Scattering electromagnetic eigenstates of a two-constituent composite and their exploitation for calculating a physical field

David J. Bergman, Parry Y. Chen, and Asaf Farhi

Raymond and Beverly Sackler School of Physics and Astronomy, Faculty of Exact Sciences, Tel Aviv University, IL-69978 Tel Aviv, Israel

(Received 6 April 2020; accepted 3 November 2020; published 4 December 2020)

The spectral representation of an electric field in a two-constituent composite medium is revisited. A theory is developed for calculating the electromagnetic (EM) eigenstates of Maxwell's equations for such a composite where the magnetic permeability, as well as the electric permittivity, have different uniform values in the two constituents. The physical electric field $\mathbf{E}(\mathbf{r})$ produced in the system either by a given incident field or by a given source current density is expanded in this set of biorthogonal eigenstates for any position $\mathbf{r}$. If the microstructure consists of a cluster of separate inclusions in a uniform host medium, then the EM eigenstates of all the isolated inclusions can also be used to calculate $\mathbf{E}(\mathbf{r})$. Once all these eigenstates are known for a given host and a given microstructure then the calculation of $\mathbf{E}(\mathbf{r})$ only involves performing three-dimensional integrals of known functions and solving sets of linear algebraic equations.

DOI: 10.1103/PhysRevA.102.063508

I. INTRODUCTION

Eigenstates of Maxwell's equations in a two-constituent composite medium were first introduced for the quasistatic regime in Ref. [1]. They were then exploited for calculating the macroscopic or bulk effective electric permittivity of a simple-cubic array of spherical inclusions in an otherwise uniform host medium [2]. Lately this theory has been applied to a discussion of the Veselago lens in the quasistatic limit [3–5]. The theory of such monochromatic eigenstates away from the quasistatic regime was first expounded in Ref. [6], based upon an integral equation formulation of Maxwell's equations. This was implemented for an isolated spherical inclusion and for interacting spherical inclusions. Closed form expressions were derived for the isolated sphere eigenstates as well as for the interactions between eigenstates of a pair of such inclusions. The expressions for the interactions which appeared in Appendix B of Ref. [6] were later corrected in Ref. [7]. The integral equation (2.7) of Ref. [6] may be problematic, due to the strong singularity of Green's tensor [Eq. (2.5) of that paper] when $\mathbf{r}' = \mathbf{r}$ [8]. However, in practice, it led to the same results that are obtained in the current article when the field sources lie outside of an inclusion. Fortunately, all the algebraic results of Ref. [6], as well as those of the current article, were obtained without using that integral equation.

In this paper, we present the basic theory of monochromatic eigenstates of Maxwell's equations in a two-constituent composite medium without using that integral equation. We also show how these eigenstates can be used to calculate the physical electric field produced by a monochromatic source current density or by a monochromatic incident field. In contrast with previous discussions, e.g., Ref. [6], where that source was required to be outside of an inclusion, it can now be either inside or outside of any inclusion, or even at an interface. Moreover, the inclusion or inclusions are no longer required to be finite: Both constituents may now extend out to infinite distances. A simple example of a composite microstructure is an isolated spherical inclusion in an otherwise uniform host medium. When a wave is incident from the host medium upon this structure, or a field source is present in it, the conventional approach to calculation of the local electric field usually requires solution of a new partial differential equation every time a new situation is discussed. By contrast, using the closed form eigenstates of this microstructure, any situation can be treated using only integrals of an eigenfunction multiplied by the incident field or the field source.

Most naturally occurring materials have a magnetic permeability μ that is very close to 1. Even ferromagnetic and anti-ferromagnetic materials exhibit this property above megahertz frequencies [9]. However, meta-materials have been synthesized where μ differs from 1 considerably, sometimes even achieving negative values [10]. This is why a spectral representation is needed for calculating the electromagnetic fields in a composite medium where both the electric permittivity and the magnetic permeability are nonuniform. We therefore develop a spectral representation for a two-constituent composite medium where both of these physical moduli have different values in the two constituents. Thus the theory presented in Sec. V assumes that the magnetic permeability, the electric permittivity, and the electric conductivity are local quantities but have different uniform values in each constituent.

