Quantum formalism for events and how time can emerge from its foundations

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(Received 26 June 2020; accepted 22 December 2020; published 21 January 2021)

Although time is one of our most intuitive physical concepts, its understanding at the fundamental level is still an open question in physics. For instance, time in quantum mechanics and general relativity are two distinct and incompatible entities. While relativity deals with events (points in spacetime), with time being observer dependent and dynamical, quantum mechanics describes physical systems by treating time as an independent parameter. To resolve this conflict, in this work, we extend the classical concept of an event to the quantum domain by defining an event as a transfer of information between physical systems. Then, by describing the universe from the perspective of a certain observer, we introduce quantum states of events with space-time-symmetric wave functions that predict the joint probability distribution of a measurement (observation) at $(t,\vec{x})$. Under these circumstances, without assuming collapse, we propose that a well-defined instant of time, like any other observable, arises from a single event, thus being an observer-dependent property. In this manner, we obtain a stationary quantum state written as a sum of a sequence of normalized states of events with increasing memories. As a consequence, a counterfactual asymmetry along this sequence of events gives rise to the flow of time as being successive "snapshots" from the observer's perspective. In this proposal, which contrasts strikingly to the view in which time is an illusion, it is the many distinguishable states in which the observer stores information that makes the existence of time possible.

DOI: 10.1103/PhysRevA.103.012219

I. INTRODUCTION

In ordinary classical and quantum mechanics (QM), time is an extrinsic parameter [1] that can be chosen arbitrarily to evaluate the state of a system. In the quantum scenario, this feature is embedded in the very definition of state, e.g., in

$$|\psi(t)\rangle=\sum_{\alpha}\psi(\alpha,t)\,|\alpha\rangle, \tag{1}$$

where $|\psi(t)\rangle$ is the state of a system $\mathcal{S}$ at the instant t and $\hat{\alpha}$ is an observable such that $\hat{\alpha}|\alpha\rangle=\alpha|\alpha\rangle$. To predict experimental outcomes, the wave function $\psi(\alpha,t)=\langle\alpha|\psi(t)\rangle$ is interpreted as the probability amplitude of measuring the system in the state $|\alpha\rangle$, *given that* the detection takes place at time t [2–6]. To avoid any confusion with other functions defined in the present work, from now on we will highlight the time-conditioned character of the Schrödinger wave function with the notation $\psi(\alpha|t)$. In this context, it is broadly familiar that time being a parameter goes against the foundations of a possible quantum theory of gravity, which should treat space and time on an equal footing [7].

With these facts in mind, it is worth summarizing the first goal of this work, which is to obtain a space-time-symmetric formalism of QM. As relativity deals with events (points in space-time), we will derive a timeless state predicting events (measurements) given by

$$||\Psi\rangle=\sum_{t,\alpha}\Psi(t,\alpha)||t,\alpha\rangle+\cdots, \tag{2}$$

with t being a variable of a timer $\mathcal{T}$ (or a macroscopic observer capable of recording t), and $\Psi(t,\alpha)$ and $||t,\alpha\rangle$ being the probability amplitude and the state of the event (t,α), respectively. Notice that instead of being represented at a specific instant of "time" as in Eq. (1), state (2) is a superposition of what is measured (α) and when this measurement takes place (t). Thus, in contrast to the time-conditioned character of the Schrödinger wave function, the modulus squared of $\Psi(t,\alpha)$ is the joint probability density of measuring α at the instant t.

As our focus are events, to obtain Eq. (2), we will not deal with a single system, and hence $||t,\alpha\rangle$ encompasses quantum states of interacting (and/or correlated) systems, including, e.g., a detector, a timer, and the system under consideration, $\mathcal{S}$. The ellipsis $(\cdots)$ in Eq. (2) represents the events following (t,α), whose moments of occurrence are recorded by different timers (or a macroscopic observer capable of recording the times of all events), thus belonging to distinct Hilbert spaces. Each of these events is normalized separately, and they are characterized by possessing increasing memories. The full state (2) is obtained in Sec. IV B, Eq. (50).

To better advocate the reasoning underlying Eq. (2), let us start by discussing two well-known assumptions that address the issue of the space-time asymmetry of QM: (i) In one of them, it is argued that a quantum formalism compatible with general relativity (GR)—in which there is no preferred independent time variable—should describe only the relative evolution between physical quantities [8,9]. As we will discuss below, the Page and Wootters (PW) formalism approached in Refs. [2–5], whose Hamiltonian constraint (a Wheeler-DeWitt-like equation) is compatible with formalisms

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about quantum gravity [10], promptly fulfills this requirement. (ii) The second hypothesis was mentioned above: As GR deals with events (points in space-time), a suitable quantum approach should predict events by modeling space and time symmetrically [9,11–14].

It is worth noticing that, unlike quantum states of physical systems, which extend indefinitely in time, an event has a finite temporal domain [11]. For instance, a QM for events should provide, even in the nonrelativistic scenario, the probability of measuring a particle in the region $x + dx$ and the interval $t + dt$. Note that by setting $\alpha = x$ in the state of Eq. (2), $||\Psi\rangle$ carries that information, in contrast to $|\psi(t)\rangle$ in Eq. (1). Despite the coherence of both conditions (i) and (ii), the attempt to reconcile at a fundamental level these two criteria [more specifically, the PW picture and (ii)] requires special attention. We address this standpoint below and more carefully in Sec. II.

In the PW picture, it is assumed that the universe is stationary, and time arises from the correlation between a system $\mathcal{C}$ that plays the role of a clock and the rest of the universe $\mathcal{R}$ [3]. The clock is conveniently chosen not to interact with the rest so that $\mathcal{R}$ evolves according to the Schrödinger equation with respect to a variable β that labels the eigenvalues of a proper clock's observable. For instance, in a universe with only two degrees of freedom, α (belonging to $\mathcal{R}$) and β, the rest $\mathcal{R}$ has a β-conditioned wave function $\psi(\alpha|\beta)$ given by Eq. (1), with $t = \beta$. Under these circumstances, β plays the same role as t in the Schrödinger prescription (1), and thus time is no longer merely a parameter but rather an observable that tracks the dynamics of $\mathcal{R}$. It is worth mentioning that in this paper, we will propose a different way of looking at the emergence of time.

Notice that as $\mathcal{C}$ is a noninteracting system, this clock cannot register the moment of an event. To obtain this information, $\mathcal{C}$ would have to interact with the systems involved in the measurement that defines an event (see Sec. II for more details). Hence, we cannot associate $\mathcal{C}$ with a time probability distribution of when a measurement happens. As a consequence, a quantum description of events [criterion (ii)] as shown in Eq. (2) cannot be derived in the PW formalism simply through the correlation between observables [criterion (i)] of $\mathcal{C}$ and $\mathcal{R}$. In this way, conditions (i) and (ii) conflict with each other.

Despite the incompleteness of the PW's relative approach to describing events, this formalism will be fundamental for this work for some reasons. For instance, for successfully eliminating the treatment of time as a parameter and proposing a stationary quantum state obeying a Wheeler-DeWitt-like equation [2–5,10], which is an expected constraint for a closed universe. Thus, the difference between our approach [Eq. (2)], which follows the criterion (ii), and the assumption (i) is that we will use the PW prescription as only a starting point; we will not focus on the correlation between $\mathcal{C}$ and $\mathcal{R}$.

The second focus of our approach is to obtain from $||\Psi\rangle$ our familiar notion of time, which time "flows" in a preferred direction. Despite the clear intuition we have about time, the nature of time at a fundamental level is one of the most intriguing puzzles of physics: Is time a fundamental entity, an illusion produced in our brain, or a property that emerges from more primitive concepts? Here, we will answer this question in favor of the latter point of view. To this end, first, we will disassociate $\mathcal{C}$ and $\mathcal{T}$ from any notion of time and focus on the events contained in $||\Psi\rangle$.

In contrast to our approach, in the PW's relative dynamics, β is called time mainly because $\psi(\alpha|\beta)$ follows the Schrödinger equation. In this manner, it is expected that this notion of time leads to inconsistencies with our daily experience. Two emblematic examples are the incompatibility between the second law of thermodynamics and the β-reversal evolution of $\psi(\alpha|\beta)$, and the unnatural interpretation in which different clock readings β's ("instants of time") are actually examples of distinct worlds [15]. From this latter viewpoint, the passage of time is a mere illusion that can arise from the perception of our memories. Unlike this approach, here we intend to address a concept of time more congruent with our common sense.

From another perspective of the problem of time, several works such as Refs. [16–18] found that as time progresses, the increasing entanglement of an object with their surroundings makes this object reach equilibrium. From this result, the arrow of time is commonly associated with an increase of correlations. However, it should be pointed out that this approach does not explain the nature of time itself but rather the reason why physical systems reach equilibrium over time.

The question of why we perceive time flowing is an open issue that we intend to address in this work. To this end, we will verify that a more fundamental description of events should take into account that an observer with many degrees of freedom can play the same role as a set of timers $\mathcal{T}$. Then, by proposing that time emerges from events, we will observe that the flow of time arises from the perspective of the observer as a consequence of a causal-like asymmetric sequence of events in $||\Psi\rangle$. Here, causal-like means a counterfactual conditional [19] that arises from the "selection" (without the need for any relationship with time) of an indefinite number of variables from a set $\{X_{(k)}\}_{k=1,2,3,\ldots}$, in which if $X_{(k)}$ is not selected, $X_{(k+1)}$ cannot be selected.

This paper is organized as follows. In Sec. II, we will present how we will extend the classical notion of events to the quantum realm. In Sec. III, we will introduce the timer $\mathcal{T}$, and we will describe events via $||\Psi\rangle$ conditioned on a specific reading β of the clock $\mathcal{C}$. The calculation of this section is an intermediate step toward the full derivation of the formalism of events. In Sec. III A, we will begin by reviewing the PW approach and explaining its importance for the formalism of events. The case of a single event will be discussed in Sec. III B, and then in Sec. III C we will extend this analysis to two causally connected events. In Sec. III D, we will verify that a system with many degrees of freedom can play the role of the timer $\mathcal{T}$. Finally, in Sec. IV, we will have the full formalism of events obtaining $||\Psi\rangle$ as showed in Eq. (2). In Secs. IV A and IV B, the case of a single event and an arbitrary number of causally connected events will be approached respectively. In Sec. IV C, we will propose how time can emerge from $||\Psi\rangle$. Lastly, the generalization for causally and noncausally connected events will be briefly addressed in Sec. IV D. In the conclusion of this paper, Sec. V, we will summarize the results.

II. EXTENDING THE CLASSICAL NOTION OF EVENTS TO THE QUANTUM DOMAIN

In a classical version of the PW picture, we have Newton's laws providing a "timeless" relationship between α and β, such as $F(\alpha,\beta)=0$. Here, the treatment of time as an independent parameter is also eliminated. Therefore, by conveniently isolating α in $F(\alpha,\beta)=0$ and with the discussion of the introduction in mind, we have the classical and quantum versions of the relative evolution between observables required in the criterion (i) of the introduction,

$$\alpha=\alpha(\beta)\ \text{(classical state)}$$

and

$$\psi=\psi(\alpha|\beta)\ \text{(quantum state)}. \tag{3}$$

For a given β, classically, we have a well-defined value of α, whereas, in the quantum scenario, we only have a probability amplitude for α.

With Eq. (3) in mind, let us turn our attention to the description of events in the PW approach. An event is a happening, for example, a classical particle $\mathcal{S}$ (belonging to $\mathcal{R}$) with position α reaching a mark on the surface on which it is moving. In the classical regime, as we know with arbitrary precision the reading of the clock at the arrival moment ($\beta\equiv t$) of the particle, by applying the first expression of Eq. (3), the event is fully described by the pair $(t,\alpha(t))$. In contrast, in the quantum domain, as both α and the arrival time t are probabilistic variables [20–24], an event should be depicted by a joint probability amplitude of t and α. Thus, the classical and quantum descriptions of events, unlike Eq. (3), should be given by

$$(t,\alpha(t))\ \text{(classical event)}$$

and

$$\Psi(t,\alpha)=\psi(\alpha|t)\,\chi(t)\ \text{(quantum event)}, \tag{4}$$

with $\Psi(t,\alpha)$ being the probability amplitude of the event (t,α), as shown in Eq. (2). By applying Bayes' rule, $|\Psi(t,\alpha)|^2$ in Eq. (4) is expressed as the probability of measuring α given that the clock reads $\beta=t$, $|\psi(\alpha|t)|^2$, multiplied by the probability for the clock to read t at the moment of the event, $|\chi(t)|^2$. Here, notice that we need a detector $\mathcal{D}$ (also belonging to $\mathcal{R}$) that, by measuring the particle $\mathcal{S}$, defines its arrival. It is worth mentioning that because of the measurement interaction, $\psi(\alpha|t)$ in Eq. (4) is, in general, different from that of the isolated situation of Eq. (1).

It should be kept in mind that the time probability amplitude $\chi(t)$ concerns the reading of the clock $\mathcal{C}$ *at the moment of the arrival*. Nevertheless, as $\mathcal{C}$ is a noninteracting system in the PW picture [3–5], $\chi(t)$ cannot be a wave function of this clock. For such a link to be possible, for instance, $\mathcal{C}$ should be coupled to the detector $\mathcal{D}$, which defines the arrival of the particle. Then, another clock, noninteracting with $\mathcal{R}$, should be taken into account to maintain the PW approach as valid. Hence, $\mathcal{R}$ would still follow the Schrödinger equation, but now with respect to an appropriate observable of this new clock. As a result, as we briefly discussed in the introduction, we verify that $\Psi(t,\alpha)$ of Eq. (4) cannot be obtained only from the correlation between $\mathcal{R}$ and $\mathcal{C}$ of the PW's approach [2,4].

In the present work, to obtain the state of Eq. (2) with $\Psi(t,\alpha)$ given by Eq. (4), we will keep $\mathcal{C}$ isolated in the PW approach and will assume that an event is a register of information about $\mathcal{S}$ performed within the rest $\mathcal{R}$ by both a detector $\mathcal{D}$ and, at firstly, a timer $\mathcal{T}$. The timer will be modeled by a Salecker-Wigner-Peres-like (SWP-like) clock [25,26], but, instead of being coupled with the system $\mathcal{S}$ under consideration [26], $\mathcal{T}$ will interact with the detector. Thus, as it will be carefully defined, $||t,\alpha\rangle$ of Eq. (2) will represent the quantum state of an event, which describes $\mathcal{CS}$ from the perspective of $\mathcal{TD}$, i.e., what $\mathcal{TD}$ records about $\mathcal{CS}$. In particular aspects, the standpoint adopted here is in line with the "relative state" formulation of QM [27], but not requiring the many-worlds (or minds) view. Besides, it is worth mentioning that in a different approach, the treatment of events as a transfer of information is also assumed in the relational interpretation of QM [28,29] (not to be confused with the relative dynamics of the PW picture and of Ref. [9]).

