

Noiseless linear amplifiers for multimode statesMingjian He^{1,*}, Robert Malaney^{1,†} and Benjamin A. Burnett^{2,‡}¹*School of Electrical Engineering and Telecommunications, The University of New South Wales, Sydney, New South Wales 2052, Australia*²*Northrop Grumman Corporation, San Diego, California 92127, USA* (Received 25 August 2020; revised 4 December 2020; accepted 21 December 2020; published 19 January 2021)

The entanglement structure between different frequency components within broadband quantum light pulses, forged at entanglement creation, represents a promising route to the practical delivery of many multipartite quantum information applications. However, the scalability of such applications is largely limited by the entanglement decoherence caused by photon loss. One promising method to combat such losses is noiseless linear amplification. However, while there have been various procedures that implement noiseless linear amplification on single-mode states, no realization has thus far been proposed for noiseless linear amplification on quantum states carrying a multimode structure. In this work we close this gap, proposing a noiseless linear amplifier (NLA) with photon catalysis (PC), namely, the PC-NLA. Constructing a multimode version of an existing NLA that uses quantum scissors (QS), the QS-NLA, we then show how the PC-NLA is compatible with the QS-NLA, even though the former uses half the physical resources of the latter. We then apply our multimode NLA frameworks to the problem of continuous-variable (CV) entanglement distillation, determining how the multimode structure of the entanglement impacts the performance of the NLAs. Different from single-mode NLA analyses, we find that a multimode NLA is only effective as a CV entanglement distillation strategy when the channel loss is beyond some threshold—a threshold largely dependent on the multimode structure. The results provided here will be valuable for real-world implementations of multipartite quantum information applications that utilize complex entanglement structure within broadband light pulses.

DOI: [10.1103/PhysRevA.103.012414](https://doi.org/10.1103/PhysRevA.103.012414)**I. INTRODUCTION**

In general, a quantum state encapsulated within a light pulse consists of multiple frequency modes (a multimode)—a reality of special importance for a plethora of continuous-variable (CV) quantum information applications that utilize high-pulse-rate laser sources. For example, entanglement between different *supermodes* (linear combinations of frequency modes) of ultrafast light pulses could allow for a practical route to higher quantum information throughput [1–3], be used as a resource for quantum computing [4–7], enable improved quantum sensing [8–11], and be used in quantum secret sharing [12]. State-of-the-art laser sources are now in the 100-MHz regime with pulse rates of 1 GHz and beyond, anticipated as mainstream in the coming years [13]. In addition to random entanglement between supermodes, the broadband nature of ultrafast light pulses represents a fertile ground for developing precisely engineered CV entanglement across frequency space [14]. Parametric down-conversion of ultrafast frequency combs provides one convenient single-step path to specifically engineered entangled-supermode states [15,16].

Like any other entangled states, multimode states suffer from decoherence over lossy channels. One route open

to compensate such loss is amplification of the states. Ideally, such amplifiers would be noiseless and linear (phase insensitive)—a goal that is impossible in a deterministic sense [17]. However, it is known that a noiseless linear amplifier (NLA) acting on quantum states is possible in a nondeterministic, or probabilistic, fashion (e.g., [18]). What remains to be determined is how to construct a probabilistic NLA for multimode states, and what the performance of such a multimode NLA is relative to a single-mode (single-frequency) NLA, especially with regard to important CV information protocols such as entanglement distillation. This task forms the focus of this work.

In order to proceed towards a multimode NLA, we consider two approaches previously developed for single-mode systems: a quantum scissors (QS) approach [18] and a photon catalysis (PC) approach [19]. QS, a technique which operates using beam splitters and single-photon detectors [20], is known to operate as an NLA under the limitation of low-energy input states [18]. PC, which requires half the number of beam splitters and single-photon detectors required by QS, is also known to act as an NLA for low-energy input states, but with higher probabilities of success relative to QS [19].¹ In order to overcome the limitation of QS-based NLAs being

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¹A cascaded application of PC was investigated in terms of entanglement distillation in [21], but found to bring insignificant enhancement relative to the single use of PC.

applicable only to low-energy inputs, the notion of parallel-processing QS was proposed [18]. Experimental realizations, at least in part, based on the ideas in [18] have been undertaken (e.g., [22–27]). Different schemes have also been proposed for implementing the NLA (e.g., [28–33]). All of the above work, however, is designed with the single-mode picture in view. To analyze the more generic picture, extensions to a multimode framework are required.

The key contributions of this work can be summarized as follows: We propose an NLA based on parallel processing of PC, the PC-NLA. We show that when applied to a range of coherent states, the PC-NLA is compatible with the QS-NLA even though the former requires fewer physical resources than the latter. We then assess how the multimode structure of states impacts the performance of the PC-NLA and QS-NLA when applied to entanglement distillation, showing how a multimode analysis can lead to outcomes quite distinct from a single-mode analysis. A multimode framework for QS is developed in order to carry out this assessment.

The rest of this paper is organized as follows. In Sec. II we present the multimode version of the QS-NLA. We then show how PC can be utilized to build the multimode PC-NLA. In Sec. III we compare the performances of the QS-NLA and the PC-NLA with regard to entanglement enhancement. In Sec. IV we compare our PC-NLA and a recently proposed cascaded processing of PC. Section V concludes our work.

II. NOISELESS LINEAR AMPLIFIERS FOR CV QUANTUM STATES

A. NLA with QS

In this section, we propose an NLA with parallel QS for multimode states. But first let us briefly review the well-studied QS operation for single-mode states.

1. Single-mode case

Let a , b , and c label the single modes involved in the QS operation. The operation is successful when a single photon at output a and zero photons at output c , $|0\rangle_c |1\rangle_a$ [as shown in Fig. 1(a)], or a single photon at output c and zero photons at output a , $|1\rangle_c |0\rangle_a$, are detected. In the Fock basis, on detecting $|0\rangle_c |1\rangle_a$ the QS operation is written as

$$\hat{M} = \langle 0|_c \langle 1|_a \hat{U}_{ac}(T_1) \hat{U}_{bc}(T_2) |0\rangle_b |1\rangle_c, \quad (1)$$

where $\hat{U}_{ac}(T_1) = \exp[\arccos(\sqrt{T_1})(\hat{a}^\dagger \hat{c} - \hat{a} \hat{c}^\dagger)]$ and $\hat{U}_{bc}(T_2) = \exp[\arccos(\sqrt{T_2})(\hat{b}^\dagger \hat{c} - \hat{b} \hat{c}^\dagger)]$ are the unitary operators of the beam splitters [shown in Fig. 1(a)], and T_1 and T_2 are their transmissivities. In the rest of the paper, we use the setting $T_1 = 1/2$ (see [18] on the reason for this) and use $T_2 = T$ for convenience. Henceforth, for the QS operation when we refer to the transmissivity T we refer to the T_2 in Fig. 1. The operator \hat{M} is then written as [32]

$$\hat{M} = \sqrt{\frac{T}{2}} |0\rangle_b \langle 0|_a + \sqrt{\frac{1-T}{2}} |1\rangle_b \langle 1|_a. \quad (2)$$

On detecting $|1\rangle_c |0\rangle_a$ the QS operation is written as

$$\hat{M}' = \sqrt{\frac{T}{2}} |0\rangle_b \langle 0|_a - \sqrt{\frac{1-T}{2}} |1\rangle_b \langle 1|_a. \quad (3)$$

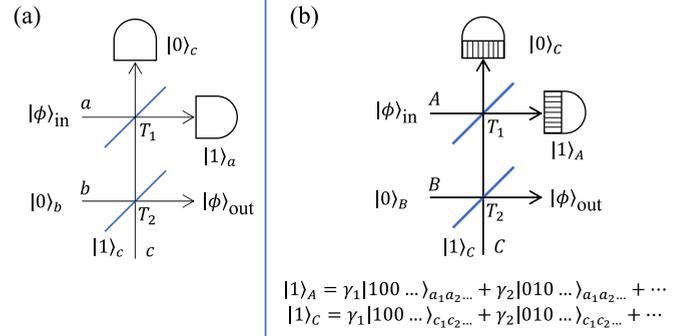


FIG. 1. QS for (a) single-mode states and (b) multimode states. For the multimode case the single-photon detector implements a joint detection on the single-mode components of the multimode state and clicks when a photon with a specific multimode structure (determined by the weighting coefficients $\gamma_1, \gamma_2, \dots$) is detected. An implementation of the multimode detector is discussed in Sec. III.