The advantages of this approach arise from the fact that the eigenstates depend only upon the microstructure and upon the electric permittivity and magnetic permeability of one of the constituents, called the host medium. Once those eigenstates are known, the physical field produced by any incident field or source current density can be found by calculating spatial integrals where the integrand is the product of an eigenfunction and the incident field or source current.

An alternative approach which has evolved recently proposes to expand the local physical field of systems like the one considered here in a set of so-called "quasinormal modes" [11]. In contrast with the eigenstates defined and described in the present article those states are not easily normalizable and apparently defy any rigorous definition [12]. One consequence of the discussion presented in the present article is that those quasinormal modes are not needed for developing a useful method for constructing a convergent expansion of the local physical field in a set of (bi-)orthogonal states in an appropriate Hilbert space.

In Sec. II, we develop the basic theory of the EM eigenstates when $\mu(\mathbf{r}) \equiv \mu_2$ has the same value everywhere. In Sec. III, we show how those eigenstates can be used for calculating a local physical field. In Sec. IV, we consider the case where the nonhost constituent is composed of a cluster of disconnected inclusions in the otherwise uniform host which fills up all the rest of space. In Sec. V, the similarly basic theory for the case where both $\mu(\mathbf{r})$ and $\kappa(\mathbf{r})$ are heterogeneous is developed. In Sec. VI, we summarize the main results and discuss some possible future extensions of the approach presented in this paper. In Appendix, we develop detailed expressions for the EM eigenstates of an isolated sphere and use them to calculate closed form expressions for the electric field produced by a monochromatic point source, when that source is either outside or inside of the sphere. We also calculate the scattered field when a plane EM wave impinges upon a perfectly conducting sphere and compare with a calculation of that field using a classic approach described in Ref. [13].

II. BASIC THEORY OF THE EM SCATTERING EIGENSTATES WHEN $\mu(\mathbf{r}) \equiv \mu_2$

In a two-constituent composite medium where the magnetic permeability equals μ_2 everywhere Maxwell's monochromatic equations can be reduced to

$$-\nabla \times (\nabla \times \mathbf{E}) + k_2^2\mathbf{E} = u^{(1)}k_2^2\theta_1\mathbf{E} - \frac{4\pi i\omega\mu_2}{c^2}\mathbf{J}_{ex},$$
$$u^{(1)} \equiv 1 - \frac{\kappa_1}{\kappa_2} \equiv \frac{1}{s^{(1)}},\ \kappa_i \equiv \epsilon_i + \frac{4\pi i\sigma_i}{\omega},\ k_2^2 \equiv \frac{\omega^2}{c^2}\kappa_2\mu_2, \quad (1)$$

or to

$$-\nabla \times (\nabla \times \mathbf{E}) + k_1^2\mathbf{E} = u^{(2)}k_1^2\theta_2\mathbf{E} - \frac{4\pi i\omega\mu_2}{c^2}\mathbf{J}_{ex},$$
$$u^{(2)} \equiv 1 - \frac{\kappa_2}{\kappa_1} \equiv \frac{1}{s^{(2)}},\ s^{(2)} = 1 - s^{(1)},\ k_1^2 \equiv \frac{\omega^2}{c^2}\kappa_1\mu_2, \quad (2)$$

where $\theta_i(\mathbf{r}) = 1$ for $\mathbf{r}$ inside the κ_i subvolume, henceforth denoted as V_i, and vanishes elsewhere, while $\mathbf{J}_{ex}$ is a monochromatic external current density; ϵ_i and σ_i, $i = 1, 2$, are the electric permittivity and electric conductivity of the two constituents. We will usually refer to κ_i as the (complex) permittivity.

Scattering eigenfunctions of Maxwell's equations satisfy Eq. (1) or Eq. (2) with $\mathbf{J}_{ex} \equiv 0$ and special values of $u^{(1)}$ or $u^{(2)}$, respectively, and behave as purely outgoing or ingoing waves at large distances with an amplitude that decreases with r at large distances. This will be elaborated upon below. The eigenfunctions $\mathbf{E}_n^{(i)}$ and their eigenvalues $u_n^{(i)}$ satisfy

$$-\nabla \times \left(\nabla \times \mathbf{E}_n^{(1)}\right) + k_2^2\mathbf{E}_n^{(1)} = u_n^{(1)}k_2^2\theta_1\mathbf{E}_n^{(1)}, \quad (3)$$

$$-\nabla \times \left(\nabla \times \mathbf{E}_n^{(2)}\right) + k_1^2\mathbf{E}_n^{(2)} = u_n^{(2)}k_1^2\theta_2\mathbf{E}_n^{(2)}. \quad (4)$$

