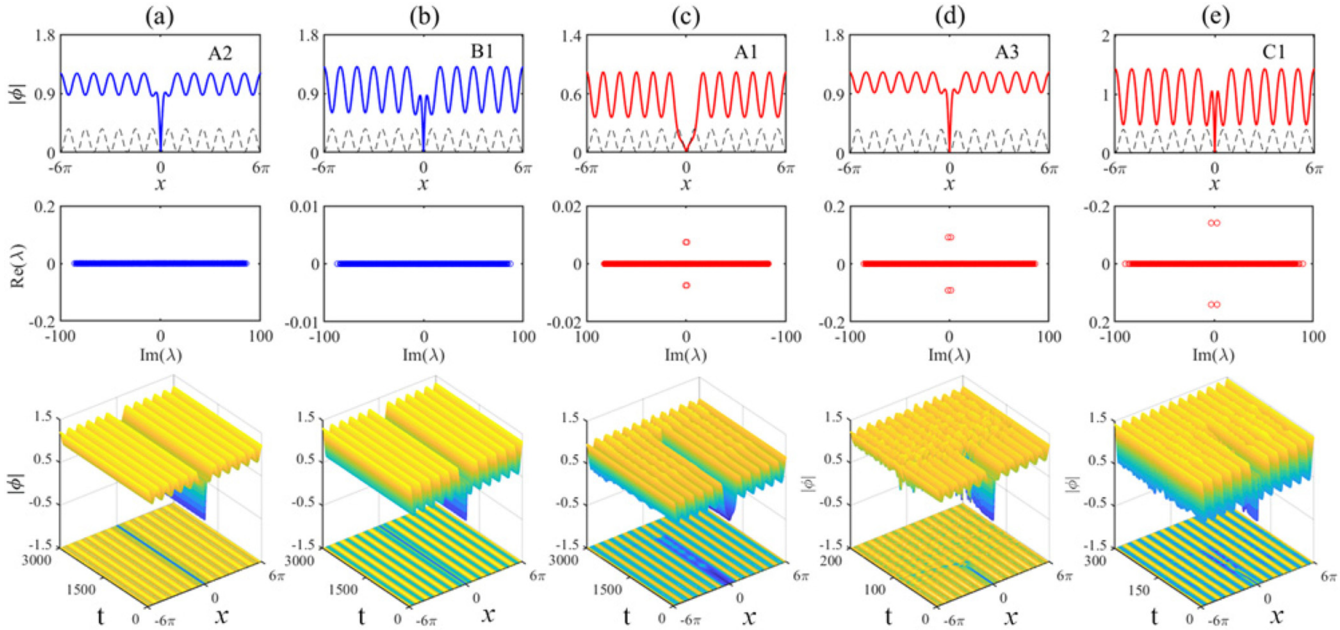

FIG. 4. Profiles (top), linear-stability spectra (middle), and evolutions (bottom) for 1D stable and unstable dark gap solitons. (a) Stable soliton at $\mu = 3.42$ in the second band gap; (b) stable soliton at $\mu = 4.78$ in the second band gap; (c) unstable solitons at $\mu = 1.82$ in the first band gap; unstable solitons at (d) $\mu = 3.62$ and (e) $\mu = 6.67$ in the second band gap. The modulation depth of the lattice: (a),(c),(d) $V_0 = 3$, (b) $V_0 = 6$, and (e) $V_0 = 9$. The contour plots for the temporal evolutions of the solitons (in the third row) are projected onto the bottom surfaces. The evolutions are shown in the bottom panels, starting with adding a small initial perturbation to the 1D dark gap solitons (the perturbed solutions), which are used throughout the paper.