Although not shown in Fig. 1(a), in the complete implementation of the QS operation, if the latter outcome occurs a phase shifting will be applied to the output state $|\phi\rangle_{\text{out}}$ to remove the phase flip. The complete implementation of the QS operation, which can be represented by an equivalent operator $\sqrt{2}\hat{M}$, is used in the rest of the paper.

A limitation for QS is that it can only operate as an NLA for weak states,² i.e., $|\psi\rangle_{\text{in}} \approx |0\rangle + \alpha |1\rangle$, where α is a complex number. A device, which we refer to as the QS-NLA, was proposed in [18] to overcome this limitation. As is depicted in Fig. 2, one major component of the QS-NLA is the N splitter, which consists of an array of beam splitters. The first N splitter, in conjunction with the vacuum ancillas, evenly divides the input state into N paths. Parallel QS operations are then applied to each path. The transmissivities T for the beam splitters in the QS operations are identical. The second N splitter adopts an inverse to the arrangement of the beam splitters in the first N splitter. The paths after the QS operations are interferometrically recombined at the second N splitter. The amplification is successful when all the output ports except the first port ($|\psi\rangle_{\text{out}}$) of the second N splitter register zero photons. The QS-NLA approaches an ideal NLA when N is large, but its success probability vanishes as N grows. For a coherent state input, the QS-NLA implements the following transformation (when $N \gg \sqrt{\frac{1-T}{T}}|\alpha|$) [18]:

$$|\alpha\rangle \rightarrow \frac{1}{\sqrt{P}} \sqrt{T}^N e^{-\frac{(1-g_s^2)|\alpha|^2}{2}} |g_s \alpha\rangle, \quad (4)$$

which has the success probability

$$P = T^N e^{-(1-g_s^2)|\alpha|^2}, \quad (5)$$

where the equivalent gain is $g_s = \sqrt{(1-T)/T}$.

The QS-NLA can be directly extended to the multimode case by replacing the single-mode QS with the multimode QS as discussed next.

²NLAs are normally considered with reference to amplification of a coherent state $|\alpha\rangle$. In this context a weak state means $|\alpha| \ll 1$.

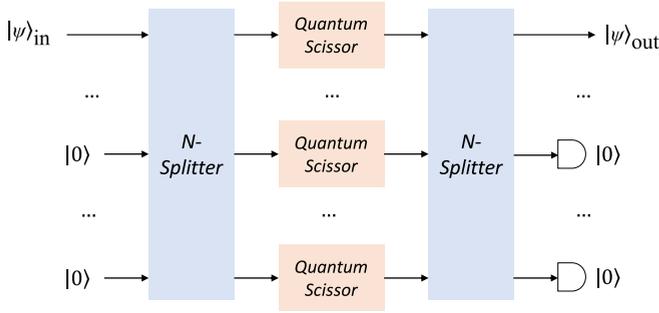


FIG. 2. The schematic for the QS-NLA.

2. Multimode case

In the terminology adopted in this work, a multimode is simply a generic collection of single modes. A supermode refers to a specific linear superposition of single modes, with a sequence of such supermodes forming an orthonormal basis. Let \hat{a}_m^\dagger be the creation operator of a single mode at a specific frequency (indexed with $m \in \{1, 2, \dots, \infty\}$),³ then a new creation operator can be defined as

$$\hat{A}^\dagger = \sum_{m=1}^{\infty} \gamma_m \hat{a}_m^\dagger, \quad (6)$$

where the γ_m 's are normalized complex weighting coefficients satisfying $\sum_{m=1}^{\infty} |\gamma_m|^2 = 1$. The composed mode created by \hat{A}^\dagger is what we refer to as a supermode.

For the multimode QS, two major components appearing in the single-mode QS need to be generalized. These are the Fock state and the beam-splitter operator. We first define the multimode Fock state as

$$|n\rangle_A = \frac{\hat{A}^{\dagger n}}{\sqrt{n!}} |0\rangle_A, \quad (7)$$

where n is a nonnegative integer,

$$|0\rangle_A = \bigotimes_{m=1}^{\infty} |0\rangle_{a_m}, \quad (8)$$

and we have used capital letters as subscripts to label the supermodes. Specifically, a multimode single-photon state can be represented by

$$|1\rangle_A = \sum_{m'=1}^{\infty} \gamma_{m'} \bigotimes_{m=1}^{\infty} |\delta_{m,m'}\rangle_{a_m}, \quad (9)$$

where $\delta_{m,m'} = 1$ for $m = m'$, and $\delta_{m,m'} = 0$ for $m \neq m'$.

We assume the beam splitter in the multimode QS operation is frequency independent such that the transmissivities associated with each single-mode component of the multimode state are identical. The multimode beam-splitter

operator can then be written as

$$\hat{U}_{AB}(T) = \bigotimes_{m=1}^{\infty} \hat{U}_{a_m b_m}(T), \quad (10)$$

where $\hat{U}_{a_m b_m}(T)$ is the single-mode beam-splitter operator coupling two single modes a_m and b_m .

As illustrated in Fig. 1(b), let A , B , and C label the supermodes involved in the QS operation. The multimode QS can then be represented by an operator (see Appendix A for details)

$$\begin{aligned} \hat{M} &= \langle 0|_C \langle 1|_A \hat{U}_{AC}(1/2) \hat{U}_{BC}(T) |0\rangle_B |1\rangle_C \\ &= \sqrt{\frac{T}{2}} |0\rangle_B \langle 0|_A + \sqrt{\frac{1-T}{2}} |1\rangle_B \langle 1|_A, \end{aligned} \quad (11)$$

where

$$|1\rangle_B = \hat{B}^\dagger |0\rangle_B = \sum_{m=1}^{\infty} \gamma_m \hat{b}_m^\dagger |0\rangle_B, \quad (12)$$

and $|1\rangle_C = \sum_{m=1}^{\infty} \gamma_m \hat{c}_m^\dagger |0\rangle_C$. The multimode QS operator reduces to the single-mode QS operator in Eq. (2) in the special case where each supermode only has one single-mode component. The complete implementation of the multimode QS operation is represented by $\sqrt{2}\hat{M}$.

Consider a multimode coherent state, which can be expressed in the multimode Fock basis by

$$\begin{aligned} |\alpha\rangle_A &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_A \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha \hat{A}^\dagger)^n}{n!} |0\rangle_A, \end{aligned} \quad (13)$$

where \hat{A}^\dagger is the supermode mode creation operator defined in Eq. (6). Similar to the single-mode case, a multimode QS-NLA can implement the following transformation,

$$|\alpha\rangle_A \rightarrow \frac{1}{\sqrt{P}} \sqrt{T}^N e^{-\frac{(1-g_s^2)|\alpha|^2}{2}} |g_s \alpha\rangle_A, \quad (14)$$

with the same success probability as in Eq. (5), where again $g_s = \sqrt{(1-T)/T}$.

B. NLA with PC

We now propose an NLA with parallel PC operations. Noticing that the single-mode case can be viewed as a special case in the multimode setting, we discuss this new amplifier in the multimode setting only. In the rest of the paper, for brevity we will omit the subscripts for the supermodes.

An NLA can be constructed with PC. We refer to this type of NLA as a PC-NLA, a schematic of which is given in Fig. 3. An advantage of the PC-NLA is that each PC module only requires one beam splitter and one single-photon detector, which is half of the apparatus required by a QS module. This reduction in apparatus can be important, especially in confined environments (e.g., on board a satellite).

We now investigate a scenario where the state to be amplified is the multimode coherent state defined in Eq. (13). The first N splitter of Fig. 3 divides the input coherent state into

³In experiments m will be an index labeling a limited number of discrete frequency bins, the bandwidth of each bin being determined by the resolution of the detectors.