It should be noted that, while the physical field $\mathbf{E}(\mathbf{r}, \omega)$ depends on the permittivities of the physical constituents κ_1, κ_2, every eigenfunction depends upon the permittivities determined by the eigenvalue. Therefore Eqs. (1) and (2) depend upon k_1 and k_2 of the two physical constituents. By contrast, in Eqs. (3) and (4), we can only choose one of these wave numbers to have the constituent physical value, while the other will depend on the appropriate eigenvalue. For example, if k_2 is chosen to have the physical value $\omega\sqrt{\kappa_2\mu_2}/c$ then k_1 must be given by $k_{1n} = k_2\sqrt{1 - u_n^{(1)}}$. Similarly, if k_1 is chosen to have the physical value $\omega\sqrt{\kappa_1\mu_1}/c$ then k_2 must have the value $k_{2n} = k_1\sqrt{1 - u_n^{(2)}}$. The asymptotic behavior of an eigenfunction will depend upon whether k_i has the physical value or the eigenvalue value k_{in}.

It is useful to note that, if the field sources are all at bounded values of r and when only V_1 extends to infinity, then for $r \gg 1/k_1$ the physical field $\mathbf{E}(\mathbf{r})$ behaves asymptotically as e^{ik_1r}/r—this follows from Eq. (2) and from the fact that this field must exhibit a finite, nonzero total outgoing electromagnetic (EM) power flux. Similarly, if only V_2 extends to infinity then it follows from Eq. (1) that the asymptotic behavior of $\mathbf{E}(\mathbf{r})$ is as e^{ik_2r}/r. If both V_1 and V_2 extend to infinity then the asymptotic behavior will be as e^{ik_ir}/r when $r \in V_i$ and $k_ir \gg 1$. If the eigenfunctions are required to exhibit a similar asymptotic behavior then they can be used to expand such a physical field.

We will now show that eigenfunctions with different eigenvalues are orthogonal. Multiplying the left-hand side (lhs) of Eq. (3) by another eigenfunction solution $\mathbf{E}_m^{(1)}$ of that equation and integrating over all space leads to

$$\int dV\mathbf{E}_m^{(1)} \cdot \left[-\nabla \times \left(\nabla \times \mathbf{E}_n^{(1)}\right) + k_2^2\mathbf{E}_n^{(1)}\right]$$
$$= \int dV\left\{\nabla \cdot \left[\mathbf{E}_m^{(1)} \times \left(\nabla \times \mathbf{E}_n^{(1)}\right)\right]\right.$$
$$\left. - \left(\nabla \times \mathbf{E}_m^{(1)}\right) \cdot \left(\nabla \times \mathbf{E}_n^{(1)}\right) + k_2^2\left(\mathbf{E}_m^{(1)} \cdot \mathbf{E}_n^{(1)}\right)\right\}. \quad (5)$$

The first integral on the right-hand-side (rhs) transforms to the following surface integral over the closed envelope of the system:

$$\oint d\mathbf{S} \cdot \left[\mathbf{E}_m^{(1)} \times \left(\nabla \times \mathbf{E}_n^{(1)}\right)\right]. \quad (6)$$

At large distances, i.e., in the radiation zone, these EM fields have the following properties—see Eq. (16.73) in Ref. [13]:

$$\mathbf{E}_m^{(1)} \perp \mathbf{n},\quad \mathbf{E}_m^{(1)}(\mathbf{r}) = \frac{e^{ik_ir}}{r}\mathbf{a}_m + O(1/r^2),\quad \mathbf{a}_m \perp \mathbf{n}, \quad (7)$$

$$\nabla \times \mathbf{E}_m^{(1)} = ik_i\mathbf{n} \times \mathbf{E}_m^{(1)} + O(1/r^2), \quad (8)$$

where $r \equiv |\mathbf{r}|$, $\mathbf{n} \equiv \mathbf{r}/|\mathbf{r}|$, $\mathbf{a}_m$ is an $O(r^0)$ vector, and the first term in the above expressions is of order $O(1/r)$. The wave number k_i is either k_{1n} or k_2, depending on whether $\mathbf{r}$ is in V_1

