

Unambiguous discrimination of fermionic states through local operations and classical communication

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The paper studies unambiguous discrimination of fermionic states through local operations and classical communication (LOCC). In the task of unambiguous discrimination, no error is tolerated but an inconclusive result is allowed. We show that contrary to the quantum case, it is not always possible to distinguish two fermionic states through LOCC unambiguously with the same success probability as if global measurements were allowed. Furthermore, we prove that we can overcome such a limit through an ancillary system made of two fermionic modes, independently of the dimension of the system, prepared in a maximally entangled state: in this case, LOCC protocols achieve the optimal success probability.

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I. INTRODUCTION

The quest for state discrimination has been thoroughly investigated in the quantum realm [1–6]. When dealing with composite systems, the peculiar nature of entanglement, which quantum systems can exhibit, gives rise to counterintuitive situations where the information is encoded in a delocalized fashion, contradicting the common wisdom according to which any information carried by a system should be encoded in its own local degrees of freedom. As a consequence, in order to discriminate a full basis of entangled pure states, a delocalized measurement is required. However, when processing distributed information, a reliable channel for the exchange of quantum systems is an expensive resource and one is thus willing to accept lower performances if it is possible to achieve a reasonable result using only local operations and classical communication (LOCC) (for a comprehensive reference on LOCC protocols, see Ref. [7]).

For this reason, discrimination of two pure states via LOCC is twice counterintuitive: indeed, classic results in quantum information theory proved that LOCC discrimination of pure states can be as good as the optimal one, both in a minimum-error scenario [8,9] and in a zero-error scenario, where one accepts an inconclusive outcome [10].

In the present work, we study the problem of optimal unambiguous discrimination of pure states via LOCC in fermionic theory. This theory describes systems, states, measurements, and transformations on fermions, or, more precisely, on fermionic modes. The interest in fermionic theory is clearly due to the fact that elementary matter fields in quantum physics are actually collections of fermionic modes. While it is well known that from a computational point of view, fermionic modes are equivalent to qubits [11], to the extent

that quantum algorithms have been devised for the simulation of scattering processes involving fermions [12], still the differences between the two theories are relevant from the point of view of the complexity of geometric structures required for such simulation: indeed, there is no local encoding of fermionic modes in qubit networks, or vice versa [13–15]. As a matter of fact, then, while the computational equivalence of qubits and fermions holds asymptotically, modulo a polynomial overhead of program complexity, for finite-size problems it is relevant to study simple information-processing tasks with fermionic modes autonomously. Among many options for the description of fermionic modes through a suitable choice of algebraic structures [16,17], it is often convenient to resort to the use of the Jordan-Wigner isomorphism [18–20], which allows one to map the algebra of N canonically anticommuting field operators into strings of Pauli operators acting on the Hilbert space of N qubits, with a parity superselection rule [21–23].

When dealing with state discrimination, where all the relevant formulas come from the application of the Born rule for the calculation of probabilities, the Jordan-Wigner representation is particularly convenient, and the only care that must be taken in representing fermions with qubits is an appropriate account for parity superselection [17], as shown in Ref. [24] where the task of optimal minimum-error discrimination was studied.

Here we take a similar approach to the problem of optimal unambiguous state discrimination. One of the key features of LOCC discrimination is that unlike the case of binary quantum state discrimination, ancilla-assisted protocols [25] prove to be useful. More precisely, in the minimum-error scenario, one entangled pair is sufficient to make optimal LOCC discrimination equivalent to its unconstrained counterpart.

The present work extends that of Ref. [24], which deals with both perfect and optimal conclusive discrimination of fermionic states. In order to distinguish between two quantum nonorthogonal preparations, the further strategy of unambiguous discrimination has been proposed by Jaeger and Shimony

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in Ref. [26]: we require the outcomes to flawlessly determine the true state. However, due to the essential nature of quantum theory, the whole protocol must make allowance for a third inconclusive result, for which no intelligence is attained from the measurement. In Refs. [10,27,28], the authors prove that restricting ourselves to LOCC protocols is no real limit to the performances of unambiguous discrimination. We briefly discuss the protocol in the fermionic realm and prove that fermionic local discrimination is typically *suboptimal* with respect to the unconstrained one, contrary to the quantum case. The necessary and sufficient conditions on the states are derived for achieving optimal local discrimination. Moreover, we prove that LOCC protocols accomplish the same discrimination performances as the unconstrained ones, once we provide an ancillary system in a maximally entangled state.

II. PRELIMINARY NOTIONS

A. Unambiguous quantum states discrimination

We are interested in discriminating between two non-normalized and pure states $\rho = p|\psi\rangle\langle\psi|$ and $\sigma = q|\phi\rangle\langle\phi|$, where $\{p, q\}$ is the prior probability distribution and $|\psi\rangle, |\phi\rangle$ are normalized vectors. In quantum theory, we know that if the two states are orthogonal, we may perfectly distinguish between the two. If we release such a condition, on the contrary, the discrimination protocol becomes probabilistic and subject to errors. We remind the reader that a quantum measurement is generally represented by a positive-operator valued measure (POVM), namely, a collection of positive operators $0 \leq S \leq I$ —called *effects*—that sums to the identity operator I . Every operator Π_i in a POVM is associated with a possible outcome i , and the probability of outcome i , provided that the measurement is performed on a system prepared in state ρ , is given by the Born rule

$$\Pr(i|\rho) = \text{Tr}[\rho\Pi_i]. \quad (1)$$

If there exist operators $0 \leq A_i, B_i \leq I$ such that $S = \sum_i A_i \otimes B_i$, we call the effect S *separable*. If a POVM is made of separable effects, then we call it separable as well, and denote with SEP the set of separable POVMs. Moreover, we recall that LOCC measurements are a proper subset of SEP POVMs [29] (for further details on LOCC protocols, see Ref. [7]).

In the following, we deal with discrimination strategies that unambiguously distinguish between the two provided states ρ and σ , though allowing for an inconclusive result. We describe the measurement through the POVM made of Π_ψ, Π_ϕ , and Π_γ , where each operator corresponds to one of the three possible outcomes. The requirement for a POVM to represent an unambiguous discrimination protocol is given by the following conditions:

$$\begin{aligned} \Pr(\phi|\psi) &= \text{Tr}[|\psi\rangle\langle\psi|\Pi_\phi] = 0, \\ \Pr(\psi|\phi) &= \text{Tr}[|\phi\rangle\langle\phi|\Pi_\psi] = 0, \end{aligned} \quad (2)$$

and $0 \leq \Pi_\psi + \Pi_\phi \leq I$. Under these circumstances, we define $\Pi_\gamma := I - \Pi_\psi - \Pi_\phi$. The success probability of the protocol is then given by

$$\mathcal{P}_s := \text{Tr}[(p|\psi\rangle\langle\psi| + q|\phi\rangle\langle\phi|)(\Pi_\psi + \Pi_\phi)]$$

or $\mathcal{P}_s = 1 - \mathcal{P}_{\text{err}}$, where $\mathcal{P}_{\text{err}} := \text{Tr}[(p|\psi\rangle\langle\psi| + q|\phi\rangle\langle\phi|)\Pi_\gamma]$ is the error probability.

In Ref. [26], the authors describe the optimal POVM that maximizes the probability of discrimination success \mathcal{P}_s for the provided states ρ, σ . Due to the effects being dominated by the identity, the set of optimal POVMs splits into two classes depending on the relationship between the scalar product $\langle\psi|\phi\rangle$ of the two preparations and the relevant quantity of

$$\Xi(p, q) := \sqrt{\frac{\min\{p, q\}}{\max\{p, q\}}}, \quad p, q > 0. \quad (3)$$

Indeed, the optimal POVM achieves

$$\mathcal{P}_s(\rho, \sigma) = 1 - 2\sqrt{pq}|\langle\psi|\phi\rangle| \quad (4)$$

for $|\langle\psi|\phi\rangle| \leq \Xi(p, q)$, whereas

$$\mathcal{P}_s(\rho, \sigma) = \max\{p, q\}(1 - |\langle\psi|\phi\rangle|^2) \quad (5)$$

otherwise. In the latter case, the probabilities are so unbalanced that it is convenient to renounce detection of the least probable state, by letting its POVM operator be null. Henceforward, we will denote such a protocol as *binary* since the measurement consists of only two effects. Consistently, the former optimal strategy will be deemed *ternary*. One may prove with ease that the two success probabilities of Eqs. (4) and (5) satisfy the following inequality:

$$\max\{p, q\}(1 - |\langle\psi|\phi\rangle|^2) \leq 1 - 2\sqrt{pq}|\langle\psi|\phi\rangle|, \quad (6)$$

where equality is achieved iff $|\langle\psi|\phi\rangle| = \Xi(p, q)$.