Without assuming any collapse, we will generalize $||\Psi\rangle$ for an indefinite number of causally connected events by considering a collection of timers and detectors ($\mathcal{TD}$s). In this context, it should be pointed out that to propose an interpretation of the emergence of time, first we will verify that the environment can play the role of a set of $\mathcal{TD}$s. The reason for this is that although the collection of innumerable $\mathcal{TD}$s will provide the ideal setting for our interpretation of time, time should emerge for more general macroscopic observers than SWP timers, for instance, the environment and the brain. Situations involving both causal and noncausal events will also be briefly discussed.

Finally, it is worth pointing out that relevant works that describe events commonly consider several "instantaneous" measurements [12,13,30]. As a result, these formalisms do not provide a state for a *single event* that treats time and any other observable α on an equal footing as in Eq. (2). On the other hand, works such as Refs. [14,23], which do consider measurements with a finite duration, do not take into account the register of the measurement time, and hence Eq. (2) is not applicable either. It is also important to stress that regarding the example of the particle reaching a point in space, there is a huge body of interesting literature about arrival times that aim to obtain time probability distribution such as $\chi(t)$ [20–24]. Nevertheless, unlike the physical scheme of Eq. (2), these works consist mostly of calculating the time of arrival either by using the wave function of a particle in the absence of measurement (i.e., the arrival time as a "property" of the particle similar to position), or introducing only a detector that by interacting with the particle defines its arrival. In addition, it should be remarked that Ref. [6] proposes the first term on the right-hand side of Eq. (2) for the particular case $\alpha=x$. However, neither the concept of events nor how to obtain it from traditional tools of QM was explored in this reference. Nevertheless, we believe that the physical connections between the results of the present paper and those of Refs. [6,31] deserve to be investigated in future works.

III. EVENTS FROM THE PERSPECTIVE OF CONDITIONED QUANTUM STATES

A. Page and Wootters approach and its importance for the formalism of events

Let us begin this section by briefly reviewing the PW formalism [2–4], with the clock $\mathcal{C}$ and the rest of the universe $\mathcal{R}$ having Hilbert spaces $\mathcal{H}_\mathcal{C}$ and $\mathcal{H}_\mathcal{R}$ respectively. It should be clear that, as we will only discuss the emergence of time in Sec. IV C, for the sake of understanding, we will frequently refer to the clock reading β as a measure of time. Thus, for convenience, let us define the possible values of $\beta = t_0, t_{(1)}, t_{(2)}, \ldots, t_{(N)}, \ldots$. The condition of an isolated universe implies that $||\Psi\rangle$ belonging to $\mathcal{H}_\mathcal{C} \otimes \mathcal{H}_\mathcal{R}$ must satisfy the Wheeler-DeWitt constraint [2–4,10]

$$\hat{H}||\Psi\rangle = 0, \tag{5}$$

where

$$\hat{H} = \hat{H}_\mathcal{C} \otimes \mathbb{1}_\mathcal{R} + \mathbb{1}_\mathcal{C} \otimes \hat{H}_\mathcal{R}. \tag{6}$$

By considering $\hat{\beta}$, with $\hat{\beta}|\beta\rangle_\mathcal{C} = \beta|\beta\rangle_\mathcal{C}$, conjugated to the clock's Hamiltonian $\hat{H}_\mathcal{C}$, $[\hat{\beta}, \hat{H}_\mathcal{C}] = i\hbar$, the static solution of Eq. (5) can be written as

$$||\Psi\rangle = \sum_{\beta=t_0} |\beta\rangle_\mathcal{C} \otimes |\psi(\beta)\rangle_\mathcal{R}, \tag{7}$$

with $|\beta\rangle_\mathcal{C} = \exp\{-i\hat{H}_\mathcal{C}(\beta - t_0)/\hbar\}|t_0\rangle_\mathcal{C}$ and

$$|\psi(\beta)\rangle_\mathcal{R} = e^{-i\hat{H}_\mathcal{R}(\beta - t_0)/\hbar}\,|\psi(t_0)\rangle_\mathcal{R}. \tag{8}$$

We choose the state $|t_0\rangle_\mathcal{C}$ of the clock to indicate the beginning of the evolution. Notice that the Schrödinger state of $\mathcal{R}$ (8) is obtained by conditioning $||\Psi\rangle$ on β; i.e., $|\psi(\beta)\rangle_\mathcal{R} = {}_\mathcal{C}\langle\beta||\Psi\rangle \in \mathcal{H}_\mathcal{R}$ is the state of $\mathcal{R}$ given that the clock $\mathcal{C}$ reads β.

It is important to notice that the description of a physical system only via Eq. (8) requires the presence of an observer external to the formalism that by measuring $\mathcal{C}$ defines the value of β and allows the conditioning $|\psi(\beta)\rangle_\mathcal{R} = {}_\mathcal{C}\langle\beta||\Psi\rangle$. In contrast, in this work, we aim to describe events without reference to an external observer. Remember that we want to describe what timers and detectors record about $\mathcal{CS}$. Thus, we will focus on the global state $||\Psi\rangle$ [and not only $|\psi(\beta)\rangle_\mathcal{R}$] that can take into account a collection of $\mathcal{TD}$s (or a macroscopic observer that plays a similar role) as the observer.

With these facts in mind, in Sec. III B we will take the first steps of the derivation of our formalism by calculating $|\psi(\beta)\rangle_\mathcal{R}$ in the context of events previously discussed. In Sec. IV, we will substitute $|\psi(\beta)\rangle_\mathcal{R}$ calculated in this Sec. III into $||\Psi\rangle$ [Eq. (7)]. Then, after some algebraic manipulations, we will obtain the full formalism of events and discuss the emergence of time. Nevertheless, still in Sec. III, we will verify that relevant information can be extracted from $|\psi(\beta)\rangle_\mathcal{R}$.

Although we disagree with the traditional interpretation of the PW picture, which time arises from the entanglement between $\mathcal{C}$ and $\mathcal{R}$, the presence of $\mathcal{C}$ and its correlation with $\mathcal{R}$ are essential for our formalism, as discussed in the introduction. The reasons for this are the following: (i) Taking into account $\mathcal{C}$ eliminates the treatment of time as an extrinsic parameter, and (ii) it is the correlation between $\mathcal{C}$ and $\mathcal{R}$ that makes $||\Psi\rangle$ encompasses the sequence of events "experienced" (measurements performed) by a certain macroscopic observer. The interpretation of the emergence of time will be based on this sequence. Furthermore, it is also the correlation between $\mathcal{C}$ and $\mathcal{R}$ that allows $\mathcal{T}$ (or a macroscopic observer) to record the reading of $\mathcal{C}$ (without interacting with it) at the time of the measurement carried out by $\mathcal{D}$. This is what makes $\mathcal{C}$ a good time measurer in our formalism.

B. $|\psi(\beta)\rangle_\mathcal{R}$ for a single event

An event will be treated as an effectively irreversible measurement carried out within $\mathcal{R}$, in which an observer records information about a system $\mathcal{S}$ with Hamiltonian $\hat{H}_\mathcal{S}$. Consider the unitary evolution (8) during the measurement of an observable $\hat{\alpha} = \sum_\alpha \alpha|\alpha\rangle_{\mathcal{S}\mathcal{S}}\langle\alpha|$ of $\mathcal{S}$ that is performed by a detector $\mathcal{D}$ (the observer). For instance, we can assume that either $\mathcal{D}$ is coupled to a large environment $\mathcal{E}$, which makes the detection effectively irreversible [23], or $\mathcal{D}$ is a macroscopic system since, in realistic measurements, detectors have a huge number of degrees of freedom. *For now*, we disregard the record of the instant of detection (there are no timers) and consider that the interaction time $\Delta t_{\mathcal{SD}}$ between the detector and the system is short enough to neglect the evolution driven by $\hat{H}_\mathcal{S}$.

Under these circumstances, with $\mathcal{R} = \mathcal{SD}$, Eq. (6) becomes $\hat{H} \approx \hat{H}_\mathcal{C} \otimes \mathbb{1}_{\mathcal{SD}} + \mathbb{1}_\mathcal{C} \otimes \hat{V}_{\mathcal{SD}}$, where $\hat{V}_{\mathcal{SD}}$ is the interaction potential between $\mathcal{S}$ and $\mathcal{D}$. The solution (8) with initial condition $|\psi(t_0)\rangle_\mathcal{R} = |0\rangle_\mathcal{S} \otimes |0\rangle_\mathcal{D}$ is

$$\begin{aligned} |0\rangle_\mathcal{S} \otimes |0\rangle_\mathcal{D} &= \left(\sum_\alpha \psi_\mathcal{S}(\alpha|t_0)\,|\alpha\rangle_\mathcal{S}\right) \otimes |0\rangle_\mathcal{D} \\ &\xrightarrow{\Delta t_{\mathcal{SD}}} \sum_\alpha \psi_\mathcal{S}(\alpha|t_0)\,|\alpha\rangle_\mathcal{S} \otimes |\alpha\rangle_\mathcal{D} \\ &= \sum_\alpha \hat{M}_\alpha\,|0\rangle_\mathcal{S} \otimes |\alpha\rangle_\mathcal{D}, \end{aligned} \tag{9}$$

where $|0\rangle_\mathcal{D}$ is the ready state of the detector $[{}_\mathcal{D}\langle\alpha|0\rangle_\mathcal{D} = \delta_{\alpha,0}]$, $|0\rangle_\mathcal{S} = |\psi(t_0)\rangle_\mathcal{S}$, $\psi_\mathcal{S}(\alpha|t_0) = \langle\alpha|0\rangle_\mathcal{S}$, and $\hat{M}_\alpha = |\alpha\rangle_{\mathcal{S}\mathcal{S}}\langle\alpha|$. As mentioned earlier, we use the notation $\psi_\mathcal{S}(\alpha|t_0)$ instead of the usual nomenclature, such as $c_\alpha(t_0)$, to emphasize the time-conditioned character of the Schrödinger amplitudes. In Eq. (9), although we neglected the evolution of $\mathcal{S}$ by $\hat{H}_\mathcal{S}$, it is important to keep in mind that $\Delta t_{\mathcal{SD}}$ is actually finite. Thus, notice that Eq. (9) guarantees that by observing the detector at some moment $\beta \gtrsim t_0 + \Delta t_{\mathcal{SD}}$, we unequivocally find out the state $|\alpha\rangle_\mathcal{S}$. Nevertheless, as will be more evident in Sec. III D, if $\mathcal{D}$ had only one degree of freedom to store information about $\mathcal{S}$, one could not claim that the detector measured $\mathcal{S}$ at some instant t before we observe of $\mathcal{D}$, i.e., in the interval $t_0 < t < \beta$. The reason for this is that before $\beta \approx t_0 + \Delta t_{\mathcal{SD}}$, the pair $\mathcal{SD}$ is in a superposition that only describes the unmeasured (detector in $|0\rangle_\mathcal{D}$) and measured (detector in $|\alpha\rangle_\mathcal{D}$) situations. As a consequence, unlike the classical picture, the time of a quantum event (or detection) only exists if a physical system somehow registers it.

With these facts in mind, we begin our approach describing the classical event $(t, \alpha(t))$ in the quantum domain by keeping the clock $\mathcal{C}$ isolated and, first, using a SWP's-like (Salecker-Wigner-Peres-like) timer $\mathcal{T}$ [25,26] synchronized with $\mathcal{C}$. We will not yet analyze the possibility of $\mathcal{D}$ (a macroscopic system) recording the instant of detection. This more general description, in which $\mathcal{T}$ can be disregarded, will be presented in Sec. III D. From now on, $\Delta t_{\mathcal{SD}}$ will not be short enough to neglect the evolution driven by $\hat{H}_{\mathcal{S}}$.

Under these circumstances, remember that $\chi(t)$ in Eq. (4) is the probability amplitude for the clock $\mathcal{C}$ to read t at the moment of the event. To obtain such an amplitude, let us couple the timer with the detector in such a way that $\mathcal{T}$ stops its counting when $\mathcal{D}$ measures $\mathcal{S}$. Thus, $\mathcal{T}$ evolves synchronized to $\mathcal{C}$ while $\mathcal{D}$ "is" in the state $|0\rangle_{\mathcal{D}}$. For simplicity, consider that $\mathcal{T}$ "instantly" recognizes the detection of $\mathcal{S}$, which means that the timescale of the interaction between the timer and the detector is significantly smaller than $\Delta t_{\mathcal{SD}}$. In this approximate picture, the probability amplitude of $\mathcal{T}$, $\chi(t)$ [see Eq. (4)], predicts the reading of $\mathcal{C}$ at the moment of the event. Now, the observer will be seen as $\mathcal{TD}$. It is worth mentioning that despite the continuous monitoring of the timer, the detector is not affected by the Zeno effect as long as it is macroscopic [32].

An interaction between $\mathcal{T}$ and $\mathcal{D}$ that models the evolution defined above is $\hat{V}_{\mathcal{TD}} = \mathbb{1}_{\mathcal{S}} \otimes \hat{H}_{\mathcal{T}} \otimes |0\rangle_{\mathcal{DD}}\langle 0|$ [26]. As every measurement is carried out internally to $\mathcal{R}$, we have

$$\hat{H} = \hat{H}_{\mathcal{C}} \otimes \mathbb{1}_{\mathcal{STD}} + \mathbb{1}_{\mathcal{C}} \otimes \hat{H}_{\mathcal{STD}}, \tag{10}$$

where $\hat{H}_{\mathcal{STD}} = \hat{H}_{\mathcal{S}} \otimes \mathbb{1}_{\mathcal{TD}} + \hat{V}_{\mathcal{SD}} + \hat{V}_{\mathcal{TD}}$. Since Eq. (10) has the same form as the PW's Hamiltonian (6), the solutions (7) and (8) still hold. Let us consider the timer's observable $\hat{T}$ such that $\hat{T}|t\rangle_{\mathcal{T}} = t|t\rangle_{\mathcal{T}}$ and $[\hat{T}, \hat{H}_{\mathcal{T}}] = i\hbar$, and the initial condition of $\mathcal{R} = \mathcal{STD}$ being

$$|\psi(t_0)\rangle_{\mathcal{R}} = |0\rangle_{\mathcal{S}} \otimes |t_0\rangle_{\mathcal{T}} \otimes |0\rangle_{\mathcal{D}}. \tag{11}$$

To prevent a dense notation, from now on, we will avoid using the symbol $\otimes$.

