Quantum coherence measures based on Fisher information with applications

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Recently various ways to quantify coherence of a quantum state have been proposed. In this paper, several reliable quantum coherence measures based on quantum Fisher information (QFI) are presented. For pure states, this coherence measure is equivalent to that based on skew information, while for mixed states, we first define two natural coherence measures via QFI and find that the two definitions are exactly identical. By virtual of the special structure of qubit, we show that the mutual unbiased bases' (MUBs) qubit coherence is constrained only by the length of the Bloch vector. Then we provide upper bounds of the coherence measure with respect to any given basis and lower bounds in the case of a set of MUBs for both pure states and mixed states of arbitrary dimension. Finally, we investigate the connection between our coherence measures and quantum metrology, thereby providing an operational meaning for these coherence measures.

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I. INTRODUCTION

Quantum coherence is one of the most fundamental features in quantum system that is not present in classical world. As an important quantum resource [1], it plays key roles in a variety of applications ranging from reference frames [2–4], quantum thermodynamics [5–9], to biological systems [10,11].

In the standard frameworks of resource theory of coherence [1], the free states or incoherent states are the states that are diagonal in a fixed basis of a d-dimensional Hilbert space H, and the free operations are incoherent operations whose Kraus operators map incoherent states to incoherent states. Therefore the coherence is not a property of state alone but is defined with respect to a given measurement basis. It also has been suggested that a suitable measure of coherence should satisfy several criteria (see details in Sec. II). Based on these frameworks, several coherence measures have been proposed to meet the above requirements [1,12–19]. In particular, Girolami [20] introduced the K-coherence measure, an intuitive quantifier for coherence, based on Wigner-Yanase skew information [21]. However, this quantifier does not satisfy one of the key features for any reasonable coherence measure, that is, it may increase under the selective incoherent operations. Actually, it is indeed a measure of asymmetry rather than a measure of coherence [22]. Fortunately, this drawback is soon remedied via simply replacing the observable by the corresponding spectral decomposition [23,24].

The quantum Fisher information (QFI) is the cornerstone of modern quantum metrology. It gives the ultimate precision bound on the estimation of a parameter encoded in a quantum state known as the Cramér-Rao bound [25]. The QFI has been also used in the description of criticality and quantum phase transitions [26], estimation of the speed limits for quantum computation [27], and detection of the entanglement in a composite system [28]. It is well known that both Wigner-Yanase skew information and QFI are natural generalizations of the classical Fisher information. Therefore, a similar problem will arise if one quantifies coherence directly by QFI itself [29]. Most recently, in the context of quantum thermodynamics, coherence cost and distillable coherence, which determine the rate of conversion of coherence in a standard pure state to general mixed states, and *vice versa*, are studied in Ref. [30]. Those authors find that, surprisingly, coherence cost of any state is determined by its QFI. Hence, a novel operational interpretation of this central quantity from the perspective of quantum metrology is revealed [30]. In this paper, we further investigate the QFI and propose several legitimate coherence measures based on it. For pure states, in view of the equivalence of QFI and Wigner-Yanase skew information, we define the coherence measure as in Refs. [23,24]. We give the upper bound of the coherence measure with respect to any given basis, while for mixed states, we propose two natural coherence measures based on QFI and find that the two definitions are exactly identical, as the quantum Fisher information can be written as the convex roof of the variance [31,32]. This finding ensures that our two coherence measures for mixed states are *bona fide*.

Another question is whether there exist complementary relations for different sets of basis. In Ref. [33] the authors examined this question for the case of mutually unbiased bases (MUBs), with respect to two well-known coherence measures based on l_1 -norm and relative entropy [1]. They found that, for the l_1-norm coherence measure, there exists an exact tradeoff relation for the qubit and a more general bound determined by quantum and classical purity, while for the relative entropy coherence measure, they obtained a nontrivially upper bound [33]. For our coherence measures, using the well-known equality [34] and inequality [35]

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regarding the index of coincidence, we derive lower bounds of the MUBs' coherence, which means that there is a tradeoff within different sets of basis for our coherence measures. Finally, in view of the fundamental role which QFI plays in the quantum metrology, we find that our coherence measures can serve as the direct bound on the uncertainty of the estimated phase. In particular, this bound is tighter than the result in Ref. [23], which in turn imposes our coherence measures an operational meaning.

