# Simple unidirectional finite-energy pulses

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We obtain a family of simple closed-form exact solutions of the three-dimensional wave equation that are free of backward components. The solutions have no singularities and possess finite energy. They describe single-cycle and subcycle pulses. Depending on two parameters, the family yields pulses with different types of localization, in particular, focused pancakes, balls, and needles.

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## I. INTRODUCTION

Progress in the generation of femtosecond and attosecond laser pulses (see, e.g., [1–6]) stimulates theoretical research related to few-cycle localized exact solutions of the wave equation without simplifying assumptions of paraxial approximation (see, e.g., [7–10]). In particular, of interest are the types of localization corresponding to focused pancakes [8], balls [9], and needles [11,12].

Yet, with few exceptions, the known closed-form localized exact solutions of the wave equation have backward-propagating components, which raised questions (see, e.g., [10,13]) of their physical realizability. This is strong motivation for addressing unidirectional waves, also called causal (see, e.g., [10]), that are free of backward components. Localized solutions of this kind were found in terms of Lommel functions in [13] and in closed form in [14,15] by inventive choices of appropriate weights for the general superposition of Bessel beams. Still, as fairly noted in [10], "few causal solutions are known, and these few are not as simple as those which have some degree of backward propagation in them."

In the present work, we obtain a family of unidirectional axisymmetric exact solutions of the wave equation, represented by an amazingly simple analytical expression. The solutions are free of singularities and have finite energy. They describe single-cycle and subcycle pulses. Depending on two parameters, the family describes pulses with different types of localization, in particular, focused pancakes, balls, and needles. The characteristic frequency of the pulses can be adjusted by scaling the parameters.

Our derivation originates from splitting a classical splash pulse (see [7]) into two unidirectional parts. Surprisingly, they both appear to be nonsingular solutions of the wave equation in free space.

## II. SPLASH PULSE

A splash pulse [7] is an axisymmetric solution of the wave equation in a linear isotropic homogeneous dispersion-free medium

$$\Box u = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial^2 u}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0, \tag{1}$$

where $c = \text{const} > 0$ is the propagation speed in the medium and $\rho^2 = x^2 + y^2$, given by

$$u = \frac{1}{(z - ct + ai)(z + ct + bi) + \rho^2}. \tag{2}$$

Here, $a$ and $b$ are free real constants. The solution (2) is nonsingular provided that

$$ab < 0. \tag{3}$$

From (2) it is seen that the disturbances propagate along both the positive and negative $z$ directions. Their relative roles depend on the ratio of the absolute values of $a$ and $b$. The bigger the ratio $|\frac{b}{a}|$ is, the smaller the contribution of the backward part is (see, e.g., [8]).

We will explicitly represent solution (2) as the difference of equally simple functions $u_+$ and $u_-$ describing pulses propagating strictly in the positive and negative $z$ directions, respectively. It is remarkable that both $u_+$ and $u_-$ satisfy the wave equation (1) in free space.

## III. SPLITTING A SPLASH PULSE INTO TWO UNIDIRECTIONAL PULSES

We represent the denominator of the function (2) as the difference of two squares,

$$(z - ct + ai)(z + ct + bi) + \rho^2 = z_*^2 - S^2, \tag{4}$$

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where

$$S = \sqrt{c^2 t_*^2 - \rho^2}, \tag{5}$$

with

$$z_* = z + \frac{i(a+b)}{2}, \quad t_* = t + \frac{i(b-a)}{2c}.$$

The condition (3) implies that $a \neq b$, and the function (5) does not vanish.

We now observe that

$$u = \frac{1}{z_*^2 - S^2} = \frac{1}{2S}\left(\frac{1}{z_* - S} - \frac{1}{z_* + S}\right) = \frac{1}{2}(u_+ - u_-), \tag{6}$$

where

$$u_+ = \frac{1}{S(z_* - S)} \tag{7}$$

and $u_- = \frac{1}{S(z_*+S)}$.

We choose the branch of the square root in (5) so that

$$S|_{\rho=0} = \sqrt{c^2 t_*^2} = ct_*. \tag{8}$$

In Sec. V, it will be shown that under such a choice (8) the function $u_+$ describes a pulse running in the positive $z$ direction, and $u_-$ runs in the opposite direction. Further we will consider solely the function $u_+$. The other family of solutions can be obtained by the substitution $z \mapsto -z$.