or V_2. Consequently, at large distances the integrand in Eq. (6) can be written as

$$\mathbf{E}_m^{(1)} \times \left(\nabla \times \mathbf{E}_n^{(1)}\right) = ik_i\left[\mathbf{E}_m^{(1)} \times \left(\mathbf{n} \times \mathbf{E}_n^{(1)}\right)\right] + O(1/r^3) = ik_i\mathbf{n}\left(\mathbf{E}_m^{(1)} \cdot \mathbf{E}_n^{(1)}\right) + O(1/r^3), \tag{9}$$

where the first term in the last expression is of order $O(1/r^2)$ and is clearly symmetric in $\mathbf{E}_m^{(1)}$ and $\mathbf{E}_n^{(1)}$. All the other contributions to the integrand of Eq. (6) are of order $O(1/r^3)$ or smaller and thus yield a vanishing result if the system envelope is taken at $r \to \infty$. Therefore the entire expression on the rhs of Eq. (5) is also symmetric in $\mathbf{E}_m^{(1)}$ and $\mathbf{E}_n^{(1)}$. Consequently we get that

$$\int dV \mathbf{E}_m^{(1)} \cdot \left[-\nabla \times \left(\nabla \times \mathbf{E}_n^{(1)}\right) + k_2^2 \mathbf{E}_n^{(1)}\right] = i \oint (d\mathbf{S} \cdot \mathbf{n})(k_{1n}\theta_1 + k_2\theta_2)\left(\mathbf{E}_m^{(1)} \cdot \mathbf{E}_n^{(1)}\right) + k_2^2 \int dV \left(\mathbf{E}_m^{(1)} \cdot \mathbf{E}_n^{(1)}\right) - \int dV \left(\nabla \times \mathbf{E}_m^{(1)}\right) \cdot \left(\nabla \times \mathbf{E}_n^{(1)}\right). \tag{10}$$

The appearance of $k_{1n}\theta_1 + k_2\theta_2$ in the surface integral takes into account that both V_1 and V_2 can reach out to infinite values of $r \equiv |\mathbf{r}|$. We note that in the case of a microstructure with a one-dimensional symmetry, i.e., parallel flat slabs or parallel cylinders, the asymptotic behavior of the fields will differ from Eqs. (7) and (8). This was discussed in detail in Refs. [5,14,15] for the quasistatic regime. A similar situation occurs when the microstructure has a three-dimensional periodicity. This was discussed in detail in Ref. [16] for the quasistatic regime. Such microstructures will not be discussed here in any detail.

In the cases discussed here, the permittivities often have an imaginary part. Therefore that imaginary part will always be positive if κ_i represents the usual type of lossy medium. However, sometimes that imaginary part will have the opposite sign. In any case, k_i will also have an imaginary part, the sign of which will depend upon which of the two square roots is used in obtaining $k_i = \pm\sqrt{k_i^2}$. In this paper, we will always choose that square root which leads to a non-negative Im k_i. This will ensure that at large distances the scattering eigenfunctions decay to zero and thus that integrals such as Eqs. (5) and (10) converge to finite values.

All the volume integrals in Eq. (10) are symmetric in m and n. From Eqs. (7) and (8), it follows that their integrands are all proportional to $(\mathbf{a}_m \cdot \mathbf{a}_n)\, e^{2ik_ir}/r^2$ at large distances. Because $dV = r^2\, dr\, d\Omega$, therefore when k_i is real these integrals can only exhibit conditional convergence when taken over all space. If the permittivities κ_i have the usual physical kind of positive real and imaginary parts, corresponding to energy dissipation, then the physical values of Re k_i and Im k_i will also be positive. In that case, the factor e^{2ik_ir} will be exponentially decreasing as $r \to \infty$ and those integrals will converge absolutely. Moreover, the surface integral will vanish when the system envelope is at infinity. When κ_i is purely real then we can proceed by adding to it a small positive imaginary part $i\delta$. This regularization will again result in absolutely convergent integrals. If we then take the limit $\delta \to 0^+$, the values obtained for those integrals will tend to well defined finite values.