II. PRELIMINARIES

A. Framework of resource theory of coherence

We first review some elementary concepts about coherence measures [1]. Given a fixed basis $\mathcal{B} = \{|i\rangle\}$, the set of incoherent state $\mathcal{I}$ is the set of state with diagonal density matrices with respect to this basis. Let C be a measure of coherence, then $C(\rho)$ must satisfy the following: (C1) $C(\rho) \geqslant 0$ for any quantum state ρ and equality holds if and only if $\rho \in \mathcal{I}$. (C2) $C(\rho)$ is monotonic under incoherent completely positive and trace-preserving mapping. (C3) $C(\rho)$ is monotonic under selective incoherent measurements on average, and (C4) $C(\rho)$ is nonincreasing under mixing of quantum states (convexity). Note that coherence measures that satisfy conditions (C3) and (C4) imply condition (C2). The most relevant class of free operations for the theory of coherence is that of incoherent operations [1], which are characterized as the set of trace-preserving completely positive maps admitting a set of Kraus operators $\{K_n\}$ such that $\sum_n K_n^\dagger K_n = I$, and for all n and $\rho \in \mathcal{I}$, $\frac{K_n\rho K_n^\dagger}{\mathrm{tr}K_n\rho K_n^\dagger} \in \mathcal{I}$.

B. Quantum Fisher information

Quantum Fisher information [25] places the fundamental limit to the accuracy of estimating an unknown parameter, playing a paramount role in quantum metrology. Accurately, assuming that we start from ρ as initial state and A as a Hermitian operator, the state evolves to $\rho(\theta)$ under the unitary dynamics of a linear interferometer $U = \exp(-iA\theta)$. A tight bound on the precision of the phase estimation is given by the Cramér-Rao bound as $\delta^2\theta \geqslant \frac{1}{NF(\rho,A)}$, where N is the repetition and $F(\rho, A) = \mathrm{tr}[\rho_\theta L_\theta^2]$ is the quantum Fisher information with the symmetric logarithmic derivative (SLD) L_θ defined by $\partial_\theta\rho_\theta = -i[A, \rho_\theta] = \frac{1}{2}(\rho_\theta L_\theta + L_\theta\rho_\theta)$ [36,37]. The QFI is intimately related to variance defined as $V(\rho, A) = \mathrm{tr}\rho A^2 - (\mathrm{tr}\rho A)^2$. For a pure state, $4V(\rho, A) = F(\rho, A)$. Furthermore, variance and QFI are dual in the sense that [31,32]

$$V(\rho, A) = \max_{\{p_i,|\varphi_i\rangle\}} \sum p_i V(|\varphi_i\rangle, A) \tag{1}$$

and

$$F(\rho, A) = 4 \min_{\{p_i,|\varphi_i\rangle\}} \sum p_i V(|\varphi_i\rangle, A), \tag{2}$$

where the max and min are taken over all pure state ensemble decompositions. If we know the spectral decomposition

$$\rho = \sum_k \lambda_k |k\rangle\langle k|,$$

with $\{|k\rangle\}$ an orthonormal basis, then the QFI can be evaluated as [25]

$$F(\rho, A) = \sum_{kl} 2\frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l}|A_{kl}|^2, \tag{3}$$

where $A_{kl} = \langle k|A|l\rangle$.

III. COHERENCE MEASURES VIA QFI AND COMPLEMENTARY RELATIONS

In this section,we will propose several coherence measures based on QFI and then investigate their properties and the corresponding complementary relations.