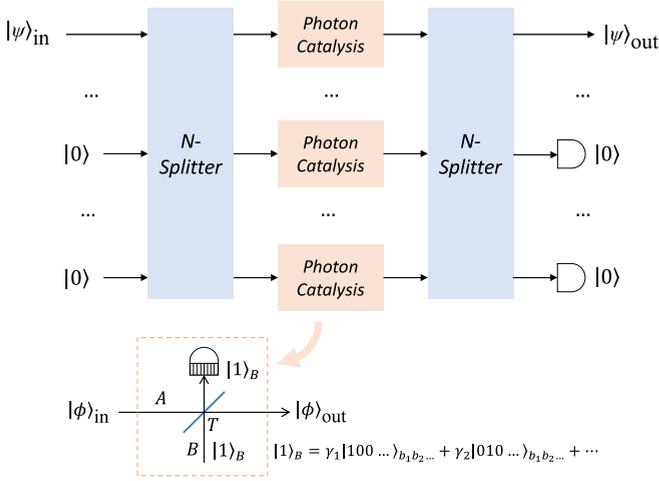


FIG. 3. The schematic for the PC-NLA.

the product state, that is,

$$|\alpha\rangle \longrightarrow |\alpha'\rangle^{\otimes N}, \quad (15)$$

where $\alpha' = \alpha/\sqrt{N}$. Parallel PC operations are then applied on each $|\alpha'\rangle$, where each PC operation can be represented by an operator [34]

$$\hat{R} = \sqrt{T} \left(-\frac{1-T}{T} \hat{A}^\dagger \hat{A} + 1 \right) \bigotimes_{m=1}^{\infty} \sqrt{T}^{\hat{a}_m^\dagger \hat{a}_m}, \quad (16)$$

and where T is now the transmissivity for the beam splitter in the PC operation. Henceforth, when we refer to PC operations, the transmissivity will be the T shown in Fig. 3 (for QS operations T will still refer to the T_2 shown in Fig. 1). The rightmost factor in the above equation satisfies the following identities:

$$\bigotimes_{m=1}^{\infty} \sqrt{T}^{\hat{a}_m^\dagger \hat{a}_m} |n\rangle = \sqrt{T}^{\hat{A}^\dagger \hat{A}} |n\rangle = \sqrt{T}^n |n\rangle. \quad (17)$$

Similar to the QS-NLA, the transmissivities T for the beam splitters in the PC operations are identical. Each PC operation will alter each $|\alpha'\rangle$ to (unnormalized)

$$\hat{R} |\alpha'\rangle = e^{-\frac{(1-T)|\alpha'|^2}{2}} \sqrt{T} \left(-\frac{1-T}{T} \alpha'' \hat{A}^\dagger + 1 \right) |\alpha''\rangle, \quad (18)$$

where $\alpha'' = \sqrt{T} \alpha'/\sqrt{N}$. The photon-catalyzed coherent state is then coherently recombined at the second N splitter. Post-selecting the state in the $|\psi\rangle_{out}$ port only when the other output ports register zero photons leads to an output state

$$|\psi\rangle_{out} = \frac{1}{\sqrt{P}} \sqrt{T}^N e^{-\frac{|\alpha|^2}{2}} \times \left(-\frac{1-T}{\sqrt{T}} \frac{\alpha}{N} \hat{A}^\dagger + 1 \right)^N e^{\sqrt{T} \alpha \hat{A}^\dagger} |0\rangle, \quad (19)$$

where P is the success probability for the PC-NLA. In the limit of $N \gg \frac{1-T}{\sqrt{T}} |\alpha|$, we have

$$\lim_{N \gg \frac{1-T}{\sqrt{T}} |\alpha|} \left(-\frac{1-T}{\sqrt{T}} \frac{\alpha}{N} \hat{A}^\dagger + 1 \right)^N e^{\sqrt{T} \alpha \hat{A}^\dagger} |0\rangle$$

$$= e^{-\frac{1-T}{\sqrt{T}} \alpha \hat{A}^\dagger} e^{\sqrt{T} \alpha \hat{A}^\dagger} |0\rangle = e^{\frac{|g_c \alpha|^2}{2}} | -g_c \alpha \rangle, \quad (20)$$

where the equivalent gain reads $g_c = (1-2T)/\sqrt{T}$. Putting everything together we conclude, in the limit of $N \gg \frac{1-T}{\sqrt{T}} |\alpha|$, the PC-NLA implements the transformation

$$|\alpha\rangle \rightarrow \frac{1}{\sqrt{P}} \sqrt{T}^N e^{-\frac{(1-g_c^2)|\alpha|^2}{2}} | -g_c \alpha \rangle, \quad (21)$$

where

$$P = T^N e^{-(1-g_c^2)|\alpha|^2}. \quad (22)$$

The PC-NLA acts as an NLA up to an irrelevant global phase. The equivalent gain of the PC-NLA satisfies $g_c > 1$ when $T < 1/4$.

From Eq. (14) and Eq. (21) it can be observed that the QS-NLA and the PC-NLA implement similar transformations. At fixed T , the only difference for the two NLAs is their equivalent gains (g_s and g_c). For an input coherent state, it can be shown that the QS-NLA always has a higher success probability than the PC-NLA when $g_s = g_c$.

C. Comparison of the two NLAs for finite N

Since the success probabilities for both QS-NLA and PC-NLA vanish as N grows, for practical purposes it is only meaningful to compare the QS-NLA and the PC-NLA when N is finite.

In the multimode Fock basis, the transformation of the QS-NLA can be expressed as [35]

$$\hat{M}_s = \sum_{n=0}^N \sqrt{T}^N \frac{N!}{(N-n)! N^n} \sqrt{\frac{1-T}{T}}^n |n\rangle \langle n|. \quad (23)$$

We note \hat{M}_s will truncate $|n\rangle$, $\forall n > N$. In the multimode Fock basis, the PC-NLA implements the transformation

$$\hat{M}_c = \sum_{n=0}^{\infty} \sqrt{T}^{N+n} \sum_{k=0}^N \binom{N}{k} \frac{n!}{(n-N+k)!} N^{k-N} p^{N-k} |n\rangle \langle n|, \quad (24)$$

where $p = \frac{T-1}{T}$. The derivation for Eq. (24) can be found in Appendix B.

We now numerically calculate (for finite N) the fidelity between a target coherent state, $|g_t \alpha\rangle$, and the state after amplification, $|\psi\rangle_{out}$. Here, g_t is the target amplification gain. Note that for the PC-NLA the target coherent state is $| -g_t \alpha \rangle$, due to the global phase shift. Since both $|g_t \alpha\rangle$ and $|\psi\rangle_{out}$ are pure states, the fidelity is simply the overlap of the two states,

$$F = |\langle g_t \alpha | \psi_{out} \rangle|^2. \quad (25)$$

The results are shown in Fig. 4. In the left column of Fig. 4 we vary the value of g_t , and for each g_t we find the T that maximizes F . The maximizations on F for the two NLAs are carried out independently. We can see for each NLA, at fixed g_t , the maximal fidelity increases as N grows. To achieve a certain level of fidelity, input coherent states with a higher amplitude require larger N . The success probability for the NLAs when achieving the maximal fidelity are illustrated in

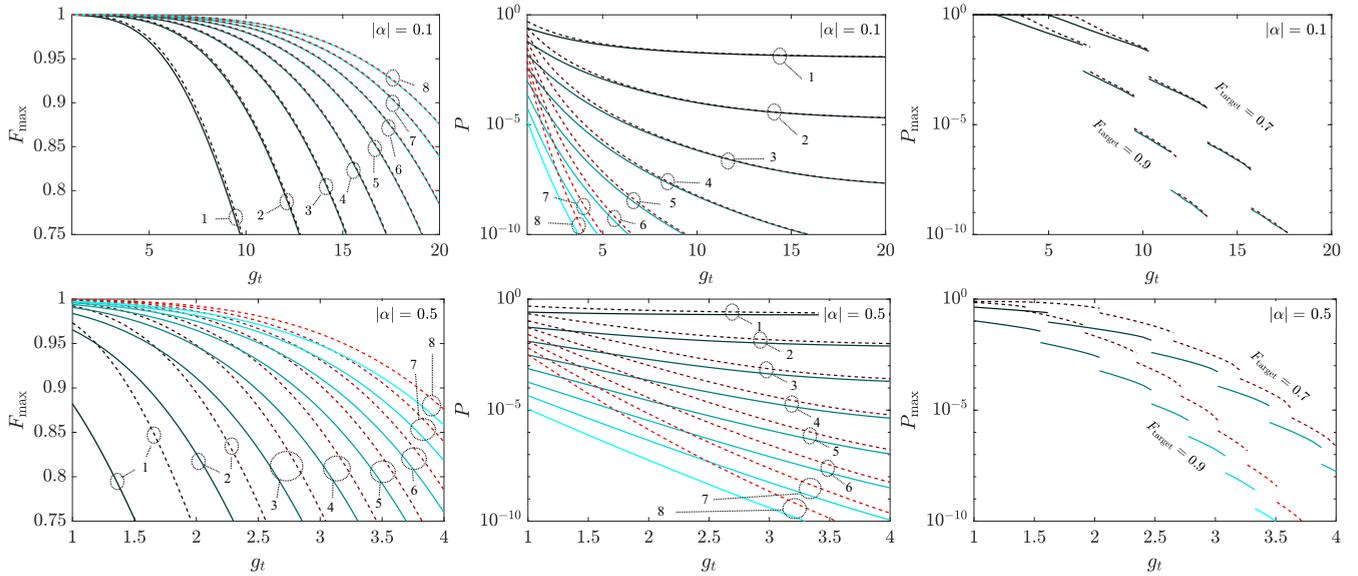


FIG. 4. Left: The maximal fidelity of the amplified states vs the target amplification gain. The numbers connected to the curves indicate the cases for different N (from $N = 1$ to $N = 8$). Middle: The corresponding success probabilities of the left column. Right: The maximal success probabilities that guarantee certain levels of fidelity F_{target} . For different F_{target} the curves in the right column are disjoint since the success probabilities are maximized in terms of N . In all figures the dashed curves correspond to the QS-NLA and the solid curves correspond to the PC-NLA.