Here we focus on states representing preparations of a bipartite system AB shared between Alice and Bob. In Ref. [10], the authors show that the optimal success probability, given by Eq. (4) or (5), depending on the circumstances, can be achieved in the bipartite scenario by an LOCC protocol. Namely, Alice has to carry out a measurement on her party in a suitably chosen orthonormal basis, and then send the outcome through a classical channel to Bob. At this stage, Bob either perfectly or unambiguously discriminates between two local states and estimates the correct result.

B. Fermionic quantum theory

The fermionic quantum theory describes states and transformations of local fermionic modes (see Refs. [11,13,14,16,17]) that satisfy the parity *superselection rule* [11,15,21–23,30,31]. A fermionic mode can be either empty or “excited” and vectors representing fermionic systems are allowed to be superimposed only if they exhibit the same parity, namely, the excitation numbers are all either even or odd. Such a rule is equivalent to requiring that local fermionic transformations are described by Kraus operators belonging to the fermionic algebra \mathfrak{F} . The generators of \mathfrak{F} are the operators φ_i , for i from 1 to N number of modes, fulfilling the canonical anticommutation relations $\{\varphi_i, \varphi_j^\dagger\} = \delta_{ij}$ and $\{\varphi_i, \varphi_j\} = \{\varphi_i^\dagger, \varphi_j^\dagger\} = 0$, $\forall i, j$. The fermionic operators enable us to construct the Fock states as $|n_1 \dots n_N\rangle := (\varphi_1^\dagger)^{n_1} \dots (\varphi_N^\dagger)^{n_N} |\Omega\rangle$, where the vacuum state $|\Omega\rangle$ is the common eigenvector of operators $\varphi_i^\dagger \varphi_i$ with null eigenvalues, and n_i corresponds to the occupation number at the i th mode, i.e., the expectation value of the operator

$\varphi_i^\dagger \varphi_i$. The linear span of all Fock states corresponds to the antisymmetric Fock space \mathcal{F} . We may denote by \mathcal{F}_e and \mathcal{F}_o those sectors of the Fock space featuring even and odd parity, respectively, with $\mathcal{F} = \mathcal{F}_e \oplus \mathcal{F}_o$. Every state (or effect) has a well-defined parity, i.e., states (and effects) satisfy the parity superselection rule.

Thanks to the Jordan-Wigner isomorphism [18–20], the Fock space \mathcal{F} of N fermionic modes is isomorphic to a N -qubit Hilbert space, by the trivial identification of the fermionic occupation number basis $|n_1 \dots n_N\rangle$ with the qubit computational basis (eigenvectors of the Pauli matrices σ_z with $1 \leq i \leq N$). The two parities of the set of states (and effects) are actually isomorphic to the $(N-1)$ -qubit states set, with even and odd pure states given by the rank-one projectors $|\psi\rangle\langle\psi|$, with the $|\psi\rangle$ normalized superposition of Fock vectors belonging to \mathcal{F}_e and \mathcal{F}_o , respectively.

Hereafter, we take advantage of the Jordan-Wigner isomorphism to handle transformations and informational protocols in the fermionic theory. In fact, the isomorphism, which maps, nonlocally, the fermionic operator algebra to an algebra of transformations on qubits, allows us to proceed with the usual quantum notation. In this paper, we will focus on fermionic state discrimination via LOCC; it is therefore fundamental to characterize LOCC for fermionic systems. In this respect, the Jordan-Wigner transformation plays a major role. Indeed, in Ref. [17], the authors show that every fermionic LOCC corresponds to a quantum LOCC on qubits by means of the Jordan-Wigner transformation \mathcal{J} . Accordingly, in the study of LOCC fermionic state discrimination, we will use the fact that in a fixed parity sector (even or odd), the results of quantum theory immediately hold in the fermionic case also.

III. FERMIONIC STATE DISCRIMINATION

Let us consider two states representing preparations of a fermionic system. The theoretical result of Sec. II A, i.e., optimal quantum unambiguous discrimination, can be applied in the fermionic case as well and this allows us to plainly reuse Eqs. (4) and (5) in the fermionic realm too.

Henceforth, we focus on the optimal discrimination strategies to distinguish two states $\rho = p|\psi\rangle\langle\psi|$ and $\sigma = q|\phi\rangle\langle\phi|$ of a fermionic bipartite system AB via LOCC protocols. First, we point out that if the two vectors $|\psi\rangle, |\phi\rangle$ feature a different parity, e.g., $|\psi\rangle \in \mathcal{F}_e(AB)$ and $|\phi\rangle \in \mathcal{F}_o(AB)$, they are perfectly discriminable, as shown in Ref. [24]. Hence, we are interested in discriminating states belonging to the same Fock space sector. Second, since the even and odd sectors are equivalent under LOCC, it is not restrictive to focus on even vectors only.

In the following, when dealing with a composite system of N fermionic modes made of two subsystems of N_1 and N_2 modes, respectively, with $N_1 + N_2 = N$, it is useful to introduce the spaces $\mathcal{H}_E := \mathcal{H}_{e1} \otimes \mathcal{H}_{e2}$ and $\mathcal{H}_O := \mathcal{H}_{o1} \otimes \mathcal{H}_{o2}$, so that $\mathcal{H}_e = \mathcal{H}_E \oplus \mathcal{H}_O$. In other words, \mathcal{H}_E and \mathcal{H}_O are the subspaces where the parities of Alice's and Bob's subsystems are both even or odd, respectively. Operators X with both support and range in \mathcal{H}_E will be denoted by X_E , and similarly operators X with both support and range in \mathcal{H}_O will be denoted by X_O . In particular, we will often use the projections P_E and P_O on \mathcal{H}_E and \mathcal{H}_O , respectively.

Let us consider, for instance, the vector $|\psi\rangle$ and introduce the bases $\{|e_i\rangle_A\}$ and $\{|o_j\rangle_A\}$ for Alice, where $|e_i\rangle_A \in \mathcal{F}_e(A)$ and $|o_j\rangle_A \in \mathcal{F}_o(A)$. Thanks to the superselection rule and the Schmidt decomposition, we can write the vector as

$$|\psi\rangle = |\psi_E\rangle + |\psi_O\rangle,$$

where $|\psi_E\rangle = \sum_i |e_i\rangle_A \otimes |e'_i\rangle_B$ and $|\psi_O\rangle = \sum_j |o_j\rangle_A \otimes |o'_j\rangle_B$ for some $|e'_i\rangle_B \in \mathcal{F}_e(B)$ and $|o'_j\rangle_B \in \mathcal{F}_o(B)$. Due to the above assumption, the scalar product between the two states $|\psi\rangle, |\phi\rangle$ reads

$$\langle\psi|\phi\rangle = \langle\psi_E|\phi_E\rangle + \langle\psi_O|\phi_O\rangle. \quad (7)$$

Before dealing with fermionic discrimination protocols, we first discuss some relevant properties of SEP effects in the fermionic theory. In order to satisfy the parity superselection rule, any SEP POVM must be made of positive operators $0 \leq S \leq I$ of the form

$$S = S_E + S_O, \quad (8)$$

where $S_E = \sum_i e_i \otimes e'_i$, $S_O = \sum_j o_j \otimes o'_j$ for $0 \leq e_i, e'_i, o_j, o'_j \leq I$. More precisely, the latter operators fulfill $\text{Supp}(e_i) \subseteq \mathcal{F}_e(A)$, $\text{Supp}(e'_i) \subseteq \mathcal{F}_e(B)$, $\text{Supp}(o_j) \subseteq \mathcal{F}_o(A)$, and $\text{Supp}(o'_j) \subseteq \mathcal{F}_o(B)$. The probability of the outcome s corresponding to the effect S , given that the fermionic system is prepared in state $\tau \in \text{St}(AB)$, is provided by the Born rule

$$\text{Pr}(s|\tau) = \text{Tr}[\tau S] = \text{Tr}[P_E \tau P_E S_E + P_O \tau P_O S_O],$$

which shows us that any separable POVM operates on the E and O parts of τ independently.