If κ_i is complex then its correct square root, used for calculating the value of k_i, is the one which results in a positive value for Im k_i. This means that the wave amplitude decreases to 0 exponentially at large distances. If Im$(\kappa_i) > 0$ and Re$(\kappa_i) > 0$, as is always the case if the i constituent is a physical dielectric material which exhibits dissipation, then Re k_i will always be positive and the factor e^{ik_ir} will be an outgoing wave. However, when ϵ_i represents a material with gain, i.e., Im $(\kappa_i) < 0$, then in order for Im k_i to be positive Re k_i will have to be negative. The factor e^{ik_ir} will then represent an incoming wave, the amplitude of which decreases exponentially to 0 at large distances. Therefore there will be no outflow of EM energy from the system. Also, the volume integrals in Eqs. (5) and (10) will then converge to finite values and the surface integral of Eq. (6) will vanish. The inflow of energy will be compensated by energy dissipation in the other constituent. The total energy must remain constant because the time dependence of all fields is periodic, i.e., $\propto e^{i\omega t}$.

It is worth noting that if V_1 does not extend out to infinity then the asymptotic behavior of the physical field $\mathbf{E}$ and the V_1 eigenfunctions is proportional to e^{ik_2r}/r. Similarly, if V_2 does not extend out to infinity then the physical field $\mathbf{E}$ and the V_2 eigenfunctions behave asymptotically as e^{ik_1r}/r. If both V_1 and V_2 extend out to infinity then the asymptotic behavior of those functions will be as e^{ik_ir}/r when r is not in V_i.

We also note that, like the eigenfunctions $\mathbf{E}_n^{(i)}$, the eigenvalues $u_n^{(i)}$ usually have complex values. If those values are translated into special values for κ_{1n} or κ_{2n}, then these too will be complex valued. Moreover, the imaginary part of the κ_{in} eigenvalue will have the opposite sign to that of the physical Im(κ) of any real material. That is because in the other constituent heat and entropy will be produced. However, because all the fields have a periodic time dependence $\propto e^{i\omega t}$, this positive rate of heat production must be compensated by a positive rate EM energy radiation or a negative rate of EM energy production and a consequent negative rate of heat production inside V_1.

From Eqs. (3) and (10), we now get

$$k_2^2 u_n^{(1)} \int dV \theta_1 \left(\mathbf{E}_m^{(1)} \cdot \mathbf{E}_n^{(1)}\right) = k_2^2 u_m^{(1)} \int dV \theta_1 \left(\mathbf{E}_n^{(1)} \cdot \mathbf{E}_m^{(1)}\right) = -i \oint (d\mathbf{S} \cdot \mathbf{n})(k_{1n}\theta_1 + k_2\theta_2)\left(\mathbf{E}_m^{(1)} \cdot \mathbf{E}_n^{(1)}\right) + k_2^2 \int dV \left(\mathbf{E}_m^{(1)} \cdot \mathbf{E}_n^{(1)}\right) - \int dV \left(\nabla \times \mathbf{E}_m^{(1)}\right) \cdot \left(\nabla \times \mathbf{E}_n^{(1)}\right). \tag{11}$$

Therefore, if $u_n^{(1)} \neq u_m^{(1)}$ then

$$\int dV \theta_1 \left(\mathbf{E}_n^{(1)} \cdot \mathbf{E}_m^{(1)}\right) = 0, \tag{12}$$

and also

$$\int dV \left(\nabla \times \mathbf{E}_m^{(1)}\right) \cdot \left(\nabla \times \mathbf{E}_n^{(1)}\right) = k_2^2 \int dV \left(\mathbf{E}_n^{(1)} \cdot \mathbf{E}_m^{(1)}\right).$$

The last result follows because the surface integral in Eq. (11) vanishes, as explained above.

When κ_1 and k_1 retain their physical values but κ_2 and k_2 are replaced by the eigenvalue values $\kappa_{2n} \equiv \kappa_1(1 - u_n^{(2)})$

and $k_{2n} \equiv k_1\sqrt{1-u_n^{(2)}}$ similar results are obtained where the indices 1 and 2 are interchanged:

$$k_1^2 u_n^{(2)} \int dV \theta_2 \big(\mathbf{E}_m^{(2)} \cdot \mathbf{E}_n^{(2)}\big) = k_1^2 u_m^{(2)} \int dV \theta_2 \big(\mathbf{E}_n^{(2)} \cdot \mathbf{E}_m^{(2)}\big)$$
$$= -i \oint (d\mathbf{S} \cdot \mathbf{n})(k_1\theta_1 + k_{2n}\theta_2)\big(\mathbf{E}_m^{(2)} \cdot \mathbf{E}_n^{(2)}\big)$$
$$+ k_1^2 \int dV \big(\mathbf{E}_m^{(2)} \cdot \mathbf{E}_n^{(2)}\big) - \int dV \big(\nabla \times \mathbf{E}_m^{(2)}\big) \cdot \big(\nabla \times \mathbf{E}_n^{(2)}\big). \tag{13}$$