the middle column of Fig. 4. When N and g_t are both fixed, the QS-NLA always provides higher maximal fidelity at larger success probabilities than the PC-NLA. In the right column of Fig. 4 we investigate the maximal success probabilities that guarantee a certain level of fidelity,

$$P_{\max} = \max_{0 < T \leq 1, 1 \leq N \leq 8} \{P\}, \text{ such that } F \geq F_{\text{target}}, \quad (26)$$

where $F_{\text{target}} < 1$ is the target fidelity. From the results shown we can conclude at fixed g_t the QS-NLA always has a higher success probability with regard to achieving a target fidelity. The difference of the probabilities of the two NLAs is insignificant when $|\alpha|$ is small. All these conclusions also hold for $N = 1$.

We also investigate the use of inefficient single-photon detectors, which can be modeled by a positive operator-valued measure (POVM) with the elements [36]

$$\begin{aligned} \hat{\Pi}_0 &= \sum_{n=0}^{\infty} (1 - \eta)^n |n\rangle \langle n|, \\ \hat{\Pi}_1 &= \sum_{n=1}^{\infty} n\eta(1 - \eta)^{n-1} |n\rangle \langle n|, \end{aligned} \quad (27)$$

where η represents the detection efficiency, and $|n\rangle$ is the multimode Fock state defined by Eq. (7). We note, different from Eqs. (11) and (18), in Eq. (27) we adopt the POVM formalism since the output states of the QS and PC will be impure when inefficient (i.e., $\eta < 1$) single-photon detectors are used. Details on the transformations of the QS and PC (with inefficient single-photon detectors) can be found in Appendix C. Let us denote $\hat{\sigma}_\eta$ as the output state with inefficient single-photon detectors, and P_η as the corresponding success probability. We use the fidelity of the output state, $F_d = \text{tr}\{\hat{\sigma}_\eta \hat{\sigma}_1\}$, and the ratio of the success probabilities, $R = P_\eta/P_1$, to quantify the impact

of the detection efficiency. In these relations, $\hat{\sigma}_1$ is the output state for the case with efficient (i.e., $\eta = 1$) single-photon detectors, and P_1 is the corresponding success probability. Our parameter space of interest is $0 < T \leq 1$ and $0 \leq |\alpha| \leq 0.5$, where the upper limit on $|\alpha|$ is set to ensure reasonable success probabilities of the QS-NLA and the PC-NLA. We find that the impact of the detection efficiency on F_d is small (<5% reduction) for both NLAs when $\eta > 0.3$. When single-photon detectors with $\eta > 0.9$ (e.g., [37]) are used, the impact is insignificant (<1% reduction) for both NLAs. The decrease of the maximal fidelity shown in Fig. 4 is also insignificant when η drops from unity to 0.9. In terms of success probability, we find both NLAs provide $\eta^N < R < 1$ for $\eta < 1$. These results illustrate the general trends expected for inefficient single-photon detectors. Resetting $\eta = 1$, we also investigate the use of on/off detectors, which can be modeled by a POVM with the elements [38]

$$\hat{\Pi}_{\text{off}} = |0\rangle \langle 0|, \quad \hat{\Pi}_{\text{on}} = \sum_{n=1}^{\infty} |n\rangle \langle n|, \quad (28)$$

where again $|n\rangle$ is the multimode Fock state. Considering the same parameter space just mentioned, we find on/off detectors have a small impact (<5% reduction) on F_d for both NLAs.

III. ENTANGLEMENT DISTILLATION

In this section we study the use of NLAs in the context of entanglement distillation over lossy channels. We first briefly review the type-II parametric down-conversion (PDC) process, which is commonly used to generate entangled states.

In reality, the PDC process does not generate a single Einstein-Podolsky-Rosen (EPR) state with two entangled

single modes, but rather an ensemble of orthogonal EPR states each consisting of two entangled supermodes (the orthogonalized Gram-Schmidt modes [14]). In the PDC process, a pump laser is first fed into a nonlinear crystal. Two correlated beams, labeled **A** and **B**, are then created. Let \hat{A}_k^\dagger and \hat{B}_k^\dagger be the creation operators of the supermodes in beams **A** and **B**, respectively, where we use the subscript $k \in \{1, 2, \dots, \infty\}$ to index the supermodes (or equivalently, the EPR states). The supermode operators satisfy the commutation relations $[\hat{A}_k, \hat{A}_{k'}^\dagger] = [\hat{B}_k, \hat{B}_{k'}^\dagger] = \delta_{k,k'}$. The output state of the PDC process, which we refer to as a PDC state, can be written as [1]

$$\begin{aligned} |\text{PDC}\rangle_{AB} &= \bigotimes_{k=1}^{\infty} \exp[G\lambda_k(\hat{A}_k^\dagger\hat{B}_k^\dagger - \hat{A}_k\hat{B}_k)] |0\rangle \\ &= \bigotimes_{k=1}^{\infty} |\text{EPR}_k\rangle, \end{aligned} \quad (29)$$

where G is the overall gain of the PDC process, the λ_k 's are normalized real-valued coefficients, and $r_k = G\lambda_k$ is the squeezing parameter for the k th EPR state.

Consider a scenario where one beam of a PDC state, say, beam **A**, is to be distributed over a pure photon-loss channel while beam **B** is kept at the transmitter. We assume the channel is frequency independent such that the supermode structure of the distributed beam is retained after passing through the channel. The channel can then be modeled by a beam splitter with transmissivity η_t . The channel attenuation in decibels is given by $-10 \log_{10} \eta_t$. We assume the number of orthogonal EPR states of the PDC state is $K = 5$.⁴

For the supermode structure of the PDC state we consider three scenarios. For the first scenario, the PDC state only contains one nontrivial EPR state, i.e., $r_k \approx 0$, $\forall k \neq 1$. For the second scenario, the PDC state contains five EPR states with squeezing parameters (r_1 to r_5) following an exponentially decaying distribution. For the third scenario, the PDC state contains five EPR states with the same amount of squeezing.

We assume an NLA (with efficient single-photon detectors) is applied to the first supermode ($k = 1$) of the received beam. We first consider an amplification strategy where the supermode to be amplified is not filtered out (spatially separated from the other supermodes) before the amplification. This strategy is commonly adopted for various multimode non-Gaussian operations, such as photon subtraction and photon addition [39–43]. We focus on this strategy also because in experiments the supermodes cannot be easily separated [44].

We use the QS-NLA as an example to explain how the NLA works. The received beam is fed into the QS-NLA and divided into N beams by the N splitter. Parallel QS operations are then applied to all N beams. In each QS operation, the multimode single-photon detector, which implements a joint detection on the frequency bins,⁵ clicks if and only if a photon

with the same multimode structure as the first supermode of the beam is detected. This also means no photon in the rest of the supermodes is detected. In this case, the transformation of the QS on the rest of the supermodes can be represented by $\langle 0| \langle 0| \hat{U}(1/2)\hat{U}(T) |0\rangle |0\rangle = |0\rangle \langle 0|$. The transformation of the QS-NLA on these supermodes is also $|0\rangle \langle 0|$ (i.e., a truncation to the vacuum state). The transformation by the QS-NLA on the first supermode is \hat{M}_s as given by Eq. (23).

Similarly, it can be shown that when a PC-NLA [\hat{M}_c as given by Eq. (24)] is applied to the first supermode the rest of the supermodes will be attenuated. The attenuation can be described by an operator $\sqrt{T}^{\hat{A}^\dagger\hat{A}+N}$.