A. Pure states

Given a preferred basis $\mathcal{K} = \{|k\rangle\}$ in n-dimensional Hilbert space, the quantum coherence of a pure state $|\varphi\rangle$ is defined as

$$C_F(|\varphi\rangle|\mathcal{K}) = \sum_k \frac{1}{4}F(|\varphi\rangle\langle\varphi|, |k\rangle\langle k|). \tag{4}$$

The constant $\frac{1}{4}$ is in fact a metric-adjusted skew information. It is clear that the above definition (4) is a *bona fide* coherence measure that satisfies all the criteria [23], since for a pure state, both the variance and QFI reduce to Wigner-Yanase skew information, which is another generalization of classical Fisher information. Recall that the coherence measure based on Wigner-Yanase skew is quantified by [23] $C(\rho|\mathcal{K}) = \sum_k I(\rho, |k\rangle\langle k|)$, where $I(\rho, |k\rangle\langle k|) = -\frac{1}{2}\mathrm{tr}\{[\rho^{\frac{1}{2}}, |k\rangle\langle k|]\}^2$ is the skew information subject to the projector $|k\rangle\langle k|$. Furthermore, Eq. (4) has an explicit formula that can be rewritten as

$$C_F(|\varphi\rangle|\mathcal{K}) = 1 - \sum_k |\langle\varphi|k\rangle|^4 = 1 - \sum_k p_k^2, \tag{5}$$

where $p_k = |\langle\varphi|k\rangle|^2$ denotes the probability of obtaining the kth result when measuring the pure state $|\varphi\rangle$. Note that the term $\sum_k p_k^2$ is called the classical purity [33], and in general, it is never less than $\frac{1}{n}$. Hence we have

$$C_F(|\varphi\rangle|\mathcal{K}) \leqslant 1 - \frac{1}{n}, \tag{6}$$

where the upper bound is saturated by the maximally coherence state $|\varphi\rangle = \frac{1}{\sqrt{n}}\sum_k |k\rangle$, which is uniformly constructed in the preferred basis. In particular, considering a set of MUBs, we can obtain a tight complementary tradeoff for coherence measure (4). Recall that two orthonormal basis sets A and B, for n-dimensional Hilbert space, are said to be MUBs if their overlaps are constant, i.e., if $|\langle a|b\rangle|^2 = n^{-1}$ for all a and b [38]. In general, the maximal number of MUBs in n dimensions is an open problem. For prime power n, there exists a complete set of $n + 1$ MUBs.

Proposition 1. Let $\{\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_M\}$ be a set of MUBs in n-dimensional Hilbert space. For any pure state $|\varphi\rangle$, we have

$$\sum_{t=1}^{M} C_F(|\varphi\rangle|\mathcal{K}_t) \geqslant M - 1 - \frac{M-1}{n}. \tag{7}$$

Proof. Let $p_{k_t} = \langle k_t|\varphi\rangle\langle\varphi|k_t\rangle$ denote the probability of obtaining the kth result when projecting the state onto the tth

MUB. Then a nice inequality was proved in Ref. [35] as

$$\sum_{t=1}^{M}\sum_{k=1}^{n} p_{k_t}^2 \leqslant \mathrm{tr}\rho^2 + \frac{M-1}{n}. \tag{8}$$

Noting that $\sum_{t=1}^{M} C_F(|\varphi\rangle|\mathcal{K}_t) = M - \sum_{t=1}^{M}\sum_{k=1}^{n} p_{k_t}^2$, we complete the proof. ■

Proposition 1 shows that, in spite of the trivially vanishing lower bound of Eq. (5), if we consider the sum of values of coherence with respect to a set of MUBs, there exists a tight complementary tradeoff. This is analogous to the statements that both the l_1-norm and relative entropy coherence measures have complementarities under MUBs [33]. Furthermore, if n is a prime power, we certainly have $n+1$ MUBs. In this case, the inequality (7) becomes an equality,

$$\sum_{t=1}^{M} C_F(|\varphi\rangle|\mathcal{K}_t) = n-1. \tag{9}$$

Here we have used the well-known identity, namely, the index of coincidence,

$$\sum_{t=1}^{n+1}\sum_{k=1}^{n} p_{k_t}^2 = 1 + \mathrm{tr}\rho^2, \tag{10}$$

which was obtained by Ivanovic [34].