For the sake of clarity, let us consider $\mathcal{S}$ being composed of a single system. To calculate $|\psi(\beta)\rangle_{\mathcal{R}}$, let us break up the evolution (8) into infinitesimal steps $\delta t = t_{(j+1)} - t_{(j)} \ll \Delta t_{\mathcal{SD}}$, so that the possible values of β and t in this discrete evolution are $t_0, t_{(1)}, t_{(2)}, \ldots, t_{(N)}, \ldots$. At the first step $t_{(1)}$, the state of $\mathcal{R} = \mathcal{STD}$ splits into two branches (system $\mathcal{S}$ measured and not measured) given by

$$\begin{aligned}|\psi(t_{(1)})\rangle_{\mathcal{R}} &= \sqrt{1-\delta p_{(1)}}\, e^{i\varphi_{(1)}} \left[\hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)}, t_0)|0\rangle_{\mathcal{S}}\right] |t_{(1)}\rangle_{\mathcal{T}}\, |0\rangle_{\mathcal{D}} \\ &\quad + \sqrt{\delta p_{(1)}} \left[\sum_{\alpha} \hat{M}_\alpha\, \hat{U}_{\mathcal{S}}^{1(1)}(t_{(1)}, t_0)|0\rangle_{\mathcal{S}}\right] |t_{(1)}\rangle_{\mathcal{T}}\, |\alpha\rangle_{\mathcal{D}},\end{aligned} \tag{12}$$

where $\varphi_{(1)}$ is the phase of the first step, and $\delta p_{(1)} \ll 1$ is the probability of the measurement taking place in the interval $[t_0, t_{(1)}]$, regardless of the outcome $|\alpha\rangle_{\mathcal{S}}$. Here, the operations $\hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)}, t_0)$ and $\hat{U}_{\mathcal{S}}^{1(1)}(t_{(1)}, t_0)$ act on $\mathcal{H}_{\mathcal{S}}$ and refer to the situations where $\mathcal{S}$ is not measured and measured in the first step, respectively. The interaction between $\mathcal{S}$ and $\mathcal{D}$ makes these operations, in general, different from $\hat{U}_{\mathcal{S}}(t_{(1)}, t_0) = \exp\{-i\hat{H}_{\mathcal{S}}(t_{(1)} - t_0)/\hbar\}$. For instance, if for each $|\alpha\rangle_{\mathcal{S}}$, the detector interacts with a different intensity, $\hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)}, t_0)$ and $\hat{U}_{\mathcal{S}}^{1(1)}(t_{(1)}, t_0)$ are no longer unitary [see the Appendix for a brief discussion of these operations and Eq. (12)]. Nevertheless, the total evolution obviously is always unitary, so that ${}_{\mathcal{R}}\langle\psi(t_{(1)})|\psi(t_{(1)})\rangle_{\mathcal{R}} = 1$. All the quantities of Eq. (12) can be calculated via Eq. (8) by appropriately defining the potentials of $\hat{H}_{\mathcal{STD}}$. By inspecting Eq. (12), first we verify that with a high probability $1 - \delta p_{(1)}$ the detector does not record any information about the system. In this case, the timer continues to evolve as an ideal quantum clock. On the other hand, with probability $\delta p_{(1)}$, the detector measures some state $|\alpha\rangle_{\mathcal{S}}$, and hence the timer stops its evolution recording $t_{(1)}$, which is the reading of $\mathcal{C}$ at this moment.

In the next step of the evolution, the unmeasured contribution of $|\psi(t_{(1)})\rangle_{\mathcal{R}}$ splits into two branches, so that

$$\begin{aligned}|\psi(t_{(2)})\rangle_{\mathcal{R}} &= \sqrt{1-\delta p_{(1)}}\, e^{i\varphi_{(1)}} \sqrt{1-\delta p_{(2)}}\, e^{i\varphi_{(2)}} \left[\hat{U}_{\mathcal{S}}^{0(2)}(t_{(2)}, t_{(1)})\, \hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)}, t_0)|0\rangle_{\mathcal{S}}\right] |t_{(2)}\rangle_{\mathcal{T}}\, |0\rangle_{\mathcal{D}} \\ &\quad + \sqrt{1-\delta p_{(1)}}\, e^{i\varphi_{(1)}} \sqrt{\delta p_{(2)}} \left[\sum_{\alpha} \hat{M}_\alpha\, \hat{U}_{\mathcal{S}}^{1(2)}(t_{(2)}, t_{(1)})\, \hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)}, t_0)\, |0\rangle_{\mathcal{S}}\right] |t_{(2)}\rangle_{\mathcal{T}}\, |\alpha\rangle_{\mathcal{D}} \\ &\quad + \sqrt{\delta p_{(1)}} \left[\hat{U}_{\mathcal{S}}(t_{(2)}, t_{(1)}) \sum_{\alpha} \hat{M}_\alpha\, \hat{U}_{\mathcal{S}}^{1(1)}(t_{(1)}, t_0)|0\rangle_{\mathcal{S}}\right] |t_{(1)}\rangle_{\mathcal{T}}\, |\alpha\rangle_{\mathcal{D}}.\end{aligned} \tag{13}$$

By inspecting the first term of Eq. (13), we verify that the probability for the system to remain unmeasured until $t_{(2)}$ is $(1 - \delta p_{(1)})(1 - \delta p_{(2)})$, and from the the second term, we observe that $\mathcal{S}$ is measured in the second step with probability $(1 - \delta p_{(1)})\delta p_{(2)}$. Finally, in the last branch of $|\psi(t_{(2)})\rangle_{\mathcal{R}}$, as the measurement happened at the previous instant $t_{(1)}$ and $\mathcal{S}$ is no longer under measurement, the information remains recorded in $\mathcal{TD}$, and $\mathcal{S}$ evolves according to $\hat{U}_{\mathcal{S}}(t_{(2)}, t_{(1)})$.

Keeping this process up to the Nth step, we have

$$|\psi(t_{(N)})\rangle_{\mathcal{R}} = \Gamma(t_{(N)}, t_0) \left[\hat{U}_{\mathcal{S}}^{0}(t_{(N)}, t_0)|0\rangle_{\mathcal{S}}\right] |t_{(N)}\rangle_{\mathcal{T}}\, |0\rangle_{\mathcal{D}} + \sum_{k=1}^{N} \chi(t_{(k)}) \left[\hat{U}_{\mathcal{S}}(t_{(N)}, t_{(k)}) \sum_{\alpha} \hat{M}_\alpha\, \hat{U}_{\mathcal{S}}^{(k)}(t_{(k)}, t_0)|0\rangle_{\mathcal{S}}\right] |t_{(k)}\rangle_{\mathcal{T}}\, |\alpha\rangle_{\mathcal{D}}, \tag{14}$$

where

$$\hat{U}_{\mathcal{S}}^{0}(t_{(N)}, t_0) = \hat{U}_{\mathcal{S}}^{0(N)}(t_{(N)}, t_{(N-1)})\, \hat{U}_{\mathcal{S}}^{0(N-1)}(t_{(N-1)}, t_{(N-2)}) \ldots \hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)}, t_0), \tag{15}$$

$$\hat{U}_{\mathcal{S}}^{(k)}(t_{(k)}, t_0) = \hat{U}_{\mathcal{S}}^{1(k)}(t_{(k)}, t_{(k-1)})\, \hat{U}_{\mathcal{S}}^{0(k-1)}(t_{(k-1)}, t_{(k-2)}) \ldots \hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)}, t_0), \tag{16}$$

$$\Gamma(t_N, t_0) = \prod_{k=1}^{N} \sqrt{1-\delta p_{(k)}}\, e^{i\varphi_{(k)}} \quad \text{and} \quad \chi(t_{(k)}) = \sqrt{\delta p_{(k)}} \prod_{\ell=1}^{k-1} \sqrt{1-\delta p_{(\ell)}}\, e^{i\varphi_{(\ell)}}. \tag{17}$$

Here, $|\Gamma(t_{\scriptscriptstyle N}, t_0)|^2$ is the probability of the system not being measured from the beginning of the process until $t_{\scriptscriptstyle (N)}$, and accordingly, $\mathcal{D}$ has no information about $\mathcal{S}$ in this branch of Eq. (14). On the other hand, in the second expression of $|\psi(t_{\scriptscriptstyle (N)})\rangle_{\mathcal{R}}$, for a given value of k, the measurement took place in the interval $[t_{\scriptscriptstyle (k-1)}, t_{\scriptscriptstyle (k)}]$ with probability $|\chi(t_{\scriptscriptstyle (k)})|^2$, regardless of the outcome $|\alpha\rangle_{\mathcal{S}}$. Let us call $\chi(t_{\scriptscriptstyle (k)})$ the probability amplitude of the clock reading $t_{\scriptscriptstyle (k)}$ at the moment of the event. It should be noted that the width of $\chi(t_{\scriptscriptstyle (k)})$ is of the order of $\Delta t_{\mathcal{SD}}$. Thus, as all states $\{|\alpha\rangle_{\mathcal{S}}\}$ are under measurement, the probability of the system not being measured is negligible when $\beta = t_{\scriptscriptstyle (N)} \gtrsim t_0 + \Delta t_{\mathcal{SD}}$, and hence $\Gamma(t_{\scriptscriptstyle (N)})$ and $\chi(t_{\scriptscriptstyle (N)})$ are ≈ 0 in Eq. (14). Unlike the ideal measurement (9), the record of the detection time by $\mathcal{T}$ allows us to claim that the event in fact occurred at some instant $t_{\scriptscriptstyle (k)}$ in the interval $[t_0, t_0 + \Delta t_{\mathcal{SD}}]$.

It is worth pointing out that if $\mathcal{S}$ is composed of many independent subsystems, we can obtain the same expression as Eq. (14) as long as we do not specify the subsystem of $\mathcal{S}$ that is measured by $\mathcal{D}$. For instance, if $\mathcal{S}$ is composed of noninteracting and distinguishable particles $\mathcal{S}_i$ (with $i = 1, 2, \ldots$), and $\mathcal{D}$ measures one subsystem at a time, the same calculation above also results in Eq. (14), but with $\hat{M}_\alpha = \sum_i \hat{M}_{\mathcal{S}_i,\alpha}$, where $\hat{M}_{\mathcal{S}_i,\alpha} = \mathbb{1}_{\mathcal{S}_1} \otimes \mathbb{1}_{\mathcal{S}_2} \otimes \cdots \otimes \hat{M}_{\alpha_i} \otimes \mathbb{1}_{\mathcal{S}_{i+1}} \ldots$. Situations involving entangled systems are the subject of a work in progress.

Let us verify that $|\psi(t_{\scriptscriptstyle (N)})\rangle_{\mathcal{R}}$ provides partial information of the first term of $||\Psi\rangle$ presented in Eq. (2) of the introduction. By calling $t = t_{\scriptscriptstyle (k)}$ and rewriting the sum over k as a sum over t, Eq. (14) evaluated after one guarantees that the measurement by $\mathcal{D}$ takes place $[\beta = t_{\scriptscriptstyle (N)} \gtrsim t_0 + \Delta t_{\mathcal{SD}}]$ becomes

$$|\psi(t_{\scriptscriptstyle (N)})\rangle_{\mathcal{R}} = \sum_{\alpha, t>t_0}^{t_{(N)}} \psi_{\mathcal{S}}(\alpha|t)\,\chi(t)\left[\hat{U}_{\mathcal{S}}(t_{\scriptscriptstyle (N)}, t)\,|\alpha\rangle_{\mathcal{S}}\,|t, \alpha\rangle_{\mathcal{TD}}\right], \tag{18}$$

where $|t, \alpha\rangle_{\mathcal{TD}} = |t\rangle_{\mathcal{T}}|\alpha\rangle_{\mathcal{D}}$, and the normalization conditions are

$$\sum_{\alpha, t>t_0}^{t_{(N)}} |\psi_{\mathcal{S}}(\alpha|t)|^2\,|\chi(t)|^2 = 1 \quad \text{and} \quad \sum_{t>t_0}^{t_{(N)}} |\chi(t)|^2 = 1. \tag{19}$$

In Eq. (18), the expression within the brackets was obtained by rewriting the state of $\mathcal{S}$ in the second branch of Eq. (14) as

$$\hat{M}_\alpha\,\hat{U}^{(k)}_{\mathcal{S}}(t_{\scriptscriptstyle (k)}, t_0)\,|0\rangle_{\mathcal{S}} = \psi_{\mathcal{S}}(\alpha|t_{\scriptscriptstyle (k)})\,|\alpha\rangle_{\mathcal{S}},$$
$$\text{with} \quad \psi_{\mathcal{S}}(\alpha|t_{\scriptscriptstyle (k)}) = {}_{\mathcal{S}}\langle\alpha|\hat{U}^{(k)}_{\mathcal{S}}(t_{\scriptscriptstyle (k)}, t_0)|0\rangle_{\mathcal{S}} \tag{20}$$

being the Schrödinger "wave function" (amplitude) of the observable $\hat{\alpha}$ of $\mathcal{S}$.

Notice that Eq. (18) is a superposition of the ideal measurements (9), with each branch describing a different instant of detection. In practice, Eq. (18) can be applied to predict outcomes obtained by an experimentalist that verifies the records in $\mathcal{TD}$ when the clock $\mathcal{C}$ reads $\beta = t_{\scriptscriptstyle (N)} \gtrsim t_0 + \Delta t_{\mathcal{SD}}$. It is noteworthy that a more rigorous analysis of this problem would require the addition, for instance, of the experimentalist's brain developing the role of a detector to the theoretical calculations. Nevertheless, in this more elaborated approach, the same predictions as Eq. (18) should be obtained by conditioning $||\Psi\rangle$ on the experimentalist's brain. The reason for this agreement is that, as the information about $\mathcal{CS}$ is already registered in $\mathcal{TD}$ before the interaction with the experimentalist, the probabilities of the event (t, α) would not change if we added her to the evolution (8).

