Quantum complementarity through entropic certainty principles

Javier M. Magan and Diego Pontello

Instituto Balseiro, Centro Atomico Bariloche S. C. de Bariloche, Rio Negro R8402AGP, Argentina

We approach the physical implications of the noncommutative nature of complementary observable algebras (COAs) from an information theoretic perspective. In particular, we derive a general *entropic certainty principle* stating that the sum of two relative entropies, naturally related to the COA, is equal to the so-called algebraic index of the associated inclusion. Uncertainty relations then arise by monotonicity of the relative entropies that participate in the underlying entropic certainty. Examples and applications are described in quantum field theories with global symmetries, where the COAs are formed by the charge-anticharge local operators (intertwiners) and the unitary representations of the symmetry group (twists), and in theories with local symmetries, where the COAs are formed by Wilson and t'Hooft loops. In general, the entropic certainty principle naturally captures the physics of order and disorder parameters, a feature that makes it a generic handle for the information theoretic characterization of quantum phases.

I. THE UNCERTAINTY PRINCIPLE: OLD AND NEW

The uncertainty principle limits the precision of potential experiments in a quantum-mechanical world. It states that the product of statistical errors is bounded away from zero. The paradigmatic example is that of position and momentum operators [1]:

$$\sigma_x \sigma_p \geqslant \hbar/2 \,, \tag{1}$$

where σ stands for the variance. Such bounds are derived and improved by allowing certain state dependence. In its generalized form, applicable to any pair of observables A and B, it reads [2,3]

$$\sigma_A^2 \sigma_B^2 \geqslant \left| \frac{1}{2}\langle\{A, B\}\rangle - \langle A\rangle\langle B\rangle \right|^2 + \left| \left\langle \frac{1}{2i}[A, B] \right\rangle \right|^2. \tag{2}$$

But the implications of quantum complementarity can be seen through different glasses as well. The most natural one is the entropy, and the emerging relations are called entropic uncertainty relations (see [4–6] for recent reviews with a historical account). A famous example is [7–9]

$$H(A) + H(B) \geqslant -2\ln c, \tag{3}$$

where $H(A)$ and $H(B)$ are the classical Shannon entropies associated to the measurements outcomes of observables A and B in a d-dimensional Hilbert space, and

$$c := \max\left\{ \left| \langle \psi_i^A | \psi_j^B \rangle \right| \,:\, i, j = 1, \ldots, d \right\} \tag{4}$$

is the maximum overlap between any two eigenvectors $|\psi_i^A\rangle$ and $|\psi_j^B\rangle$ of A and B.

Entropic approaches are motivated for different reasons. First, in some cases, entropic uncertainty relations are stronger than the conventional ones [4]. Second, they have a precise operational meaning in the context of information theory [10]. Finally, they suggest further generalizations of the uncertainty principle, such as including memories (side information), to which the observer has access too [11–14]. A well-studied example is

$$H(A|M) + H(B|M) \geqslant -\ln c, \tag{5}$$

where $H(A|M)$ and $H(B|M)$ are the conditional entropies with respect to the memory M. From the previous relations, it can also be derived that

$$I(A, M) + I(B, M) \leqslant \ln(d^2 c), \tag{6}$$

where $I(A, B) := S(A) + S(B) - S(A \cup B)$ stands for the mutual information. Intuitively, the uncertainty principle constrains the amount of information that M can have about a certain observable A, given the amount it has about a complementary one B. A nice way to prove this relation uses the monotonicity of the relative entropy [12]. Below, we use the monotonicity of the relative entropy in a different way to obtain a large family of uncertainty relations.

It is worth remarking that entropic uncertainty relations are an active area of research with many applications in modern physics, ranging from entanglement witnessing [15–17] to quantum cryptography [6].

In this paper, we explore two avenues or consequences of quantum complementarity. On one hand, instead of two observables, we consider two complementary operator algebras (COAs) $\mathcal{A}$ and $\mathcal{B}$. These are two observable subalgebras containing at least some operators which do not commute with each other. These algebras can be considered the "laboratories" (specifying the set of allowed experiments) associated with different observers. On the other hand, instead of inequalities, we seek to find "entropic certainties." This is inspired by recent results in the context of quantum field theory (QFT), where the first example of an entropic certainty principle was discovered [18]. Such an example, together with other applications, is described in the last section.

II. RELATIVE ENTROPY AND COMPLEMENTARY OBSERVABLE ALGEBRAS

Instead of entropies, our principle will use a different information measure: the relative entropy. This quantity measures the distinguishability between two states with respect to a given algebra $\mathcal{M}$. For a finite dimensional algebra $\mathcal{M}$, it is defined by

$$S_{\mathcal{M}}(\omega|\phi) := \text{Tr}_{\mathcal{M}}[\rho^{\omega}(\ln \rho^{\omega} - \ln \rho^{\phi})], \quad (7)$$

where $\text{Tr}_{\mathcal{M}}$ is the canonical trace on $\mathcal{M}$, and ρ^{ω} and ρ^{ϕ} are the density matrices representing the underlying states (see [19] and Appendix B).

Using relative entropy is convenient from several perspectives. First, all other information quantities can be derived from it. For example, for a full matrix algebra acting on a finite d-dimensional Hilbert space the entropy is

$$S_{\mathcal{M}}(\omega) := -\text{Tr}_{\mathcal{M}}(\rho^{\omega} \ln \rho^{\omega}) = \ln d - S_{\mathcal{M}}(\omega|\tau), \quad (8)$$

where $\tau := \mathbf{1}/d$ is the maximally mixed density matrix. Second, relative entropy shows monotonicity under general quantum channels and restrictions onto subalgebras [19]. Finally, relative entropy is well defined across different types of algebras, including type-III von Neumann algebras appearing in QFT.

To motivate our entropic certainty principle, notice that in (8) the maximally mixed state τ can be defined by composing ω with a map $\varepsilon : \mathcal{M} \to \mathbf{1}$, defined by $\varepsilon(m) := \tau(m)\mathbf{1}$. This rewriting leads to

$$S_{\mathcal{M}}(\omega) = \ln d - S_{\mathcal{M}}(\omega|\omega \circ \varepsilon), \quad (9)$$

and suggests further generalizations. First, the map $\varepsilon : \mathcal{M} \to \mathbf{1}$ is one example of a whole space of such maps. In addition, instead of the identity as the target algebra, we could choose any subalgebra $\mathcal{N} \subset \mathcal{M}$. The maps $\varepsilon : \mathcal{M} \to \mathcal{N}$ are called conditional expectations [19]. They are positive, linear, and unital maps from an algebra $\mathcal{M}$ to a subalgebra $\mathcal{N}$ satisfying

$$\varepsilon(n_1 \, m \, n_2) = n_1 \varepsilon(m) n_2, \quad \forall m \in \mathcal{M}, \, \forall n_1, n_2 \in \mathcal{N}. \quad (10)$$

These maps are the mathematical definition of what restricting our observational abilities means. Examples are tracing out part of the system or retaining the neutral part of a subalgebra under the action of a certain symmetry group. If $\mathcal{M} = \mathcal{N} \vee \mathcal{A}$ is the algebra generated by $\mathcal{N}$ and certain algebra $\mathcal{A}$, we say the conditional expectation "kills" $\mathcal{A}$. Notice also that the conditional expectation can be used to lift a state $\omega_{\mathcal{N}}$ in $\mathcal{N}$ to a state $\omega = \omega_{\mathcal{N}} \circ \varepsilon$ in $\mathcal{M}$.

In this context $S_{\mathcal{M}}(\omega|\omega \circ \varepsilon)$ measures the amount of information in $\mathcal{A}$ which is erased by ε. It takes into account side correlations with $\mathcal{N}$, which is the subalgebra left invariant by ε. In this light, complementarity refers to the algebra $\mathcal{A}$, while $\mathcal{N}$ plays the role of quantum memory. What remains to be answered is who plays the role of the COA. A canonical candidate arises from the following diagram:

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\varepsilon} & \mathcal{N} \\
\updownarrow{\prime} & & \updownarrow{\prime} \\
\mathcal{M}' & \xleftarrow{\varepsilon'} & \mathcal{N}'.
\end{array}
\quad (11)
$$

In this diagram, going vertically takes the algebras to its commutants, while horizontally in the arrow direction means restricting to the target subalgebra. If ε kills the algebra $\mathcal{A} \subset \mathcal{M}$, the *dual conditional expectation* ε' kills the COA $\tilde{\mathcal{A}} \subset \mathcal{N}'$. Notice that $\tilde{\mathcal{A}}$ does not commute with $\mathcal{A}$.

As an example, take $\mathcal{M}$ as the Abelian algebra $\mathcal{X}$ generated by the position operator, and a conditional expectation that kills the full $\mathcal{M} = \mathcal{A} =: \mathcal{X}$. We then obtain

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\varepsilon} & \mathbf{1} \\
\updownarrow{\prime} & & \updownarrow{\prime} \\
\mathcal{X} & \xleftarrow{\varepsilon'} & \mathcal{X} \vee \mathcal{P}.
\end{array}
\quad (12)
$$

We conclude that the COA of $\mathcal{X}$ is $\mathcal{P}$, the algebra generated by the momentum operator, as expected.