We parametrize $u_+$ by the free real constants

$$\zeta = (a+b)/2, \qquad \tau = (b-a)/2c, \tag{9}$$

where

$$\tau \neq 0, \tag{10}$$

so that

$$z_* = z + i\zeta, \qquad t_* = t + i\tau. \tag{11}$$

The condition (3) is equivalent to

$$|\zeta| < c|\tau|, \tag{12}$$

which ensures the nonsingularity of both $u_+$ and $u_-$ simultaneously. When considering only $u_+$, the condition of nonsingularity can be relaxed to read

$$\begin{cases} \zeta < c\tau & \text{if } \tau > 0, \\ \zeta > c\tau & \text{if } \tau < 0, \end{cases} \tag{13}$$

which will be shown in Sec. VI. The condition (13) can be easily rewritten in the form

$$c\tau(c\tau - \zeta) > 0 \tag{14}$$

or

$$\frac{\zeta}{\tau} < c. \tag{15}$$

Note that any linear combination of $u_+$ and $u_-$ is a novel solution of (1) except for the case of their difference. It is not unidirectional, however, unless it is proportional either to $u_+$ or to $u_-$.

## IV. PROOF THAT $u_+$ SATISFIES THE WAVE EQUATION

We now prove by direct calculation the nonobvious fact that function (7) satisfies the wave equation (1).

Having in mind the relation

$$\frac{\partial S}{\partial t} = \frac{c^2 t_*}{S}, \tag{16}$$

we start by differentiating $u_+$ with respect to $t$,

$$\frac{\partial u_+}{\partial t} = -\frac{z_* - 2S}{S^2(z_* - S)^2}\frac{c^2 t_*}{S} = \frac{1}{z_*}\left(-\frac{1}{S^2} + \frac{1}{(z_* - S)^2}\right)\frac{c^2 t_*}{S}, \tag{17}$$

then

$$\frac{1}{c^2}\frac{\partial^2 u_+}{\partial t^2} = \frac{1}{z_*}\left(\frac{2}{S^3} + \frac{2}{(z_* - S)^3}\right)\frac{c^2 t_*^2}{S^2} + \frac{1}{z_*}\left(-\frac{1}{S^2} + \frac{1}{(z_* - S)^2}\right)\frac{1}{S} - \frac{1}{z_*}\left(-\frac{1}{S^2} + \frac{1}{(z_* - S)^2}\right)\frac{c^2 t_*^2}{S^3}. \tag{18}$$

Similarly,

$$\frac{\partial S}{\partial \rho} = -\frac{\rho}{S},$$

$$\rho\frac{\partial u_+}{\partial \rho} = -\frac{1}{z_*}\left(-\frac{1}{S^2} + \frac{1}{(z_* - S)^2}\right)\frac{\rho^2}{S}, \tag{19}$$

and

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u_+}{\partial \rho}\right) = \frac{1}{z_*}\left(\frac{2}{S^3} + \frac{2}{(z_* - S)^3}\right)\frac{\rho^2}{S^2} - \frac{1}{z_*}\left(-\frac{1}{S^2} + \frac{1}{(z_* - S)^2}\right)\frac{2}{S} - \frac{1}{z_*}\left(-\frac{1}{S^2} + \frac{1}{(z_* - S)^2}\right)\frac{\rho^2}{S^3}. \tag{20}$$

Further we find

$$\frac{\partial u_+}{\partial z} = -\frac{1}{S(z_* - S)^2} \tag{21}$$

and

$$\frac{\partial^2 u_+}{\partial z^2} = \frac{2}{S(z_* - S)^3}. \tag{22}$$

From (18), (20), and (22), with (5) taken into account, we get

$$\Box u_+ = 0. \tag{23}$$

## V. UNIDIRECTIONALITY OF $u_+$

Rewrite $u_+$ [Eq. (7)] as follows:

$$u_+ = \frac{1}{S[z - \operatorname{Re} S + i(\zeta - \operatorname{Im} S)]}, \tag{24}$$

with Re and Im standing for the real and imaginary parts of a complex number, respectively.