Under such circumstances, given that the experimentalist observes $\mathcal{TD}$ when $\beta = t_{\scriptscriptstyle (N)} \gtrsim t_0 + \Delta t_{\mathcal{SD}}$, the probability for her to verify that the detector measured $\mathcal{S}$ in $|\alpha\rangle_{\mathcal{S}}$ with the clock $\mathcal{C}$ reading t (and $\mathcal{T}$ registering t) is

$$P(t, \alpha) = |{}_{\mathcal{TD}}\langle t, \alpha|\psi(t_{\scriptscriptstyle (N)})\rangle_{\mathcal{R}}|^2 = |\psi_{\mathcal{S}}(\alpha|t)|^2\,|\chi(t)|^2. \tag{21}$$

This probability is the modulus squared of the amplitude (4) introduced in Sec. II: the probability of measuring α given that the clock reads $\beta = t$, $|\psi(\alpha|t)|^2$, multiplied by the probability for the clock to read t at the moment of the event, $|\chi(t)|^2$. In Eq. (18), as $\hat{U}_{\mathcal{S}}(t_{\scriptscriptstyle (N)}, t)$ only acts on $\mathcal{S}$, it does not influence the probability distribution (21). Also, the probability of measuring $|\alpha\rangle_{\mathcal{S}}$ regardless of the instant of detection can be calculated as

$$\begin{aligned} P(\alpha) &= \mathrm{Tr}\Big\{|\psi(t_{\scriptscriptstyle (N)})\rangle_{\mathcal{R}\,\mathcal{R}}\langle\psi(t_{\scriptscriptstyle (N)})| \otimes |\alpha\rangle_{\mathcal{D}\,\mathcal{D}}\langle\alpha|\Big\} \\ &= \sum_{t>t_0} P(t, \alpha) = \sum_{t>t_0} |\psi_{\mathcal{S}}(\alpha|t)|^2\,|\chi(t)|^2. \end{aligned} \tag{22}$$

Because $\chi(t) \approx 0$ when $t \approx t_0 + \Delta t_{\mathcal{SD}} < t_{\scriptscriptstyle (N)}$, $P(\alpha)$ does not depend on $t_{\scriptscriptstyle (N)}$, and thus the sum over t can be extended to infinity. The dependence of $P(\alpha)$ on the measurement interaction is verified in both the time probability amplitude $\chi(t)$ and $\psi_{\mathcal{S}}(\alpha|t)$. Nevertheless, if the measurement is "instantaneous," as in the ideal case (9) [$\chi(t)$ with negligible width], we can take $\psi_{\mathcal{S}}(\alpha|t) \approx \psi_{\mathcal{S}}(\alpha|t_0)$, and hence the probability of observing α in Eq. (22) becomes $P(\alpha) \approx |\psi_{\mathcal{S}}(\alpha|t_0)|^2$, as expected. Here, it should be used that $\sum_{t>t_0} |\chi(t)|^2 = 1$.

Lastly, let us turn our attention to the state of $\mathcal{R}$ when the clock reads an arbitrary value of β, as illustrated in Eq. (14). By using the notation of Eq. (18), $|\psi(\beta)\rangle_{\mathcal{R}} = {}_{\mathcal{C}}\langle\beta||\Psi\rangle$ becomes

$$\begin{aligned} |\psi(\beta)\rangle_{\mathcal{R}} = {}& \Gamma(\beta, t_0)\left[\hat{U}^{0}_{\mathcal{S}}(\beta, t_0)|0\rangle_{\mathcal{S}}\,|\beta, 0\rangle_{\mathcal{TD}}\right] \\ &+ \sum_{\alpha, t>t_0}^{\beta} \chi(t)\,\psi_{\mathcal{S}}(\alpha|t)\left[\hat{U}_{\mathcal{S}}(\beta, t)|\alpha\rangle_{\mathcal{S}}\,|t, \alpha\rangle_{\mathcal{TD}}\right]. \end{aligned} \tag{23}$$

For future discussions, it is essential to keep in mind that $|t, \alpha\rangle_{\mathcal{TD}}$ should not be recognized only as the physical state of $\mathcal{TD}$, but mainly by the information about $\mathcal{CS}$ that this state stores. Thus, as $\beta > t$, we will refer to the state $|t, \alpha\rangle_{\mathcal{TD}}$ in the past as the memory of the event "the system $\mathcal{S}$ in the state $|\alpha\rangle_{\mathcal{S}}$ with the clock $\mathcal{C}$ reading t." It is important to note that this particular interpretation is valid since the detector registers information about $\mathcal{S}$ by interacting directly with it, and the timer instantly recognizes the detection of $\mathcal{S}$. On this basis, by focusing on $|t, \alpha\rangle_{\mathcal{TD}}$ [information about $\mathcal{CS}$ recorded in $\mathcal{TD}$], we say that Eq. (23) aims to describe $\mathcal{CS}$ from the perspective of $\mathcal{TD}$.

C. $|\psi(\beta)\rangle_{\mathcal{R}}$ for two causally connected events

In this section, we will extend the calculation above to two events causally connected by assuming two consecutive measurements in the same system $\mathcal{S}$. Here, causality means that the occurrence of the second event depends on the happening of the first event. In this regard, we consider two pairs of $\mathcal{TD}$ so that, similarly to Eq. (11), the initial condition is

$$|\psi(t_0)\rangle_{\mathcal{R}} = |0\rangle_{\mathcal{S}}\, |t_0, 0\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_0, 0\rangle_{\mathcal{T}_2\mathcal{D}_2}, \tag{24}$$

where $\mathcal{T}_1\mathcal{D}_1$ and $\mathcal{T}_2\mathcal{D}_2$ register the measurements 1 and 2 respectively. From now on, for the sake of understanding, let us consider the simplest case in which the operators defined in Eq. (12) satisfy $\hat{U}^{0(j)}_{\mathcal{S}}(t_{(j+1)}, t_{(j)}) \approx \hat{U}^{1(j)}_{\mathcal{S}}(t_{(j+1)}, t_{(j)}) \approx \hat{U}_{\mathcal{S}}(t_{(j+1)}, t_{(j)})$ (assume the same for the evolution associated with the second measurement). For two causally related events, the first step of the evolution (8) is the same as that of Eq. (12), but with the presence of the state $|t_{(1)}, 0\rangle_{\mathcal{T}_2\mathcal{D}_2}$ at the end of each branch of $|\psi(t_{(1)})\rangle_{\mathcal{R}}$. It should be warned not to confuse the labels 1 and 2 with (1) and (2): $j = 1, 2$ specify the events, while (j) the possible values of β, t_1, and t_2 in the discrete evolution $(t_{(1)}, t_{(2)}, \ldots)$. For simplicity, let us assume that the second measurement starts right after the end of the first one.

Under these circumstances, the second step of our schematic evolution is

$$\begin{aligned}
|\psi(t_{(2)})\rangle_{\mathcal{R}} &= \sqrt{1-\delta p_{1(1)}}\, e^{\varphi_{1(1)}} \sqrt{1-\delta p_{1(2)}}\, e^{\varphi_{2(2)}} \Big[\hat{U}_{\mathcal{S}}(t_{(2)}, t_0)\, |0\rangle_{\mathcal{S}}\Big]\, |t_{(2)}, 0\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_{(2)}, 0\rangle_{\mathcal{T}_2\mathcal{D}_2} \\
&+ \sqrt{1-\delta p_{1(1)}}\, e^{\varphi_{1(1)}} \sqrt{\delta p_{1(2)}} \Big[\sum_{\alpha_1} \hat{M}_{\alpha_1}\, \hat{U}_{\mathcal{S}}(t_{(2)}, t_0)\, |0\rangle_{\mathcal{S}}\Big]\, |t_{(2)}, \alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_{(2)}, 0\rangle_{\mathcal{T}_2\mathcal{D}_2} \\
&+ \sqrt{\delta p_{1(1)}} \sqrt{1-\delta p_{2(2,1)}}\, e^{\varphi_{2(2)}} \Big[\hat{U}_{\mathcal{S}}(t_{(2)}, t_{(1)}) \sum_{\alpha_1} \hat{M}_{\alpha_1}\, \hat{U}_{\mathcal{S}}(t_{(1)}, t_0)\, |0\rangle_{\mathcal{S}}\Big]\, |t_{(1)}, \alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_{(2)}, 0\rangle_{\mathcal{T}_2\mathcal{D}_2} \\
&+ \sqrt{\delta p_{1(1)}} \sqrt{\delta p_{2(2,1)}} \Big[\sum_{\alpha_2} \hat{M}_{\alpha_2}\, \hat{U}_{\mathcal{S}}(t_{(2)}, t_{(1)}) \sum_{\alpha_1} \hat{M}_{\alpha_1}\, \hat{U}_{\mathcal{S}}(t_{(1)}, t_0)\, |0\rangle_{\mathcal{S}}\Big]\, |t_{(1)}, \alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_{(2)}, \alpha_2\rangle_{\mathcal{T}_2\mathcal{D}_2},
\end{aligned} \tag{25}$$

Here, the first two branches come from the evolution of the unmeasured contribution of $|\psi(t_{(1)})\rangle_{\mathcal{R}}$. The first term of $|\psi(t_{(2)})\rangle_{\mathcal{R}}$ illustrates the situation where no events happen during the clock reading interval $[t_0, t_{(2)}]$. In the second one, the first event takes place at the second step when the clock reads $t = t_{(2)}$. The last two contributions in Eq. (25) result from the evolution of the branch of $|\psi(t_{(1)})\rangle_{\mathcal{R}}$ in which the first event took place at $t_{(1)}$. In the third (fourth) expression of (25), the second measurement does not happen (happens) at $t_{(2)}$, and $\delta p_{2(2,1)}$ is the probability of the second event occurring with the clock reading $t_{(2)}$ regardless of the outcome α_2, given that the first measurement took place at $t_{(1)}$.

Keep doing the evolution until $\beta = t_{(N)}$, we obtain a similar expression to (23) given by

$$\begin{aligned}
|\psi(\beta)\rangle_{\mathcal{R}} &= \Gamma_1(\beta, t_0)\Big[\hat{U}_{\mathcal{S}}(\beta, t_0)\, |0\rangle_{\mathcal{S}}\, |\beta, 0\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |\beta, 0\rangle_{\mathcal{T}_2\mathcal{D}_2}\Big] \\
&+ \sum_{\substack{\alpha_1 \\ t_1 > t_0}}^{\beta} \Gamma_2(\beta, t_1)\, \psi_{\mathcal{S}}(\alpha_1|t_1)\, \chi_1(t_1)\Big[\hat{U}_{\mathcal{S}}(\beta, t_1)\, |\alpha_1\rangle_{\mathcal{S}}\, |t_1, \alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |\beta, 0\rangle_{\mathcal{T}_2\mathcal{D}_2}\Big] \\
&+ \sum_{\substack{\alpha_2 \\ t_2 > t_1}}^{\beta} \sum_{\substack{\alpha_1 \\ t_1 > t_0}}^{t_1<\beta} \psi_{\mathcal{S}}(\alpha_2|t_2; \alpha_1, t_1)\, \chi_2(t_2|t_1)\, \psi_{\mathcal{S}}(\alpha_1|t_1)\, \chi_1(t_1)\Big[\hat{U}_{\mathcal{S}}(\beta, t_2)|\alpha_2\rangle_{\mathcal{S}}\, |t_1, \alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_2, \alpha_2\rangle_{\mathcal{T}_2\mathcal{D}_2}\Big],
\end{aligned} \tag{26}$$

where

$$\psi_{\mathcal{S}}(\alpha_2|t_2; \alpha_1, t_1) = {}_{\mathcal{S}}\langle\alpha_2|\hat{U}_{\mathcal{S}}(t_2, t_1)|\alpha_1\rangle_{\mathcal{S}} \tag{27}$$

is the Schrödinger wave function of $\mathcal{S}$ at the instant t_2 in the case where the first event (t_1, α_1) took place. Hence, $|\psi_{\mathcal{S}}(\alpha_2|t_2; \alpha_1, t_1)|^2$ is the probability of the system having been measured in the state $|\alpha_2\rangle_{\mathcal{S}}$ given that the clock read t_2 and that $\mathcal{S}$ was measured in $|\alpha_1\rangle_{\mathcal{S}}$ with $\mathcal{C}$ reading t_1. Finally, the time probability amplitudes of the first and second events are

$$\chi_1(t_{(k)}) = \sqrt{\delta p_{1(k)}} \prod_{q=1}^{k-1} \sqrt{1-\delta p_{1(q)}}\, e^{i\varphi_{1(q)}}$$

and

$$\chi_2(t_{(\ell)}|t_{(k)}) = \sqrt{\delta p_{(\ell,k)}} \prod_{r=k+1}^{\ell-1} \sqrt{1-\delta p_{2(r,k)}}\, e^{i\varphi_{2(r,k)}}. \tag{28}$$

Let us carefully analyze Eqs. (26)–(28). In the first expression of Eq. (26), no events happen between $[t_0, \beta]$, and thus both $\mathcal{T}_1\mathcal{D}_1$ and $\mathcal{T}_2\mathcal{D}_2$ have no information about $\mathcal{S}$. In the second term of Eq. (26), $|\alpha_1, t_1\rangle_{\mathcal{D}_1\mathcal{T}_1}|0, \beta\rangle_{\mathcal{D}_2\mathcal{T}_2}$ represents the situation in which the system $\mathcal{S}$ was in the state $|\alpha_1\rangle_{\mathcal{S}}$ with the clock reading t_1, and no event happens from t_1 to β. Besides, $|\chi_1(t_{(k)})|^2$ is the probability of the first event having happened with the clock reading $t_{(k)}$, regardless of the outcome α_1. Finally, the last line of $|\psi(\beta)\rangle_{\mathcal{R}}$ describes the case where the two events occurred within the interval $]t_0, \beta]$. Thus, $|\alpha_1, t_1\rangle_{\mathcal{D}_1\mathcal{T}_1}|\alpha_2, t_2\rangle_{\mathcal{D}_2\mathcal{T}_2}$ means "$\mathcal{S}$ was in the state $|\alpha_1\rangle_{\mathcal{S}}$ with the clock reading t_1, and in the state $|\alpha_2\rangle_{\mathcal{S}}$ at t_2." Also, $|\chi_2(t_{(\ell)}|t_{(k)})|^2$ is the probability of the second measurement having taken place at time $t_{(\ell)}$, given that the first measurement happened at time $t_{(k)} < t_{(\ell)}$, independent of the outcome. This causal dependence can be verified by observing the sum over t_2 in Eq. (26), in which $t_2 > t_1$, as well as the product operator on the right side of Eq. (28), which begins at $k+1$. It ensures that the second event takes place only after the first one. The joint probability of these two events is the modulus squared of the wave function outside the brackets of the last sum of Eq. (26),

$$P(\alpha_1, t_1; \alpha_2, t_2) = |\psi_{\mathcal{S}}(\alpha_2|t_2; \alpha_1, t_1)|^2\,|\chi_2(t_2|t_1)|^2 \times|\psi_{\mathcal{S}}(\alpha_1|t_1)|^2|\chi_1(t_1)|^2. \tag{29}$$

Finally, it should be noticed that the conditioned state (26) has an indefinite number of events, so that if an experimentalist decides to check the records in $\mathcal{TD}$ at this moment ($\beta = t_{(N)}$), she can observe none, one, or two events. Although Eq. (26) can predict observations of an external experimentalist, as discussed earlier, a rigorous analysis would demand the conditioning of $||\Psi\rangle$ on the state of the experimentalist's brain and not on $|\beta\rangle_{\mathcal{C}}$. Thus, as $\mathcal{C}$ is by definition isolated, in Sec. IV, we will describe events through $||\Psi\rangle$ and not $|\psi(\beta)\rangle_{\mathcal{R}}$. But first, let us investigate how to deal with events in more elementary circumstances, i.e., without resorting to an artificial laboratory timer to register the detection time.