Let $\hat{\rho}_{\text{tot}} = \bigotimes_{k=1}^5 \hat{\rho}_k$ be the density operator for the PDC state after the distribution and amplification, where $\hat{\rho}_k$ is the density operator for the k th distributed (and possibly amplified) EPR state. We compare the QS-NLA and the PC-NLA in terms of the maximal logarithmic negativity of $\hat{\rho}_{\text{tot}}$, which is defined as

$$E_{\text{in}}^{\text{tot}} = \max_{0 < T \leq 1} \left\{ \sum_{k=1}^5 E_{\text{in}}[\hat{\rho}_k] \right\}, \quad (30)$$

where the logarithmic negativity for each $\hat{\rho}_k$ is defined as [47]

$$E_{\text{in}}[\hat{\rho}_k] = \log_2[1 + 2\epsilon(\hat{\rho}_k)], \quad (31)$$

where $\epsilon(\hat{\rho}_k)$ stands for the absolute value of the sum of negative eigenvalues of the partially transposed $\hat{\rho}_k$. At each channel attenuation level, the maximization in $E_{\text{in}}^{\text{tot}}$ is performed on the transmissivity T of the beam splitters in the NLAs. We note in both NLAs, T , which is the only adjustable parameter, determines the amplification gain.

The results are illustrated in Fig. 5. For the first scenario (only one nontrivial EPR state), at fixed N , the PC-NLA provides larger $E_{\text{in}}^{\text{tot}}$ than the QS-NLA when the channel attenuation is below certain thresholds (i.e., less photon loss). The two NLAs show similar performances when the attenuation is above these thresholds. Both NLAs can achieve a certain level of $E_{\text{in}}^{\text{tot}}$, independent of the channel attenuation level. The truncation effect of the QS-NLA degrades $E_{\text{in}}^{\text{tot}}$ significantly when the initial squeezing of the EPR state is large. For the second and third scenarios (more than one nontrivial EPR state), the increase of E_{in} from the amplified EPR state is negated by the decrease of E_{in} from the other EPR states, making the thresholds above which the NLAs can enhance $E_{\text{in}}^{\text{tot}}$ higher than the first scenario.

For the amplification strategy where the supermode to be amplified is first filtered out before the amplification, the rest of the supermodes remain unchanged after the amplification process. Under this strategy the performances of each NLA will be less dependent on the supermode structure of the PDC states.

In summary, the NLAs can only enhance the entanglement when the channel attenuation is above some thresholds (i.e., more photon loss). These thresholds depend on the supermode

⁴In experiments K can be considered as the number of frequency-resolved bins for the specific detector used.

⁵The measurement of a supermode can be realized by a multipixel detection method (e.g., [5,14,45,46]), in which the supermode is fanned out by a diffraction grating onto homodyne detectors, each having a resolution of one frequency bin (i.e., one pixel). The joint

single-photon detection on the frequency bins can be implemented by a similar method, in which the homodyne detectors are replaced with mode-nonselective single-photon detectors.

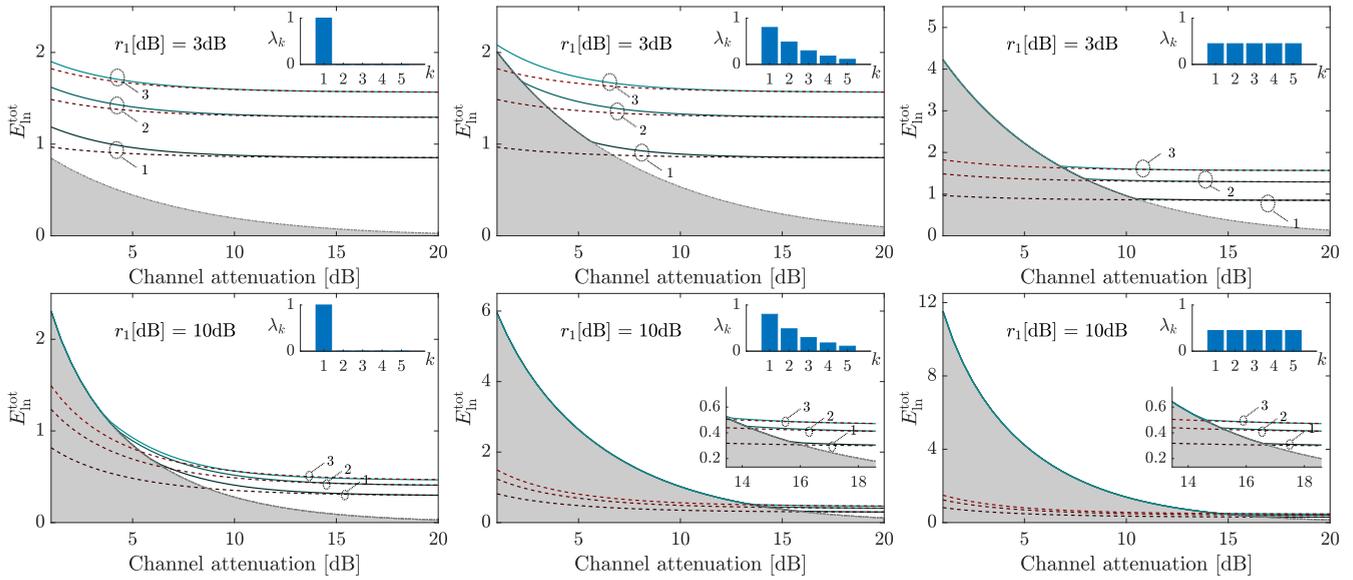


FIG. 5. The maximized logarithmic negativity of the amplified states vs the channel attenuation. The dashed curves correspond to the QS-NLA and the solid curves correspond to the PC-NLA. The value of N is indicated by the number connected to the curves. The edge of the gray area (also marked as a dash-dotted gray curve) represents the reference case without NLAs. The squeezing in decibels is calculated by $r_1[\text{dB}] \approx 8.67r_1 = 8.67G\lambda_1$. The insets illustrate the supermode structure of the PDC state.

structure, the squeezing levels, and the number of amplifying units (QS or PC) N in the NLAs. In general, the thresholds increase as the entanglement of the initial PDC states increases.

IV. COMPARISON OF THE PC-NLA AND THE CASCADED PROCESSING OF PC

In this section, we compare the performance of our PC-NLA (the parallel processing of PC) and the cascaded processing of PC proposed in [21] in the context of entanglement distillation of EPR states. To better study the impact of the two processes on EPR states, we restrict ourselves to the scenario where the PDC state only contains one EPR state. We note our conclusions in this section will also apply to the scenario where the PDC state contains more than one EPR state.

We first generalize the cascaded PC to the multimode setting so as to compare the two processes. Recall that the PC operation, which we repeat here for completeness, can be represented by

$$\hat{R} = \sqrt{T} \left(-\frac{1-T}{T} \hat{A}^\dagger \hat{A} + 1 \right) \sqrt{T}^{\hat{A}^\dagger \hat{A}}. \quad (32)$$

The cascaded PC, which contains N repetitions of the same PC operation, can then be represented by

$$\hat{R}^N = \sqrt{T}^N \left(-\frac{1-T}{T} \hat{A}^\dagger \hat{A} + 1 \right)^N \sqrt{T}^{N\hat{A}^\dagger \hat{A}}, \quad (33)$$

where we have used the fact that $\sqrt{T}^{\hat{A}^\dagger \hat{A}}$ and $\hat{A}^\dagger \hat{A}$ commute.

In Figs. 6 and 7 we compare the logarithmic negativity for the two processes in the absence of channel losses. For the two processes the beam splitter transmissivity T is optimized individually so as to maximize E_{in} . In Fig. 6 the maximal logarithmic negativity, $E_{\text{in}}^{\text{max}}$, is plotted against the initial squeezing r of the EPR state. In Fig. 7 we plot $E_{\text{in}}^{\text{max}}$ against the num-

ber of PC operations in each process. From the figures we can see the parallel PC can achieve much higher $E_{\text{in}}^{\text{max}}$ than the cascaded PC. The difference between $E_{\text{in}}^{\text{max}}$ for the two processes becomes more significant as N grows. However, there is a trade-off between the success probabilities and the achievable $E_{\text{in}}^{\text{max}}$. The cascaded PC can be orders of magnitude

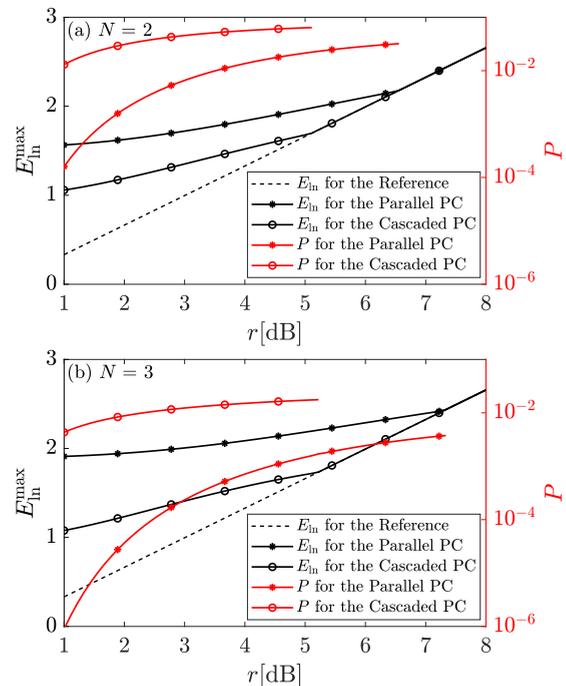


FIG. 6. The maximal logarithmic negativity $E_{\text{in}}^{\text{max}}$ (black curves) and the success probability P (red curves) against r [dB] for (a) $N = 2$ and (b) $N = 3$. The black dashed curve represents the reference case without any PC operations.