B. Mixed states

It has been shown that QFI itself is not a valid measure of the coherence for a mixed state because it can increase under an incoherent operation [29]. Consequently, in Ref. [39] one remedied this problem by introducing an auxiliary incoherent operation before preparing the quantum states in the definition of a coherence measure with respect to QFI for pure states, and generalized it to mixed states via the convex roof construction. However, along with the approach proposed in Ref. [23] and the aforementioned coherence measure (4), while for mixed states, we can also present two natural coherence measures based on QIF without involving the specific quantum incoherent operation.

Definition 2 (Direct generalization of coherence measure via QFI for a mixed state). The quantum coherence of state ρ in preferred basis $\mathcal{K} = \{|k\rangle\}$ can be directly quantified by

$$C_F^D(\rho|\mathcal{K}) = \sum_k \frac{1}{4} F(\rho, |k\rangle\langle k|). \tag{11}$$

Definition 3 (Convex roof construction of coherence measure for a mixed state). The quantum coherence of state ρ in preferred basis $\mathcal{K} = \{|k\rangle\}$ can be quantified by convex roof construction,

$$C_F^C(\rho|\mathcal{K}) = \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_F(|\varphi_i\rangle|\mathcal{K}), \tag{12}$$

where the minimization is taken over all pure state decompositions of the state $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$.

It was shown [40] that the function used to define the pure state coherence measure (5) is real symmetric concave on the probability simplex, and consequently, Eq. (12) must be a coherence measure due to the convex roof construction. At first glance, there is a big gap between the two definitions. The former can be seen as a natural remedy of the drawback which is presented in the original coherence measure with respect to an observable [29]. The latter is the usual approach to deal with the case of mixed states if there exists a well-defined coherence measure for pure states [17,40]. It should be noted that Definition 3 is strikingly different from that in Ref. [39], in which they prepared the quantum state via a prior incoherent operation. However, we observe that by taking advantage of the extremal properties of the variance and QFI, our two definitions are indeed equivalent.

Proposition 4. For any state ρ in preferred basis $\mathcal{K} = \{|k\rangle\}$, $C_F^D(\rho|\mathcal{K}) = C_F^C(\rho|\mathcal{K})$.

Proof.

$$\begin{aligned} C_F^C(\rho|\mathcal{K}) &= \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_F(|\varphi_i\rangle|\mathcal{K}) \\ &= \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i \sum_k V(|\varphi_i\rangle, |k\rangle\langle k|) \\ &= \sum_k \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i V(|\varphi_i\rangle, |k\rangle\langle k|) \\ &= \sum_k \frac{1}{4} F(\rho, |k\rangle\langle k|) \\ &= C_F^D(\rho|\mathcal{K}). \end{aligned}$$

The fourth equality is from Eq. (2). ■

It is of great interest to note that these two definitions, though having different expressions, are essentially equivalent. First, from the viewpoint of computation, we can avoid the tedious optimization process in Definition 3 using Definition 2. In fact, $C_F^D(\rho|\mathcal{K})$ can be analytically derived by Eq. (3). Second, this statement instead gives us an alternative way to verify that the coherence measure in Definition 2 is also a legitimate coherence measure for mixed states without any prior preparation.

Theorem 5. $C_F^C(\rho|\mathcal{K})$ is a valid coherence measure.

Proof. We first show that it satisfies the condition (C1). It is obvious that $C_F^C(\rho|\mathcal{K}) \geqslant 0$. If ρ is incoherent, then it can be written as $\rho = \sum_k p_k|k\rangle\langle k|$. Thus $C_F^C(\rho|\mathcal{K}) = 0$ follows from $\sum_k p_k V(|k\rangle, |k\rangle\langle k|) = 0$. On the other hand, if ρ is not diagonal in the computational basis $\mathcal{K} = \{|k\rangle\}$, then for any decomposition $\rho = \sum_k p_k|\varphi_k\rangle\langle\varphi_k|$, there always exists at least a state $|\varphi_i\rangle$ not in $\mathcal{K}$. As a result, $C_F^C(\rho|\mathcal{K}) > 0$ follows from $V(|\varphi_i\rangle, |k\rangle\langle k|) > 0$. The convexity (C4) is from the convex roof construction of $C_F^C(\rho|\mathcal{K})$. As mentioned in Sec. II A, it is sufficient to verify the condition (C3). An incoherent operation Λ is a completely positive trace-preserving map such that