We can now establish that as in quantum theory, in fermionic theory SEP and LOCC unambiguous discrimination also achieve the same performances.

Theorem 1. Let $\rho := p|\psi\rangle\langle\psi|$ and $\sigma := q|\phi\rangle\langle\phi|$ be two pure and non-normalized states for $p, q \geq 0$ and $p + q = 1$. The optimal SEP unambiguous discrimination is implementable through LOCC, i.e., $\mathcal{P}_s^{\text{SEP}} = \mathcal{P}_s^{\text{LOCC}}$, and its success probability reads

$$\mathcal{P}_s^{\text{SEP}} = \text{Pr}(E)\mathcal{P}_s(\rho_E, \sigma_E) + \text{Pr}(O)\mathcal{P}_s(\rho_O, \sigma_O), \quad (9)$$

where $\text{Pr}(E) := \text{Tr}[(\rho + \sigma)P_E]$, P_E is the projector onto \mathcal{H}_E , $\rho_E := P_E \rho P_E / \text{Pr}(E)$, $\sigma_E := P_E \sigma P_E / \text{Pr}(E)$, and the same definitions apply for the O sector.

Proof. We now require that the three elements of the POVM are separable, i.e., $\Pi_\psi, \Pi_\phi, \Pi_\gamma \in \text{SEP}$. From Eq. (8), a necessary condition for separability is that the operators can be written as $\Pi_\psi = \Pi_\psi^E + \Pi_\psi^O$, $\Pi_\phi = \Pi_\phi^E + \Pi_\phi^O$ and $\Pi_\gamma = \Pi_\gamma^E + \Pi_\gamma^O$, and thus the conditions for unambiguous discrimination of Eq. (2) read

$$\text{Tr}[p|\psi_E\rangle\langle\psi_E| \Pi_\phi^E] + \text{Tr}[p|\psi_O\rangle\langle\psi_O| \Pi_\phi^O] = 0$$

and

$$\text{Tr}[q|\phi_E\rangle\langle\phi_E| \Pi_\psi^E] + \text{Tr}[q|\phi_O\rangle\langle\phi_O| \Pi_\psi^O] = 0.$$

Since all the operators involved are positive, the terms $\text{Tr}[p|\psi_i\rangle\langle\psi_i| \Pi_\phi^i]$ and $\text{Tr}[q|\phi_i\rangle\langle\phi_i| \Pi_\psi^i]$ for $i = E, O$ must be null altogether. In other words, the optimization procedure runs independently on the E and O sectors, and the optimal strategy corresponds then to first measuring the projectors P_E, P_O and, depending on the outcome, optimally distinguishing

between ρ_E, σ_E or ρ_O, σ_O , respectively. Both steps are locally implementable through SEP; therefore they lead us to the success probability of Eq. (9).

As proved in Ref. [10], in quantum theory $\mathcal{P}_s = \mathcal{P}_s^{\text{SEP}} = \mathcal{P}_s^{\text{LOCC}}$. Namely, there exists a LOCC quantum protocol for distinguishing between ρ_i, σ_i for $i = E, O$, such that its success probability equals the optimal one. On the other hand, as shown in Ref. [17], LOCC POVMs on a fixed parity sector correspond to LOCC fermionic POVMs in the Jordan-Wigner representation, and thus the quantum LOCC protocol provides a fermionic LOCC protocol. Since the optimal unambiguous discrimination between states belonging to the same E or O sector is LOCC implementable, we achieve $\mathcal{P}_s^{\text{SEP}} = \mathcal{P}_s^{\text{LOCC}}$ in the fermionic case as well. ■

Theorem 1 provides us a key result to determine the necessary and sufficient condition for optimal unambiguous discrimination of SEP and LOCC protocols in the fermionic theory. Since we pointed out at the beginning of the section that the optimal unconstrained fermionic discrimination protocol is the quantum one, in the following we will compare the unconstrained success probabilities of Eqs. (4) or (5) to that of SEP protocols given by Eq. (9). Under the particular hypotheses, the SEP optimal strategy achieves the same performance as the unconstrained one, which indeed proves optimality. Furthermore, Lemma 1 tells us that if a separable protocol is optimal, then there always exists a LOCC one that achieves the same success probability, thus inextricably linking the performances of the two classes. It is not restrictive to focus only on the SEP discrimination strategies, which is a real advantage since their mathematical definition is much clearer than that of LOCC [7].

The next lemma introduces the most significant difference with respect to quantum theory, proving a necessary condition for a pair of fermionic states to be optimally discriminable through LOCC POVMs.

Lemma 1. Necessary condition. Let $\rho := p|\psi\rangle\langle\psi|$ and $\sigma := q|\phi\rangle\langle\phi|$ be two pure and non-normalized states, with $p, q \geq 0$ and $p + q = 1$. The discrimination protocol through SEP is optimal only if

$$\arg\langle\psi_E|\phi_E\rangle = \arg\langle\psi_O|\phi_O\rangle \quad (10)$$

or if any scalar product $\langle\psi_E|\phi_E\rangle, \langle\psi_O|\phi_O\rangle$ is null.

Proof. Both success probabilities in Eqs. (4) and (5) are functions of $|\langle\psi|\phi\rangle| = \sqrt{|\langle\psi|\phi\rangle|^2}$ or

$$\begin{aligned} |\langle\psi|\phi\rangle|^2 &= |\langle\psi_E|\phi_E\rangle|^2 + |\langle\psi_O|\phi_O\rangle|^2 \\ &\quad + 2|\langle\psi_E|\phi_E\rangle||\langle\psi_O|\phi_O\rangle|\cos\Delta, \end{aligned}$$

where $\Delta = \arg\langle\psi_E|\phi_E\rangle - \arg\langle\psi_O|\phi_O\rangle$. From the expressions in Eqs. (4) and (5), it is clear that for fixed $|\psi_i\rangle, |\phi_i\rangle$ such that $\langle\psi_i|\phi_i\rangle \neq 0$, with $i = E, O$, the unconstrained success probability is a function $\mathcal{P}_s(\Delta)$, whose minimum is achieved for $\Delta \in 2\pi\mathbb{Z}$. On the other hand, $\mathcal{P}_s^{\text{SEP}}$ in Eq. (9) is independent of the phase shift Δ . Since $\mathcal{P}_s^{\text{SEP}} \leq \mathcal{P}_s(\Delta)$ for any value of Δ , the SEP discrimination for $\langle\psi_i|\phi_i\rangle \neq 0$, with $i = E, O$, can achieve the performances of the optimal one only if $\Delta \in 2\pi\mathbb{Z}$. The case where $\langle\psi_E|\phi_E\rangle = 0$ or $\langle\psi_O|\phi_O\rangle = 0$ is straightforward since the component of $|\langle\psi|\phi\rangle|$ depending on Δ vanishes and one has $\mathcal{P}_s^{\text{SEP}} = \mathcal{P}_s(\Delta)$. In conclusion, the SEP protocol achieves optimal performances only if the

unconstrained one cannot take advantage from the relative phase of the E and O parts. ■

Necessary and sufficient conditions for optimal LOCC discrimination

Based on Lemma 1, we now assume the complex arguments of the two scalar products being equal, as in Eq. (10), and proceed to derive the necessary and sufficient conditions for optimal discrimination through LOCC POVMs. As in the quantum case, the optimal unconstrained discrimination of the states $\{\rho, \sigma\}$ can be ternary or binary. Moreover, in the fermionic case, one has a broader range of cases since the ternary and binary strategies could be applied to distinguish the states $\{\rho_E, \sigma_E\}$ and $\{\rho_O, \sigma_O\}$ in the even and odd sector, respectively. In the following, we consider all possible cases.