Having the COAs $\mathcal{A}$ and $\tilde{\mathcal{A}}$, we might expect an uncertainty relation of the form

$$S_{\mathcal{M}}(\omega|\omega \circ \varepsilon) + S_{\mathcal{N}'}(\omega|\omega \circ \varepsilon') \leqslant \ln \lambda, \quad (13)$$

where λ is a constant to be determined. This is analogous to Eq. (6) but generalized in different ways. The difference is that, given ε, we have some freedom when choosing ε'. In this paper, we show we can choose it so as to obtain the *entropic certainty principle*

$$S_{\mathcal{M}}(\omega|\omega \circ \varepsilon) + S_{\mathcal{N}'}(\omega|\omega \circ \varepsilon') = \ln \lambda, \quad (14)$$

where λ is a fixed number and ω an arbitrary pure state. This is a generalization of the known equality of entanglement entropy associated to commutant observable algebras in a global pure state to noncommutative COA.

III. THE SPACE OF CONDITIONAL EXPECTATIONS

To approach this problem, we need to understand the space of conditional expectations $C(\mathcal{M}, \mathcal{N})$ for a generic inclusion of algebras $\mathcal{N} \subset \mathcal{M}$. This was studied in [20–23], but here we use a different approach. Irrespective of the inclusion, both algebras have the following general form:

$$\mathcal{M} \cong \bigoplus_{j=1}^{z_{\mathcal{M}}} M_{m_j}(\mathbb{C}) \otimes \mathbf{1}_{m_j'}, \quad \mathcal{N} \cong \bigoplus_{k=1}^{z_{\mathcal{N}}} M_{n_k}(\mathbb{C}) \otimes \mathbf{1}_{n_k'}, \quad (15)$$

where $M_m(\mathbb{C})$ is the full matrix algebra of $m \times m$ complex matrices. The minimal central projectors of the algebra $\mathcal{M}$ are $P_j^{\mathcal{M}} := \mathbf{1}_{m_j} \otimes \mathbf{1}_{m_j'}$, with $j = 1, \dots, z_{\mathcal{M}}$. And similarly for the algebra $\mathcal{N}$. We define the subalgebras $\mathcal{M}_k := P_k^{\mathcal{N}} \mathcal{M} P_k^{\mathcal{N}}$.

Given these algebras, generic inclusions $\mathcal{N} \subset \mathcal{M}$ are characterized by the number of times μ_{kj} the factor $M_{n_k}(\mathbb{C})$ of $\mathcal{N}$ is included in the factor $M_{m_j}(\mathbb{C})$ of $\mathcal{M}$ [24–26]. They must satisfy $m_j = \sum_{k=1}^{z_{\mathcal{N}}} n_k \times \mu_{kj}$.

To understand the space $C(\mathcal{M}, \mathcal{N})$ it is useful to split $\mathcal{N} \subset \mathcal{M}$ in two steps:

$$\mathcal{M} \supset \bigoplus_{k=1}^{z_{\mathcal{N}}} M_{n_k}(\mathbb{C}) \otimes \left(\bigoplus_{j=1}^{z_{\mathcal{M}}} M_{\mu_{kj}}(\mathbb{C}) \otimes \mathbf{1}_{m_j'} \right) \supset \mathcal{N}, \quad (16)$$

where we must also have $n_k' = \sum_{j=1}^{z_{\mathcal{M}}} \mu_{kj} \times m_j'$. The hint towards $C(\mathcal{M}, \mathcal{N})$ is that any $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ arises by composing a conditional expectation from $\mathcal{M}$ to the intermediate

algebra, which turns out to be unique, and a conditional expectation from such intermediate algebra to $\mathcal{N}$, the space of which can be simply derived. The following lemma determines $C(\mathcal{M}, \mathcal{N})$ completely.

Lemma 1. Let $\mathcal{N} \subset \mathcal{M}$ be a generic inclusion of finite dimensional algebras, and $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ a conditional expectation. Then, there exist unique conditional expectations $\varepsilon_k \in C(\mathcal{M}_k, \mathcal{N}_k)$ such that

$$\varepsilon(A) = \bigoplus_{k=1}^{z_{\mathcal{N}}} \varepsilon_k(A_k), \qquad A_k := P_k^{\mathcal{N}} A P_k^{\mathcal{N}} \in \mathcal{M}_k. \qquad (17)$$

These conditional expectations are uniquely determined by states $\rho_k^\varepsilon = \bigoplus_{j=1}^{z_{\mathcal{M}}} p_{kj}^\varepsilon \rho_{kj}^\varepsilon$ on $\mathcal{M}_k \cap \mathcal{N}' \simeq \bigoplus_{j=1}^{z_{\mathcal{M}}} M_{\mu_{kj}}(\mathbb{C})$ (with $\sum_{j=1}^{z_{\mathcal{M}}} p_{kj}^\varepsilon = 1$) through

$$\varepsilon_k(A_k) := \left(\sum_{j=1}^{z_{\mathcal{M}}} p_{kj}^\varepsilon \operatorname{Tr}\left(\rho_{kj}^\varepsilon C_{kj}\right) \right) \left(B_k \otimes \mathbf{1}_{n_k'}\right), \qquad (18)$$

where $A_k := B_k \otimes (\bigoplus_{j=1}^{z_{\mathcal{M}}} C_{kj} \otimes \mathbf{1}_{m_j'})$, $B_k \in M_{n_k}(\mathbb{C})$, and $C_{kj} \in M_{\mu_{kj}}(\mathbb{C})$. Equation (18) is extended to a general element $A_k \in \mathcal{M}_k$ by linearity.

Proof. See Appendix A. ∎

Therefore, the conditional expectations ε and ε' are parametrized by states ρ_k^ε on $\mathcal{M}_k \cap \mathcal{N}'$, and $\tilde{\rho}_j^\varepsilon$ on $\mathcal{N}_j' \cap \mathcal{M}$, respectively, where $\mathcal{N}_j' := P_j^{\mathcal{M}} \mathcal{N}' P_j^{\mathcal{M}}$.

Before we finish this section, let us make some definitions. Given ε parametrized by ρ_k^ε, let $\{t_{l,jk} : l = 1, \ldots, \mu_{kj}\}$ be the eigenvalues of ρ_{kj}^ε. Then, we define

$$\lambda_{jk}(\varepsilon) := \sum_{l=1}^{\mu_{kj}} \frac{1}{t_{l,jk}} \quad \text{and} \quad \lambda_j(\varepsilon) := \sum_{k=1}^{z_{\mathcal{N}}} \frac{\lambda(\varepsilon_{jk})}{p_{kj}^\varepsilon}. \qquad (19)$$

In addition, we define the space $\hat{C}(\mathcal{M}, \mathcal{N}) \subset C(\mathcal{M}, \mathcal{N})$ to be the one formed by those conditional expectations for which $\lambda_j(\varepsilon) := \lambda(\varepsilon)$ is independent of j.

IV. ENTROPIC CERTAINTIES AND THE ALGEBRAIC INDEX

The following theorem, which is the main result of the present paper, holds for "connected inclusions" $\mathcal{N} \subset \mathcal{M}$, which satisfy the extra property $\mathcal{Z}(\mathcal{M}) \cap \mathcal{Z}(\mathcal{N}) = \mathbb{1}$ [25]. Nonconnected inclusions are direct sums of connected ones, and (14) holds for each term of the sum independently. We consider them in Appendix B.

Theorem 1. Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a connected inclusion of finite dimensional algebras. Then, for every $\varepsilon \in \hat{C}(\mathcal{M}, \mathcal{N})$ there exists a unique $\varepsilon' \in \hat{C}(\mathcal{M}, \mathcal{N})$ such that

$$S_{\mathcal{M}}(\omega | \omega \circ \varepsilon) + S_{\mathcal{N}'}(\omega | \omega \circ \varepsilon') = \ln(\lambda) \qquad (20)$$

holds for any global pure state ω on $\mathcal{B}(\mathcal{H})$, and where $\lambda := \lambda(\varepsilon) \equiv \lambda(\varepsilon')$ is the algebraic index of the conditional expectations, to be described below. Moreover, the conditional expectation ε_0 that minimizes λ always exists, it is unique, and $\varepsilon_0 \in \hat{C}(\mathcal{M}, \mathcal{N})$. The *minimal index* $\lambda(\varepsilon_0) =: [\mathcal{M} : \mathcal{N}]$ only depends on the inclusion $\mathcal{N} \subset \mathcal{M}$.

Proof. See Appendix B. ∎

As stated, $\lambda \geqslant 1$, defined in (19), is the algebraic index of the conditional expectations. This index has been extensively

studied in the mathematical and physics communities. The first notion was proposed by Jones in the context of inclusions of type-II_1 subfactors [24]. It was later noticed independently by Kosaki [27] and Longo [28] that the index was most naturally associated with a conditional expectation, and both were able to extend the definition to type-III algebras. We review the definition and several examples in Appendix C.