$$\Lambda(\rho) = \sum_n K_n \rho K_n^{\dagger}$$

with the Kraus operators K_n satisfying $\sum_n K_n^{\dagger} K_n = I$ and $\sum_n K_n \mathcal{I} K_n^{\dagger} \subset \mathcal{I}$. Let $\rho_n = \frac{K_n\rho K_n^{\dagger}}{p_n}$ and $p_n = \mathrm{tr}[K_n\rho K_n^{\dagger}]$. For a pure state, the monotonicity requirement of (C2) is satisfied obviously [23]. For a mixed state ρ, suppose that $\rho = \sum_i p_i|\varphi_i\rangle\langle\varphi_i|$ is the optimal decomposition that achieves the minimum in the definition $C_F^C(\rho|\mathcal{K})$, that is,

$$C_F^C(\rho|\mathcal{K}) = \sum_i p_i C_F(|\varphi_i\rangle|\mathcal{K}).$$

It remains to prove that for incoherent operation Λ, there must be

$$C_F^C(\rho|\mathcal{K}) \geqslant \sum_n p_n C_F^C(\rho_n|\mathcal{K}).$$

Note that

$$\begin{aligned}\rho_n &= \frac{K_n\rho K_n{}^\dagger}{p_n}\\ &= \sum_i \frac{p_i}{p_n} K_n|\varphi_i\rangle\langle\varphi_i|K_n{}^\dagger\\ &= \sum_i \frac{p_i}{p_n} p_{\text{in}}\rho_{\text{in}},\end{aligned}$$

where $p_{\text{in}} = \text{tr}[K_n|\varphi_i\rangle\langle\varphi_i|K_n{}^\dagger]$ and $\rho_{\text{in}} = \frac{K_n|\varphi_i\rangle\langle\varphi_i|K_n{}^\dagger}{p_{\text{in}}}$. Thus $p_n = \sum_i p_i p_{\text{in}}$, and it follows that

$$\begin{aligned}C_F^C(\rho|\mathcal{K}) &= \sum_i p_i C_F(|\varphi_i\rangle|\mathcal{K})\\ &\geqslant \sum_i p_i \sum_n p_{\text{in}} C_F(\rho_{\text{in}}|\mathcal{K})\\ &= \sum_n p_n \sum_i \frac{p_i p_{\text{in}}}{p_n} C_F(\rho_{\text{in}}|\mathcal{K}))\\ &\geqslant \sum_n p_n C_F^C\left(\sum_i \frac{p_i p_{\text{in}}}{p_n}\rho_{\text{in}}|\mathcal{K}\right)\\ &= \sum_n p_n C_F^C(\rho_n|\mathcal{K}),\end{aligned}$$

where the first inequality comes from the fact that the monotonicity requirement of (C3) has been satisfied for the pure state [23] and the last inequality is due to the convexity of coherence measure. ■

We now investigate if there exist restrictions on the degree of the coherence when more than one basis is considered. In order to demonstrate our approach explicitly, we start from the case of the qubit. Recall that the Bloch representation of a qubit is