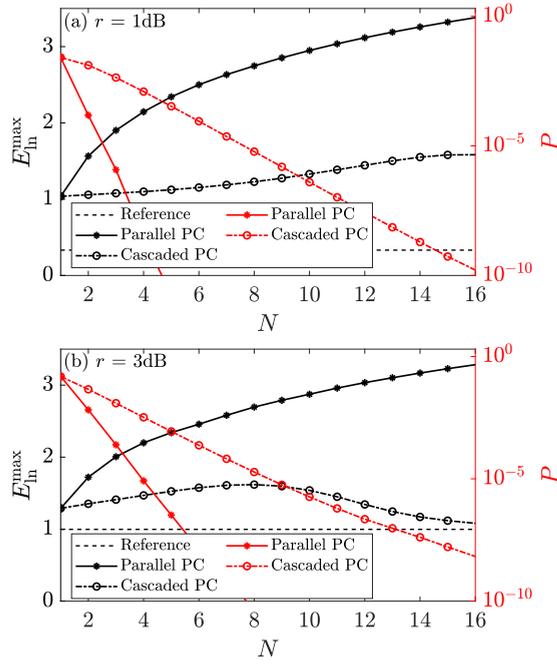


FIG. 7. The maximal logarithmic negativity $E_{\text{Ln}}^{\text{max}}$ (black curves) and the success probability P (red curves) against N for (a) $r = 1$ dB and (b) $r = 3$ dB. The black dashed curve represents the reference case without any PC operations.

more successful than the parallel PC in terms of probability at the price of a drop in $E_{\text{Ln}}^{\text{max}}$. The success probabilities for both processes are unity when $T = 1$.

We attempt to better explain why for the cascaded PC the entanglement increases insignificantly (and even decreases) as N grows. From Eq. (32) it can be observed that the operator for PC can be factorized into $\sqrt{T}(-\frac{1-T}{T}\hat{A}^\dagger\hat{A} + 1)$ and $\sqrt{T}^{\hat{A}^\dagger\hat{A}}$. The latter factor is the operator for a noiseless linear attenuator, which only decreases the entanglement of an EPR state [48]. When multiple cascaded PC operations are applied, the attenuation factor scales as $\sqrt{T}^{N\hat{A}^\dagger\hat{A}}$ [the rightmost factor in Eq. (33)]. The detrimental impact of the attenuation factor will eventually negate the entanglement. For the parallel PC the attenuation factor is independent of N . Therefore, the entanglement grows steadily as N grows.

V. CONCLUSIONS

In this work, we proposed an NLA that uses parallel processing of PC, the PC-NLA. We also constructed a multimode version of an existing NLA that uses QS, namely, the QS-NLA. We showed that when applied to a range of coherent states, the PC-NLA, which can be built with much simpler linear optics, is compatible with the QS-NLA. In the context of entanglement distillation of PDC states, we found that both NLAs can enhance the entanglement when the channel attenuation is above certain thresholds. Distinct from the single-mode NLA analysis, these thresholds largely depend on the supermode structure of the PDC states. An interesting finding is that the two NLAs can maintain certain levels of entanglement, independent of the channel attenuation level.

We also compare the PC-NLA with the cascaded processing of PC, showing that the PC-NLA can distill more entanglement, albeit at lower success probabilities. Our results will be important for next-generation real-world implementations of multipartite quantum information applications that utilize broadband pulses of lights.

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APPENDIX A: THE DERIVATION FOR EQ. (11)

In this section, for conciseness we omit the subscripts that label the spatial modes (a, b, c , etc.) but keep the subscripts that index the single modes ($m \in \{1, 2, \dots, \infty\}$). We can show

$$\begin{aligned}
 \hat{M} &= \langle 0|_C \langle 1|_A \hat{U}_{AC}(T_1) \hat{U}_{BC}(T_2) |0\rangle_B |1\rangle_C \\
 &= \sum_{m'=1}^{\infty} \gamma_{m'}^* \gamma_{m'} \hat{M}_{m'}^{(11)} \bigotimes_{m=1, m \neq m'}^{\infty} \hat{M}_m^{(00)} \\
 &\quad + \sum_{m'=1, m''=1, m' \neq m''}^{\infty} \gamma_{m'}^* \gamma_{m''} [\hat{M}_{m'}^{(10)} \otimes \hat{M}_{m''}^{(01)}] \\
 &\quad \times \bigotimes_{m=1, m \neq m', m \neq m''}^{\infty} \hat{M}_m^{(00)},
 \end{aligned} \tag{A1}$$

where $(\cdot)^*$ stands for the complex conjugate, and

$$\hat{M}_m^{(n_1 n_2)} := \langle 0|_m \langle n_1|_m \hat{U}_m(T_1) \hat{U}_m(T_2) |0\rangle_m |n_2\rangle_m, \tag{A2}$$

for $n_1, n_2 \in \{0, 1\}$. After some algebraic manipulations, we find

$$\begin{aligned}
 \hat{M}_m^{(11)} &= \sqrt{(1-T_1)T_2} |0\rangle_m \langle 0|_m + \sqrt{T_1(1-T_2)} |1\rangle_m \langle 1|_m, \\
 \hat{M}_m^{(00)} &= |0\rangle_m \langle 0|_m, \\
 \hat{M}_m^{(10)} &= \sqrt{T_1} |0\rangle_m \langle 1|_m, \\
 \hat{M}_m^{(01)} &= \sqrt{1-T_2} |1\rangle_m \langle 0|_m.
 \end{aligned} \tag{A3}$$

Substituting Eq. (A3) into Eq. (A1) it follows that

$$\begin{aligned}
 \hat{M} &= \sqrt{(1-T_1)T_2} \bigotimes_{m=1}^{\infty} |0\rangle_m \langle 0|_m \\
 &\quad + \sum_{m'=1}^{\infty} \gamma_{m'}^* \gamma_{m'} \sqrt{T_1(1-T_2)} \bigotimes_{m=1}^{\infty} |\delta_{m,m'}\rangle_m \langle \delta_{m,m'}|_m \\
 &\quad + \sum_{m'=1, m''=1, m' \neq m''}^{\infty} \gamma_{m'}^* \gamma_{m''} \sqrt{T_1(1-T_2)} \\
 &\quad \times \bigotimes_{m=1}^{\infty} |\delta_{m,m'}\rangle_m \langle \delta_{m,m'}|_m \\
 &= \sqrt{(1-T_1)T_2} |0\rangle_B \langle 0|_A + \sqrt{T_1(1-T_2)} |1\rangle_B \langle 1|_A.
 \end{aligned} \tag{A4}$$

Setting $T_1 = 1/2$ we arrive at Eq. (11). Similarly, we can also show that

$$\begin{aligned} \hat{M}' &= \langle 1|_C \langle 0|_A \hat{U}_{AC}(T_1) \hat{U}_{BC}(T_2) |0\rangle_B |1\rangle_C \\ &= \sqrt{T_1 T_2} |0\rangle_B \langle 0|_A - \sqrt{(1-T_1)(1-T_2)} |1\rangle_B \langle 1|_A. \end{aligned} \quad (\text{A5})$$

APPENDIX B: THE DERIVATION FOR EQ. (24)