clusters arranged with several fundamental modes. The idea of generating dark gap soliton clusters is derived from the experimentally observed nonlinear self-trapping in Bose-condensed ^{87}Rb atoms—the phenomenon that is referred to as the truncated nonlinear Bloch waves (the so-called gap waves), that is, a novel type of bright gap states besides the bright gap solitons [47–49]. The striking difference between the bright gap solitons and gap waves is that the former were only created with a few number of atoms in ^{87}Rb condensate placed in an optical lattice (e.g., about 250 atoms) [39]; in contrast, the latter can be generated experimentally for arbitrarily large initial atom numbers [49]. It is relevant to emphasize that the truncated nonlinear Bloch waves also have been broadly explored in other nonlinear periodic systems [50,51]. Our systematic simulations confirm that their dark counterparts, i.e., the dark gap soliton clusters, can be supported by our model. The dependence $N(\mu)$ for the dark gap soliton clusters formed in shallow periodic potential with lattice depth $V_0 = 3$ is accumulated in Fig. 5(a), where the anti-VK stability criterion $dN/d\mu > 0$ prevails. Also shown is the corresponding stability and instability area, expressed as Re(λ) versus μ, in the same panel, demonstrating that such clusters are completely unstable in the second band gap, although they are stable inside some regions of the first band gap. By slowly increasing the lattice depth, e.g., we set $V_0 = 6$ to make the optical lattice become a medium shallow one, we can see from Fig. 5(b) that the soliton clusters have a much wider stability region in the first band gap of the underlying linear spectrum, even though they cannot be stable localized modes in the second gap either. The fact that the dark gap soliton clusters are strongly unstable can be explained by the interaction between their fundamental counterparts (dark gap solitons), owing to the stability region for the gap soliton clusters always being narrower than their fundamental gap solitons which, as described above, are tough to be stabilized in the second band gap. The stability and instability characteristics are also verified by our direct perturbed

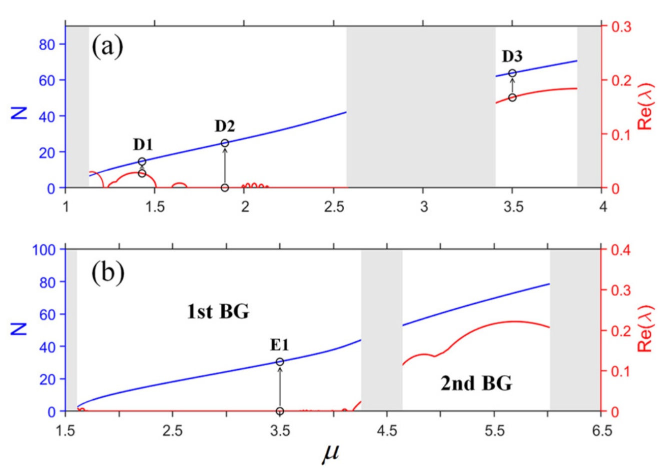

FIG. 5. The chemical potential μ vs number of atoms N (left, blue y axis) curve and maximal real part of the eigenvalues Re(λ) (right, red y axis) curve for 1D matter-wave dark gap soliton clusters under quintic nonlinearity. The lattice's modulation depth (a) $V_0 = 3$ and (b) $= 6$; the dark gap soliton clusters composed of five fundamental dark gap solitons within the range of $x \in [-10\pi, 10\pi]$ are presented. The linear Bloch bands are shaded gray. The structure feature and dynamics of the dark gap soliton clusters marked by circles (D1, D2, D3) are depicted in Fig. 6.