D. The environment as a timer (and detector)

As discussed in Sec. II, to formulate our interpretation of the emergence of time, a more general macroscopic observer should also be able to play the role of the set of detectors and timers. The reason for this is that although a collection of innumerable $\mathcal{TD}$s will provide the ideal setting for our interpretation of time (as we will see in the next section), time should not emerge only in situations involving SWP timers. Thus, now instead of using $\mathcal{T}$s, we will consider the environment monitoring the detector $\mathcal{D}$. In this section, we will still continue dealing with $|\psi(\beta)\rangle_{\mathcal{R}}$ and using some previous results.

Let us start with a single detector "continuously" interacting with the environment around it. In this situation, any macroscopic change in the detector's state is "instantaneously" recognized by the environment through a measurement-like process similar to Eq. (9). It means that the timescale of the interaction between the system and detector [$\Delta t_{\mathcal{SD}}$ in Eq. (9)] is significantly longer than the time $\delta t_{\mathcal{DE}}$ required for the environment to read information about $\mathcal{S}$ stored in the detector [32].

Under these circumstances, at every interval $\delta t_{\mathcal{DE}}$, a subsystem of the environment acquires information about the detector and then rapidly departs from it, not interacting with the detector again. This interaction defines the pointer states of the detector [33]. As $\delta t_{\mathcal{DE}}$ is the shortest timescale of the problem, it is convenient to assume that the steps δt of the evolution (8) is approximately $\delta t_{\mathcal{DE}}$. Thus, at every δt, the detector interacts with a new degree of freedom of the environment in such a way that the detector always finds a subsystem of the environment in the same initial state. This kind of interaction, but in a completely different context, can be found in Ref. [32].

Let us consider the evolution for the case of a single event of Sec. III B, with the environment interacting with $\mathcal{D}$ and in the absence of $\mathcal{T}$. In this context, the initial state (11) becomes

$$|\psi(t_0)\rangle_{\mathcal{R}} = |0\rangle_{\mathcal{S}}\,|0\rangle_{\mathcal{D}}\,|r_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}, \tag{30}$$

where $|r_{(1)}\rangle, |r_{(2)}\rangle, \ldots$ are the ready states of the subsystems of the environment that eventually acquire information about $\mathcal{S}$ via interaction with $\mathcal{D}$. As in the circumstances of this section, the evolution of $\mathcal{S}$ and $\mathcal{D}$ is the same as the one in Sec. III B, let us exclusively focus on the evolution of the environment. In the first step of the evolution of $\mathcal{R}$, given by Eq. (12), the state of the environment changes according to

$$|r_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}} \to |0_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}},\ |\alpha_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}, \tag{31}$$

where at $t_{(1)}$, $|\psi(t_{(1)})\rangle_{\mathcal{R}}$ becomes a superposition of the two last states. In the branch of $|\psi(t_{(1)})\rangle_{\mathcal{R}}$ where there is no detection, the environment records the state $|0\rangle_{\mathcal{D}}$ of the detector, and thus the state of the environment is $|0_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}$. In the other branch of Eq. (12), the detector measures the system and instantly transmits this information to the environment's subsystem (1). As a result, the state of the environment is $|\alpha_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}$. In both branches of Eq. (12), the dispersion of the subsystem (1) makes the detector interact with the environment's degree of freedom (2) at $t_{(2)}$.

From $t_{(1)}$ to $t_{(2)}$ [see Eq. (13)], the environment makes the transitions

$$\begin{aligned} |0_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}} &\to |0_{(1)}, 0_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}\ ,\\ &\quad\ |0_{(1)}, \alpha_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}};\ \text{ and}\\ |\alpha_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}} &\to |\alpha_{(1)}, \alpha_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}, \end{aligned} \tag{32}$$

where $|\psi(t_{(2)})\rangle_{\mathcal{R}}$ becomes a superposition of the three states on the right-hand side of Eq. (32). Here, the branch $|0_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}$ splits into two parts. In one of them, the detector remains without measuring the system, and hence the environment "visualizes" the detector again in its initial state, becoming $|0_{(1)}, 0_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}$. In the other branch of $|\psi(t_{(2)})\rangle_{\mathcal{R}}$ coming from $|0_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}$, the detector measures the system, and then the environment's state becomes $|0_{(1)}, \alpha_{(2)}, r_{(3)}\ldots\rangle_{\mathcal{E}}$. Note that in Eq. (32), the information about $\mathcal{S}$ is stored in the subsystem (2) of the environment, whereas the subsystem (1), which is away from the detector at $t_{(2)}$, remains in the state $|0_{(1)}\rangle_{\mathcal{E}}$. Regarding the branch $|\alpha_{(1)}, r_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}$ at $t_{(1)}$, as we are dealing with a single event, the environment changes to $|\alpha_{(1)}, \alpha_{(2)}, r_{(3)}, \ldots\rangle_{\mathcal{E}}$ at $t_{(2)}$. The information about $\mathcal{S}$ recorded in the detector $\mathcal{D}$ is now also stored in the environment's subsystem (2).

Keeping this evolution until the step N, in the branch of $|\psi(t_{(N)})\rangle_{\mathcal{R}}$ where the measurement happened at the instant $t_{(k)}$, the environment has state

$$|t_{(k)},\alpha\rangle_{\mathcal{E}} \equiv |0_{(1)},0_{(2)},\dots,0_{(k-1)},\alpha_{(k)},\dots,\alpha_{(N)},r_{(N+1)},r_{(N+2)},\dots\rangle_{\mathcal{E}}. \tag{33}$$

It is readily verified that the position (k) of the first subsystem of $\mathcal{E}$ that registers α indicates that the clock $\mathcal{C}$ read $t=t_{(k)}$ at the moment of the event. Therefore, for different clock readings $t_{(k)}$ at the instant of the event, $\mathcal{E}$ records the state of $\mathcal{S}$ in distinct arrangements. As the states $\{|0_{(1)},0_{(2)},\dots,\alpha_{(k)},\dots,\alpha_{(N)},r_{(N+1)},\dots\rangle_{\mathcal{E}}\}_{k=1,2,\dots,N}$ are orthogonal to each other, they unambiguously register the events of the set $\{(t_{(k)},\alpha)\}_{k=1,2,\dots,N}$. In other words, by measuring the environment subsystems, it is possible, in principle, to find out the state $|\alpha\rangle_{\mathcal{S}}$ of $\mathcal{S}$ and the clock reading when $\mathcal{S}$ was in this state. For instance, if $\alpha=\pm$ and the system was in $|+\rangle_{\mathcal{S}}$ ($|-\rangle_{\mathcal{S}}$) at $t_{(2)}$ ($t_{(3)}$), the state of the environment at $t=t_{(N)}$ is

$$|t_{(2)},+\rangle_{\mathcal{E}} \equiv |0_{(1)},+_{(2)},+_{(3)},\dots,+_{(N)},r_{(N+1)},\dots\rangle_{\mathcal{E}}$$
$$(\,|t_{(3)},-\rangle_{\mathcal{E}} \equiv |0_{(1)},0_{(2)},-_{(3)},\dots,-_{(N)},0_{(N+1)},\dots\rangle_{\mathcal{E}}). \tag{34}$$

Notice that we could consider the environment interacting directly with $\mathcal{S}$, and thereby also disregard the detector in the measurement process and still obtain the same state for $|\psi(\beta)\rangle_{\mathcal{R}}$ as the one with $\mathcal{TD}$ of Sec. III B. For example, in the case where a particle passes through a cloud chamber [34], we could use, in principle, the formalism presented here to describe this particle from the perspective of the supersaturated vapor of water inside the chamber.

Therefore, when the information of the moment of the event is registered in the degrees of freedom of $\mathcal{E}$, the conditioned state of $\mathcal{R}$ should be given by equations such as (23) and (26), but with $\mathcal{TD}$ (or collection of $\mathcal{TD}$s) being replaced by $\mathcal{DE}$. Nevertheless, because usually we do not have access to the environment's degrees of freedom, we do not detect $P(t,\alpha)$, but $P(\alpha)$ as written in Eq. (22). Besides, notice that if $\Delta t_{\mathcal{SD}}$ is short enough, Eq. (18) with $\mathcal{E}$ substituting $\mathcal{T}$ becomes

$$|\psi(t_{(N)})\rangle_{\mathcal{R}} \approx \sum_{\alpha} \psi_{\mathcal{S}}(\alpha|t_0)\,|\alpha\rangle_{\mathcal{S}}\,|\alpha\rangle_{\mathcal{D}}\left[\sum_{k=1}^{N}\chi(t_{k})\,|t_{(k)},\alpha\rangle_{\mathcal{E}}\right]. \tag{35}$$

In this simpler scenario, the inaccessibility of the information about $\mathcal{S}$ stored in $\mathcal{E}$ allows us to conveniently define $|\alpha\rangle_{\mathcal{E}} \equiv \sum_{k=1}^{N}\chi(t_k)\,|t_{(k)},\alpha\rangle_{\mathcal{E}}$ so that

$$|\psi(t_{(N)})\rangle_{\mathcal{R}} = \sum_{\alpha}\psi_{\mathcal{S}}(\alpha|t_0)\,|\alpha\rangle_{\mathcal{S}}\,|\alpha\rangle_{\mathcal{D}}\,|\alpha\rangle_{\mathcal{E}} \tag{36}$$

becomes the measurement (9) taking into account the environment-induced decoherence. In this case, $P(\alpha)\approx |\psi_{\mathcal{S}}(\alpha|t_0)|^2$, as expected.

Finally, notice that the generalization of the environment's state for an arbitrary number of events is straightforward. For instance, in the case of two events that happened at the instants $t_{(k)}$ and $t_{(\ell)}$, the state of the environment for $\beta=t_{(N)}$ is

$$|\,t_{(k)}\,,\alpha_1\rangle_{\mathcal{E}_1}|t_{(\ell)},\alpha_2\rangle_{\mathcal{E}_2} \equiv \big|0_{(1)},\dots,\alpha_{1(k)},\dots,\alpha_{1(\ell)},\dots,\alpha_{1(N)},r_{(N+1)},\dots\rangle_{\mathcal{E}_1} \otimes |0_{(1)},\dots,0_{(k)},\dots,\alpha_{2(\ell)},\dots,\alpha_{2(N)},r_{(N+1)},\dots\rangle_{\mathcal{E}_2}. \tag{37}$$

Here, $\mathcal{E}_1$ and $\mathcal{E}_2$ represent the environment around the detector $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. After the discussion of this Sec. III about events via the conditioned state $|\psi(\beta)\rangle_{\mathcal{R}}$ (with either timers and environment), let us now formalize a more general description in terms of $||\Psi\rangle$.

IV. EVENTS AND THE EMERGENCE OF TIME FROM THE PERSPECTIVE OF $||\Psi\rangle$

A. $||\Psi\rangle$ for a single event

Before starting the description of events via $||\Psi\rangle$, it is important to bear in mind that in Sec. III, we obtained a joint probability distribution for events as discussed in the introduction and Sec. II, but we still resorted to a relative approach between $\mathcal{R}$ and $\mathcal{C}$, $|\psi(\beta)\rangle_{\mathcal{R}}$. Thus, as remarked in Sec. III, using $|\psi(\beta)\rangle_{\mathcal{R}}$ requires an observer (external to formalism) that measures $\mathcal{C}$ to define β and thus allow the conditioning $|\psi(\beta)\rangle_{\mathcal{R}} = {}_{\mathcal{C}}\langle\beta||\Psi\rangle$. Now, unlike Sec. III, we want to describe an event without reference to an external observer. Thus, as in this section we will consider a single event, the focus now is exclusively on describing $\mathcal{CS}$ from the perspective of $\mathcal{TD}$ (or a macroscopic observer that plays a similar role). Hence, as $\mathcal{TD}$ does not interact with the clock $\mathcal{C}$, we will not condition $||\Psi\rangle$ on a specific state $|\beta\rangle_{\mathcal{C}}$.

It should be remembered that, without discussing the nature of time, we will continue referring to β and t as measures of time until Sec. IV B. Let us begin by calculating $||\Psi\rangle$ for a single event. By substituting Eq. (23) into Eq. (7) with $\mathcal{R}=\mathcal{SDT}$, we obtain

$$||\Psi\rangle = \sum_{\beta\geqslant t_0}\Gamma(\beta,t_0)\Big[|\beta\rangle_{\mathcal{C}}\,\hat{U}_{\mathcal{S}}(\beta,t_0)|0\rangle_{\mathcal{S}}\,|\beta,0\rangle_{\mathcal{TD}}\Big] + \sum_{\beta\geqslant t_0}\sum_{\substack{\alpha\\ t>t_0}}^{\beta}\chi(t)\,\psi_{\mathcal{S}}(\alpha|t)\Big[|\beta\rangle_{\mathcal{C}}\hat{U}_{\mathcal{S}}(\beta,t)|\alpha\rangle_{\mathcal{S}}|t,\alpha\rangle_{\mathcal{TD}}\Big]. \tag{38}$$

Here, $||\Psi\rangle$ takes into account both the occurrence and nonoccurrence of the event in the interval $[t_0,\beta]$ of the clock reading, for $\beta=t_0,t_{(1)},t_{(2)},\dots$. Now, we will analyze Eq. (38) carefully to obtain a suitable notation for describing events.

A relevant feature of $||\Psi\rangle$ can be verified by rewriting the second term of Eq. (38) using the relation $\sum_{\beta\geqslant t_0}\sum_{t>t_0}^{\beta} = \sum_{t>t_0}\sum_{\beta\geqslant t}$. By highlighting only this term and separating the contribution of $\beta=t$ from the sum over β, Eq. (38) becomes

$$||\Psi\rangle = \dots + \sum_{\substack{\alpha\\ t>t_0}}\chi(t)\,\psi_{\mathcal{S}}(\alpha|t)\,|t\rangle_{\mathcal{C}}|\alpha\rangle_{\mathcal{S}}\,|t,\alpha\rangle_{\mathcal{TD}} + \sum_{\substack{\alpha\\ t>t_0}}\sum_{\beta>t}\chi(t)\,\psi_{\mathcal{S}}(\alpha|t)|\beta\rangle_{\mathcal{C}}\hat{U}_{\mathcal{S}}(\beta,t)|\alpha\rangle_{\mathcal{S}}\,|t,\alpha\rangle_{\mathcal{TD}}. \tag{39}$$

By analyzing the first term of Eq. (39), first we observe the correlation between the state of $\mathcal{CS}$ and the information acquired by $\mathcal{TD}$. Let us call this α-time agreement as the quantum event (t,α) whose state is defined as

$$||t,\alpha\rangle \equiv |t,\alpha\rangle_{\mathcal{CS}}\, |t,\alpha\rangle_{\mathcal{TD}}. \tag{40}$$

Besides that, as discussed earlier, since the modulus squared of $\psi(\alpha|t)\chi(t)$ [coefficient of the first term of Eq. (39)] is the probability of (t,α), we will call this amplitude the wave function $\Psi(t,\alpha)$ of this event. In Eq. (40), we use the two bars to indicate that this state contains the pairs $\mathcal{TD}$ and $\mathcal{CS}$ recording the same information. Note that although $\mathcal{C}$ is noninteracting, the correlation between $\mathcal{C}$ and $\mathcal{R}$ that comes from the PW solution [Eq. (7)] allows $\mathcal{T}$ (or a macroscopic observer) to register the reading of $\mathcal{C}$ at the time of the measurement performed by $\mathcal{D}$. Thus, the state $||t,\alpha\rangle$ expresses what $\mathcal{TD}$ records about $\mathcal{CS}$.