In order to use Eqs. (4) and (5) for calculating $\mathcal{P}_s(\rho_E, \sigma_E)$ and $\mathcal{P}_s(\rho_O, \sigma_O)$, it is convenient to introduce the conditional probability distributions $\{p_i, q_i\}$, and the normalized states $|\tilde{\psi}_i\rangle := |\psi_i\rangle/\|\psi_i\|$, $|\tilde{\phi}_i\rangle := |\phi_i\rangle/\|\phi_i\|$, with $i = E, O$, such that

$$\begin{aligned} \rho_i &= p_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|, & p_i &:= \text{Tr}[\rho_i] = \frac{p\|\psi_i\|^2}{p\|\psi_i\|^2 + q\|\phi_i\|^2}, \\ \sigma_i &= q_i |\tilde{\phi}_i\rangle\langle\tilde{\phi}_i|, & q_i &:= \text{Tr}[\sigma_i] = \frac{q\|\phi_i\|^2}{p\|\psi_i\|^2 + q\|\phi_i\|^2}, \\ & & i &= E, O. \end{aligned} \quad (11)$$

Given the probability distribution $\{p_i, q_i\}$, the condition for the optimal discrimination strategy between ρ_i and σ_i being binary rather than ternary becomes $|\langle\tilde{\psi}_i|\tilde{\phi}_i\rangle| \leq \Xi(p_i, q_i)$. One can prove with ease that if both $p_E \geq q_E$ and $p_O \geq q_O$, then we have $p \geq q$. The same relation applies for the reverse and strict ordering.

Hereafter, we assume that at least one scalar product between $\langle\psi_E|\phi_E\rangle, \langle\psi_O|\phi_O\rangle$ is non-null. Otherwise, we have $\langle\psi|\phi\rangle = \langle\psi_E|\phi_E\rangle + \langle\psi_O|\phi_O\rangle = 0$ and the unambiguous discrimination problem reduces to a perfect discrimination one, which was solved in Ref. [24]. Moreover, if both states $|\psi\rangle, |\phi\rangle$ belong to the same E or O sector, e.g., $|\psi\rangle = |\psi_E\rangle$ and $|\phi\rangle = |\phi_E\rangle$, one can straightforwardly apply the corresponding results in quantum theory [10,27,28]. Indeed, since in that case one has $|\langle\psi|\phi\rangle| = |\langle\psi_E|\tilde{\phi}_E\rangle|$, $\text{Pr}(O) = 0$ and $\text{Pr}(E) = 1$, Eq. (9) leads to $\mathcal{P}_s = \mathcal{P}_s^{\text{SEP}}$.

1. Ternary case

We begin with the unconstrained discrimination being ternary.

Theorem 2. Ternary case. Let $\rho := p|\psi\rangle\langle\psi|$ and $\sigma := q|\phi\rangle\langle\phi|$ be two pure and non-normalized states, with $p, q \geq 0$ and $p + q = 1$. If $|\psi\rangle$ and $|\phi\rangle$ satisfy

$$|\langle\psi|\phi\rangle| \leq \Xi(p, q),$$

then the discrimination protocol through SEP is optimal if and only if $\arg\langle\psi_E|\phi_E\rangle = \arg\langle\psi_O|\phi_O\rangle$ and

$$|\langle\tilde{\psi}_i|\tilde{\phi}_i\rangle| \leq \Xi(p_i, q_i) \quad (12)$$

for both $i = E, O$. See Eqs. (3) and (11) for the definition of Ξ and $|\tilde{\psi}_i\rangle, |\tilde{\phi}_i\rangle, p_i, q_i, i = E, O$.

Proof. (\Rightarrow) In Lemma 1, we have already proved that Eq. (10) is a necessary condition for optimal SEP discrimination. Hence, let us consider the case where Eq. (12) is not satisfied, i.e., the optimal discrimination strategy is binary in at least one of the E or O sectors. Without loss of generality, we assume that $|\langle \psi_E | \phi_E \rangle| > \Xi(p_E, q_E)$, and compare the unconstrained success probability of Eq. (4) with

$$\mathcal{P}_s^{\text{SEP}} = \Pr(E) \max\{p_E, q_E\} (1 - |\langle \tilde{\psi}_E | \tilde{\phi}_E \rangle|^2) + \Pr(O) (1 - 2\sqrt{p_O q_O} |\langle \tilde{\psi}_O | \tilde{\phi}_O \rangle|).$$

The above relation arises from Lemma 1 where we substituted Eq. (5) for $\mathcal{P}_s(\rho_E, \sigma_E)$ and Eq. (4) for $\mathcal{P}_s(\rho_O, \sigma_O)$. Since from Eq. (6) the binary case is strictly less performing than the ternary, one has

$$\begin{aligned} \mathcal{P}_s^{\text{SEP}} &< \Pr(E) (1 - 2\sqrt{p_E q_E} |\langle \tilde{\psi}_E | \tilde{\phi}_E \rangle|) \\ &\quad + \Pr(O) (1 - 2\sqrt{p_O q_O} |\langle \tilde{\psi}_O | \tilde{\phi}_O \rangle|) \\ &= 1 - 2\sqrt{pq} (|\langle \psi_E | \phi_E \rangle| + |\langle \psi_O | \phi_O \rangle|) \\ &\leq \mathcal{P}_s(\rho, \sigma), \end{aligned}$$

with the latter inequality being due to the triangle inequality. The above relation applies as well when the binary discrimination occurs on the O sector or on both.

(\Leftarrow) Suppose now that the states fulfill Eqs. (10), and (12). Using again Lemma 1, where we substituted Eq. (4) for both $\mathcal{P}_s(\rho_E, \sigma_E)$ and $\mathcal{P}_s(\rho_O, \sigma_O)$, the optimal success probability for a separable discrimination protocol reads

$$\begin{aligned} \mathcal{P}_s^{\text{SEP}} &= 1 - 2\sqrt{pq} (|\langle \psi_E | \phi_E \rangle| + |\langle \psi_O | \phi_O \rangle|) \\ &= 1 - 2\sqrt{pq} |\langle \psi | \phi \rangle| = \mathcal{P}_s, \end{aligned}$$

where the last equality one is due to (12) [the triangle inequality achieves equality if and only if Eq. (10) is satisfied]. ■

2. Binary case

We are now left with the case where the optimal unconstrained strategy is binary. We point out that if $|\langle \psi | \phi \rangle| > \Xi(p, q)$, then $p \neq q$ and, analogously, if $|\langle \tilde{\psi}_i | \tilde{\phi}_i \rangle| > \Xi(p_i, q_i)$, one has $p_i \neq q_i$. In the binary scenario, $|\langle \psi | \phi \rangle| > \Xi(p, q)$ and we take $p \neq q$ hereafter.

Before providing the necessary and sufficient conditions for optimal discrimination through LOCC POVMs, we prove the following lemma.

Lemma 2. Let $\rho := p |\psi\rangle\langle\psi|$ and $\sigma := q |\phi\rangle\langle\phi|$ be two pure and non-normalized states, with $p, q \geq 0$ and $p + q = 1$. If $|\psi\rangle$ and $|\phi\rangle$ satisfy $|\langle \tilde{\psi}_i | \tilde{\phi}_i \rangle| \leq \Xi(p_i, q_i)$ for $i = E$ and O , then $|\langle \psi | \phi \rangle| \leq \Xi(p, q)$.