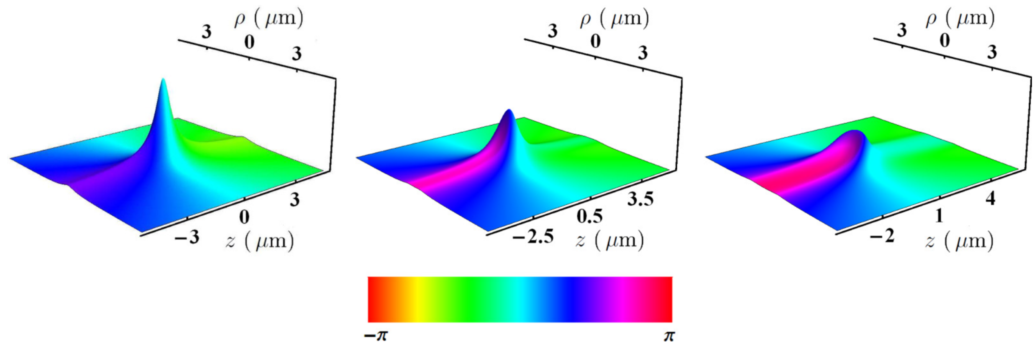


FIG. 1. Modulus of $u_+$ at $t = 0,\ 0.5/c,\ 1/c\ \mu$s ($\zeta = 1/12\,\mu$m, $c\tau = 5/12\,\mu$m).

The only dependence on $z$ of $u_+$ is given by the real part of the second factor in the denominator, which is $z - \operatorname{Re} S$. This accounts for a propagation along the $z$ axis with "speed" $\frac{\partial(\operatorname{Re} S)}{\partial t}$ dependent on $\rho$ and $t$. Thus, the pulse will be moving in the positive $z$ direction if the speed $\frac{\partial(\operatorname{Re} S)}{\partial t}$ is positive, that is, if $\operatorname{Re} S(t,\rho)$ is monotonically increasing with $t$ at each fixed $\rho$. Note that $z = \operatorname{Re} S$ corresponds to a single maximum of $|u_+|$ at each $\rho$, which in this case will also be moving with time in the positive $z$ direction.

We now formally show that $\operatorname{Re} S$ is indeed a monotonic function of $t$ at each fixed $\rho$. As seen from (5) and (8), the value of $\arg S$ at each $\rho$ differs from $\arg t_*$ by less than $\frac{\pi}{2}$. Therefore $|\arg(\frac{ct_*}{S})| < \frac{\pi}{2}$, and

$$\operatorname{Re}\left(\frac{ct_*}{S}\right) > 0\,. \tag{25}$$

Comparing inequality (25) with (16) shows that $\operatorname{Re}(\frac{\partial S}{\partial t}) > 0$ at each $\rho$, and thus, $\operatorname{Re} S$ is increasing monotonically with $t$.

Figure 1 shows the modulus of $u_+$ at three successive instants of time. The plots in Fig. 1 are colored according to the phase of $u_+$.

## VI. NONSINGULARITY OF $u_+$

Since, as noted above, $S$ is never zero, the denominator of (24) vanishes only if

$$\operatorname{Re} S = z, \qquad \operatorname{Im} S = \zeta\,. \tag{26}$$

We prove that under condition (13) the second equality in (26) cannot hold and thus the condition (13) guarantees the nonsingularity of $u_+$.

Introducing the notation $\operatorname{Re} S = p$, $\operatorname{Im} S = q$, that is, $S = p + iq$, we rewrite the squared complex-domain equation (5) as a system of real relations,

$$p^2 - q^2 + \rho^2 = c^2(t^2 - \tau^2),$$
$$pq = c^2 t\tau.$$

Finding $t = pq/c^2\tau$ from the second relation and substituting this into the first, after simple algebra we arrive at

$$q^2 = c^2\tau^2 \frac{p^2 + \rho^2 + c^2\tau^2}{p^2 + c^2\tau^2} = c^2\tau^2\left(1 + \frac{\rho^2}{p^2 + c^2\tau^2}\right),$$

hence

$$|q| = |\operatorname{Im} S| \geqslant c|\tau|. \tag{27}$$

This result is independent of the choice of the branch of the square root in (5). Consequently, (27) implies that the condition (12) guarantees the absence of singularities for both $u_+$ and $u_-$ simultaneously.