It should be remembered that in the previous sections, from $|\psi(\beta)\rangle_{\mathcal{R}}$, we obtained the probability for $\mathcal{D}$ to measure $|\alpha\rangle_{\mathcal{S}}$ at time t given that $\mathcal{C}$ reads $\beta > t$. In that approach, an experimentalist had to measure $\mathcal{C}$ to guarantee the reading β. Because $\beta > t$, we referred to $|t,\alpha\rangle_{\mathcal{TD}}$ in $|\psi(\beta)\rangle_{\mathcal{R}}$ as an event that occurred in the past, i.e., before the experimentalist measures $\mathcal{C}$. In contrast, now to understand the state of Eq. (40) better, first we should remove the experimentalist from the physical picture and assume $\mathcal{TD}$ as the only observer. Then, visualize $\mathcal{C}$ as a clock that always keeps running, while $\mathcal{T}$ stops running at the moment of the measurement. Under such circumstances, the agreement $(\beta = t)$ between $\mathcal{C}$ and $\mathcal{T}$ in $||t,\alpha\rangle$ represents $\mathcal{CS}$ at the moment of the observation performed by $\mathcal{TD}$, and hence $||t,\alpha\rangle$ will be referred in the present as "from the perspective of $\mathcal{TD}$, the system $\mathcal{S}$ has the property α [is in the state $|\alpha\rangle_{\mathcal{S}}$] and the clock $\mathcal{C}$ reads t." Note that another observer uncorrelated to $\mathcal{TD}$ cannot affirm the same about $\mathcal{CS}$. Thus, $|\Psi(t,\alpha)|^2$ is the probability of the system being in the state $|\alpha\rangle_{\mathcal{S}}$ and the clock reading t from $\mathcal{TD}$'s point of view. As remarked before, unlike the classical treatment for events, Eq. (40) shows that a quantum event happens with the exchange of information between physical systems, so that the event can be seen as the very act of registering this information.

Now, by inspecting the second contribution of Eq. (39), we verify that in the product state $|\beta\rangle_{\mathcal{C}}\hat{U}_{\mathcal{S}}(\beta,t)|\alpha\rangle_{\mathcal{S}}|t,\alpha\rangle_{\mathcal{TD}}$, the reading of $\mathcal{C}$ is evaluated after the occurrence of the event (t,α), i.e., $\beta > t$. Thus, similarly to the last section, the state $|t,\alpha\rangle_{\mathcal{TD}}$ in the second expression of Eq. (39) is a memory of the event (t,α) that occurred with the clock reading $\beta = t$. Notice that there is a sum over $\beta > t$ in Eq. (39) because, in our model, the event remains recorded in the detector and timer after its occurrence. Finally, let us turn our attention to the first term of Eq. (38), which concerns the physical scenario before the occurrence of the event (t,α). In this branch, the modulus squared of $\Gamma(\beta,t_0)$ provides the probability for the event not to happen in the interval $[t_0,\beta]$. Thus, consider $\Psi(t,0) \equiv \Gamma(t,t_0)$ and, in the spirit of Eq. (40), define

$$||\beta,0\rangle \equiv |\beta\rangle_{\mathcal{C}}\, \hat{U}_{\mathcal{S}}(\beta,t)|0\rangle_{\mathcal{S}}\, |\beta,0\rangle_{\mathcal{TD}} \tag{41}$$

as the state associated with the absence of events. In Eq. (41), the clock and the timer are synchronized, but $\mathcal{D}$ has no information about $\mathcal{S}$. Also, notice that $||t,\alpha\rangle$ and $||\beta,0\rangle$ are orthogonal to each other.

By using the states (40) and (41), we will rewrite $||\Psi\rangle$ by joining the first term of Eq. (38) and the first one of Eq. (39)—which refer to the nonoccurrence and occurrence of the event respectively—in the same sum. To this end, first, let us make the convenient change of variables: $\beta \to t_1$ in the first expression of Eq. (38), and $t \to t_1$ and $\beta \to t_2$ in Eq. (39). Besides that, consider $\alpha \to \alpha_1$. Now, by using Eqs. (40) and (41), and the first terms of Eqs. (38) and (39) expressed in a sum over $t_{(1)}$, $||\Psi\rangle$ of Eq. (38) becomes

$$\begin{aligned} ||\Psi\rangle = &\sum_{\substack{a_1\\ t_1\geqslant t_0}} \Psi(t_1,a_1)\, ||t_1,a_1\rangle \\ &+ \sum_{t_2>t_1}\sum_{\substack{\alpha_1\\ t_1>t_0}} \chi(t_1)\, \psi(\alpha_1|t_1) \\ &\times\Big[|t_2\rangle_{\mathcal{C}}\, \hat{U}_{\mathcal{S}}(t_2,t_1)|\alpha_1\rangle_{\mathcal{S}}\, |t_1,\alpha_1\rangle_{\mathcal{TD}}\Big], \end{aligned} \tag{42}$$

with $a_1 = 0$ and $a_1 = \alpha_1$ describing the nonoccurrence and occurrence of the event respectively.

Lastly, the second contribution of Eq. (42), which is the branch associated with memory, should be written in the same notation as the first one. In other words, this branch should be expressed in such a way that it represents the absence of a second event with the memory of the first event. To this end, consider a collection of detectors and timers such that $\mathcal{TD} = \mathcal{T}_1\mathcal{D}_1\,\mathcal{T}_2\mathcal{D}_2\,\mathcal{T}_3\mathcal{D}_3\ldots$. Now, let us consider this collection as being the observer. Because we are still dealing with a single event, these additional detectors do not measure $\mathcal{S}$. They will be useful only to obtain the desired notation. By assuming that each timer has the same initial condition as $\mathcal{T}_1$ and is synchronized with $\mathcal{C}$, the states (40) and (41) in the presence of the additional detectors and timers become $||t_1,\alpha_1\rangle \equiv |t_1,\alpha_1\rangle_{\mathcal{CS}}\, |t_1,\alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_1,0\rangle_{\mathcal{T}_2\mathcal{D}_2}\, |t_1,0\rangle_{\mathcal{T}_3\mathcal{D}_3}\ldots$ and $||t_1,0\rangle \equiv |t_1\rangle_{\mathcal{C}}\, \hat{U}_{\mathcal{S}}(t_1,t_0)|\alpha_1\rangle_{\mathcal{S}}\, |t_1,0\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_1,0\rangle_{\mathcal{T}_2\mathcal{D}_2}\, |t_1,0\rangle_{\mathcal{T}_3\mathcal{D}_3}\ldots$, respectively. Notice that in both cases, the new detectors have no information about $\mathcal{S}$. Also, despite the presence of numerous $\mathcal{T}_j\mathcal{D}_j$, as our focus is on events, the state of the event (t_1,α_1) is still represented by the single ket $||t_1,\alpha_1\rangle$.

Under the circumstances assumed above, by following the same notation as the first term of Eq. (42), the memory state [ket within the brackets in Eq. (42)] is rewritten as

$$\begin{aligned} ||t_1,\alpha_1;t_2,0\rangle \equiv\ & |t_2\rangle_{\mathcal{C}}\, \hat{U}_{\mathcal{S}}(t_2,t_1)|\alpha_1\rangle_{\mathcal{S}} \\ &\otimes|t_1,\alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_2,0\rangle_{\mathcal{T}_2\mathcal{D}_2}\, |t_2,0\rangle_{\mathcal{T}_3\mathcal{D}_3}\ldots, \end{aligned} \tag{43}$$

where $\mathcal{T}_1\mathcal{D}_1$ records the event (t_1,α_1) and $\mathcal{T}_2\mathcal{D}_2\,\mathcal{T}_3\mathcal{D}_3\ldots$ do not have information about $\mathcal{S}$ with the clock reading t_2. Therefore, with the additional timers and detectors, $||t_1,\alpha_1;t_2,0\rangle$ can be seen (in a similar way to $||t_1,0\rangle$) as the state related to the absence of a second event in the interval $]t_1,t_2]$, and the memory of the first event (t_1,α_1). On the other hand, the state related to the occurrence of the second event—which must have zero amplitude since we are still dealing with a single event—can be written (in a similar way to $||t_1,\alpha\rangle$) as

$$\begin{aligned} &||t_1,\alpha_1;t_2,\alpha_2\rangle \\ &\quad\equiv |t_1,\alpha_1\rangle_{\mathcal{CS}}\, |t_1,\alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\, |t_2,\alpha_2\rangle_{\mathcal{T}_2\mathcal{D}_2}\, |t_2,0\rangle_{\mathcal{T}_3\mathcal{D}_3}\ldots, \end{aligned} \tag{44}$$

with the α-time agreement happening between $\mathcal{CS}$ and $\mathcal{T}_2\mathcal{D}_2$, and including the memory of the first event (t_1, α_1).

Finally, according to Eq. (42), the probability amplitudes associated with the states (43) and (44) are $\Psi(t_1, \alpha_1; t_2, 0) = \Psi(t_1, \alpha_1) = \chi(t_1)\,\psi(\alpha_1|t_1)$ and $\Psi(t_1, \alpha_1; t_2, \alpha_2) = 0$, respectively. Then, by taking into account $\mathcal{T}_2\mathcal{D}_2\,\mathcal{T}_3\mathcal{D}_3\ldots$, the global state $||\Psi\rangle$ of Eq. (42) can be rewritten as

$$||\Psi\rangle = \sum_{\substack{a_1 \\ t_1 \geqslant t_0}} \Psi(t_1, a_1)\,||t_1, a_1\rangle + \sum_{\substack{a_2 \\ t_2 > t_1}} \sum_{\substack{\alpha_1 \\ t_1 > t_0}} \Psi(t_1, \alpha_1; t_2, a_2)\,||t_1, \alpha_1; t_2, a_2\rangle, \quad (45)$$

where $a_2 = 0$ [the memory contribution of Eq. (42)] can now be seen as the absence of the second event and $a_2 = \alpha_2$ as the occurrence of the second event. Here, $\Psi(t_1, \alpha_1; t_2, 0)$ is the probability amplitude of the second event not happening in the interval $]t_1, t_2]$ and the first event (t_1, α_1) having occurred. Notice that by Bayes' rule, this distribution can be written as $|\Psi(t_1, \alpha_1; t_2, 0)|^2 = P(t_2, 0|t_1, \alpha_1)\,|\Psi(t_1, \alpha_1)|^2$, with $P(t_2, 0|t_1, \alpha_1) = 1$. Hence, as expected, since we are considering a single event, the probability of the second event not occurring is always 1, regardless of the value of t_2. It should be noted that as the event states are orthogonal to each other, we have

$$\begin{aligned} &\langle t_1, a_1||t_1', a_1'\rangle = \delta_{t_1,t_1'}\,\delta_{a_1,a_1'}; \\ &\langle t_1, \alpha_1; t_2, a_2||t_1', \alpha_1'; t_2', a_2'\rangle = \delta_{t_1,t_1'}\,\delta_{\alpha_1,\alpha_1'}\,\delta_{t_2,t_2'}\,\delta_{a_2,a_2'}; \text{ and} \\ &\langle t_1, a_1||t_1, \alpha_1; t_2, a_2\rangle = 0, \end{aligned} \quad (46)$$

which imply that $\langle t_1, a_1||\Psi\rangle = \Psi(t_1, a_1)$ and $\langle t_1, \alpha_1; t_2, a_2||\Psi\rangle = \Psi(t_1, \alpha_1; t_2, a_2)$.

Before concluding this section, it is worth pointing out that $||\Psi\rangle$ can be grouped into two orthogonal contributions, the one that describes the happening of the event and the other that refers to the absence of the event:

$$||\Psi\rangle = ||\Psi_{\text{event}}\rangle + ||\Psi_{\text{no event}}\rangle, \quad (47)$$

where, as $\Psi(t_1, \alpha_1; t_2, \alpha_2) = 0$,

$$||\Psi_{\text{event}}\rangle = \sum_{\substack{a_1 \\ t_1 \geqslant t_0}} \Psi(t_1, \alpha_1)\,||t_1, \alpha_1\rangle \quad (48)$$

being the state of Eq. (2) proposed in the introduction, and

$$||\Psi_{\text{no event}}\rangle = \sum_{t_1 \geqslant t_0} \Psi(t_1, 0)\,||t_1, 0\rangle + \sum_{t_2 > t_1} \sum_{\substack{\alpha_1 \\ t_1 > t_0}} \Psi(t_1, \alpha_1; t_2, 0)\,||t_1, \alpha_1; t_2, 0\rangle. \quad (49)$$

Notice that by performing position measurements, i.e., by setting $|\alpha_1\rangle = |x\rangle$ in Eq. (48), we have a space-time-symmetric description of a single event. By keeping in mind the procedure to obtain Eq. (45), we can readily extend it for the case of multiple events.