Proof. Thanks to the triangle inequality applied to Eq. (7), we know that

$$|\langle \psi | \phi \rangle| \leq \|\psi_E\| \|\phi_E\| |\langle \tilde{\psi}_E | \tilde{\phi}_E \rangle| + \|\psi_O\| \|\phi_O\| |\langle \tilde{\psi}_O | \tilde{\phi}_O \rangle|,$$

and, due to our hypothesis,

$$|\langle \psi | \phi \rangle| \leq \|\psi_E\| \|\phi_E\| \Xi(p_E, q_E) + \|\psi_O\| \|\phi_O\| \Xi(p_O, q_O).$$

Furthermore, we point out that the quantities $\Xi(p_i, q_i)$ for $i = E, O$ do satisfy

$$\begin{aligned} \|\psi_i\| \|\phi_i\| \Xi(p_i, q_i) &= \|\psi_i\| \|\phi_i\| \Xi(p \|\psi_i\|^2, q \|\phi_i\|^2) \\ &= \begin{cases} \frac{q \|\phi_i\|^2}{\sqrt{qp}} & \text{for } p \|\psi_i\|^2 \geq q \|\phi_i\|^2 \\ \frac{p \|\psi_i\|^2}{\sqrt{pq}} & \text{for } p \|\psi_i\|^2 < q \|\phi_i\|^2 \end{cases} \\ &= \frac{\min\{p \|\psi_i\|^2, q \|\phi_i\|^2\}}{\sqrt{pq}}, \end{aligned}$$

so the sum of the above expressions leads us to

$$\begin{aligned} |\langle \psi | \phi \rangle| &\leq \frac{1}{\sqrt{pq}} \sum_{i=E, O} \min\{p \|\psi_i\|^2, q \|\phi_i\|^2\} \\ &\leq \frac{\min\{p, q\}}{\sqrt{pq}} = \Xi(p, q), \end{aligned}$$

and the thesis follows. ■

The previous result tells us that when $|\langle \psi | \phi \rangle| > \Xi(p, q)$, at least one of the two discrimination protocols for $\{\rho_E, \sigma_E\}$ or $\{\rho_O, \sigma_O\}$ has to be binary. In particular, either (A) they are both binary or (B) one is binary and the other is ternary. We analyze case (A) in Theorem 3 and Proposition 1, whereas case (B) is discussed in Theorem 4.

Theorem 3. Binary case A. Let $\rho := p |\psi\rangle\langle\psi|$ and $\sigma := q |\phi\rangle\langle\phi|$ be two pure and non-normalized states with $p, q \geq 0$, $p + q = 1$, fulfilling

$$|\langle \psi | \phi \rangle| > \Xi(p, q).$$

If we further assume that $p_E, q_E, p_O, q_O > 0$ and

$$|\langle \tilde{\psi}_i | \tilde{\phi}_i \rangle| \leq \Xi(p_i, q_i), \quad (13)$$

for either $i = E$ or O , then SEP discrimination is optimal if and only if $\arg \langle \psi_E | \phi_E \rangle = \arg \langle \psi_O | \phi_O \rangle$, and

$$|\langle \psi | \phi \rangle| - \Xi(p, q) = \Lambda_{\bar{i}}, \quad (14)$$

where $\bar{i} = O, E$ for $i = E, O$, respectively. The quantity Ξ and $|\tilde{\psi}_i\rangle, |\tilde{\phi}_i\rangle, p_i, q_i, i = E, O$, are defined in Eqs. (3) and (11), respectively, while

$$\Lambda_i := \sqrt{\frac{\max\{p \|\psi_i\|^2, q \|\phi_i\|^2\}}{\max\{p, q\}}} [|\langle \tilde{\psi}_i | \tilde{\phi}_i \rangle| - \Xi(p_i, q_i)].$$

Proof. Without loss of generality, we assume that Eq. (13) holds for $i = E$. Therefore, due to Lemma 2, one has $|\langle \tilde{\psi}_O | \tilde{\phi}_O \rangle| > \Xi(p_O, q_O)$. We now compare the unconstrained success probability \mathcal{P}_s of Eq. (5) to

$$\begin{aligned} \mathcal{P}_s^{\text{SEP}} &= \Pr(E) (1 - 2\sqrt{p_E q_E} |\langle \tilde{\psi}_E | \tilde{\phi}_E \rangle|) \\ &\quad + \max\{p \|\psi_O\|^2, q \|\phi_O\|^2\} (1 - |\langle \tilde{\psi}_O | \tilde{\phi}_O \rangle|^2), \end{aligned}$$

of Eq. (9) in Lemma 1, where we replaced the expressions of Eqs. (4) and (5) for $\mathcal{P}_s(\rho_E, \sigma_E)$ and $\mathcal{P}_s(\rho_O, \sigma_O)$, respectively. If the two scalar products $\langle \psi_E | \phi_E \rangle$ and $\langle \psi_O | \phi_O \rangle$ have the same complex argument, the condition for optimality $\mathcal{P}_s = \mathcal{P}_s^{\text{SEP}}$ can be written in the form

$$ax^2 + by^2 + cx + dy + fxy + g = 0, \quad (15)$$

where $x = |\langle \tilde{\psi}_E | \tilde{\phi}_E \rangle|$, $y = |\langle \tilde{\psi}_O | \tilde{\phi}_O \rangle|$, and

$$\begin{aligned} a &= -\max\{p, q\} \|\psi_E\|^2 \|\phi_E\|^2, \\ b &= \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} - \max\{p, q\} \|\psi_O\|^2 \|\phi_O\|^2, \\ c &= 2\sqrt{pq} \|\psi_E\| \|\phi_E\|, \\ d &= 0, \\ f &= -2 \max\{p, q\} \|\psi_E\| \|\phi_E\| \|\psi_O\| \|\phi_O\|, \\ g &= \max\{p, q\} - \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} - \Pr(E). \end{aligned} \quad (16)$$

Since $a \neq 0$ by hypothesis, we can solve Eq. (15) as a second degree equation in x , as explicitly done in Appendix A. Upon dividing the solution, whose explicit expression can be found in Eq. (A3), by $-\max\{p, q\} \|\psi_E\| \|\phi_E\|$, one obtains the thesis.

On the other hand, if we consider the case where solving Eq. (13) is fulfilled for $i = O$, Eq. (15) in the variable y (see Appendix A) with the appropriate substitution of parameters, then we obtain the solution in Eq. (A4), instead. For a full derivation of the quantity Λ_i , see Appendix B. ■

In Theorem 3, we assume all four vectors $|\psi_E\rangle$, $|\psi_O\rangle$, $|\phi_E\rangle$, and $|\phi_O\rangle$ are non-normalized, but with the norm strictly greater than zero. Since we are interested in discriminating between two states ρ , σ , we must require that at least one probability among p_E , p_O and q_E , q_O is non-null. If both states belong to the same sector, i.e., either $p_E = q_E = 0$ or $p_O = q_O = 0$, then the protocol reduces to the quantum one, and therefore it is optimally LOCC implementable. On the contrary, if they belong to different sectors, they are orthogonal and thus perfectly distinguishable even with LOCC; see Ref. [24]. Hereafter, we discuss the cases that are not included in Theorem 3, namely, where only one of the probabilities p_E , p_O , q_E , q_O is equal to zero.

Proposition 1. Binary case A. Let $\rho := p|\psi\rangle\langle\psi|$ and $\sigma := q|\phi\rangle\langle\phi|$ be two pure and non-normalized states for $p, q \geq 0$ and $p + q = 1$ fulfilling

$$|\langle\psi|\phi\rangle| > \Xi(p, q).$$

If either p_E , q_E , p_O , or q_O is null, then SEP discrimination is optimal if and only if any of the following conditions applies:

- (1) $p_i \geq q_i$ for both $i = E, O$,
 - (2) $q_i \geq p_i$ for both $i = E, O$,
 - (3) $q_i = 0$, $p_i < q_i$ and $|\langle\tilde{\psi}_i|\tilde{\phi}_i\rangle| = \min\{1, p/q\}$,
 - (4) $p_i = 0$, $q_i < p_i$ and $|\langle\tilde{\psi}_i|\tilde{\phi}_i\rangle| = \min\{1, q/p\}$,
- with $|\tilde{\psi}_i\rangle$, $|\tilde{\phi}_i\rangle$, p_i , q_i , $i = E, O$, defined as in Eq. (11).