Under the specific choice of the branch (8) we attain additional values of free constants allowing the nonsingularity of $u_+$. From (8) we see that $q = \operatorname{Im} S$ has the same sign as $\tau$ at $\rho = 0$. Since $q$ is a continuous function of $\rho$, inequality (27) together with the assertion $\tau \neq 0$ (10) implies that the sign of $q$ is preserved for all values of $\rho$. Thus, we obtain

$$\operatorname{Im} S \geqslant c\tau \ \text{ if } \tau > 0,$$
$$\operatorname{Im} S \leqslant c\tau \ \text{ if } \tau < 0,$$

which is, in view of (13), incompatible with the second condition of singularity in (26). This proves that the function $u_+$ is nonsingular whenever condition (13) holds.

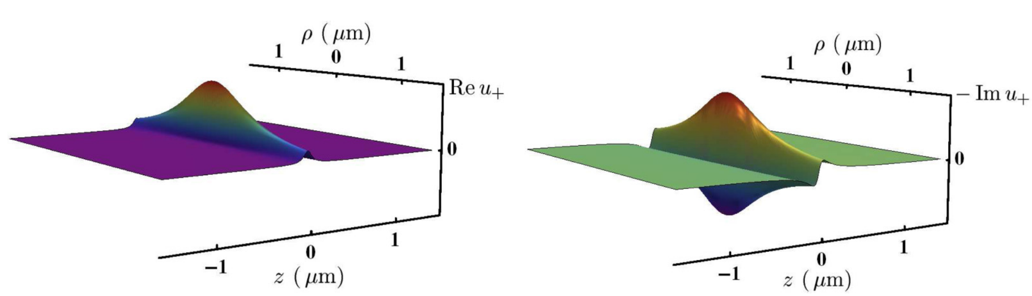


FIG. 2. Pancake pulses at $t = 0$ ($\zeta = 24.995\,\mu$m, $c\tau = 25.005\,\mu$m).

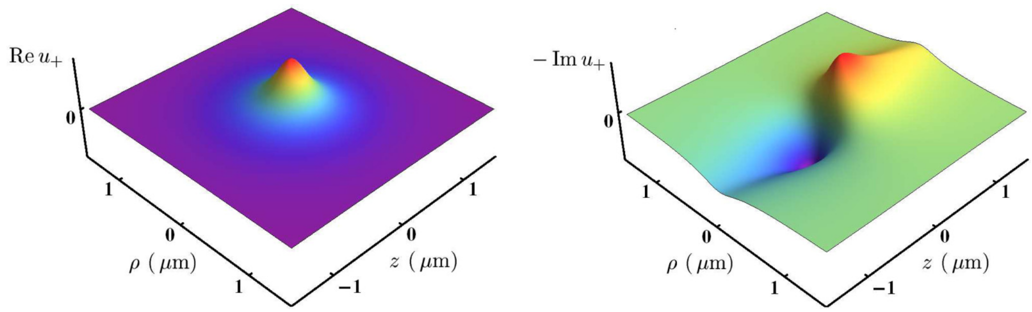


FIG. 3. Ball pulses at $t = 0$ ($\zeta = 0$, $c\tau = 0.25\,\mu$m).

## VII. FINITENESS OF ENERGY OF $u_+$

In this section, by a simple qualitative reasoning, we will see that the energy is finite. For an explicit calculation of energy of $u_+$, the reader is referred to the Appendix.

The energy of $u_+$ is obtained by integrating over the whole space the energy density

$$\mathcal{E} = \frac{1}{2}\left(\frac{1}{c^2}\left|\frac{\partial u_+}{\partial t}\right|^2 + |\nabla u_+|^2\right). \tag{28}$$

First, we show that $u_+$ is square integrable. Consider the integral of $|u_+|^2$ over the space, with $dV = dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz$ denoting a volume element,

$$\iiint_{\mathbb{R}^3} |u_+|^2 dV = \int_{-\pi}^{\pi} d\phi \int_0^{\infty} \rho\,d\rho \int_{-\infty}^{+\infty} \frac{dz}{|S|^2|z_* - S|^2}.$$