In this section, for conciseness we omit the subscripts for the supermodes. We consider an N splitter that implements the following transformation:

$$[\hat{A}_1, \hat{A}_2, \dots, \hat{A}_N]_{\text{out}}^T = U[\hat{A}_1, \hat{A}_2, \dots, \hat{A}_N]_{\text{in}}^T, \quad (\text{B1})$$

where U is an $N \times N$ unitary matrix whose entries satisfy $u_{1j} = u_{i1} = 1/\sqrt{N}$, $\forall i, j \in \{1, 2, \dots, N\}$, and \hat{A}_1 to \hat{A}_N are the annihilation operators of the supermodes in the N paths. For an input multimode Fock state $|n\rangle$, the ensemble state after the first N splitter can be written as

$$\begin{aligned} |\psi'\rangle &= \frac{N^{-\frac{n}{2}} (\hat{A}_1^\dagger + \hat{A}_2^\dagger + \dots + \hat{A}_N^\dagger)^n}{\sqrt{n!}} |0\rangle \\ &= \frac{N^{-\frac{n}{2}}}{\sqrt{n!}} \sum_{n_1+n_2+\dots+n_N=n} \binom{n}{n_1, n_2, \dots, n_N} \\ &\quad \times \prod_{i=1}^N \hat{A}_i^{\dagger n_i} |0\rangle, \end{aligned} \quad (\text{B2})$$

where the sum is taken over all combinations of nonnegative integer indices n_1 through n_N such that the sum of all n_i is n , and

$$\binom{n}{n_1, n_2, \dots, n_N} = \frac{n!}{n_1! n_2! \dots n_N!}. \quad (\text{B3})$$

The state after the PC operations can be written as

$$\begin{aligned} |\psi''\rangle &= \frac{N^{-\frac{n}{2}}}{\sqrt{n!}} \sum_{n_1+n_2+\dots+n_N=n} \binom{n}{n_1, n_2, \dots, n_N} \\ &\quad \times \prod_{i=1}^N \sqrt{T} q_n \hat{A}_i^{\dagger n_i} |0\rangle, \end{aligned} \quad (\text{B4})$$

where

$$q_{n_i} = \left(-\frac{1-T}{T} n_i + 1 \right) \sqrt{T}^{n_i}. \quad (\text{B5})$$

The output state, which is the postselected state after the second N splitter, can be written as

$$|\psi\rangle_{\text{out}} = \sqrt{T}^N \frac{c}{N^n} |n\rangle, \quad (\text{B6})$$

where

$$c = \sum_{n_1+n_2+\dots+n_N=n} \binom{n}{n_1, n_2, \dots, n_N} \prod_{i=1}^N q_{n_i}. \quad (\text{B7})$$

After some algebraic manipulations, we find

$$c = \sqrt{T}^n \sum_{k=0}^N \binom{N}{k} \frac{n!}{(n-N+k)!} N^{n-N+k} \left(\frac{T-1}{T} \right)^{N-k}. \quad (\text{B8})$$

Putting Eq. (B8) into Eq. (B6), we arrive at Eq. (24).

APPENDIX C: QS AND PC WITH INEFFICIENT SINGLE-PHOTON DETECTORS AND ON/OFF DETECTORS

With inefficient single-photon detectors, the transformation for a QS operation can be written as

$$\hat{\rho}_{\text{out}} = \frac{1}{P_{\eta, \text{QS}}} \sum_{n=1, n'=0}^{\infty} n\eta(1-\eta)^{n'+n-1} \hat{M}_{n,n'} \hat{\rho}_{\text{in}} \hat{M}_{n,n'}^\dagger, \quad (\text{C1})$$

where $P_{\eta, \text{QS}}$ is the success probability, η is the detection efficiency, $\hat{\rho}_{\text{in}}$ and $\hat{\rho}_{\text{out}}$ are the density operators for the input and output states, respectively, and

$$\begin{aligned} \hat{M}_{n,n'} &= (-1)^{n'} 2^{-\frac{n+n'-1}{2}} (n-n') \sqrt{\frac{(n+n'-1)!}{n!n'}} \\ &\quad \times \sqrt{T} |0\rangle_B \langle n+n'-1|_A \\ &\quad + (-1)^{n'} 2^{-\frac{n+n'-1}{2}} \sqrt{\frac{(n+n')!}{n!n'}} \sqrt{1-T} |1\rangle_B \langle n+n'|_A. \end{aligned} \quad (\text{C2})$$

For a PC operation the corresponding transformation can be written as

$$\hat{\rho}_{\text{out}} = \frac{1}{P_{\eta, \text{PC}}} \sum_{n=1}^{\infty} [n\eta(1-\eta)^{n-1}] \hat{R}_n \hat{\rho}_{\text{in}} \hat{R}_n^\dagger, \quad (\text{C3})$$

where $P_{\eta, \text{PC}}$ is the success probability, and

$$\hat{R}_n = \sqrt{T} \frac{(-1)^n}{\sqrt{n!}} \sum_{l=0}^{\min(1, n)} (-n)^l (t\hat{A}^\dagger \hat{A})^{1-l} (\sqrt{t}\hat{A})^{n-1} \sqrt{T}^{\hat{A}^\dagger \hat{A}}, \quad (\text{C4})$$

where $t = \frac{1-T}{T}$.

With on/off detectors, the transformation for a QS operation can be written as

$$\hat{\rho}_{\text{out}} = \frac{1}{P_{\text{on/off, QS}}} \sum_{n=1}^{\infty} \hat{M}_{n,0} \hat{\rho}_{\text{in}} \hat{M}_{n,0}^\dagger, \quad (\text{C5})$$

where $P_{\text{on/off, QS}}$ is the success probability, and

$$\hat{M}_{n,0} = 2^{-\frac{n-1}{2}} (\sqrt{n}\sqrt{T} |0\rangle_B \langle n-1|_A + \sqrt{1-T} |1\rangle_B \langle n|_A). \quad (\text{C6})$$

With on/off detectors, the transformation for a PC operation can be written as

$$\hat{\rho}_{\text{out}} = \frac{1}{P_{\text{on/off, PC}}} \sum_{n=1}^{\infty} \hat{R}_n \hat{\rho}_{\text{in}} \hat{R}_n^\dagger, \quad (\text{C7})$$

where $P_{\text{on/off, PC}}$ is the success probability, and \hat{R}_n is the operator given by Eq. (C4).