Probably, the most notable application of index theory to quantum physics concerns a discovery by Longo [28] in the context of the algebraic approach to superselection sectors in QFT, developed by Doplicher, Haag, and Roberts [29–32]. He found that the dimension d_r of a superselection sector, associated to a representation r of G and characterized by an endomorphism ρ_r of the observable algebra $\mathcal{O}$, is related to the minimal index of the inclusion $\rho_r(\mathcal{O}) \subset \mathcal{O}$ by means of

$$[\mathcal{O} : \rho_r(\mathcal{O})] = d_r^2. \qquad (21)$$

Also, considering the field algebra $\mathcal{F}$, including all charged operators, and the observable (neutral respect to G) subalgebra $\mathcal{O} \subset \mathcal{F}$, one obtains [33]

$$[\mathcal{O} : \rho(\mathcal{O})]^{\frac{1}{2}} = [\mathcal{F} : \mathcal{O}] = \sum_r d_r^2, \qquad (22)$$

where $\rho \simeq \bigoplus d_r \rho_r$, r runs over the irreducible representations of G, and d_r are their dimensions.

There is also an old application of index theory to information theory [34]. It works as follows. In the context of inclusion of subfactors, the index can be defined by the so-called Pimsner-Popa bound [34]:

$$\varepsilon(m^+) \geqslant \lambda^{-1} m^+, \qquad \forall m^+ \in \mathcal{M}_+. \qquad (23)$$

In the original references [27,28], Kosaki and Longo showed their index definitions imply the bound. This bound can be used to constrain relative entropies. We first notice that for two normal states ω and ω' on $\mathcal{M}$ satisfying $\omega \geqslant \mu \omega'$, we have that $S_{\mathcal{M}}(\omega | \omega') \leqslant \ln \mu^{-1}$ [19]. Therefore,

$$S_{\mathcal{M}}(\omega | \omega \circ \varepsilon) \leqslant \ln \lambda. \qquad (24)$$

Further similar applications have appeared recently in [35,36]. Relation (24) is explicit in the entropic certainty relation (20), since $S_{\mathcal{N}'}(\omega | \omega \circ \varepsilon') \geqslant 0$. But (20) further improves such bounds by unraveling the cause for the departure from saturation.

V. APPLICATIONS

We begin with two physical examples which have been discussed extensively in [18,37]. First, we consider QFTs with global symmetries. These are both those theories for which G acts equally in all space, and those for which the charged operators are local operators [32]. The existence of the unitary representation U_g of G suggests the existence of local representations τ_g^A, where A is the space domain of support of the operator, satisfying the group algebra

$$\tau_g^A \tau_{g'}^A = \tau_{gg'}^A. \qquad (25)$$

The construction of τ_g^A in the lattice is trivial. In the continuum it is more subtle, but it can be done [38–40]. The operators τ_g^A are called *twists*. If G is non-Abelian, τ_g^A are not invariant

under G. We can construct invariant combinations by averaging over the group, resulting in one invariant twist $\tau_{[g]}^A$ per equivalence class of G [18,37].

On the other hand, the local charged operators $V_{r,i}^A$, localized in region A and transforming according to the irreducible representation r as $\tau_g^A V_{r,i}^A (\tau_g^A)^{-1} = \sum_{j=1}^{d_r} \mathcal{R}_{ij}^r(g) V_{r,j}^A$, suggest the construction of nonlocal neutral operators, called *intertwiners*, formed by contracting two charged operators localized on different regions:

$$\mathcal{I}_r^{AB} := \sum_{i=1}^{d_r} V_{r,i}^A V_{r,i}^{B\,\dagger}. \tag{26}$$

Crucially $[\mathcal{I}_r, \tau_{[g]}^A] \neq 0$, since $\tau_{[g]}^A$ acts as a group transformation on A, but as the identity on B. Then, the meaningful COA is formed by the twists τ^A and the intertwiners $\mathcal{I}^{AB}$. Consider now two disjoint regions A and B, with neutral (observable) algebras $\mathcal{O}_A$ and $\mathcal{O}_B$, and its complementary region $(AB)'$ with algebra $\mathcal{O}_{(AB)'}$. Defining $\mathcal{O}_{AB} \equiv \mathcal{O}_A \vee \mathcal{O}_B$ the COA diagram (12) reads

$$\begin{array}{ccc} \mathcal{O}_{AB} \vee \mathcal{I}_{AB} & \xrightarrow{\;\varepsilon\;} & \mathcal{O}_{AB} \\ \updownarrow \prime & & \updownarrow \prime \\ \mathcal{O}_{(AB)'} & \xleftarrow{\;\varepsilon'\;} & \mathcal{O}_{(AB)'} \vee \tau^A, \end{array} \tag{27}$$

where ε kills the intertwiners and ε' kills the twists. The minimal index appearing in the entropic certainty becomes

$$\lambda_{\min} = \sum_r d_r^2 = |G|, \tag{28}$$

which coincides with the expression for the topological entanglement entropy [41,42]. This coincidence is explained in [37]. Our main contribution here, expanding the results found in [18], is that this proof uses only the neutral algebra.

The last feature is crucial when considering theories with local symmetries. In this scenario and concentrating in four spacetime dimensions, we have Wilson and t'Hooft loops, located on rings, associated with the center of the group [37]. It is well known that these algebras do not commute. An interesting COA arises by the algebra of Wilson loops W^R in a ring R and the algebra of t'Hooft loops $T^{R'}$ in the complementary ring R'. The COA diagram reads

$$\begin{array}{ccc} \mathcal{O}_R \vee W_R & \xrightarrow{\;\varepsilon\;} & \mathcal{O}_R \\ \updownarrow \prime & & \updownarrow \prime \\ \mathcal{O}_{R'} \vee W_{R'} & \xleftarrow{\;\varepsilon'\;} & \mathcal{O}_{R'} \vee W_{R'} \vee T_{R'}. \end{array} \tag{29}$$

The entropic certainty follows with an index equal to the dimension of the center [37].

In the first case, the intertwiners are natural order parameters, while the twists can be thought of as disorder operators. In the second case, the Wilson loop is the order parameter, while the t'Hooft loop is the disorder one. The entropic certainty thus captures, in a quantitative manner, the interplay between order and disorder parameters, the commutation relations of which are crucial for the characterization of quantum phases.

The previous observation suggests that the complementarity between order and disorder parameters can be always framed by such entropic certainty principles. Uncertainty relations can be derived by using monotonicity of the relative entropy under general quantum channels or algebra restrictions. In the general case,

$$S_{\mathcal{M}}[\mathcal{E}(\omega)|\mathcal{E}(\omega \circ \varepsilon)] + S_{\mathcal{N}'}[\mathcal{E}'(\omega)|\mathcal{E}'(\omega \circ \varepsilon')] \leqslant \ln \lambda,$$

$$S_{\tilde{\mathcal{M}}}(\omega|\omega \circ \varepsilon) + S_{\tilde{\mathcal{N}}'}(\omega|\omega \circ \varepsilon') \leqslant \ln \lambda, \tag{30}$$

where $\mathcal{E}$ and $\mathcal{E}'$ are any two quantum channels, and $\tilde{\mathcal{M}} \subset \mathcal{M}$ and $\tilde{\mathcal{N}}' \subset \mathcal{N}'$. At the present time, we do not know what is the class of uncertainty relations that can be proven in this way. They certainly expand the uncertainty relations proved in [12] with monotonicity of relative entropy in unexplored directions. It would be interesting if all uncertainty relations could be derived from such entropic certainties. We leave this as an open problem for the near future. Further uncertainties arise in the case where the global state ω considered is not pure. These are considered in Appendix B.

VI. DISCUSSION AND FUTURE PROSPECTS

In this paper, we have explored implications of quantum complementarity. Most importantly we have found a formulation in which, instead of uncertainty principles, one naturally finds entropic certainty principles. Our main result is Theorem 1. From this generic result, one obtains entropic uncertainty principles by using the monotonicity of the relative entropy. From a more physical perspective, our principle captures the interplay between order and disorder parameters in quantum theories. This has been described for QFTs with global and local symmetries.

Several open problems and questions are left for the future. The first concerns the consideration of more general quantum channels, instead of conditional expectations. We wonder if (14) holds for some a suitably defined λ associated with the quantum channel. This might provide a path to extend the notion of index to more generic quantum channels. On the other hand, the validity of (14) only for conditional expectations could suggest they play a distinguished role in the description of quantum complementarity. The second problem concerns the extension of the proof to general infinite dimensional von Neumann algebras. Finally, a deeper understanding of how the entropic certainty discerns and characterizes phases of quantum matter is needed. In particular, it would be interesting how (14) responds under dualities. Duality transformations typically map order parameters to disorder parameters, and we expect an interesting interplay when combining this feature with the entropic certainty.