Integrating with respect to $\phi$ and $z$, we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} d\phi \int_0^{\infty} \rho\,d\rho \int_{-\infty}^{+\infty} \frac{dz}{|S|^2|z_* - S|^2}\\
&= 2\pi \int_0^{\infty} \frac{\rho\,d\rho}{|S|^2} \int_{-\infty}^{+\infty} \frac{dz}{(z - \operatorname{Re} S)^2 + (\zeta - \operatorname{Im} S)^2}\\
&= 2\pi^2 \int_0^{\infty} \frac{\rho\,d\rho}{|S|^2|\zeta - \operatorname{Im} S|},
\end{aligned} \tag{29}$$

where we have used the fact that

$$\int_{-\infty}^{\infty} \frac{dZ}{Z^2 + C^2} = \frac{\pi}{C} \quad \text{for } C > 0 \tag{30}$$

(see [16]). Since the integrand in (29) has no singularities, the convergence of the integral depends solely on its far-distance behavior. At any fixed $t$ and large $\rho$, we have $|S| \sim \rho$ and $|\operatorname{Im} S| \sim \rho$. Therefore, as $\rho$ tends to infinity, the integrand on the right-hand side of (29) behaves as $\rho^{-2}$, so the integral (29) converges.

Differentiation makes $u_+$ more attenuating at infinity; therefore, the fact that $u_+$ is square integrable allows us to conclude that the energy is finite. This also applies, e.g., to electromagnetic energy of pulses that can be derived on the basis of $u_+$ using standard procedures.

## VIII. PANCAKE, BALL, AND NEEDLE PULSES

Consider examples of various kinds of pulses described by $u_+$.

Pancake pulses are obtained when $\zeta \approx c\tau$ (see Fig. 2). Pancake pulses are used, e.g., in [17].

Ball pulses correspond to the case $|\zeta| \ll c|\tau|$ (see Fig. 3). Ball pulses might be useful in applications requiring ultrahigh energy density at the focus (see, e.g., [18,19]).

When $|\zeta| \gg c|\tau|$ and $\tau\zeta < 0$, needle pulses are obtained (see Fig. 4). Various applications of needle pulses are listed in [12].

In Figs. 2–4, the rainbow coloring is adjusted to the height of each pulse.

## IX. CHARACTERISTIC FREQUENCY

We now define a characteristic frequency of the pulse (7) in terms of its instantaneous frequency, which roughly describes its temporal properties but, nevertheless, is helpful in estimating the range of the electromagnetic spectrum to which the pulse belongs. As noted in [20], in nonstationary cases "the definition of instantaneous frequency is controversial, application-related, and empirically assessed." From several definitions of the characteristic frequency [19,20], we choose the one based on the theory of an analytic signal (see, e.g., [21]).

Consider the time profile at the focal point $\rho = 0$, $z = 0$:

$$\begin{aligned}
u_+|_{\rho=0,z=0} &= -\frac{1}{c^2(t + i\tau)[t + i(\tau - \zeta/c)]}\\
&= \frac{\tau(\tau - \zeta/c) - t^2}{c^2(t^2 + \tau^2)[t^2 + (\tau - \zeta/c)^2]}\\
&\quad + i\frac{(2\tau - \zeta/c)t}{c^2(t^2 + \tau^2)[t^2 + (\tau - \zeta/c)^2]}\\
&= \operatorname{Re} u_+(t) + i \operatorname{Im} u_+(t) = |u_+(t)|e^{i\phi(t)},
\end{aligned} \tag{31}$$

where $\operatorname{Re} u_+(t) = \operatorname{Re} u_+|_{z=0,\rho=0}$, $\operatorname{Im} u_+(t) = \operatorname{Im} u_+|_{z=0,\rho=0}$, $\phi(t)$ is the argument of the complex function $u_+|_{z=0,\rho=0}$, and $|u_+(t)|$ is its modulus. From (31),

$$\begin{aligned}
\phi(t) &= \pi - \arg(t + i\tau) - \arg[t + i(\tau - \zeta/c)]\\
&= \pi - \left(\operatorname{arccot}\frac{t}{\tau} + \operatorname{arccot}\frac{t}{\tau - \zeta/c}\right).
\end{aligned} \tag{32}$$