Proof. Without loss of generality, we consider the case where $q_E = 0$, i.e., $\|\phi_E\| = 0$ and $\|\phi_O\| = 1$, namely, in Eq. (A2) a, c, d, f , and $|\langle\psi_E|\phi_E\rangle|$ are null. Thus, we are left with the following condition for optimal SEP discrimination:

$$b|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2 + g = 0.$$

Now, if $p_O \geq q_O$, we necessarily have $p \geq q$; then, $b = g = 0$ and thus SEP discrimination is always optimal. On the other hand, if $p_O < q_O$, i.e., $p\|\psi_O\|^2 < q$, we have

$$\begin{aligned} b &= \begin{cases} q - p\|\psi_O\|^2, & p \geq q \\ q\|\psi_E\|^2, & p < q, \end{cases} \\ g &= \begin{cases} -(q - p\|\psi_O\|^2), & p \geq q \\ -p\|\psi_E\|^2, & p < q. \end{cases} \end{aligned} \quad (17)$$

The condition for optimality in the hypothesis of $p_O < q_O$ is then

$$|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| = \min\{1, p/q\}.$$

We can analogously evaluate Eq. (A2) for the remaining cases and the thesis follows. ■

Finally, we deal with the necessary and sufficient condition for optimal SEP discrimination if the protocol is binary for both the E and O sectors.

Theorem 4. Binary case B. Let $\rho := p|\psi\rangle\langle\psi|$ and $\sigma := q|\phi\rangle\langle\phi|$ be two pure and non-normalized states for $p, q \geq 0$, $p + q = 1$ fulfilling

$$|\langle\psi|\phi\rangle| > \Xi(p, q)$$

and

$$|\langle\tilde{\psi}_i|\tilde{\phi}_i\rangle| > \Xi(p_i, q_i) \quad (18)$$

for both $i = E, O$. Then, SEP discrimination is optimal if and only if

$$\|\psi_E\| \|\phi_O\| |\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| - \|\psi_O\| \|\phi_E\| |\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| = \Gamma_E^- + \Gamma_O^- \quad (19)$$

for $p > q$, otherwise if and only if

$$\|\psi_O\| \|\phi_E\| |\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = \|\psi_E\| \|\phi_E\| |\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| \pm (\Gamma_E^+ + \Gamma_O^+), \quad (20)$$

where

$$\Gamma_i^\pm := \sqrt{\frac{\Pr(i)}{\max\{p, q\}}} \sqrt{(p_i - q_i)^\pm (1 - |\langle\tilde{\psi}_X|\tilde{\phi}_X\rangle|^2)},$$

and $(\cdot)^\pm$ denote the positive and negative parts. The quantity Ξ and $|\tilde{\psi}_i\rangle$, $|\tilde{\phi}_i\rangle$, p_i , q_i , $i = E, O$, are defined in Eqs. (3) and (11), respectively.

Remark 1. We observe that for $p > q$, one has $\Gamma_E^- \Gamma_O^- = 0$, and for $p < q$, $\Gamma_E^+ \Gamma_O^+ = 0$. The reason is that if we assume, e.g., $p > q$, we have either $p_E > q_E$, $p_O > q_O$; $p_E < q_E$, $p_O > q_O$; or $p_E > q_E$, $p_O < q_O$. Hence, at least one of the factors $(p_i - q_i)^\pm$ for $i = E, O$ is null. The same argument applies for the case $p < q$.

Proof. (Theorem 4). Thanks to Lemma 1, we know that

$$\begin{aligned} \mathcal{P}_s^{\text{SEP}} &= \max\{p\|\psi_E\|^2, q\|\phi_E\|^2\} (1 - |\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle|^2) \\ &\quad + \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} (1 - |\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2), \end{aligned} \quad (21)$$

which has to be compared to Eq. (5) for the unconstrained case. The condition of optimality $\mathcal{P}_s = \mathcal{P}_s^{\text{SEP}}$ may be rewritten in the form of Eq. (A2), where c, d are null and

$$\begin{aligned} a &= \max\{p\|\psi_E\|^2, q\|\phi_E\|^2\} - \max\{p, q\} \|\psi_E\|^2 \|\phi_E\|^2, \\ b &= \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} - \max\{p, q\} \|\psi_O\|^2 \|\phi_O\|^2, \\ f &= -2 \max\{p, q\} \|\psi_E\| \|\phi_E\| \|\psi_O\| \|\phi_O\|, \\ g &= \max\{p, q\} - \max\{p\|\psi_E\|^2, q\|\phi_E\|^2\} \\ &\quad - \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\}. \end{aligned}$$

The solutions satisfy either Eq. (A3) or (A4), namely,

$$\begin{aligned} 2a|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| &= -f|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| \\ &\quad \pm \sqrt{(f^2 - 4ab)|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2 - 4ag} \end{aligned} \quad (22)$$

or

$$2b|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| = -f|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| \pm \sqrt{(f^2 - 4ab)|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle|^2 - 4bg} \quad (23)$$

We now have

$$a = \begin{cases} p\|\psi_E\|^2\|\phi_O\|^2, & p_E > q_E \\ \|\phi_E\|^2(q - p\|\psi_E\|^2), & p_E < q_E, \end{cases}$$

$$b = \begin{cases} p\|\psi_O\|^2\|\phi_E\|^2, & p_O > q_O \\ \|\phi_O\|^2(q - p\|\psi_O\|^2), & p_O < q_O, \end{cases}$$

$$g = \begin{cases} 0, & p_E > q_E, p_O > q_O \\ p\|\psi_O\|^2 - q\|\phi_O\|^2, & p_E > q_E, p_O < q_O \\ p\|\psi_E\|^2 - q\|\phi_E\|^2, & p_E < q_E, p_O > q_O. \end{cases}$$

Moreover, we estimate the following quantity in the radicands of Eqs. (22) and (23):

$$f^2 - 4ab = \begin{cases} 0, & p_E > q_E, p_O > q_O \\ 4ag, & p_E > q_E, p_O < q_O \\ 4bg, & p_E < q_E, p_O > q_O. \end{cases}$$

We now consider the three cases compatible with the requirements discussed in the text, viz., that $p_E \neq q_E$, $p_O \neq q_O$ due to the hypotheses in Eq. (18) and that $p_E > q_E$ or $p_O > q_O$, due to $p > q$. Thus, we first point out that for $p_E > q_E$, $p_O > q_O$, both Eqs. (22) and (23) reduce to

$$\|\psi_E\|\|\phi_O\||\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = \|\psi_O\|\|\phi_E\||\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|.$$

Indeed, in such a case, we have that $\Gamma_E^-, \Gamma_O^- = 0$ and the thesis follows. Second, for $p_E > q_E$, $p_O < q_O$, we consider the expression of Eq. (22) and rewrite it as

$$a|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = -\frac{f}{2}|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| \pm \sqrt{ag(|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2 - 1)}$$

or, equivalently,

$$\|\psi_E\|\|\phi_O\||\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = \|\psi_O\|\|\phi_E\||\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| \pm \frac{1}{\sqrt{p}}\sqrt{(q\|\phi_O\|^2 - p\|\psi_O\|^2)(1 - |\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2)}. \quad (24)$$

Due to our hypotheses, we have that $\Gamma_E^- = 0$ and, thus, Eq. (24) reduces to the thesis. In the case where $p_E < q_E$, $p_O > q_O$, we rewrite Eq. (23) as

$$b|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| = -\frac{f}{2}|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| \pm \sqrt{bg(|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle|^2 - 1)},$$

such that

$$\|\psi_O\|\|\phi_E\||\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| = \|\psi_E\|\|\phi_O\||\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| \pm \frac{1}{\sqrt{p}}\sqrt{(q\|\phi_E\|^2 - p\|\psi_E\|^2)(1 - |\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle|^2)}. \quad (25)$$