APPENDIX A: THE SPACE OF CONDITIONAL EXPECTATIONS

Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be an inclusion of algebras. A linear map $\varepsilon : \mathcal{M} \to \mathcal{N}$ is called a conditional expectation if it is

positive, is unital, and satisfies the bimodule property

$$\varepsilon(B_1 A B_2) = B_1 \varepsilon(A) B_2, \quad \forall A \in \mathcal{M} \text{ and } \forall B_1, B_2 \in \mathcal{N}. \tag{A1}$$

Conditional expectations are completely positive maps and, hence, special cases of quantum channels.

We want to characterize the space of all conditional expectations, denoted by $C(\mathcal{M}, \mathcal{N})$, for finite dimensional algebras. To such an end, we will use two building blocks given by Lemmas 2 and 3.

Lemma 2. Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be an inclusion of algebras, $\{P_1^{\mathcal{N}}, \ldots, P_{z_{\mathcal{N}}}^{\mathcal{N}}\}$ the minimal projectors of $\mathcal{Z}(\mathcal{N})$, and $\varepsilon \in C(\mathcal{M}, \mathcal{N})$. Let us define the algebras

$$\mathcal{M}_k := P_k^{\mathcal{N}} \mathcal{M} P_k^{\mathcal{N}}, \quad \mathcal{N}_k := P_k^{\mathcal{N}} \mathcal{N} P_k^{\mathcal{N}}. \tag{A2}$$

Then, there exist unique conditional expectations $\varepsilon_k \in C(\mathcal{M}_k, \mathcal{N}_k)$ such that

$$\varepsilon(A) = \bigoplus_{k=1}^{z_{\mathcal{N}}} \varepsilon_k (P_k^{\mathcal{N}} A P_k^{\mathcal{N}}). \tag{A3}$$

Conversely, given any set of conditional expectations $\varepsilon_k \in C(\mathcal{M}_k, \mathcal{N}_k)$, the above formula defines a conditional expectation $\varepsilon \in C(\mathcal{M}, \mathcal{N})$.

Proof. Let $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ and $A \in \mathcal{M}$. Then, using the bimodule property of the conditional expectation we obtain

$$\varepsilon(A) = \varepsilon \left(\sum_{k,k'=1}^{z_{\mathcal{N}}} P_k^{\mathcal{N}} A P_{k'}^{\mathcal{N}} \right) = \sum_{k=1}^{z_{\mathcal{N}}} P_k^{\mathcal{N}} \varepsilon (P_k^{\mathcal{N}} A P_k^{\mathcal{N}}) P_k^{\mathcal{N}}. \tag{A4}$$

The last term naturally defines $\varepsilon_k : \mathcal{M}_k \to \mathcal{N}_k$ by means of

$$\varepsilon_k (P_k^{\mathcal{N}} A P_k^{\mathcal{N}}) := P_k^{\mathcal{N}} \varepsilon (P_k^{\mathcal{N}} A P_k^{\mathcal{N}}) P_k^{\mathcal{N}}. \tag{A5}$$

Using that ε is a conditional expectation, a straightforward computation shows that $\varepsilon_k \in C(\mathcal{M}_k, \mathcal{N}_k)$ for all $k = 1, \ldots, z_{\mathcal{N}}$.

Conversely, given conditional expectations $\varepsilon_k \in C(\mathcal{M}_k, \mathcal{N}_k)$ $(k = 1, \ldots, z_{\mathcal{N}})$, it is easy to show that $\varepsilon : \mathcal{M} \to \mathcal{N}$ defined as (A3) is a conditional expectation in $C(\mathcal{M}, \mathcal{N})$. ∎

Now, we need to consider the case when $\mathcal{N}$ is a factor. In this case, we must have that

$$\mathcal{N} \cong M_n(\mathbb{C}) \otimes \mathbb{1}_{n'}, \quad \mathcal{M} \cong M_n(\mathbb{C}) \otimes \mathcal{A}, \tag{A6}$$

where $\mathcal{A} \subset M_{n'}(\mathbb{C})$ is some finite dimensional subalgebra. In the most general case, we have that

$$\mathcal{A} \cong \bigoplus_{j=1}^{z_{\mathcal{A}}} M_{a_j}(\mathbb{C}) \otimes \mathbb{1}_{a'_j}, \tag{A7}$$

and hence

$$\mathcal{M} \cong \bigoplus_{j=1}^{z_{\mathcal{A}}} M_n(\mathbb{C}) \otimes M_{a_j}(\mathbb{C}) \otimes \mathbb{1}_{a'_j} \cong \bigoplus_{j=1}^{z_{\mathcal{A}}} M_{n \times a_j}(\mathbb{C}) \otimes \mathbb{1}_{a'_j}. \tag{A8}$$

In this scenario, we have the following lemma.

Lemma 3. Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ as in (A6)–(A8). Then, any conditional expectation $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ is of the form

$$\varepsilon(B \otimes A) = \varphi_\varepsilon(A) (B \otimes \mathbb{1}_{n'}), \quad \forall B \in M_n(\mathbb{C}), \forall A \in \mathcal{A}, \tag{A9}$$

where φ_ε is a state on $\mathcal{A}$. Conversely, any state φ_ε on $\mathcal{A}$ defines a conditional expectation by means of (A9) for simple elements, and it is extended by linearity for more general ones.

Proof. Let $\varepsilon \in C(\mathcal{M}, \mathcal{N})$, $A \in \mathcal{A}$, and $B \in M_n(\mathbb{C})$. Then,

$$\varepsilon(\mathbb{1}_n \otimes A) \cdot (B \otimes \mathbb{1}_{n'}) = \varepsilon[(\mathbb{1}_n \otimes A) \cdot (B \otimes \mathbb{1}_{n'})]$$
$$= \varepsilon[(B \otimes \mathbb{1}_{n'}) \cdot (\mathbb{1}_n \otimes A)]$$
$$= (B \otimes \mathbb{1}_{n'}) \cdot \varepsilon(\mathbb{1}_n \otimes A), \tag{A10}$$

which means that $\varepsilon(\mathbb{1}_n \otimes A) \in \mathcal{N}'$. Since $\varepsilon \in C(\mathcal{M}, \mathcal{N})$, we also have that $\varepsilon(\mathbb{1}_n \otimes A) \in \mathcal{N}$, and hence $\varepsilon(\mathbb{1}_n \otimes A) \in \mathcal{Z}(\mathcal{N}) = \mathbb{1}_n \otimes \mathbb{1}_{n'}$, for all $A \in \mathcal{A}$. Then, there exists $\varphi_\varepsilon : \mathcal{A} \to \mathbb{C}$ such that

$$\varepsilon(\mathbb{1}_n \otimes A) = \varphi_\varepsilon(A) (\mathbb{1}_n \otimes \mathbb{1}_{n'}), \quad \forall A \in \mathcal{A}, \tag{A11}$$

and (A9) automatically holds. To end, we have to show that the map φ_ε is a state on $\mathcal{A}$. It is clear that φ_ε is linear since ε is linear. We also have that

$$\varphi_\varepsilon(\mathbb{1}_{n'}) (\mathbb{1}_n \otimes \mathbb{1}_{n'}) = \varepsilon(\mathbb{1}_n \otimes \mathbb{1}_{n'}) = \mathbb{1}_n \otimes \mathbb{1}_{n'}, \tag{A12}$$

and hence $\varphi_\varepsilon(\mathbb{1}_{n'}) = 1$. Finally, given $A \in \mathcal{A}$, we have

$$\varphi_\varepsilon(AA^\dagger) (\mathbb{1}_n \otimes \mathbb{1}_{n'}) = \varepsilon(\mathbb{1}_n \otimes AA^\dagger)$$
$$= \varepsilon[(\mathbb{1}_n \otimes A) \cdot (\mathbb{1}_n \otimes A)^\dagger] \geqslant 0, \tag{A13}$$

which implies that $\varphi_\varepsilon(AA^\dagger) \geqslant 0$ for all $A \in \mathcal{A}$.