- [1] A. Christ, C. Lupo, and C. Silberhorn, Exponentially enhanced quantum communication rate by multiplexing continuous-variable teleportation, *New J. Phys.* **14**, 083007 (2012).
- [2] N. Hosseinidehaj and R. Malaney, Multimode entangled states in the lossy channel, in *2017 IEEE 85th Vehicular Technology Conference (IEEE, Piscataway, NJ, 2017)*, pp. 1–5.
- [3] R. Kumar, X. Tang, A. Wonfor, R. Penty, and I. White, Continuous variable quantum key distribution with multi-mode signals for noisy detectors, *J. Opt. Soc. Am. B: Opt. Phys.* **36**, B109 (2019).
- [4] N. C. Menicucci, S. T. Flammia, and O. Pfister, One-Way Quantum Computing in the Optical Frequency Comb, *Phys. Rev. Lett.* **101**, 130501 (2008).
- [5] S. Armstrong, J.-F. Morizur, J. Janousek, B. Hage, N. Treps, P. K. Lam, and H.-A. Bachor, Programmable multimode quantum networks, *Nat. Commun.* **3**, 1 (2012).
- [6] G. Ferrini, J.-P. Gazeau, T. Coudreau, C. Fabre, and N. Treps, Compact Gaussian quantum computation by multi-pixel homodyne detection, *New J. Phys.* **15**, 093015 (2013).
- [7] M. Chen, N. C. Menicucci, and O. Pfister, Experimental Realization of Multipartite Entanglement of 60 Modes of a Quantum Optical Frequency Comb, *Phys. Rev. Lett.* **112**, 120505 (2014).
- [8] Q. Zhuang, Z. Zhang, and J. H. Shapiro, Distributed quantum sensing using continuous-variable multipartite entanglement, *Phys. Rev. A* **97**, 032329 (2018).
- [9] Y. Xia, Q. Zhuang, W. Clark, and Z. Zhang, Repeater-enhanced distributed quantum sensing based on continuous-variable multipartite entanglement, *Phys. Rev. A* **99**, 012328 (2019).
- [10] X. Guo, C. R. Breum, J. Borregaard, S. Izumi, M. V. Larsen, T. Gehring, M. Christandl, J. S. Neergaard-Nielsen, and U. L. Andersen, Distributed quantum sensing in a continuous-variable entangled network, *Nat. Phys.* **16**, 281 (2020).
- [11] M. Gessner, A. Smerzi, and L. Pezzè, Multiparameter squeezing for optimal quantum enhancements in sensor networks, *Nat. Commun.* **11**, 1 (2020).
- [12] Y. Cai, J. Roslund, G. Ferrini, F. Arzani, X. Xu, C. Fabre, and N. Treps, Multimode entanglement in reconfigurable graph states using optical frequency combs, *Nat. Commun.* **8**, 1 (2017).
- [13] T. Fortier and E. Baumann, 20 years of developments in optical frequency comb technology and applications, *Commun. Phys.* **2**, 1 (2019).
- [14] R. Medeiros de Araújo, J. Roslund, Y. Cai, G. Ferrini, C. Fabre, and N. Treps, Full characterization of a highly multimode entangled state embedded in an optical frequency comb using pulse shaping, *Phys. Rev. A* **89**, 053828 (2014).
- [15] J. Roslund, R. M. De Araujo, S. Jiang, C. Fabre, and N. Treps, Wavelength-multiplexed quantum networks with ultrafast frequency combs, *Nat. Photonics* **8**, 109 (2014).
- [16] S. Gerke, J. Sperling, W. Vogel, Y. Cai, J. Roslund, N. Treps, and C. Fabre, Full Multipartite Entanglement of Frequency-Comb Gaussian States, *Phys. Rev. Lett.* **114**, 050501 (2015).
- [17] C. M. Caves, Quantum limits on noise in linear amplifiers, *Phys. Rev. D* **26**, 1817 (1982).
- [18] T. C. Ralph and A. Lund, Nondeterministic noiseless linear amplification of quantum systems, in *9th International Conference on Quantum Communication, Measurement and Computing*, AIP Conf. Proc. No. 1110 (AIP, New York, 2009), pp. 155–160.
- [19] S. Zhang and X. Zhang, Photon catalysis acting as noiseless linear amplification and its application in coherence enhancement, *Phys. Rev. A* **97**, 043830 (2018).
- [20] D. T. Pegg, L. S. Phillips, and S. M. Barnett, Optical State Truncation by Projection Synthesis, *Phys. Rev. Lett.* **81**, 1604 (1998).
- [21] Y. Mardani, A. Shafiei, M. Ghadimi, and M. Abdi, Continuous-variable entanglement distillation by cascaded photon replacement, *Phys. Rev. A* **102**, 012407 (2020).
- [22] F. Ferreyrol, M. Barbieri, R. Blandino, S. Fossier, R. Tualle-Brouri, and P. Grangier, Implementation of a Nondeterministic Optical Noiseless Amplifier, *Phys. Rev. Lett.* **104**, 123603 (2010).
- [23] A. Zavatta, J. Fiurášek, and M. Bellini, A high-fidelity noiseless amplifier for quantum light states, *Nat. Photonics* **5**, 52 (2011).
- [24] H. M. Chrzanowski, N. Walk, S. M. Assad, J. Janousek, S. Hosseini, T. C. Ralph, T. Symul, and P. K. Lam, Measurement-based noiseless linear amplification for quantum communication, *Nat. Photonics* **8**, 333 (2014).
- [25] C. N. Gagatsos, J. Fiurášek, A. Zavatta, M. Bellini, and N. J. Cerf, Heralded noiseless amplification and attenuation of non-Gaussian states of light, *Phys. Rev. A* **89**, 062311 (2014).
- [26] A. E. Ulanov, I. A. Fedorov, A. A. Pushkina, Y. V. Kurochkin, T. C. Ralph, and A. Lvovsky, Undoing the effect of loss on quantum entanglement, *Nat. Photonics* **9**, 764 (2015).
- [27] J. Y. Haw, J. Zhao, J. Dias, S. M. Assad, M. Bradshaw, R. Blandino, T. Symul, T. C. Ralph, and P. K. Lam, Surpassing the no-cloning limit with a heralded hybrid linear amplifier for coherent states, *Nat. Commun.* **7**, 1 (2016).
- [28] M. Mičuda, I. Straka, M. Miková, M. Dušek, N. J. Cerf, J. Fiurášek, and M. Ježek, Noiseless Loss Suppression in Quantum Optical Communication, *Phys. Rev. Lett.* **109**, 180503 (2012).
- [29] H.-J. Kim, S.-Y. Lee, S.-W. Ji, and H. Nha, Quantum linear amplifier enhanced by photon subtraction and addition, *Phys. Rev. A* **85**, 013839 (2012).
- [30] S. Yang, S. L. Zhang, X. B. Zou, S. W. Bi, and X. L. Lin, Improving noiseless linear amplification for optical quantum communication with quadrature squeezing, *Phys. Rev. A* **87**, 024302 (2013).
- [31] J. Zhao, J. Y. Haw, T. Symul, P. K. Lam, and S. M. Assad, Characterization of a measurement-based noiseless linear amplifier and its applications, *Phys. Rev. A* **96**, 012319 (2017).
- [32] L. Hu, M. Al-amri, Z. Liao, and M. S. Zubairy, Entanglement improvement via a quantum scissor in a realistic environment, *Phys. Rev. A* **100**, 052322 (2019).
- [33] M. Winnel, N. Hosseinidehaj, and T. C. Ralph, Generalized quantum scissors for noiseless linear amplification, *Phys. Rev. A* **102**, 063715 (2020).
- [34] M. He, R. Malaney, and J. Green, Global entanglement distribution with multi-mode non-Gaussian operations, *IEEE J. Sel. Areas Commun.* **38**, 528 (2020).
- [35] J. Dias and T. C. Ralph, Quantum repeaters using continuous-variable teleportation, *Phys. Rev. A* **95**, 022312 (2017).
- [36] J. Eisert, D. E. Browne, S. Scheel, and M. Plenio, Distillation of continuous-variable entanglement with optical means, *Ann. Phys.* **311**, 431 (2004).
- [37] F. Marsili, V. B. Verma, J. A. Stern, S. Harrington, A. E. Lita, T. Gerrits, I. Vayshenker, B. Baek, M. D. Shaw, R. P. Mirin *et al.*, Detecting single infrared photons with 93% system efficiency, *Nat. Photonics* **7**, 210 (2013).
- [38] A. Kitagawa, M. Takeoka, M. Sasaki, and A. Chefles, Entanglement evaluation of non-Gaussian states generated by photon

- subtraction from squeezed states, *Phys. Rev. A* **73**, 042310 (2006).
- [39] V. A. Averchenko, V. Thiel, and N. Treps, Nonlinear photon subtraction from a multimode quantum field, *Phys. Rev. A* **89**, 063808 (2014).
- [40] V. Averchenko, C. Jacquard, V. Thiel, C. Fabre, and N. Treps, Multimode theory of single-photon subtraction, *New J. Phys.* **18**, 083042 (2016).
- [41] M. Walschaers, C. Fabre, V. Parigi, and N. Treps, Statistical signatures of multimode single-photon-added and -subtracted states of light, *Phys. Rev. A* **96**, 053835 (2017).
- [42] M. Walschaers, Y.-S. Ra, and N. Treps, Mode-dependent-loss model for multimode photon-subtracted states, *Phys. Rev. A* **100**, 023828 (2019).
- [43] Y.-S. Ra, A. Dufour, M. Walschaers, C. Jacquard, T. Michel, C. Fabre, and N. Treps, Non-Gaussian quantum states of a multimode light field, *Nat. Phys.* **16**, 144 (2020).
- [44] N. Huo, Y. Liu, J. Li, L. Cui, X. Chen, R. Palivela, T. Xie, X. Li, and Z. Y. Ou, Direct Temporal Mode Measurement for the Characterization of Temporally Multiplexed High Dimensional Quantum Entanglement in Continuous Variables, *Phys. Rev. Lett.* **124**, 213603 (2020).
- [45] W. N. Plick, F. Arzani, N. Treps, E. Diamanti, and D. Markham, Violating Bell inequalities with entangled optical frequency combs and multipixel homodyne detection, *Phys. Rev. A* **98**, 062101 (2018).
- [46] Y. Cai, J. Roslund, V. Thiel, C. Fabre, and N. Treps, Quantum enhanced measurement of an optical frequency comb, [arXiv:2003.05833](https://arxiv.org/abs/2003.05833).
- [47] G. Vidal and R. F. Werner, Computable measure of entanglement, *Phys. Rev. A* **65**, 032314 (2002).
- [48] S. Yang, S. L. Zhang, X. B. Zou, S. W. Bi, and X. L. Lin, Continuous-variable entanglement distillation with noiseless linear amplification, *Phys. Rev. A* **86**, 062321 (2012).