The same argument as in the previous case applies here: $\Gamma_O^- = 0$ since $p_O > q_O$, and Eq. (25) becomes the thesis. Exactly the same steps can be taken for $p < q$. We only point out that one has to consider Eq. (22) when $p_E < q_E$, $p_O > q_O$ and Eq. (23) for $p_E > q_E$, $p_O < q_O$, instead. The final result is the condition in the statement. ■

IV. ANCILLA-ASSISTED DISCRIMINATION

In this section, we prove that the limits to the performances of unambiguous LOCC discrimination in fermionic theory may be overcome by taking advantage of an ancillary system shared by Alice and Bob. Let us take the simplest sharable system, i.e., two local fermionic modes, in the pure and normalized state,

$$|\omega\rangle := a|00\rangle + b|11\rangle, \quad |a|, |b| \neq 0. \quad (26)$$

We are now interested in distinguishing between the two pure and non-normalized states, $\rho' := \rho \otimes |\omega\rangle\langle\omega|$, $\sigma' := \sigma \otimes |\omega\rangle\langle\omega|$. In the following result, we prove that any pair ρ, σ can be optimally discriminated via LOCC if and only if the ancillary system is in a *maximally entangled* state.

Theorem 5. We can always optimally and unambiguously discriminate through SEP every pair of pure and non-normalized states $\rho := p|\psi\rangle\langle\psi|$, $\sigma := q|\phi\rangle\langle\phi|$, for $p, q \geq 0$ and $p + q = 1$, by means of an ancillary system in the state $|\omega\rangle$ as in Eq. (26), if and only if

$$|\omega\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\varphi}|11\rangle), \quad \varphi \in [0, 2\pi).$$

Proof. We can write $\rho' = p|\psi'\rangle\langle\psi'|$ and $\sigma' = q|\phi'\rangle\langle\phi'|$ for

$$|\psi'\rangle := |\psi\rangle \otimes |\omega\rangle = |\psi'_E\rangle + |\psi'_O\rangle,$$

$$|\phi'\rangle := |\phi\rangle \otimes |\omega\rangle = |\phi'_E\rangle + |\phi'_O\rangle$$

with $|\psi'_E\rangle = a|\psi_E00\rangle + b|\psi_O11\rangle$, $|\psi'_O\rangle = b|\psi_E11\rangle + a|\psi_O00\rangle$, $|\phi'_E\rangle = a|\phi_E00\rangle + b|\phi_O11\rangle$, and $|\phi'_O\rangle = b|\phi_E11\rangle + a|\phi_O00\rangle$. Hence, the scalar products of the E and O parts read

$$\langle\psi'_E|\phi'_E\rangle = |a|^2\langle\psi_E|\phi_E\rangle + |b|^2\langle\psi_O|\phi_O\rangle,$$

$$\langle\psi'_O|\phi'_O\rangle = |a|^2\langle\psi_O|\phi_O\rangle + |b|^2\langle\psi_E|\phi_E\rangle.$$

(\Rightarrow) Lemma 1 requires Eq. (10) to be satisfied by the states for SEP discrimination to be optimal. Namely,

$$\arg\langle\psi'_E|\phi'_E\rangle = \arg\langle\psi'_O|\phi'_O\rangle,$$

which may be rewritten as

$$(|a|^4 - |b|^4)(e^{i\Delta} - e^{-i\Delta}) = 0, \quad (27)$$

where $\Delta = \arg\langle\psi_E|\phi_E\rangle - \arg\langle\psi_O|\phi_O\rangle$; see Ref. [32]. The above expression is satisfied by any Δ if and only if $|a|^2 = |b|^2 = 1/2$, i.e., for a maximally entangled ancilla.

(\Leftarrow) We now suppose $|a|^2 = |b|^2 = 1/2$ and prove sufficiency. The new states satisfy the following properties:

(1) $\langle\psi'|\phi'\rangle = \langle\psi|\phi\rangle$, hence $|\langle\psi'|\phi'\rangle| \leq \Xi(p, q)$ if and only if $|\langle\psi|\phi\rangle| \leq \Xi(p, q)$.

(2) $\|\psi'_E\| = \|\psi'_O\| = \|\phi'_E\| = \|\phi'_O\| = 1/\sqrt{2}$.

(3) $\Pr'(E) = \text{Tr}[(\rho' + \sigma')P_E] = 1/2 = \Pr'(O)$.

(4) $p'_E = \text{Tr}[\rho'P_E]/\Pr'(E) = p = p'_O$ and $q'_E = q'_O = q$.

Thus, $\Xi(p, q) = \Xi(p'_E, q'_E) = \Xi(p'_O, q'_O)$.

(5) $\langle\psi'_E|\phi'_E\rangle = 1/2(\langle\psi_E|\phi_E\rangle + \langle\psi_O|\phi_O\rangle) = \langle\psi'_O|\phi'_O\rangle$. [See Eq. (3) for the definition of Ξ .] Thanks to properties (2) and (5), we have that

$$\langle\psi'|\phi'\rangle = \langle\psi'_E|\phi'_E\rangle + \langle\psi'_O|\phi'_O\rangle = 2\langle\psi'_E|\phi'_E\rangle$$

$$= 2\|\psi'_E\|\|\phi'_E\|\langle\tilde{\psi}'_E|\tilde{\phi}'_E\rangle = \langle\tilde{\psi}'_E|\tilde{\phi}'_E\rangle,$$

$$\langle\psi'|\phi'\rangle = 2\langle\psi'_O|\phi'_O\rangle = \langle\tilde{\psi}'_O|\tilde{\phi}'_O\rangle.$$

The above results lead us to the fact that either $|\langle \psi | \phi \rangle|$, $|\langle \psi' | \phi' \rangle|$, $|\langle \tilde{\psi}'_E | \tilde{\phi}'_E \rangle|$, and $|\langle \tilde{\psi}'_O | \tilde{\phi}'_O \rangle|$ are all smaller than $\Xi(p, q)$ or they are all greater. We observe that if the unconstrained discrimination between the two original states ρ, σ is ternary, then Theorem 2 ensures us that SEP discrimination between ρ' and σ' is optimal. Otherwise, if $|\langle \psi | \phi \rangle| > \Xi(p, q)$, we have to further prove the validity of either Eq. (19) or (20). However, property (4) tells us that either $p_E, p'_E, p'_O > q_E, q'_E, q'_O$ or vice versa. Therefore, the quantities $\Gamma_i^\pm = 0$ for both $i = E$ and O , as we have shown in Remark 1. Eventually, both equations

$$\|\psi'_E\| \|\phi'_O\| |\langle \tilde{\psi}'_E | \tilde{\phi}'_E \rangle| = \|\psi'_O\| \|\phi'_E\| |\langle \tilde{\psi}'_O | \tilde{\phi}'_O \rangle|$$

and

$$\|\psi'_O\| \|\phi'_E\| |\langle \tilde{\psi}'_E | \tilde{\phi}'_E \rangle| = \|\psi'_E\| \|\phi'_O\| |\langle \tilde{\psi}'_O | \tilde{\phi}'_O \rangle|$$

are satisfied by the states ρ' and σ' , which proves sufficiency. \blacksquare

V. DISCUSSION

As we have seen so far, the behavior of fermionic unambiguous discrimination strategies genuinely differs from their quantum counterparts when we focus on their performances under locality restriction. Fermionic protocols optimally distinguish two states through LOCC only if strict conditions on the preparations are fulfilled. However, we remark that the relationship between the LOCC and SEP classes still remains unchanged in terms of binary discrimination and, as we have seen in Lemma 1, the two classes achieve an identical maximum probability of success in unambiguous discrimination.

We proved that the limits of fermionic LOCC discrimination are completely overcome if we take advantage of an ancillary system. A rather striking result, which echoes a similar constraint in the case of minimum-error discrimination [24], is that the ancilla is required to be maximally entangled in order to attain optimality for every pair of preparations.