Conversely, given a state φ on $\mathcal{A}$, let us show that (A9) defines a conditional expectation $\varepsilon \in C(\mathcal{M}, \mathcal{N})$. Given $A_i \in \mathcal{A}$, $B_i \in M_n(\mathbb{C})$, and $c_i \in \mathbb{C}$ $(i = 1, \ldots, N)$, then

$$\varphi \left(\sum_{i=1}^N c_i A_i \right) (B_1 \otimes \mathbb{1}_{n'}) = \sum_{i=1}^N c_i [\varphi(A_i)(B_1 \otimes \mathbb{1}_{n'})], \tag{A14}$$

$$\varphi(A_1) \left[\left(\sum_{i=1}^N c_i B_i \right) \otimes \mathbb{1}_{n'} \right] = \sum_{i=1}^N c_i [\varphi(A_1)(B_i \otimes \mathbb{1}_{n'})], \tag{A15}$$

which means that the map $(B, A) \mapsto \varphi(A) (B \otimes \mathbb{1}_{n'})$ is a bilinear map from the Cartesian product $M_n(\mathbb{C}) \times \mathcal{A}$ into $M_n(\mathbb{C}) \otimes \mathbb{1}_{n'}$, and hence defines a unique linear map

$$\varepsilon(B \otimes A) := \varphi(A) (B \otimes \mathbb{1}_{n'}) \tag{A16}$$

from $M_n(\mathbb{C}) \otimes \mathcal{A}$ into $M_n(\mathbb{C}) \otimes \mathbb{1}_{n'}$. Given $B \in \mathcal{N}$ and $C \in \mathcal{M}$, then

$$B := \tilde{B} \otimes \mathbb{1}_{n'}, \quad \text{with } \tilde{B} \in M_n(\mathbb{C}), \tag{A17}$$

$$C := \sum_{i=1}^N \tilde{B}_i \otimes \tilde{A}_i, \quad \text{with } \tilde{B}_i \in M_n(\mathbb{C}), \text{ and } \tilde{A}_i \in \mathcal{A}. \tag{A18}$$

This implies that

$$\varepsilon(CB) = \sum_{i=1}^N \varphi(\tilde{A}_i) [(\tilde{B}_i \tilde{B}) \otimes \mathbb{1}_{n'}]$$
$$= \left[\sum_{i=1}^N \varphi(\tilde{A}_i)(\tilde{B}_i \otimes \mathbb{1}_{n'}) \right] (\tilde{B} \otimes \mathbb{1}_{n'})$$
$$= \varepsilon(C)B. \tag{A19}$$

Finally, since any positive linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is completely positive, then the map

$$\text{id}_n \otimes \varphi : M_n(\mathbb{C}) \otimes \mathcal{A} \to M_n(\mathbb{C}) \otimes \mathbb{C} \cong M_n(\mathbb{C}), \quad \text{(A20)}$$

given by

$$(\text{id}_n \otimes \varphi)(B \otimes A) := \varphi(A)B, \quad \text{(A21)}$$

is positive. Furthermore, the inclusion map $\iota_{n,n'} : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes \mathbb{1}_{n'}$ given by $\iota_{n,n'}(B) = B \otimes \mathbf{1}_{n'}$ is trivially positive. Then, we have that

$$\varepsilon = \iota_{n,n'} \circ (\text{id}_n \otimes \varphi) \quad \text{(A22)}$$

is positive since it results from the composition of two positive linear transformations. ∎

Finally, combining Lemmas 2 and 3, we obtain Lemma 1 in the paper. Our approach to the space of conditional expectations $C(\mathcal{M}, \mathcal{N})$ is more physically oriented, and, certainly, it makes easier the proof of Theorem 1. The space $C(\mathcal{M}, \mathcal{N})$ was first studied long ago by Umegaki using different techniques [20–23]. The final result provides a different parametrization of the space, but equivalent to ours.

APPENDIX B: PROOF OF THE ENTROPIC CERTAINTY PRINCIPLE (THEOREM 1)

In this Appendix, we provide the proof of Theorem 1. We present here the equation for the convenience of the reader:

$$S_{\mathcal{M}}(\omega|\omega \circ \varepsilon) + S_{\mathcal{N}'}(\omega|\omega \circ \varepsilon') = \ln \lambda. \quad \text{(B1)}$$

We start by recalling the generic form of the algebras $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$:

$$\mathcal{M} \cong \bigoplus_{j=1}^{z_{\mathcal{M}}} M_{m_j}(\mathbb{C}) \otimes \mathbb{1}_{m'_j}, \quad \mathcal{N} \cong \bigoplus_{k=1}^{z_{\mathcal{N}}} M_{n_k}(\mathbb{C}) \otimes \mathbb{1}_{n'_k}, \quad \text{(B2)}$$

Automatically, we have for their commutants

$$\mathcal{M}' \cong \bigoplus_{j=1}^{z_{\mathcal{M}}} \mathbb{1}_{m_j} \otimes M_{m'_j}(\mathbb{C}), \quad \mathcal{N}' \cong \bigoplus_{k=1}^{z_{\mathcal{N}}} \mathbb{1}_{n_k} \otimes M_{n'_k}(\mathbb{C}). \quad \text{(B3)}$$

To prove the theorem we compute each relative of (B1) separately. Let us focus on the first one:

$$S_{\mathcal{M}}(\omega|\omega \circ \varepsilon) = -S_{\mathcal{M}}(\omega) - \text{Tr}_{\mathcal{M}}[\rho^\omega \ln(\rho^{\omega \circ \varepsilon})], \quad \text{(B4)}$$

where ρ^ω and $\rho^{\omega \circ \varepsilon}$ are the density matrices of the states ω and $\omega \circ \varepsilon$ on the algebra $\mathcal{M}$. The von Neumann entropy $S_{\mathcal{M}}(\omega)$ in (B4) will later cancel out with an equivalent term coming from the second relative entropy of (B1). To analyze the other term, we need an expression for the density matrix $\rho^{\omega \circ \varepsilon}$. This density matrix is defined as the unique element in $\mathcal{M}$ satisfying

$$\text{Tr}_{\mathcal{M}}(\rho^{\omega \circ \varepsilon}A) = \omega[\varepsilon(A)] = \text{Tr}_{\mathcal{M}}[\rho^\omega \varepsilon(A)], \quad \text{(B5)}$$

for all $A \in \mathcal{M}$. According to Lemma 2, we can decompose the conditional expectation ε as in (A3):

$$\text{Tr}_{\mathcal{M}}[\rho^\omega \varepsilon(A)] = \text{Tr}_{\mathcal{M}}\left(\rho^\omega \bigoplus_{k=1}^{z_{\mathcal{N}}} \varepsilon_k(A_k)\right) = \sum_{k=1}^{z_{\mathcal{N}}} \text{Tr}_{\mathcal{M}_k}[\rho_k^\omega \varepsilon_k(A_k)], \quad \text{(B6)}$$

where we have defined the operators $A_k := P_k^{\mathcal{N}} A P_k^{\mathcal{N}}$ and $\rho_k^\omega := P_k^{\mathcal{N}} \rho^\omega P_k^{\mathcal{N}}$. Let us now assume that

$$A_k := B_k \otimes \bigoplus_{j=1}^{z_{\mathcal{M}}} C_{kj}, \quad B_k \in M_{n_k}(\mathbb{C}), \quad C_{kj} \in M_{\mu_{kj}}(\mathbb{C}). \quad \text{(B7)}$$

Then, according to Lemma 3, there exist states φ_k^ε on $\bigoplus_{j=1}^{z_{\mathcal{M}}} M_{\mu_{kj}}(\mathbb{C}) \cong \mathcal{M}_k \cap \mathcal{N}'$ such that

$$\varepsilon_k\left(B_k \otimes \bigoplus_{j=1}^{z_{\mathcal{M}}} C_{kj}\right) =: \varphi_k^\varepsilon\left(\bigoplus_{j=1}^{z_{\mathcal{M}}} C_{kj}\right)(B_k \otimes \mathbf{1}_{n'_k}), \quad \text{(B8)}$$

where $n'_k := \sum_{j=1}^{z_{\mathcal{M}}} \mu_{kj}$.[1] Then, plugging (B8) into (B6) we have that

$$\text{Tr}_{\mathcal{M}}[\rho^\omega \varepsilon(A)] = \sum_{k=1}^{z_{\mathcal{N}}} \text{Tr}_{\mathcal{M}_k}\left[\rho_k^\omega \varepsilon_k\left(B_k \otimes \bigoplus_{j=1}^{z_{\mathcal{M}}} C_{kj}\right)\right]$$

$$= \sum_{k=1}^{z_{\mathcal{N}}} \text{Tr}_{\mathcal{N}_k}[\text{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}(\rho_k^\omega)B_k] \text{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}\left(\rho_k^\varepsilon \bigoplus_{j=1}^{z_{\mathcal{M}}} C_{kj}\right), \quad \text{(B9)}$$

where $\rho_k^\varepsilon \in \bigoplus_{j=1}^{z_{\mathcal{M}}} M_{\mu_{kj}}(\mathbb{C})$ are the density matrices corresponding to the states φ_k^ε on $\mathcal{M}_k \cap \mathcal{N}'$, and $\text{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}(\rho_k^\omega) \in M_{n_k}(\mathbb{C})$. Then, it follows that

$$\text{Tr}_{\mathcal{M}}[\rho^\omega \varepsilon(A)] = \text{Tr}_{\mathcal{M}}\left[\left(\bigoplus_{k=1}^{z_{\mathcal{N}}} \text{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}(\rho_k^\omega) \otimes \rho_k^\varepsilon\right)A\right], \quad \text{(B10)}$$

---

[1]The numbers n'_k defined in this way coincide with the ones introduced in the paper once we set $m'_j = 1$, which is correct since we are working in the canonical representation of $\mathcal{M}$.