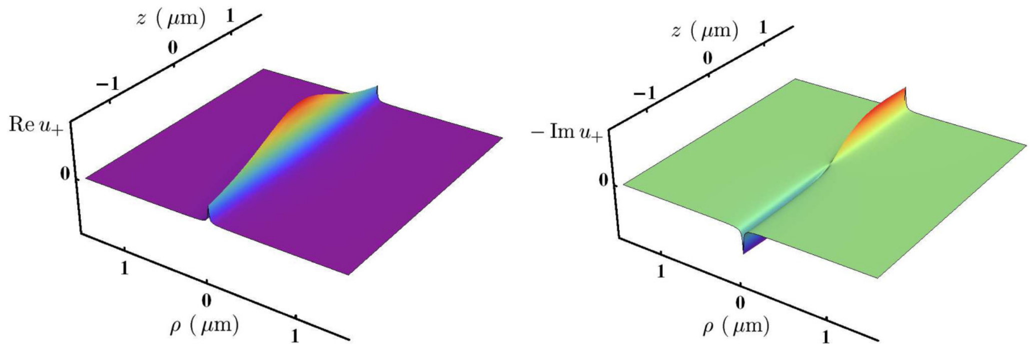


FIG. 4. Needle pulses at $t = 0$ ($\zeta = -0.995\,\mu$m, $c\tau = 0.005\,\mu$m).

Borrowing from [20] the definition of the instantaneous frequency at time instant $t$

$$f(t) = \frac{1}{2\pi}\left|\frac{d\phi(t)}{dt}\right|, \tag{33}$$

where we introduce the modulus to avoid negative values of frequency that would otherwise occur when $\tau < 0$, from (32) we obtain

$$f(t) = \frac{1}{2\pi}\left|\frac{\tau}{t^2+\tau^2} + \frac{\tau - \zeta/c}{t^2 + (\tau - \zeta/c)^2}\right|. \tag{34}$$

We accept as a characteristic frequency of the pulse the maximal value of the instantaneous frequency at $t = 0$:

$$f(0) = \frac{1}{2\pi}\left|\frac{1}{\tau} + \frac{1}{\tau - \zeta/c}\right|. \tag{35}$$

Note that the scaling of the free parameters $\zeta$ and $\tau$ allows for varying the characteristic frequency, thus making possible obtaining pulses not only in the optical domain but in any other domain as well.

From (31), one can see that the imaginary and real parts of $u_+$ describe single-cycle and subcycle pulses at the focus, respectively. (Recall that a single-cycle pulse is a bipolar pulse with approximately equal magnitudes of its positive and negative parts, whereas a subcycle pulse has a dominant part of one polarity; see, e.g., [22].) On propagation, a single-cycle pulse becomes subcycle and vice versa.

Figure 5 shows the time profiles $\operatorname{Re} u_+(t)$ and $\operatorname{Im} u_+(t)$ [see (31)] at the focus for the cases of pancakes, balls, and needles considered in the previous section. The points $t = \pm T/2$, where $T = 1/f(0)$ is the characteristic cycle duration, are marked by dots. Since we take the maximal value of the instantaneous frequency, the cycle duration $T$ is underestimated (that is, approximately half as much according to Fig. 5).

## X. CONCLUSIONS

We derived a family of unidirectional closed-form exact solutions of the wave equation, which are free of singularities and have finite energy. They describe single-cycle and subcycle pulses with the characteristic frequency adjustable by scaling the free parameters. By a simple analytical expression, this family of solutions covers a rich variety of pulses with different types of localization such as focused pancakes, balls, and needles.

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## APPENDIX: CALCULATION OF ENERGY OF $u_+$

We will calculate the energy of $u_+$ at $t = 0$. By the law of conservation of energy, at other instants of time the energy will be the same. For definiteness, assume that $\tau > 0$. Let $s = \operatorname{Im} S|_{t=0} = \sqrt{(c\tau)^2 + \rho^2}$. Using (17), (19), and (21), we find the energy density (28) at $t = 0$:

$$\begin{aligned}\mathcal{E}|_{t=0} &= \frac{1}{2}|u|^2_{t=0}\left[\frac{z^2 + (2s-\zeta)^2}{s^2[z^2 + (s-\zeta)^2]} + \frac{1}{z^2 + (s-\zeta)^2}\right] \\ &= \frac{1}{2}\left[\frac{1}{s^4[z^2 + (s-\zeta)^2]} + \frac{4}{s^2[z^2+(s-\zeta)^2]^2} \right. \\ &\quad \left. - \frac{2\zeta}{s^3[z^2 + (s-\zeta)^2]^2}\right]. \end{aligned} \tag{A1}$$