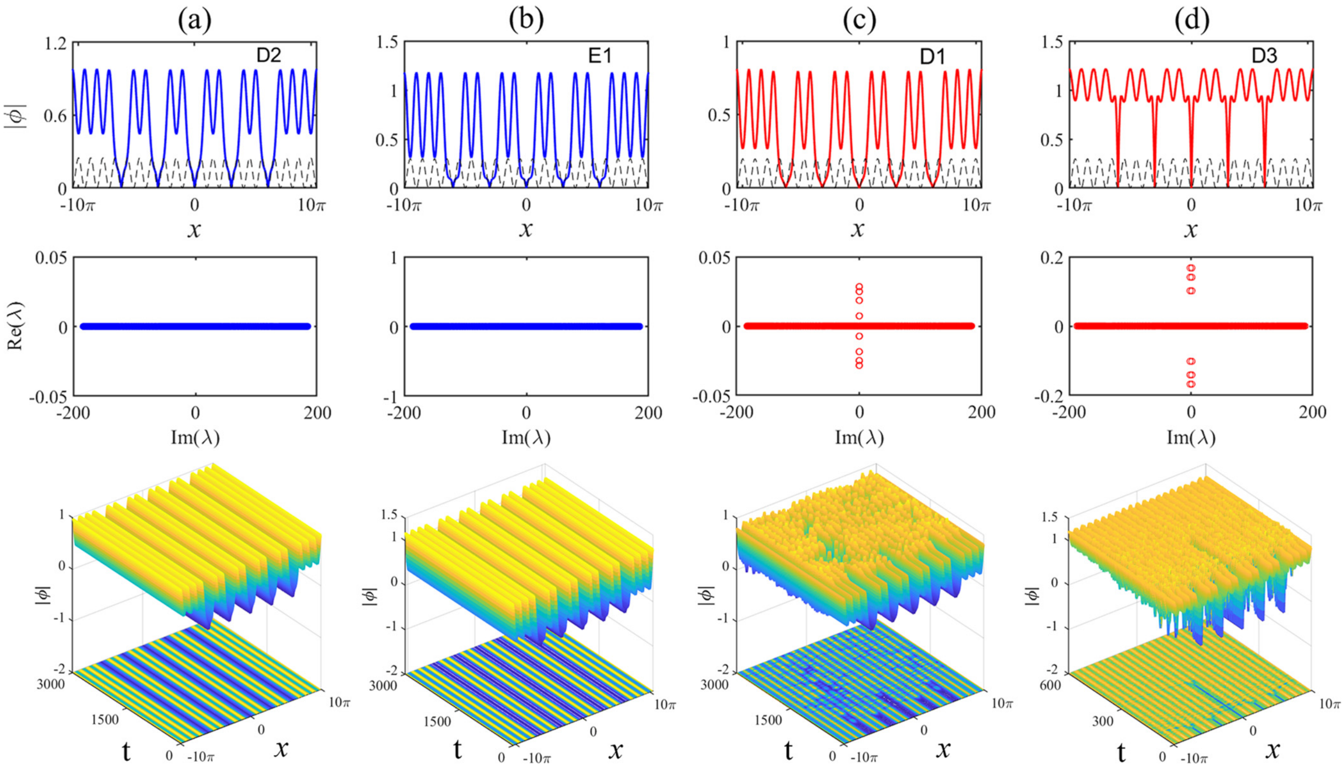

FIG. 6. Profiles (top), linear-stability spectra (middle), and evolutions (bottom) of the 1D stable and unstable five-soliton family of dark gap soliton clusters. (a),(b) Stable soliton clusters at $\mu = 1.9$ and $\mu = 3.5$ in the first finite band gap; (c) unstable soliton cluster at $\mu = 1.9$ in the first band gap; (d) unstable cluster at $\mu = 3.5$ in second band gap. The modulation depth of the lattice: (a),(c),(d) $V_0 = 3$, (b) $V_0 = 6$. The contour plots for the temporal evolutions of the soliton clusters (at the base) are mapped onto the bottom surfaces.

simulations shown in Fig. 6. Profiles of the dark gap soliton clusters constructed by five fundamental gap solitons, with spacing ($\triangle$) between adjacent dark solitons equaling three times the period of the optical lattice ($\triangle_{latt} = \pi$), which is to say, $\triangle = 3\pi$, are also depicted in the same figure. Our simulations demonstrate that such soliton clusters can be robustly stable objects only when $\triangle \geqslant 2\pi$, resembling the findings in the cubic model reported recently [53]. Like their fundamental counterparts shown in Fig. 4, the dynamics evolutions of dark gap soliton clusters also prove that the stable modes keep their excellent coherence, while the unstable solutions transform themselves into oscillating modes and merge into the nonlinear Bloch waves background in the end, according to the bottom panels in Fig. 6.