which implies that

$$\rho^{\omega \circ \varepsilon} = \bigoplus_{k=1}^{z_{\mathcal{N}}} \mathrm{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}\left(\rho_k^\omega\right) \otimes \rho_k^\varepsilon \in \bigoplus_{k=1}^{z_{\mathcal{N}}} \mathcal{M}_k \subset \mathcal{M}. \tag{B11}$$

We now need to find the restriction of the state ω on $\mathcal{N}$, the density matrix of which is denoted by $\rho^{\omega_{\mathcal{N}}}$. Given $A_k \in M_{n_k}(\mathbb{C}) \simeq \mathcal{N}_k$, then

$$\mathrm{Tr}_{\mathcal{N}}\left(\rho^{\omega_{\mathcal{N}}}\bigoplus_{k=1}^{z_{\mathcal{N}}} A_k\right) = \sum_{k=1}^{z_{\mathcal{N}}} \mathrm{Tr}_{\mathcal{N}_k}\left[\mathrm{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}\left(\rho_k^\omega\right)A_k\right] = \mathrm{Tr}_{\mathcal{N}}\left[\left(\bigoplus_{k=1}^{z_{\mathcal{N}}} \mathrm{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}\left(\rho_k^\omega\right)\right)\left(\bigoplus_{k=1}^{z_{\mathcal{N}}} A_k\right)\right], \tag{B12}$$

which implies that $\mathrm{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}(\rho_k^\omega) = \rho_k^{\omega_{\mathcal{N}}}$. Then, (B11) becomes

$$\rho^{\omega \circ \varepsilon} = \bigoplus_{k=1}^{z_{\mathcal{N}}} \rho_k^{\omega_{\mathcal{N}}} \otimes \rho_k^\varepsilon. \tag{B13}$$

Now, we are in conditions to compute the second term of Eq. (B4). After some algebraic manipulations, we find

$$\mathrm{Tr}_{\mathcal{M}}[\rho^\omega \ln(\rho^{\omega \circ \varepsilon})] = -S_{\mathcal{N}}(\omega) + \sum_{k=1}^{z_{\mathcal{N}}} \mathrm{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}\left[\mathrm{Tr}_{\mathcal{N}_k}\left(\rho_k^\omega\right)\ln\left(\rho_k^\varepsilon\right)\right], \tag{B14}$$

where the operators $\mathrm{Tr}_{\mathcal{N}_k}(\rho_k^\omega) \in \bigoplus_{j=1}^{z_{\mathcal{M}}} M_{\mu_{kj}}(\mathbb{C})$. Finally, the relative entropy (B4) can be written

$$S_{\mathcal{M}}(\omega|\omega \circ \varepsilon) = -S_{\mathcal{M}}(\omega) + S_{\mathcal{N}}(\omega) - \sum_{k=1}^{z_{\mathcal{N}}} \mathrm{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}\left[\mathrm{Tr}_{\mathcal{N}_k}\left(\rho_k^\omega\right)\ln\left(\rho_k^\varepsilon\right)\right]. \tag{B15}$$

The second relative entropy on the left-hand side of (B1) can be computed in a similar way. In the end, we arrive to an equivalent expression as in (B15) but for the inclusion of algebras $\mathcal{M}' \subset \mathcal{N}'$, where these algebras are as in (B3). In this case, we introduce the algebras

$$\mathcal{M}'_j := P_j^{\mathcal{M}} \mathcal{M}' P_j^{\mathcal{M}} \cong \mathbb{1}_{m_j} \otimes M_{m'_j}(\mathbb{C}), \tag{B16}$$

$$\mathcal{N}'_j := P_j^{\mathcal{M}} \mathcal{N}' P_j^{\mathcal{M}} \cong \left(\bigoplus_{k=1}^{z_{\mathcal{N}}} M_{\mu_{kj}}(\mathbb{C})\right) \otimes M_{m'_j}(\mathbb{C}), \tag{B17}$$

where $P_j^{\mathcal{M}}$ $(j = 1, \ldots, z_{\mathcal{M}})$ are the minimal central projectors of $\mathcal{M}$. The density matrix of the state ω on the algebra $\mathcal{N}'$ is denoted by $\tilde{\rho}^\omega \in \bigoplus_{k=1}^{z_{\mathcal{N}}} M_{n'_k}(\mathbb{C})$, and we define the operators

$$\tilde{\rho}_j^\omega := P_j^{\mathcal{M}} \tilde{\rho}^\omega P_j^{\mathcal{M}} \in \mathcal{N}'_j. \tag{B18}$$

Again, using Lemmas 2 and 3, we have that

$$\varepsilon' = \bigoplus_{j=1}^{z_{\mathcal{M}}} \varepsilon'_j, \quad \text{with} \quad \varepsilon'_j \in C(\mathcal{N}'_j, \mathcal{M}'_j), \quad \varepsilon'_j\left[\left(\bigoplus_{k=1}^{z_{\mathcal{N}}} C'_{kj}\right) \otimes B'_j\right] = \mathrm{Tr}_{\mathcal{N}'_j \cap \mathcal{M}}\left(\tilde{\rho}_j^{\varepsilon'} \bigoplus_{k=1}^{z_{\mathcal{N}}} C'_{kj}\right)(\mathbf{1}_{m_j} \otimes B'_j), \tag{B19}$$

for all $C'_{kj} \in M_{\mu_{kj}}(\mathbb{C})$ and $B'_j \in M_{m'_j}(\mathbb{C})$, where we have that $m_j = \sum_{k=1}^{z_{\mathcal{N}}} \mu_{kj}$ for the consistency of the inclusion. The density matrices $\tilde{\rho}_j^{\varepsilon'}$ represent states on $\mathcal{N}'_j \cap \mathcal{M} \cong \bigoplus_{k=1}^{z_{\mathcal{N}}} M_{\mu_{kj}}(\mathbb{C})$.

Following algebraic manipulations analogous to the ones applied to the first term, the second term on the left-hand side of (B1) reads

$$S_{\mathcal{N}'}(\omega|\omega \circ \varepsilon') = -S_{\mathcal{N}'}(\omega) + S_{\mathcal{M}'}(\omega) - \sum_{j=1}^{z_{\mathcal{M}}} \mathrm{Tr}_{\mathcal{N}'_j \cap \mathcal{M}}\left[\mathrm{Tr}_{\mathcal{M}'_j}\left(\tilde{\rho}_j^\omega\right)\ln\left(\tilde{\rho}_j^{\varepsilon'}\right)\right]. \tag{B20}$$

We now conclude that

$$S_{\mathcal{M}}(\omega|\omega \circ \varepsilon) + S_{\mathcal{N}'}(\omega|\omega \circ \varepsilon') = -\sum_{k=1}^{z_{\mathcal{N}}} \mathrm{Tr}_{\mathcal{M}_k \cap \mathcal{N}'}\left[\mathrm{Tr}_{\mathcal{N}_k}\left(\rho_k^\omega\right)\ln\left(\rho_k^\varepsilon\right)\right] - \sum_{j=1}^{z_{\mathcal{M}}} \mathrm{Tr}_{\mathcal{N}'_j \cap \mathcal{M}}\left[\mathrm{Tr}_{\mathcal{M}'_j}\left(\tilde{\rho}_j^\omega\right)\ln\left(\tilde{\rho}_j^{\varepsilon'}\right)\right], \tag{B21}$$

where we have used that $S_{\mathcal{M}}(\omega) = S_{\mathcal{M}'}(\omega)$ and $S_{\mathcal{N}}(\omega) = S_{\mathcal{N}'}(\omega)$ since the state ω is pure in the global algebra $\mathcal{B}(\mathcal{H})$.

In order to simplify the right-hand side of Eq. (B21), we notice that

$$\text{Tr}_{\mathcal{N}_k}\big(\rho_k^\omega\big) \in \bigoplus_{j=1}^{z_\mathcal{M}} M_{\mu_{kj}}(\mathbb{C}) \Rightarrow \text{Tr}_{\mathcal{N}_k}\big(\rho_k^\omega\big) = \bigoplus_{j=1}^{z_\mathcal{M}} \rho_{kj}^\omega, \tag{B22}$$

$$\text{Tr}_{\mathcal{M}'_j}\big(\tilde{\rho}_j^\omega\big) \in \bigoplus_{k=1}^{z_\mathcal{N}} M_{\mu_{kj}}(\mathbb{C}) \Rightarrow \text{Tr}_{\mathcal{M}'_j}\big(\tilde{\rho}_j^\omega\big) = \bigoplus_{k=1}^{z_\mathcal{N}} \tilde{\rho}_{jk}^\omega. \tag{B23}$$

In addition, a straightforward computation, like the one we did in Eq. (B12), shows that $\rho_{kj}^\omega = \tilde{\rho}_{jk}^\omega$ and

$$\rho^{\omega_{\mathcal{N}'\cap\mathcal{M}}} := \bigoplus_{j=1}^{z_\mathcal{M}} \bigoplus_{k=1}^{z_\mathcal{N}} \rho_{kj}^\omega \in \bigoplus_{j=1}^{z_\mathcal{M}} \bigoplus_{k=1}^{z_\mathcal{N}} M_{\mu_{kj}}(\mathbb{C}) \tag{B24}$$

is the density matrix of the state ω on the algebra $\mathcal{N}' \cap \mathcal{M}$.