The comparison between the current results and those pertaining to the unambiguous discrimination of quantum states through LOCC leaves room for some remarks. In Ref. [10], the authors prove the existence of a particular basis for Alice that allows Bob to either perfectly or unambiguously

distinguish between two local states, thus achieving the optimal unconstrained performance. However, in the fermionic case, Theorems 2–4 and Proposition 1 show that such a basis does not exist in those cases where the conditions of the theorems are not met. Most notably, if we consider a pair of fermionic states for which the LOCC unambiguous discrimination is strictly suboptimal, the strategy of Ref. [10] would inevitably involve vectors for Alice or Bob that are forbidden by the parity superselection rule.

In this paper, we obtained the condition for optimal unambiguous discrimination through LOCC of fermionic states. Our proof is based on the comparison between the success probability of the constrained and unconstrained strategies. We may wonder if another proof could be derived where the conditions on the states were expressed in a more algebraic form, as those developed for the conclusive case of Ref. [24]. This is left as an open question for further development.

A natural sequel of the present work would address the conditions for unambiguous discrimination through local resources in the case of *mixed* fermionic states. Nonetheless, such a task remains unresolved even in the quantum realm [33–36], where strict conditions on the preparations have been derived only for particular circumstances. From full rigorous development, we would expect the conditions on the density matrices in the fermionic case to be stronger than in the quantum one. Since we have already observed that some further requirements arise to unambiguously distinguish between two pure fermionic states, we have evidence that mixed fermionic preparations are subject to more stringent prerequisites. In Ref. [34], Chefles derives a necessary and sufficient condition for the feasibility of unambiguous discrimination of mixed quantum states through LOCC, i.e., for a non-null success probability. However, the superselection rule cannot but increase the chances of having states not unambiguously distinguishable to any extent.

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APPENDIX A: GENERAL CONDITION FOR SEP DISCRIMINATION

In Theorems 3 and 4 and in Proposition 1, we derive the necessary and sufficient conditions for SEP unambiguous discriminations by taking advantage of Theorem 1. Indeed, the latter states that the optimal SEP discrimination protocol is LOCC implementable with success probability

$$\mathcal{P}_s^{\text{SEP}} = \Pr(E) \mathcal{P}_s(\rho_E, \sigma_E) + \Pr(O) \mathcal{P}_s(\rho_O, \sigma_O). \quad (\text{A1})$$

We then compare the success probability of the SEP protocol given by Eq. (9) with the unconstrained one of Eq. (5). Moreover, since we know from Lemma 1 that we can optimally distinguish between two states only if

$$\arg \langle \psi_E | \phi_E \rangle = \arg \langle \psi_O | \phi_O \rangle,$$

in the proofs of the aforementioned results we take for granted that such a condition is satisfied. As a result, we have that the SEP discrimination is optimal if and only if

$$a |\langle \tilde{\psi}'_E | \tilde{\phi}'_E \rangle|^2 + b |\langle \tilde{\psi}'_O | \tilde{\phi}'_O \rangle|^2 + c |\langle \tilde{\psi}'_E | \tilde{\phi}'_E \rangle| + d |\langle \tilde{\psi}'_O | \tilde{\phi}'_O \rangle| + f |\langle \tilde{\psi}'_E | \tilde{\phi}'_E \rangle| |\langle \tilde{\psi}'_O | \tilde{\phi}'_O \rangle| + g = 0, \quad (\text{A2})$$

where the parameters a, b, c, d, f , and g depend on the case under scrutiny. In the most general case, Eq. (A2) has the following solutions:

$$-2a|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = f|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| + c \pm \sqrt{(f^2 - 4ab)|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2 + (2cf - 4ad)|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| + c^2 - 4ag} \quad \text{for } a \neq 0, \quad (\text{A3})$$

$$-2b|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| = f|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| + d \pm \sqrt{(f^2 - 4ab)|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle|^2 + (2df - 4bc)|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| + d^2 - 4bg} \quad \text{for } b \neq 0. \quad (\text{A4})$$

In Theorems 3 and 4 and in Proposition 1, we take advantage of the additional hypothesis on the states to further simplify the above expressions.

The choice between Eqs. (A3) and (A4) is only based on a simplification and convenience criterion in the analysis of the discrimination cases.

APPENDIX B: DERIVATION OF QUANTITY Λ_i

In Theorem 3, we covered the case where the SEP discrimination is binary on either the E or O sector and ternary in the other. In particular, in the proof of the theorem, we assumed $|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| \leq \Xi(p_E, q_E)$ and $|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| > \Xi(p_O, q_O)$, so that the necessary and sufficient condition for optimal SEP discrimination reads

$$2a|\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = -f|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| - c \pm \sqrt{(f^2 - 4ab)|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2 + 2cf|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| + c^2 - 4ag}.$$

The values for parameters a, b, c, f , and g are expressed in Eq. (16) so that

$$f^2 - 4ab = 4\max^2\{p, q\}\|\psi_E\|^2\|\phi_E\|^2\|\psi_O\|^2\|\phi_O\|^2 + 4\max\{p, q\}\|\psi_E\|^2\|\phi_E\|^2(\max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} - \max\{p, q\}\|\psi_O\|^2\|\phi_O\|^2).$$

Since we assumed $p_E, q_E \neq 0$, i.e., $\|\psi_E\|, \|\phi_E\| \neq 0$, we may reformulate the above expression as

$$\|\psi_E\|\|\phi_E\||\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = -\|\psi_O\|\|\phi_O\||\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| + \sqrt{\frac{pq}{\max^2\{p, q\}}} \pm \sqrt{\frac{\Theta}{\max\{p, q\}}},$$

for

$$\Theta = \alpha|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle|^2 + \beta|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| + \gamma,$$

where we define

$$\begin{aligned} \alpha &= \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\}, \\ \beta &= -2\sqrt{pq}\|\psi_O\|\|\phi_O\|, \\ \gamma &= \max\{p, q\} - \Pr(E) - \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} + \frac{pq}{\max\{p, q\}}. \end{aligned}$$

We prove with ease that

$$\frac{pq}{\max\{p, q\}} = \min\{p, q\},$$

so that we find

$$\|\psi_E\|\|\phi_E\||\langle\tilde{\psi}_E|\tilde{\phi}_E\rangle| = -\|\psi_O\|\|\phi_O\||\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| + \Xi(p, q) \pm \sqrt{\frac{\Theta}{\max\{p, q\}}}$$

and

$$\gamma = \max\{p, q\} + \min\{p, q\} - \Pr(E) - \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} = \Pr(O) - \max\{p\|\psi_O\|^2, q\|\phi_O\|^2\} = \min\{p\|\psi_O\|^2, q\|\phi_O\|^2\}.$$

Now we can rewrite the above condition as

$$|\langle\psi_E|\phi_E\rangle| + |\langle\psi_O|\phi_O\rangle| - \Xi(p, q) = \pm \sqrt{\frac{\Theta}{\max\{p, q\}}},$$

and remembering that by the hypotheses of Theorem 3, one has $|\langle\psi_E|\phi_E\rangle| + |\langle\psi_O|\phi_O\rangle| = |\langle\phi|\psi\rangle| > \Xi(p, q)$, we obtain

$$|\langle\psi|\phi\rangle| - \Xi(p, q) = \sqrt{\frac{\Theta}{\max\{p, q\}}}.$$

The term Θ eventually reads

$$\begin{aligned}\Theta &= (\sqrt{\max(p\|\psi_O\|^2, q\|\phi_E\|^2)}|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| - \sqrt{\min(p\|\psi_O\|^2, q\|\phi_O\|^2)})^2 \\ &= \sqrt{\max\{p\|\psi_O\|^2, q\|\phi_O\|^2\}}[|\langle\tilde{\psi}_O|\tilde{\phi}_O\rangle| - \Xi(p_O, q_O)],\end{aligned}$$

and we finally achieve Eq. (14).

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$$\arg(z+w) = \frac{1}{2i}[\ln(z+w) - \ln(\bar{z}+\bar{w})].$$

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