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}, \quad (13)$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denotes three Pauli qubit observables and $\vec{r} = (r_1, r_2, r_3)$ is the corresponding Bloch vector. It is known that the three Pauli matrices are mutually unbiased, in the sense that the distribution of any one of these observables is uniform for any eigenstates of the others. Let $\{|\pm_i\rangle\langle\pm_i|\}$, $i = x, y, z$ be the corresponding eigenstates of three Pauli matrices. From the definition of QFI, we have

$$F(\rho, |\pm_x\rangle\langle\pm_x|) = \text{tr}\rho L_{\pm_x}^2, \quad (14)$$

where $L_{\pm_x}$ is the SLD that satisfies the following linear equations [41]:

$$-i[|\pm_x\rangle\langle\pm_x|, \rho] = \tfrac{1}{2}\{L_{\pm_x}, \rho\} \quad (15)$$

and

$$L_{\pm_x} = L_{\pm_x}^\dagger. \quad (16)$$

For the sake of solving the equations, we introduce a set of orthogonal basis in the Hilbert-Schmidt space as $A_i = \{\frac{I}{\sqrt{2}}, \frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}\}$, $i = 0, 1, 2, 3$. Then the eigenstates can be expanded with respect to A_i as

$$|\pm_x\rangle\langle\pm_x| = \sum_{i=0}^{3} a_{ki}A_i \quad (17)$$

Substituting (13) and (17) into (15) and (16), and by straightforward calculation, we have

$$r_2\sigma_z - r_3\sigma_y = \{L_{+_x}, \rho\} \quad (18)$$

and

$$r_3\sigma_y - r_2\sigma_z = \{L_{-_x}, \rho\}. \quad (19)$$

Assume that the corresponding SLDs are

$$L_{\pm_x} = c_1\sigma_y + c_2\sigma_z, \quad (20)$$

Insert (20) into (18) and (19), we have $L_{+_x} = -r_3\sigma_y + r_2\sigma_z$ and $L_{-_x} = r_3\sigma_y - r_2\sigma_z$. Hence the direct generalization of coherence measure via QFI with respect to the σ_x is

$$C_F^D(\rho|\sigma_x) = \sum_\pm \frac{1}{4}\text{tr}\rho L_{\pm_x}^2 = \frac{1}{2}(r_2^2 + r_3^2). \quad (21)$$

The coherence measure via QFI with respect to the σ_y, σ_z can be derived by similar calculation. According to the aforementioned arguments, the complementary relations for MUBs coherence can be given as follows.

Theorem 6. For any qubit density matrix ρ represented by (13), we have

$$\sum_{i=x,y,z} C_F^D(\rho|\sigma_i) = \|\vec{r}\|^2. \quad (22)$$

Clearly Theorem 6 sets a constraint on the usefulness of the state as a coherence resource. There is a tradeoff about coherence measure with respect to different sets of basis. The equality (22) also indicates that the qubit coherence for a given basis is constrained not only by the coherence of MUB, but also by the length of the Bloch vector $\vec{r}$. In particular, for $\vec{r} = 0$, i.e., the maximal mixed state, all coherence must vanish. Note also that Eq. (22) is consistent with Eq. (9), since for pure states, the Bloch vector satisfies $\|\vec{r}\| = 1$.

Actually, Eq. (21) provides us a method to derive the analytical value of coherence measure via QFI. For arbitrary dimension n, it is always difficult to solve Eqs. (15) and (16) since the scale of equations will increase dramatically. However, using the special form of Definitions 2 and 3 and Eq. (4), we can evaluate the QFI coherence measure for a mixed state for arbitrary dimensions.

Proposition 7. For arbitrary quantum state ρ and preferred basis $\mathcal{K} = \{|k\rangle\}$, we have

$$0 \leqslant C_F^C(\rho|\mathcal{K}) \leqslant 1 - \frac{1}{n}. \quad (23)$$

The lower bound is saturated by the state that is diagonal with respect to the preferred basis $\mathcal{K} = \{|k\rangle\}$, while the upper bound is saturated by the maximally coherent mixed state that is written as $\rho = \sum_i p_i |i_+\rangle\langle i_+|$ with a fixed spectrum $\{p_i\}$, where $|i_+\rangle$ denotes a mutually unbiased basis with respect to the preferred basis $\mathcal{K} = \{|k\rangle\}$, i.e., $|\langle k|i_+\rangle|^2 = \frac{1}{n}$.