We also recall that the density matrices ρ_k^ε and $\tilde{\rho}_j^{\varepsilon'}$, which define the conditional expectation ε and ε', can be written conveniently as

$$\rho_k^\varepsilon = \bigoplus_{j=1}^{z_\mathcal{M}} p_{kj}^\varepsilon \, \rho_{kj}^\varepsilon, \quad \tilde{\rho}_j^{\varepsilon'} = \bigoplus_{k=1}^{z_\mathcal{N}} \tilde{p}_{jk}^{\varepsilon'} \, \tilde{\rho}_{jk}^{\varepsilon'}, \tag{B25}$$

where ρ_{kj}^ε and $\tilde{\rho}_{jk}^{\varepsilon'}$ are normalized and the leftover probabilities add up to 1. Substituting these previous expressions and doing some algebraic manipulations, (B21) becomes

$$S_\mathcal{M}(\omega|\omega\circ\varepsilon) + S_{\mathcal{N}'}(\omega|\omega\circ\varepsilon') = -\sum_{j,k} \text{Tr}_{M_{\mu_{kj}}(\mathbb{C})}\big\{\rho_{jk}^\omega\big[\ln\big(p_{kj}^\varepsilon\,\rho_{kj}^\varepsilon\big) + \ln\big(\tilde{p}_{jk}^{\varepsilon'}\,\tilde{\rho}_{jk}^{\varepsilon'}\big)\big]\big\}. \tag{B26}$$

Given this last expression, if expression (B1) holds, there must exist a positive number $\lambda > 0$, independent of j and k, such that

$$\ln\big(p_{kj}^\varepsilon\,\rho_{kj}^\varepsilon\big) + \ln\big(\tilde{p}_{jk}^{\varepsilon'}\,\tilde{\rho}_{jk}^{\varepsilon'}\big) = -\ln(\lambda)\,\mathbf{1}_{\mu_{kj}}, \tag{B27}$$

for all $j = 1,\dots,z_\mathcal{M}$ and $k = 1,\dots,z_\mathcal{N}$. To complete the proof, we must show that given $\varepsilon \in \hat{C}(\mathcal{M},\mathcal{N})$, parametrized by p_{kj}^ε and ρ_{kj}^ε, we can choose $\tilde{p}_{jk}^{\varepsilon'}$ and $\tilde{\rho}_{jk}^{\varepsilon'}$, parametrized by $\varepsilon' \in \hat{C}(\mathcal{M},\mathcal{N})$, satisfying (B27).

Let $\{t_{l,jk}\}$ and $\{\tilde{t}_{l,jk}\}$ be the eigenvalues of ρ_{kj}^ε and $\tilde{\rho}_{jk}^{\varepsilon'}$, respectively ($l = 1,\dots,\mu_{kj}$). For (B27) to hold, the operator $\tilde{\rho}_{jk}^{\varepsilon'}$ must be diagonal in the same basis as ρ_{kj}^ε. Then, we must have

$$p_{kj}^\varepsilon\,\tilde{p}_{jk}^{\varepsilon'}\,\tilde{t}_{l,jk}\,t_{l,jk} = \frac{1}{\lambda}, \quad \forall l = 1,\dots,\mu_{kj}, \tag{B28}$$

which implies

$$p_{kj}^\varepsilon\,\tilde{p}_{jk}^{\varepsilon'} = p_{kj}^\varepsilon\,\tilde{p}_{jk}^{\varepsilon'}\sum_{l=1}^{\mu_{kj}} \tilde{t}_{l,jk} = \frac{1}{\lambda}\sum_{l=1}^{\mu_{kj}} \frac{1}{t_{l,jk}} \equiv \frac{1}{\lambda}\lambda_{jk}(\varepsilon). \tag{B29}$$

From this last expression we obtain

$$1 = \sum_{k=1}^{z_\mathcal{N}} \tilde{p}_{jk}^{\varepsilon'} = \frac{1}{\lambda}\sum_{k=1}^{z_\mathcal{N}} \frac{\lambda_{jk}}{p_{kj}^\varepsilon}, \tag{B30}$$

which is only possible if $\sum_{k=1}^{z_\mathcal{N}} \frac{\lambda_{jk}}{p_{kj}^\varepsilon} \equiv \lambda$ is independent of the subindex j. The set of conditional expectations satisfying this condition was called $\hat{C}(\mathcal{M},\mathcal{N}) \subset C(\mathcal{M},\mathcal{N})$ in the paper. In this case, the theorem is satisfied provided the dual conditional expectation is parametrized by

$$\tilde{p}_{jk}^{\varepsilon'} := \frac{1}{\lambda}\frac{\lambda(\varepsilon_{jk})}{p_{kj}^\varepsilon}, \quad \tilde{t}_{l,jk} := \frac{1}{\lambda}\frac{1}{p_{kj}^\varepsilon\,\tilde{p}_{jk}^{\varepsilon'}\,t_{l,jk}}. \tag{B31}$$

A straightforward computation shows

$$\lambda(\varepsilon') = \lambda(\varepsilon) = \lambda. \tag{B32}$$

To end this Appendix, we generalize (B1) to the case of nonconnected inclusions of algebras. In this case, $\mathcal{N} \subset \mathcal{M}$ can be uniquely decomposed as

$$\mathcal{M} = \bigoplus_{i=1}^{z_c} \mathcal{M}_i, \quad \mathcal{N} = \bigoplus_{i=1}^{z_c} \mathcal{N}_i, \quad \mathcal{N}_i \subset \mathcal{M}_i, \tag{B33}$$

where $\mathcal{M}_i := E_i\mathcal{M}E_i, \mathcal{N}_i := E_i\mathcal{N}E_i$, and $\{E_i : i = 1,\dots,z_c\}$ are the minimal projectors of the common center $\mathcal{Z}(\mathcal{M}) \cap \mathcal{Z}(\mathcal{N})$. A conditional expectation $\varepsilon \in C(\mathcal{M},\mathcal{N})$ can be uniquely decomposed as $\varepsilon = \bigoplus_i \varepsilon_i$, where $\varepsilon_i \in C(\mathcal{M}_i,\mathcal{N}_i)$. Then, we define $\hat{C}(\mathcal{M},\mathcal{N})$ to be set formed by all conditional expectations $\varepsilon = \bigoplus_i \varepsilon_i$ such that $\varepsilon_i \in \hat{C}(\mathcal{M}_i,\mathcal{N}_i)$ for all $i = 1,\dots,z_c$.

Corollary 1. Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a general inclusion of finite dimensional algebras. Then, for every $\varepsilon \in \hat{C}(\mathcal{M},\mathcal{N})$ there exists a unique $\varepsilon' \in \hat{C}(\mathcal{M},\mathcal{N})$ such that

$$S_\mathcal{M}(\omega|\omega\circ\varepsilon) + S_{\mathcal{N}'}(\omega|\omega\circ\varepsilon') = \sum_{i=1}^{z_c} \omega(E_i)\ln(\lambda_i) \tag{B34}$$

holds for any global pure state ω on $\mathcal{B}(\mathcal{H})$, and where $\lambda_i := \lambda(\varepsilon_i) \equiv \lambda(\varepsilon'_i)$ is the algebraic index of the "partial" conditional expectation ε_i.

Proof. The proof consists of splitting the relative entropies on (B34) into relative entropies on the algebras $\mathcal{M}_i$ and $\mathcal{N}'_i$, and apply Theorem 1 to each connected inclusion $\mathcal{N}_i \subset \mathcal{M}_i$. ∎

APPENDIX C: THE ALGEBRAIC INDEX OF INCLUSION OF ALGEBRAS

Given an inclusion of von Neumann type-II$_1$ subfactors $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$, Jones proposed [24] an algebraic index $[\mathcal{M} : \mathcal{N}] \geqslant 1$, which "measures," in a certain sense, the size of $\mathcal{N}$ inside $\mathcal{M}$. The generalization to all types of algebras was developed independently by Kosaki [27] and Longo [28]. They found that the algebraic index is most naturally attached to a conditional expectation $\varepsilon \in C(\mathcal{M}, \mathcal{N})$, instead of an inclusion of subfactors $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$.