B. $||\Psi\rangle$ for causally connected events

In Sec. IV A, we used the conditioned state for a single event [see Eq. (23)] to obtain Eq. (45) via Eq. (7). Here, we will use this procedure and the conditioned state for two events [see Eq. (26)] to help us to understand the generalization of $||\Psi\rangle$ to the case of $\mathcal{N}$ events causally connected. By considering $\mathcal{N}$ consecutive measurements on the same system $\mathcal{S}$, it is readily verified that the state (45) is extended to

$$\begin{aligned} ||\Psi\rangle = &\sum_{\substack{a_1 \\ t_1 \geqslant t_0}} \Psi(t_1, a_1)\,||t_1, a_1\rangle \\ &+ \sum_{\substack{a_2 \\ t_2 > t_1}} \sum_{\substack{\alpha_1 \\ t_1 > t_0}} \Psi(t_1, \alpha_1; t_2, a_2)\,||t_1, \alpha_1; t_2, a_2\rangle \\ &\vdots \\ &+ \sum_{\substack{a_{\mathcal{N}} \\ t_{\mathcal{N}} > t_{\mathcal{N}-1}}} \cdots \sum_{\substack{\alpha_2 \\ t_2 > t_1}} \sum_{\substack{\alpha_1 \\ t_1 > t_0}} \Psi(t_1, \alpha_1; t_2, \alpha_2; \ldots; t_{\mathcal{N}}, a_{\mathcal{N}})\,||t_1, \alpha_1; t_2, \alpha_2; \ldots; t_{\mathcal{N}}, a_{\mathcal{N}}\rangle \\ &+ \sum_{\substack{a_{\mathcal{N}+1} \\ t_{\mathcal{N}+1} > t_{\mathcal{N}}}} \sum_{\substack{\alpha_n \\ t_{\mathcal{N}} > t_{\mathcal{N}-1}}} \cdots \sum_{\substack{\alpha_2 \\ t_2 > t_1}} \sum_{\substack{\alpha_1 \\ t_1 > t_0}} \Psi(t_1, \alpha_1; t_2, \alpha_2; \ldots; t_{\mathcal{N}}, \alpha_{\mathcal{N}}; t_{\mathcal{N}+1}, a_{\mathcal{N}+1})\,||t_1, \alpha_1; t_2, \alpha_2; \ldots; t_{\mathcal{N}}, \alpha_{\mathcal{N}}; t_{\mathcal{N}+1}, a_{\mathcal{N}+1}\rangle. \end{aligned} \quad (50)$$

Here, the first line refers to the first event, the second line to the second event, and so on. Note that the wave functions related to the first event are the same as that of Eq. (45), with $\Psi(t_1, \alpha_1) = \psi_S(\alpha_1|t_1)\chi_1(t_1)$ [with $\chi_1(t_1) = \chi(t_1)$] and $\Psi(t_1, 0) = \Gamma_1(t_1, t_0)$ [$\Gamma_1(t_1, t_0) = \Gamma(t_1, t_0)$]. On the other hand, unlike in Eq. (45), the wave functions associated with the second event can be obtained from the third and fourth terms of Eq. (26), which are

$$\begin{aligned} &\Psi(t_1, \alpha_1; t_2, 0) = \Gamma_2(t_2, t_1)\,\Psi(t_1, \alpha_1), \text{ and} \\ &\Psi(t_1, \alpha_1; t_2, \alpha_2) = \psi_{\mathcal{S}}(\alpha_2|t_2; t_1, \alpha_1)\chi_2(t_2|t_1)\,\Psi(t_1, \alpha_1), \end{aligned} \quad (51)$$

with $\psi_S(\alpha_2|t_2;t_1,\alpha_1)$ and $\chi_2(t_2|t_1)$ given by Eq. (28). The wave function of the second event $\Psi(t_1,\alpha_1;t_2,\alpha_2)$ is the joint probability amplitude of $\mathcal{S}$ having been in the state $|\alpha_1\rangle_{\mathcal{S}}$ with the clock $\mathcal{C}$ reading t_1, and being in $|\alpha_2\rangle_{\mathcal{S}}$ with the clock $\mathcal{C}$ reading t_2. Besides, the wave function $\Psi(t_1,\alpha_1;t_2,0)$ is the probability amplitude of $\mathcal{S}$ having been in $|\alpha_1\rangle_{\mathcal{S}}$ with $\mathcal{C}$ reading t_1, multiplied by the amplitude of the second event not happening in the clock reading interval $]t_1,t_2]$.

Let us analyze the event e, where $\Psi(t_1,\alpha_1;t_2,\alpha_2;\dots;t_e,a_e)=\langle t_1,\alpha_1;t_2,\alpha_2;\dots;t_e,a_e||\Psi\rangle$. Note that on each line in Eq. (50), a_e appears only in the last label of both the wave function and its respective quantum state, indicating the nonoccurrence ($a_e=0$) and occurrence ($a_e=\alpha_e$) of the eth event. Thus, for $a_e=\alpha_e$, the quantum state of the eth event is

$$\begin{aligned}&||t_1,\alpha_1;\dots;t_{e-1},\alpha_{e-1};t_e,\alpha_e\rangle=|t_e,\alpha_e\rangle_{\mathcal{CS}}\\&\otimes|t_1,\alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\ \dots\ |t_{e-1},\alpha_{e-1}\rangle_{\mathcal{T}_{e-1}\mathcal{D}_{e-1}}\\&\otimes|t_e,\alpha_e\rangle_{\mathcal{T}_e\mathcal{D}_e}\ \dots\ |t_e,0\rangle_{\mathcal{T}_{\mathcal{N}+1}\mathcal{D}_{\mathcal{N}+1}},\end{aligned}\tag{52}$$

where the agreement between the states of the clock + system and timer + detector happens with $\mathcal{CS}$ and $\mathcal{T}_e\mathcal{D}_e$. Thus, as expected, the event associated with the line e is the pair (t_e,α_e), and, since $t_e>t_{e-1}>\dots>t_1$, the events $(t_1,\alpha_1;t_2,\alpha_2;\dots;t_{e-1},\alpha_{e-1})$ are memories recorded in the set $\mathcal{TD}$. In addition, for $a_e=0$, we have

$$\begin{aligned}&||t_1,\alpha_1;\dots;t_{e-1},\alpha_{e-1};t_e,0\rangle=|t_e\rangle_{\mathcal{C}}\ \hat{U}_{\mathcal{S}}(t_e,t_{e-1})|\alpha_{e-1}\rangle_{\mathcal{S}}\\&\otimes|t_1,\alpha_1\rangle_{\mathcal{T}_1\mathcal{D}_1}\ \dots\ |t_{e-1},\alpha_{e-1}\rangle_{\mathcal{T}_{e-1}\mathcal{D}_{e-1}}\\&\otimes\big|t_e,0\rangle_{\mathcal{T}_e\mathcal{D}_e}\ \dots\ |t_e,0\rangle_{\mathcal{T}_{\mathcal{N}+1}\mathcal{D}_{\mathcal{N}+1}},\end{aligned}\tag{53}$$

which depicts the ignorance of $\mathcal{D}_e$ about $\mathcal{S}$, with $\mathcal{C}$ reading t_e. In this manner, this state represents the nonoccurrence of the event e in the interval $]t_{e-1},t_e]$, and the memory of the events before t_e.

Turning our attention to the wave functions, for $a_e=\alpha_e$, the probability amplitude of (t_e,α_e) with memories $(t_1,\alpha_1)\dots(t_{e-1},\alpha_{e-1})$ is

$$\begin{aligned}\Psi(t_1,\alpha_1;\dots;t_e,\alpha_e)&=\psi_S(\alpha_e|t_e;t_{e-1},\alpha_{e-1})\,\chi_e(t_e|t_{e-1})\\&\quad\times\Psi(t_1,\alpha_1;\dots;t_{e-1},\alpha_{e-1}),\end{aligned}\tag{54}$$

where $\psi_S(\alpha_e|t_e;t_{e-1},\alpha_{e-1})={}_{\mathcal{S}}\langle\alpha_e|\hat{U}_{\mathcal{S}}(t_e,t_{e-1})|\alpha_{e-1}\rangle_{\mathcal{S}}$. Lastly, for $a_e=0$,

$$\Psi(t_1,\alpha_1;\dots;t_e,0)=\Gamma_e(t_e,t_{e-1})\,\Psi(t_1,\alpha_1;\dots;t_{e-1},\alpha_{e-1})\tag{55}$$

is the probability amplitude of the memories $(t_1,\alpha_1)\dots(t_{e-1},\alpha_{e-1})$, multiplied by the probability amplitude of the eth event not having occurred within $]t_{e-1},t_e]$. Furthermore, by the very formulation of the problem, each event e has normalization condition given by

$$\sum_{\substack{\alpha_e\\t_e>t_{e-1}}}\dots\sum_{\substack{\alpha_1\\t_1>t_0}}|\Psi(t_1,\alpha_1;\dots;t_e,\alpha_e)|^2=1.\tag{56}$$

At this point, it is essential to emphasize that $||\Psi\rangle$ in Eq. (45) is written as a sum of a sequence of normalized events that differ from each other by the amount of memory stored in the basis states of these events. This structure will be fundamental in Sec. IV C for our interpretation of the emergence of time.

To conclude the description of causally connected events, consider that the detector registers more than one degree of freedom of the system, for example, an observable $\hat{\alpha}$ of the particle and its position $\hat{\vec{x}}$. Thus, we can predict, for instance, the event "a particle with spin up in the region d^3x around the position $\vec{x}$ and in the interval $[t,t+dt]$." Rewriting Eq. (50) more compactly, now we have

$$||\Psi\rangle=\sum_{e=1}^{\mathcal{N}+1}\sum_{x_1^\mu,\alpha_1}^{x_1^0<x_2^0}\dots\sum_{x_{e-1}^\mu,\alpha_{e-1}}^{x_{e-1}^0<x_e^0}\sum_{x_e^\mu,a_e}\Psi\big(x_1^\mu,\alpha_1;\dots;x_{e-1}^\mu,\alpha_{e-1};x_e^\mu,a_e\big)\big||x_1^\mu,\alpha_1;\dots;x_{e-1}^\mu,\alpha_{e-1};x_e^\mu,a_e\big\rangle,\tag{57}$$

where $\mu=0,1,2,3$, with $x^0=t$, $x^1=x$, $x^2=y$, and $x^3=z$. Notice that time is treated on an equal footing with position and any other physical observable. Equation (57) will be particularly important when we generalize the formalism to an arbitrary number of causally and noncausally connected events.

Finally, it is worth analyzing the notation defined above for the case where the environment develops the role of the set of timers. Let us focus only on the second event since the generalization for the eth event is straightforward. For instance, the state of the environment for the situation where the first event happened with the clock reading $t_{1(k)}$ and the second event has not happened in the interval $]t_{1(k)},t_{2(\ell)}]$ is

$$\begin{aligned}&||t_{1(k)},\alpha_1\ ;\ t_{2(\ell)},0\rangle\equiv|t_{2(\ell)}\rangle_{\mathcal{C}}\hat{U}_{\mathcal{S}}(t_{2(\ell)},t_{1(k)})\,|\alpha_1\rangle_{\mathcal{S}}\\&\otimes|\alpha_1\rangle_{\mathcal{D}_1}\,|0_{(1)},\dots,\alpha_{1(k)},\dots,\alpha_{1(\ell)},r_{(\ell+1)},\dots\rangle_{\mathcal{E}_1}\\&\otimes\,|0\rangle_{\mathcal{D}_2}\,|0_{(1)},\dots,0_{(k)},\dots,0_{(\ell)},r_{(\ell+1)},\dots\rangle_{\mathcal{E}_2}.\end{aligned}\tag{58}$$

Note that $\mathcal{E}_{2(\ell)}$ is the subsystem of the environment that is in "contact" with $\mathcal{D}_2$ at $t_{2(\ell)}$ registering this detector in its initial state $|0\rangle_{\mathcal{D}_2}$. On the other hand, the state of the second event happening at $t_{2(\ell)}$, with a memory from $t_{1(k)}$, is

$$\begin{aligned}&||t_{1(k)},\alpha_1;\ t_{2(\ell)},\alpha_2\rangle\equiv|t_{2(\ell)},\alpha_2\rangle_{\mathcal{CS}}\\&\otimes\,|\alpha_1\rangle_{\mathcal{D}_1}\,|0_{(1)},\dots,\alpha_{1(k)},\dots,\alpha_{1(\ell)},r_{(\ell+1)},\dots\rangle_{\mathcal{E}_1}\\&\otimes\,|\alpha_2\rangle_{\mathcal{D}_2}\,|0_{(1)},\dots,0_{(k)},\dots,\alpha_{2(\ell)},r_{(\ell+1)},\dots\rangle_{\mathcal{E}_2}.\end{aligned}\tag{59}$$

As discussed in Sec. III D, from Eqs. (58) and (59), we also verify that the arrangement in which the environment stores information about $\mathcal{S}$ determines the readings of the clock $\mathcal{C}$ at the moment of the events. In this more fundamental scenario, instead of referring to the clock reading, an event can also be characterized by both the information stored in the observer and the internal arrangement used to register this information. Finally, it worth remarking that a relevant situation occurs when an event arises from the acquisition of information by the observer about its surrounding environment, as happens with our perceptions of the external world. With the formalism of this section in mind, now we will discuss the emergence of time based on causally connected events.

C. Time emerging from $||\Psi\rangle$

As we discussed in the introduction, the assumption that time simply arises from the correlation between the rest of the universe $\mathcal{R}$ and the clock $\mathcal{C}$ yields serious disagreements with our personal experiences. Thus, although for the sake of comprehension we referred to β and t_e as measures of time, we will now disregard such a relation. Instead, we will investigate the possibility of time emerging from $||\Psi\rangle$ by associating the nature of an instant of time with a single event. In this context, as the existence of an event requires a physical object (the observer) to store information about other systems, time should also depend on this gain of information to exist. From this interpretation, the flow of time will emerge from an asymmetric sequence of events of $||\Psi\rangle$ associated with the same observer. Let us discuss this standpoint carefully.

First, by assuming that an instant of time arises from a single event, the flow of time should be related to events of the same observer that occur with probability 1. This feature implies that the universe from the point of view of this observer (e.g., the set $\mathcal{TD}$, $\mathcal{DE}$, or $\mathcal{E}$) is characterized by the existence of *all* these events. In addition, for such events to produce a flow with a preferred direction, specific asymmetries are needed along a particular sequence of these events in such a way that such asymmetries give rise to the notion of "passage" and "motion" for the observer. Hence, besides the probability of each event being 1, the flow should emerge from a sequence of events such as that of Eq. (50), where the occurrence of a given (t, α) means that the other events with lower values of t (not time) also happen. Note that, without referring to time, the initial condition and the Hamiltonian can generate this kind of sequence.

More specifically, notice that the events of Eq. (50) follow the causal-like sequence described in the introduction: Without any reference to time, in Eq. (50), the constraint $\hat{H}||\Psi\rangle = 0$ "selects" events such that if (t_e, α_e) is not selected, (t_{e+1}, α_{e+1}) cannot be selected. This behavior can be seen in the gradual increase of the observer's memory along the growth of t. Under such circumstances, let us say that the causal-like asymmetry along a particular sequence of events (the one with growing memory) associated with the same observer gives rise to a "flow" of events (or time) from the observer's point of view. It should be noted that in this approach the β-reversal symmetry (not time) of QM still holds.

With the discussion above in mind, notice that for the events of Eq. (50) to bring about the flow of time, the observer (the environment or the set $\mathcal{TD}$) had to perform measurements by registering information in different degrees of freedom. Thus, the states of distinct events are orthogonal to each other [see Eq. (46)] and can be normalized separately as in Eq. (56). These features together with the correlation between $\mathcal{C}$ and $\mathcal{R}$ of the PW approach [Eq. (7)] yield a one-to-one relationship between the possible readings of $\mathcal{C}$ and the distinguishable states of the observer at the moment of the event. Then, since the arrangement employed to store information has longer memories for higher values of the clock reading, the growth of $\beta = t$ works as a good track of the flow of time. Finally, also notice that each event can occur for β ranging from $-\infty$ to ∞ (with $\chi = 0$ for $\beta = t < t_0$), giving rise to a single instant of time from the observer's perspective. This characteristic is similar to the position x of a particle, which has a definite value within $-\infty < x < \infty$ when it is measured.