Proof. By Definition 3,

$$\begin{aligned} C_F^C(\rho|\mathcal{K}) &= \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_F(|\varphi_i\rangle|\mathcal{K}) \\ &= \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i \Big(1 - \sum_k |\langle k|\varphi_i\rangle|^4\Big) \\ &= 1 - \max_{\{p_i,p_{ik}\}} \sum_{ik} p_i p_{ik}^2, \end{aligned} \tag{24}$$

where $p_{ik} = |\langle k|\varphi_i\rangle|^2$. The equality $\sum_k p_{ik} = 1$ for any i immediately yields

$$\frac{1}{n} \leqslant \max_{\{p_i,p_{ik}\}} \sum_{ik} p_i p_{ik}^2 \leqslant 1,$$

where the lower bound is satisfied when all $p_{ik} = \frac{1}{n}$ for any fixed spectrum $\{p_i\}$ and the upper bound is satisfied when for any i, there is only one index k such that $p_{ik} = 1$. ■

Furthermore, summing over Eq. (24) for a set of MUBs, we have the following complementarity relation of coherence measure via QFI.

Theorem 8. For an arbitrary n-dimensional quantum state ρ and a set of MUBs $\{\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_M\}$, we have

$$\sum_{t=1}^{M} C_F^C(\rho|\mathcal{K}_t) \geqslant M - \frac{M-1}{n} - \mathrm{tr}\rho^2. \tag{25}$$

Proof. By Definition 3 again, we obtain

$$\begin{aligned} \sum_{t=1}^{M} C_F^C(\rho|\mathcal{K}_t) &= \sum_{t=1}^{M} \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i C_F(|\varphi_i\rangle|\mathcal{K}_t) \\ &= \sum_{t=1}^{M} \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i \Big(1 - \sum_{k_t} |\langle k_t|\varphi_i\rangle|^4\Big) \\ &= M - \max_{\{p_i,p_{ik_t}\}} \sum_{ikt} p_i p_{ik_t}^2 \\ &= M - \max_{\{p_i,p_{ik_t}\}} \sum_i p_i \sum_{kt} p_{ik_t}^2 \\ &\geqslant M - \frac{M-1}{n} - \mathrm{tr}\rho^2. \end{aligned} \tag{26}$$

In the last step, we have used the fact that for any i, $\sum_{t=1}^{M} \sum_{k=1}^{n} p_{ik_t}^2 \leqslant \mathrm{tr}\rho^2 + \frac{M-1}{n}$. ■

IV. APPLICATIONS IN QUANTUM METROLOGY

In Ref. [23] the author discussed how to relate the coherence measure via skew information with a specific quantum metrology scheme. In such a scheme, we can present a tighter bound for quantum metrology by our coherence measure via QFI.

The scheme is described as follows. For any n-dimensional state ρ and preferred basis $\mathcal{K} = \{|k\rangle\}$, one replaces parameter θ and operator A by φ_k and $|k\rangle\langle k|$ in the Preliminaries, respectively. Then based on quantum parameter estimation theory [25], the uncertainty of estimated phase $\delta\varphi_k$ is limited by the Cramér-Rao bound as

$$(\delta\varphi_k)^2 \geqslant \frac{1}{NF(\rho, |k\rangle\langle k|)}. \tag{27}$$

In order to relate $\delta\varphi_k$ to the coherence measure via skew information, one [23] considered the optimal variance which achieves the Cramér-Rao bound,

$$\big(\delta\varphi_k^0\big)^2 = \frac{1}{NF(\rho, |k\rangle\langle k|)}. \tag{28}$$

It was shown in Refs. [25,42] that this bound can always be reached asymptotically by maximum likelihood estimation and a projective measurement in the eigenbasis of the symmetric logarithmic derivative operator. Combining with the inequality [43]

$$I(\rho, |k\rangle\langle k|) \leqslant \frac{F(\rho, |k\rangle\langle k|)}{4} \leqslant 2I(\rho, |k\rangle\langle k|), \tag{29}$$

where $I(\rho, |k\rangle\langle k|)$ is the skew information, one obtained [23]

$$4NI(\rho, |k\rangle\langle k|) \leqslant \frac{1}{\big(\delta\varphi_k^0\big)^2} \leqslant 8NI(\rho, |k\rangle\langle k|). \tag{30}$$