One may wonder about the impact of three-body loss on the predicted dark gap solitons; we have considered such effect by introducing the loss term, which is expressed by $ik|\psi|^4\psi$ (here, k is the coefficient), with which the stationary solutions and the direct perturbed simulations (2) can be performed. We show two examples of dark gap solitons under the action of this loss term in Fig. 7, from which one can see that the three-body loss affects the stability of the dark gap solitons, where the instability increases with increasing k. In fact, with a small k, weak instability exists in the predicted dark gap solitons, which is still stable over $t = 1000$. In dense BECs with small loss, the dark gap soliton can keep its form for quite a while during the evolution. Although a bigger loss will quickly destroy its stability, this is an issue that deserves to be studied in detail in the future; adding a small cubic nonlinearity to balance such loss may be an alternative way. On the other hand, such loss term can be ignored in the context of nonlinear optics with transparent optical media. Although the optics lattices in BECs and photonic crystals and lattices in optics share some similarities, e.g., the existence of forbidden band gaps and the possible nonlinear localized waves, they have some differences; the distinctive feature is that the depth, periodic, and structure of the optical lattices can be tuned freely, while the physical parameters of periodic optical materials are generally unalterable.

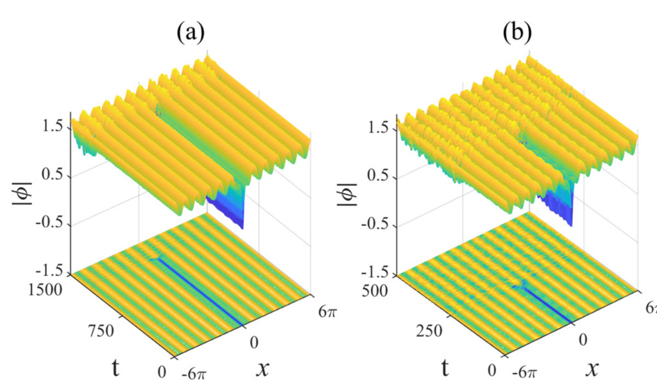

FIG. 7. Evolutions of the 1D dark gap solitons under different three-body loss with loss coefficient (a) $k = 0.001$ and (b) $k = 0.002$. Other parameters are $V_0 = 3$ and $\mu = 3.42$.

IV. CONCLUSION

The optical lattice has been introduced as a powerful toolbox for manipulating the emergent nonlinear phenomena in Bose-Einstein condensates (BECs), wherein the experimental observation of bright matter-wave gap solitons marks a major breakthrough. The dark gap solitons, however, have not been observed and the underlying physics remains largely unexplored; in particular, their existence and properties in dense ultracold atoms loaded onto external periodic potential (optical lattice). We have dug deep into such issue by reporting on theoretical analysis and systematic numerical simulations of dark gap solitons for dense BECs with repulsive three-body interaction (quintic nonlinearity) by using the mean-field theory, that is, the Gross-Pitaevskii or nonlinear Schrödinger equation with quintic nonlinear term. Two families of dark gap solitons, such as the fundamental solitons and soliton clusters, in optical lattices possessing shallow, medium shallow, and moderately deep lattices (modulation depths) are found and their stability and instability regions in the underlying linear band-gap structures are identified by means of linear-stability analysis and direct perturbed simulations (both methods achieved consistent results). The stability regions for both fundamental dark gap solitons and soliton clusters within the first band gap are narrow at a shallow lattice and become wider when increasing lattice depth (e.g., at a medium shallow lattice), while both classes of dark gap modes cannot be stabilized in the second band gap, posing a challenge to the survey of dark gap solitons in higher band gaps. On the contrary, the dark gap solitons lying in the second gap predicted in dilute BECs with repulsive atom-atom interactions (cubic nonlinearity) can be stable objects.

The physical settings considered here can be realized for matter waves in dense BECs trapped by an optical lattice and for light waves in periodic optical materials with quintic nonlinearity. We thus expect that the predicted dark gap solitons can be observed in experiments on ultracold atoms and beyond. Our results provide theoretical insight into the physics of dark gap solitons in nonlinear atomic and optical media with quintic nonlinearity, and put forward a different problem for finding stable dark gap solitons within the second band gap, which is a key issue to be explored next. From a fundamental point of view, in addition to the quintic model introduced here, the cubic-quintic model, representing the BECs with both two-body and three-body interactions and the nonlinear optical medium having Kerr and quintic nonlinearity, has presently been widely researched [56,58,73,74], while the formation and properties of dark gap solitons in such model have yet to be deeply explored.

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