It is readily seen that the proposal above is coherent with our personal experience of time. Here, one should consider our brain playing a role similar to that of the set $\mathcal{TD}$, so that we perceive all events contained in $||\Psi\rangle$ associated with our brain. Now, by taking $\mathcal{E}$ as the *observed* system, the events become our perceptions that come from the acquisition of information stored by our brain about the universe around us. In this context, similarly to the conclusion above, the sequence of causal-like events contained in $||\Psi\rangle$, which follows the increase of memory, is responsible for our perception of the flow of time. At this point, we verify the epistemic asymmetry of time, where the past is remembered, whereas the future is inferred. It is worth pointing out that our approach is compatible with works in philosophy and neuroscience of consciousness concerning our perception of motion, which takes place in discrete processing frames or snapshots [35]. Nevertheless, notice that the asymmetric sequence of events are properties of QM itself, and thus the flow of time can be, in principle, associated with observers with many degrees of freedom, not necessarily a conscious being.

It is important to observe that from the perspective of the environment $\mathcal{E}$ (see Sec. III D), the shortest time interval is not given by the distance $(t_{e+1} - t_e)$ between two consecutive measurements of $\mathcal{S}$. Remember that at every step $\delta t_{\mathcal{DE}}$, the environment records information about the detector ($|0\rangle_{\mathcal{D}}$ or $|\alpha\rangle_{\mathcal{D}}$), and thus a new event (or instant of time) takes place from the environment's point of view. Under these circumstances, the events associated with the highest frequency measurements ($\mathcal{E}$ observing $\mathcal{D}$) work as a background time for those events with lower frequency measurements ($\mathcal{E}$ observing $\mathcal{S}$ via $\mathcal{D}$). On this basis, e.g., in Eq. (59), $k\delta t_{\mathcal{DE}} = (t_{(k)} - t_0)$ is the time interval (or the number of measurements of $\mathcal{D}$ by $\mathcal{E}$) for $\mathcal{E}$ to observe $\mathcal{S}$ with a well-defined α.

We conclude this section by highlighting that in this paper we abdicated the concept of an external parameter t that flows generating events and motion (Newtonian time), as well as the interpretation of time arising from correlations (PW time). Instead, here, we analyze the viewpoint in which time and its flow emerge from an asymmetric sequence of orthogonal events in $||\Psi\rangle$ belonging to the same observer.

D. $||\Psi\rangle$ for arbitrary events

As mentioned earlier, the generalization of Eq. (57) for cases where $\mathcal{S}$ is composed of entangled systems is the subject of a work in progress. Thus, in this section, we still deal with measurements of uncorrelated systems but assuming noncausally connected events. To this end, let $\mathcal{M}$ be independent systems ($s = 1, 2, \ldots, \mathcal{M}$) with each one of them submitted to $\mathcal{N}_s - 1$ consecutive measurements. The measurements between different values of s are independent of each other, and thus the noncausal (spacelike) events are identified by the set $\{s\}$, so that each event is characterized by the pair $(x^{\mu}_{se_s}, \alpha_{se_s})$, where $e_s = 1, 2, \ldots, \mathcal{N}_s$ represent the causally related events for a given s. Under these circumstances, as each measurement can be of a different observable, Eq. (57) can assume the form

$$||\Psi\rangle = \sum_{s=1}^{\mathcal{M}} \sum_{e_s=1}^{\mathcal{N}_s} \sum_{x^{\mu}_{s1},\alpha_{s1}}^{x^0_{s1}<x^0_{s2}} \cdots \sum_{x^{\mu}_{se_s},a_{se_s}} \Psi\big(\big\{x^{\mu}_{s1},\alpha_{s1};\ldots;x^{\mu}_{se_s},a_{se_s}\big\}_{s=1}^{\mathcal{M}}\big) \bigotimes_{s=1}^{\mathcal{M}} \big|\big|x^{\mu}_{s1},a_{s1};\ldots;x^{\mu}_{se_s},a_{se_s}\big\rangle_s, \tag{60}$$

where $\{x^{\mu}_{s1},\alpha_{s1};\ldots;x^{\mu}_{se_s},a_{se}\}_{s=1}^{\mathcal{M}} = \{\{x^{\mu}_{11},\alpha_{11};\ldots;x^{\mu}_{1e_s},a_{1e_s}\},\ldots,\{x^{\mu}_{\mathcal{M}1},\alpha_{\mathcal{M}1};\ldots;x^{\mu}_{\mathcal{M}e_s},a_{\mathcal{M}e_s}\}\}$, and the function $\Psi(\ldots)$ is separable for different values of s. A more detailed discussion of this state will be carried out in future works. In Eq. (60), for convenience, we isolate causally connected events in the same ket $||\rangle$. Notice that, in contrast to the causal events for a given s, where always $t_{se_s} > t_{se_s-1}$, for different values of s, there is no correlation between the times t_{se_s}. Moreover, note that in Eq. (60), for fixed values of e_s and s, we have the contribution of e_s sums $(s1, s2, \ldots, se_s)$. The reason for this is that for a given event e_s, we have $e_s - 1$ memories referring to previous events that must occur for the event e_s to happen.

V. CONCLUSION

First, this work proposed a quantum formalism that treats time on an equal footing with any other observable. To this end, as relativity deals with events describing space and time symmetrically, we extended the classical concept of events to the quantum realm. Here, an event in QM was treated as a transfer of information between physical systems via a (noncollapse) measurement. We started our approach by following the Page and Wootters formalism with a noninteracting clock $\mathcal{C}$ correlated with the rest of the universe $\mathcal{R}$, where the events take place.

Then, we verified that $\mathcal{C}$, being isolated, could not register the moment at which a measurement happens. We also remarked that the description of events only via the PW's correlation between $\mathcal{R}$ and $\mathcal{C}$ (given by $|\psi(\beta)\rangle_{\mathcal{R}} = {}_{\mathcal{C}}\langle\beta||\Psi\rangle$) requires the presence of an observer external to the formalism: By measuring $\mathcal{C}$, this observer defines the value of β, and hence allows the conditioning ${}_{\mathcal{C}}\langle\beta||\Psi\rangle$. In contrast to the PW's correlation approach, in this work, we aimed to describe events by both registering the moment at which the measurement occurs and without resorting to an external observer.

With the above facts in mind, we focused on the global state $||\Psi\rangle$ (and not just $|\psi(\beta)\rangle_{\mathcal{R}}$), with $\mathcal{R}$ encompassing the system of interest $\mathcal{S}$, a detector $\mathcal{D}$, and, *at first*, a timer $\mathcal{T}$ coupled with $\mathcal{D}$. Here, the only observer was $\mathcal{TD}$. $\mathcal{T}$ was modeled by a Salecker-Wigner-Peres-like timer to make the formalism as transparent as possible. Remember that assuming a more general observer was fundamental for the interpretation of the emergence of time. From the correlation between $\mathcal{C}$ and $\mathcal{R}$ of the PW approach, $\mathcal{T}$ could register (by stopping its counting via a coupling with $\mathcal{D}$) the reading of the clock $\beta = t$ at the moment at which $\mathcal{D}$ measures an observable $\hat{\alpha}$ of $\mathcal{S}$.

Under the circumstances above, the record of information about $\mathcal{CS}$ in $\mathcal{TD}$ defined the single event (x^{μ},α). This event was described by a joint probability amplitude $\Psi(x^{\mu},\alpha)$ and represented by $||x^{\mu},\alpha\rangle$, which was interpreted as the state of $\mathcal{CS}$ from the perspective of $\mathcal{TD}$. In other words, $||x^{\mu},\alpha\rangle$ means that from $\mathcal{TD}$'s point of view, $\mathcal{S}$ is in the state $|\alpha\rangle$ at x^{μ}. It is worth mentioning that another observer not correlated with $\mathcal{TD}$ cannot affirm the same about S at x^{μ}. From this result, we calculated how the probabilities of a given observable could be affected by the detection time distribution when the measurement process is not instantaneous.

We extended this approach for an arbitrary number of causally connected events by assuming a collection of $\mathcal{TD}$s (set $\mathcal{TD}$). Here, the correlation between $\mathcal{C}$ and $\mathcal{R}$ proposed by the PW formalism allowed $||\Psi\rangle$ to contain the sequence of events (measurements) that the observer, given by the set $\mathcal{TD}$, "experiences" (performs). Thus, the global state $||\Psi\rangle$ was written as a sum of all possible events predicted by a Wheeler-DeWitt-like equation, with each normalized event as a superposition of the detection times. The treatment for noncausal events was also briefly addressed. Finally, to give an interpretation for the emergence of time, a key point was to verify that a more general macroscopic observer could play the role of the set $\mathcal{TD}$. In this regard, we showed that the environment monitoring the detector can record information about $\mathcal{C}$ at the moment of occurrence of events in distinguishable states.

In this context, our second goal was to obtain the nature of time from the formalism of events. To that end, we first disassociated the observables of the clocks and timers with the concept of time and then assumed that an instant of time emerges from a single event. As mentioned above, this assumption was made possible by the observer's ability to store information in orthogonal states, which also provided a one-to-one relationship between the readings of the noninteracting clock $\mathcal{C}$ and the states of the observer at the moments of events. This relationship makes the clock $\mathcal{C}$ a good measurer of time. In this way, the flow of time emerged from the observer's perspective via a causal-like sequence of events in $||\Psi\rangle$ associated with this observer.

A reason for this interpretation of the flow of time came from the fact that $||\Psi\rangle$ predicts each event with probability 1, and so all these events were assumed to be "real" from the observer's perspective. Besides that, the occurrence of a given event depended on the happening of other ones with less memory and lower values of the clock reading. Thus, the causal-like sequence mentioned above encompassed all the normalized events in $||\Psi\rangle$ that presented an increase of the observer's memory along the growth of t. From our conscious perception standpoint, the passage of time was approached as our experience of all these causal-like events (and their memories) in $||\Psi\rangle$ related to our brain.

ACKNOWLEDGMENTS

I thank Prof. Vlatko Vedral and his group, Frontiers of Quantum Physics, for the fruitful discussions that contributed significantly to the development of this work. I also thank Rodolfo Brandao and Aditya Iyer for the discussions about the emergence of time, and Philipp Höhn for presenting me excellent references related to the subject of this work. I acknowledge financial support from Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) through its program GPCT - 17/2016 (Grant No. 88887.312745/2018-00) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) through its program 12/2017 (Grant No. 311043/2017-8). I also acknowledge the support of Wolfson College, University of Oxford.

APPENDIX: CALCULATION OF $|\psi(t_{(1)})\rangle_{\mathcal{R}}$

In this work, as we do not intend to solve Eq. (8) for a specific Hamiltonian $\hat{H}_{\mathcal{R}} = \hat{H}_{\mathcal{STD}}$ (10), the evolved state $|\psi(t_{(N)})\rangle_{\mathcal{R}}$ (14) should represent the measurement process introduced in Sec. III B as generally as possible. Therefore, let us verify that Eq. (12) can describe, for instance, a measurement in which the detector interacts differently with distinct states $|\alpha\rangle_{\mathcal{S}}$, i.e.,

$$\begin{aligned}|\psi(t_{(1)})\rangle_{\mathcal{R}} = &\left[\sum_{\alpha}\sqrt{1-\delta p_{\alpha(1)}}\, e^{i\varphi_{\alpha(1)}}\,\hat{M}_{\alpha}\,\hat{U}_{\mathcal{S}}(t_{(1)},t_0)\right]\\ &\times|0\rangle_{\mathcal{S}}\,|t_{(1)}\rangle_{\mathcal{T}}\,|0\rangle_{\mathcal{D}}\\ &+\left[\sum_{\alpha}\sqrt{\delta p_{\alpha(1)}}\, e^{i\tilde{\varphi}_{\alpha(1)}}\,\hat{M}_{\alpha}\,\hat{U}_{\mathcal{S}}(t_{(1)},t_0)\right]\\ &\times|0\rangle_{\mathcal{S}}\,|t_{(1)}\rangle_{\mathcal{T}}\,|\alpha\rangle_{\mathcal{D}},\end{aligned} \tag{A1}$$

where $\delta p_{\alpha(1)} \ll 1$ is a probability related to the measurement of $|\alpha\rangle_{\mathcal{S}}$, $\varphi_{\alpha(1)}$ and $\tilde{\varphi}_{\alpha(1)}$ are phases associated with the unmeasured and measured situations respectively, and $\hat{U}_{\mathcal{S}}(t_{(1)},t_0) = \exp\{-i\hat{H}_{\mathcal{S}}(t_{(1)}-t_0)/\hbar\}$. It is worth mentioning that rewriting Eq. (A1) in the form of Eq. (12) is extremely useful for clarity and understanding of the formalism of events that we present in this paper. To this end, we have $\delta p_{(1)}$ of Eq. (12) defined as

$$\delta p_{(1)} = \sum_{\alpha}\delta p_{\alpha(1)}\,|_{\mathcal{S}}\langle\alpha|\hat{U}_{\mathcal{S}}(t_{(1)},t_0)|0\rangle_{\mathcal{S}}|^2, \tag{A2}$$

which is the probability of the measurement taking place in the interval $[t_0, t_{(1)}]$ regardless of the outcome α. Thus, Eq. (A1) can be written in the form of Eq. (12) by defining

$$\hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)},t_0) = \sum_{\alpha}\sqrt{\frac{1-\delta p_{\alpha(1)}}{1-\delta p_{(1)}}}\, e^{i(\varphi_{\alpha(1)}-\varphi_{(1)})}\,\hat{M}_{\alpha}\,\hat{U}_{\mathcal{S}}(t_{(1)},t_0) \tag{A3}$$

and

$$_{\mathcal{S}}\langle\alpha|\hat{U}_{\mathcal{S}}^{1(1)}(t_{(1)},t_0)|0\rangle_{\mathcal{S}} = \sqrt{\frac{\delta p_{\alpha(1)}}{\delta p_{(1)}}}\, e^{i\tilde{\varphi}_{\alpha(1)}}\; {}_{\mathcal{S}}\langle\alpha|\hat{U}_{\mathcal{S}}(t_{(1)},t_0)|0\rangle_{\mathcal{S}}. \tag{A4}$$

From Eqs. (A3) and (A4), we observe that for each time step of the evolution, the operators $\hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)},t_0)$ and $\hat{U}_{\mathcal{S}}^{1(1)}(t_{(1)},t_0)$ must be updated. Nevertheless, notice that if the detector interacts equally with all states $|\alpha\rangle_{\mathcal{S}}$, $\delta p_{\alpha(1)} = \delta p_{(1)}$, and thus we can have $\hat{U}_{\mathcal{S}}^{0(1)}(t_{(1)},t_0) = \hat{U}_{\mathcal{S}}^{1(1)}(t_{(1)},t_0) = \hat{U}_{\mathcal{S}}(t_{(1)},t_0)$.

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