To explain the definition, let us first describe the space of weights $P(\mathcal{M}, \mathcal{N})$. A *weight* $\eta \in P(\mathcal{M}, \mathcal{N})$ is an unbounded (and un-normalized) positive map $\eta : \mathcal{M} \to \mathcal{N}$ with dense domain in $\mathcal{M}_+$ (the positive subspace of $\mathcal{M}$) satisfying the bimodule property (A1). In particular, we have that $C(\mathcal{M}, \mathcal{N}) \subset P(\mathcal{M}, \mathcal{N})$. Connes established a canonical bijection between $P(\mathcal{M}, \mathcal{N})$ and $P(\mathcal{N}', \mathcal{M}')$ (see [43]). However, this bijection, in general, does not map $C(\mathcal{M}, \mathcal{N})$ into $C(\mathcal{N}', \mathcal{M}')$. Still, what remains true is that for a conditional expectation $\varepsilon \in C(\mathcal{M}, \mathcal{N})$, with ϵ^{-1} being the previous canonical Connes inverse (not necessarily a conditional expectation itself), we have that $A\varepsilon^{-1}(\mathbf{1})A^\dagger = \varepsilon^{-1}(\mathbf{1})$ for any unitary $A \in \mathcal{M}$ and, hence, $\varepsilon^{-1}(\mathbf{1}) \in \mathcal{Z}(\mathcal{M})$ whenever it is finite.

Let us now start by assuming $\mathcal{M}$ is a factor. In this case

$$\varepsilon^{-1}(\mathbf{1}) = \lambda(\varepsilon)\,\mathbf{1}, \quad 1 \leqslant \lambda(\varepsilon) \leqslant +\infty. \tag{C1}$$

The number $\lambda(\varepsilon)$ is called the index of the conditional expectation ε. This definition is due to Kosaki [27]. If there exists $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ such that $\lambda(\varepsilon) < +\infty$, we say that $\mathcal{N} \subset \mathcal{M}$ is an inclusion of factors of *finite index*. In this case, it was shown in [28] that there exists a unique conditional expectation $\varepsilon_0 \in C(\mathcal{M}, \mathcal{N})$ such that

$$\lambda(\varepsilon_0) = \min \{\lambda(\varepsilon) : \varepsilon \in C(\mathcal{M}, \mathcal{N})\}. \tag{C2}$$

The number $\lambda(\varepsilon_0)$ coincides with the algebraic index $[\mathcal{M} : \mathcal{N}]$ defined by Jones whenever the relative commutant $\mathcal{N}' \cap \mathcal{M}$ is a factor. In such cases, the conditional expectation that minimizes the index is the one that preserves the trace on $\mathcal{M}$. In the most general case, they do not coincide. If $\mathcal{N} \subset \mathcal{M}$ is not of a finite index, we simply set $[\mathcal{M} : \mathcal{N}] = +\infty$. It was also shown by Kosaki [27] that this definition implies, and indeed is equivalent to, the Pimsner-Popa bound (see [34] and (23) in the paper).

For an inclusion of finite index, given $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ we can define the dual conditional expectation $\varepsilon' \in C(\mathcal{N}', \mathcal{M}')$ by means of

$$\varepsilon'(\cdot) := \frac{1}{\lambda(\varepsilon)} \varepsilon^{-1}(\cdot). \tag{C3}$$

Notice that $\lambda(\varepsilon) = \lambda(\varepsilon')$ for all $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ with finite index.

Probably the simplest example to illustrate these ideas is the case of an inclusion of finite dimensional factors:

$$\mathcal{N} := M_n(\mathbb{C}) \otimes \mathbb{1}_d \otimes \mathbb{1}_{m'}, \tag{C4}$$

$$\mathcal{M} := M_n(\mathbb{C}) \otimes M_d(\mathbb{C}) \otimes \mathbb{1}_{m'}. \tag{C5}$$

According to the discussion in Appendix A, any $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ is determined by a state $\rho^\varepsilon \in M_d(\mathbb{C})$ on $\mathcal{N}' \cap$

$\mathcal{M} \simeq M_d(\mathbb{C})$ by means of

$$\varepsilon(A \otimes B \otimes \mathbf{1}_{m'}) = \text{Tr}_{M_d(\mathbb{C})}(\rho^\epsilon B)(A \otimes \mathbf{1}_d \otimes \mathbf{1}_{m'}). \tag{C6}$$

As shown in [28], the index of ε is

$$\lambda(\varepsilon) = \sum_{j=1}^{d} \frac{1}{t_j}, \tag{C7}$$

where t_j are the eigenvalues of ρ^ϵ. If ρ^ϵ is not invertible, then $\lambda(\varepsilon) = +\infty$. A straightforward computation shows that the conditional expectation ε_0 that minimizes the index is the one having all equal eigenvalues $t_j = 1/d$. In this case,

$$\lambda(\varepsilon_0) = [\mathcal{M} : \mathcal{N}] = d^2. \tag{C8}$$

The more general case of an inclusion of algebras with centers was developed in [25]. Let $P_j^\mathcal{M}$ ($j = 1, \ldots, z_\mathcal{M}$) be the minimal projectors of $\mathcal{Z}(\mathcal{M})$. According to our discussion above, we must have that

$$\Lambda(\varepsilon) := \varepsilon^{-1}(\mathbf{1}) = \sum_{j=1}^{z_\mathcal{M}} c_j P_j^\mathcal{M}, \quad 0 \leqslant c_j \leqslant +\infty. \tag{C9}$$

In this case, the index is an operator belonging to the center of $\mathcal{M}$. According to the characterization of the space $C(\mathcal{M}, \mathcal{N})$ discussed in Appendix A, we have that [25]

$$c_j = \sum_{k=1}^{z_\mathcal{N}} p_{jk}^{-1} \lambda(\varepsilon_{jk}), \tag{C10}$$

where $\lambda(\varepsilon_{jk})$ are the "partial" indices of the conditional expectations $\varepsilon_{jk} \in C(\mathcal{M}_{jk}, \mathcal{N}_{jk})$. Notice that since $\mathcal{N}_{jk}$ and $\mathcal{M}_{jk}$ are factors, $\lambda(\varepsilon_{jk})$ is a number. In the case where $\mathcal{M}_{jk}$ is finite dimensional (and hence $\mathcal{N}_{jk}$), $\lambda(\varepsilon_{jk})$ can be computed according to (C7). To obtain a number from (C10), we can take its operator norm:

$$\lambda(\varepsilon) := \|\Lambda(\varepsilon)\| = \max_{j=1,\ldots,z_\mathcal{M}} \{c_j\}. \tag{C11}$$

In the particular case when all the constants (C10) are independent of $j = 1, \ldots, z_\mathcal{M}$, we have that

$$\Lambda(\varepsilon) = \lambda(\varepsilon)\,\mathbf{1}_\mathcal{H}, \tag{C12}$$

$$\lambda(\varepsilon) = c_j = \sum_{k=1}^{z_\mathcal{N}} p_{jk}^{-1} \lambda(\varepsilon_{jk}), \quad \forall j = 1, \ldots, z_\mathcal{M}, \tag{C13}$$

i.e. the index is a scalar. Furthermore, if $\lambda(\varepsilon) < +\infty$, we can define the dual conditional expectation $\varepsilon' \in C(\mathcal{N}', \mathcal{M}')$ as in (C3). It follows that $\lambda(\varepsilon) = \lambda(\varepsilon')$ and ε' also satisfies the previous equation (C12). In these lines, it is useful to define

$$\hat{C}(\mathcal{M}, \mathcal{N}) := \{\varepsilon \in C(\mathcal{M}, \mathcal{N}) : \lambda(\varepsilon) < +\infty,$$
$$\varepsilon \text{ satisfies } (C12)\}. \tag{C14}$$

The following key theorem, proven in [25], shows that $\hat{C}(\mathcal{M}, \mathcal{N})$ is nonempty and it contains the conditional expectation that minimizes the index norm (C11).

Theorem 2. Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a connected inclusion, $\varepsilon \in C(\mathcal{M}, \mathcal{N})$ a conditional expectation with finite norm

index (C11), and $\varepsilon_{jk} \in C(\mathcal{M}_{jk}, \mathcal{N}_{jk})$ the "partial" conditional expectations determined by ε. Then, there exists $\tilde{\varepsilon} \in \hat{C}(\mathcal{M}, \mathcal{N})$ (with scalar index), having the same "partial" conditional expectations $\varepsilon_{jk} \in C(\mathcal{M}_{jk}, \mathcal{N}_{jk})$, such that

$$\lambda(\tilde{\varepsilon}) \leqslant \lambda(\varepsilon). \tag{C15}$$

This theorem takes us to the last following lemma.

Lemma 4. Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be a connected inclusion of finite norm index (C11). Then, there exists a conditional expectation $\varepsilon_0 \in C(\mathcal{M}, \mathcal{N})$ such that

$$\lambda(\varepsilon_0) = \min \{\lambda(\varepsilon) : \varepsilon \in C(\mathcal{M}, \mathcal{N})\} =: [\mathcal{M} : \mathcal{N}]. \tag{C16}$$

Moreover, we have $\varepsilon_0 \in \hat{C}(\mathcal{M}, \mathcal{N})$, and its partial conditional expectations $\varepsilon_{0,jk} \in \hat{C}(\mathcal{M}_{jk}, \mathcal{N}_{jk})$ are the ones which minimize the index for the inclusion of factors $\mathcal{N}_{jk} \subset \mathcal{M}_{jk}$.

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