Summing Eq. (30) over k, it has [23]

$$4NC(\rho|\mathcal{K}) \leqslant \sum_k \frac{1}{\big(\delta\varphi_k^0\big)^2} \leqslant 8NC(\rho|\mathcal{K}), \tag{31}$$

where $C(\rho|\mathcal{K}) = \sum_k I(\rho, |k\rangle\langle k|)$ is the coherence measure via skew information with respect to the preferred basis $\mathcal{K} = \{|k\rangle\langle k|\}$. Subsequently, they defined $\sum_k \frac{1}{(\delta\varphi_k^0)^2} = \frac{1}{(\Delta\varphi_k^0)^2}$, and used the fact that the practical measurement strategy cannot be as ideal as we expect theoretically, i.e., $\delta\varphi_k \geqslant \Delta\varphi_k^0$, finally yielding two relations as [23]

$$\frac{1}{(\delta\varphi_k)^2} \leqslant 8NI(\rho, |k\rangle\langle k|) \tag{32}$$

and

$$\sum_k \frac{1}{(\delta\varphi_k)^2} \leqslant 8NC(\rho|\mathcal{K}). \tag{33}$$

It is obvious that the lower and upper bounds in (32) and (33) are not tight enough, since the derivations are based on two premises of Eqs. (28) and (29). Here, following the same scheme, and using coherence measure $C_F^D(\rho|\mathcal{K})$, we can immediately present a more direct and tighter bound for the above quantum metrology scheme.

Theorem 9. The uncertainty of the estimated phase can be bound as

$$\frac{1}{(\delta\varphi_k)^2} \leqslant NF(\rho, |k\rangle\langle k|) \tag{34}$$

and

$$\sum_k \frac{1}{(\delta\varphi_k)^2} \leqslant 4NC_F^D(\rho|\mathcal{K}). \tag{35}$$

Furthermore, according to Proposition 7, we have

$$\sum_k \frac{1}{(\delta\varphi_k)^2} \leqslant 4N\Big(1 - \frac{1}{n}\Big). \tag{36}$$

Proof. The statement follows from Eq. (27) and Definition 2. ■

It is obvious that Eq. (35) sets a sharper bound than that of (33) due to (29). That is to say, our coherence measure $C_F^D(\rho|\mathcal{K})$ which is based on QFI is more intrinsically related to quantum metrology.

V. CONCLUSION

In summary, we have presented several reliable coherence measures via QFI which is the fundamental concept of quantum metrology. We have shown that, for pure states, this coherence measure is equivalent to that defined by skew information [23], therefore it satisfies all the requirements for quantifying coherence. By using the notable relations of the index of coincidence [34,35], we have derived a tight complementary tradeoff under a set of MUBs. In the case that n is prime power, we have obtained an equality, thus constraining the usefulness of the state as a coherence measure in a given basis. That is, if the coherence is minimal with respect to a given basis, it must be larger than a quantity with respect to another basis.

For mixed states, we first have proposed two natural coherence measures via QFI and then shown that these two definitions are exactly identical because the quantum Fisher information itself can be seen as the convex roof of the variance [31,32]. Furthermore, both of the coherence measures are *bona fide* coherence measures satisfying all the criteria. By the Bloch representation of a qubit, we have obtained an analytical expression for the coherence measure in the eigenbasis of three Pauli qubit observables. It is interesting that the sum of the coherence measure is determined by the length of the Bloch vector. The complementary relations for an arbitrary dimensional mixed state under MUBs have also been studied. In the end, due to the important role of QFI in the quantum metrology scheme, we have related the uncertainty of the estimated phase to our coherence measures. It sets a tighter bound than Ref. [23] as expected, since the well-known Cramér-Rao bound [25] is connected with QFI in a more straightforward manner than skew information. Furthermore, this relationship instead imposes our coherence measures with an operational meaning.

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