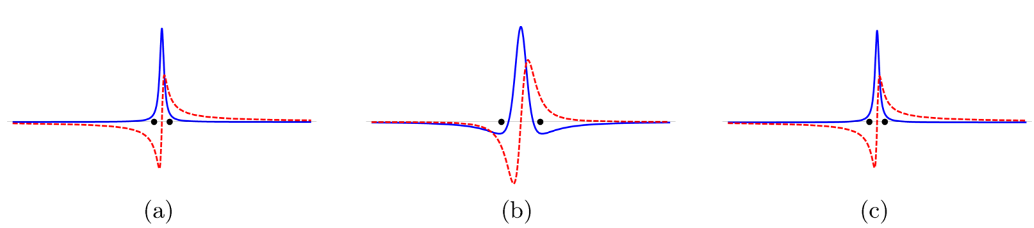


FIG. 5. $\operatorname{Re} u_+|_{z=0,\rho=0}$ (blue solid line) and $\operatorname{Im} u_+|_{z=0,\rho=0}$ (red dashed line) for the (a) pancake, (b) ball, and (c) needle cases.

Integrating (A1) over the whole space, using (30), and taking into account that

$$\int_{-\infty}^{\infty} \frac{dZ}{(Z^2 + C^2)^2} = \frac{\pi}{2C^3} \text{ for } C > 0$$

(see [16]), we calculate the energy

$$\begin{aligned}
E &= \iiint_{\mathbb{R}^3} \mathcal{E}|_{t=0} dV \\
&= \frac{1}{2} \int_{-\pi}^{\pi} d\phi \int_{0}^{\infty} \rho\, d\rho \int_{-\infty}^{+\infty} \left[ \frac{1}{s^4[z^2 + (s-\zeta)^2]} + \frac{4}{s^2[z^2 + (s-\zeta)^2]^2} - \frac{2\zeta}{s^3[z^2 + (s-\zeta)^2]^2} \right] dz \\
&= \pi \left[ \int_{c\tau}^{\infty} \frac{s\, ds}{s^4} \int_{-\infty}^{+\infty} \frac{dz}{z^2 + (s-\zeta)^2} + 4 \int_{c\tau}^{\infty} \frac{s\, ds}{s^2} \int_{-\infty}^{+\infty} \frac{dz}{[z^2 + (s-\zeta)^2]^2} - 2\zeta \int_{c\tau}^{\infty} \frac{s\, ds}{s^3} \int_{-\infty}^{+\infty} \frac{dz}{[z^2 + (s-\zeta)^2]^2} \right] \\
&= \pi \left[ \int_{c\tau}^{\infty} \frac{ds}{s^3} \frac{\pi}{s-\zeta} + 4 \int_{c\tau}^{\infty} \frac{ds}{s} \frac{\pi}{2(s-\zeta)^3} - 2\zeta \int_{c\tau}^{\infty} \frac{ds}{s^2} \frac{\pi}{2(s-\zeta)^3} \right] \\
&= \frac{\pi^2}{\zeta} \int_{c\tau}^{\infty} \left[ \frac{1}{(s-\zeta)^3} - \frac{1}{s^3} \right] ds \\
&= \frac{\pi^2}{2\zeta} \left[ \frac{1}{(c\tau - \zeta)^2} - \frac{1}{(c\tau)^2} \right] \\
&= \frac{\pi^2 (2c\tau - \zeta)}{2(c\tau - \zeta)^2 (c\tau)^2}.
\end{aligned}$$

One can easily show that in the general case where $\tau$ is not assumed to be positive the energy $E$ is given by

$$E = \frac{\pi^2 |2c\tau - \zeta|}{2(c\tau - \zeta)^2 (c\tau)^2}. \tag{A2}$$

It is seen from (A2) that the energy of the pulse grows as $\zeta \to c\tau$.


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