Therefore, if $u_n^{(2)} \neq u_m^{(2)}$ then

$$\int dV \theta_2 \big(\mathbf{E}_n^{(2)} \cdot \mathbf{E}_m^{(2)}\big) = 0 \tag{14}$$

and

$$\int dV \big(\nabla \times \mathbf{E}_m^{(2)}\big) \cdot \big(\nabla \times \mathbf{E}_n^{(2)}\big) = k_1^2 \int dV \big(\mathbf{E}_n^{(2)} \cdot \mathbf{E}_m^{(2)}\big).$$

Defining the V_i scalar product of two vector fields by

$$\langle \mathbf{F}|\mathbf{E}\rangle_i \equiv \int dV \theta_i (\mathbf{F} \cdot \mathbf{E}) = \langle \mathbf{E}|\mathbf{F}\rangle_i, \tag{15}$$

we get that the eigenfunctions $\mathbf{E}_n^{(i)}$ form a biorthogonal set in the Hilbert space of vector functions in the subvolume V_i. Since the integrand in the scalar product of a complex field with itself is $\mathbf{E}^2$, which is not positive definite in general, therefore the scalar products $\langle \mathbf{E}_n^{(i)}|\mathbf{E}_n^{(i)}\rangle_i$ are not ensured to be nonzero. Therefore this property needs to be verified explicitly in each case. As shown after Eqs. (58) and (59) below, even when the normalization integrals vanish this does not prevent using these eigenfunctions for expanding the physical field.

We now note that all the eigenfunctions with eigenvalues $u_n^{(1)} \neq 1$, $u_n^{(2)} \neq 1$ will be divergence-free in both V_1 and V_2, though not at the V_1, V_2 interface. In fact, at that interface their normal component will usually be discontinuous. Therefore we should be able to exploit $\mathbf{E}_n^{(i)}$ for expanding any physical field $\mathbf{E}(\mathbf{r})$ inside V_i if $\mathbf{J}_{ex} = 0$ there. Obviously, where $\mathbf{J}_{ex} \neq 0$ the field $\mathbf{E}(\mathbf{r})$ will not be divergence-free and we can therefore not use the eigenfunctions defined there for expanding the physical field. As we shall show in Sec. III, we will nevertheless be able to use the divergence-free eigenfunctions to calculate that field everywhere. These eigenstates were first introduced in Ref. [6] using an approach based upon a Green tensor.

The eigenstates fall into three classes. (1) All the eigenstates for which $u_n^{(1)} \neq 1$ or $u_n^{(2)} \neq 1$. In that case, $\nabla \cdot \mathbf{E}_n^{(i)} = 0$ both inside V_1 and inside V_2.

(2) Divergence-free eigenstates for which $u_n^{(i)} = 1$ and $\kappa_{in} = 0$. Each of these states satisfies $\nabla \times \mathbf{E}_n^{(i)} = \nabla \psi_n^{(i)}$ and $\nabla^2 \psi_n^{(i)} = 0$ inside V_i for some scalar function $\psi_n^{(i)}(\mathbf{r})$.

(3) Curl-free or longitudinal eigenstates $\mathbf{E}_n^{(i)} = \nabla \phi_n^{(i)}$ for $\mathbf{r} \in V_i$. These vector fields all have eigenvalues $s_n^{(i)} = u_n^{(i)} = 1$ and $\kappa_{in} = 0$. Because the magnetic field $\mathbf{H}_n^{(i)}(\mathbf{r})$ is given by

$$\mathbf{H}_n^{(i)} = \frac{c}{i\omega\mu_2}\big(\nabla \times \mathbf{E}_n^{(i)}\big),$$

therefore $\mathbf{H}_n^{(i)} = 0$ for $\mathbf{r} \in V_i$. Since the tangential component of $\mathbf{H}_n^{(i)}$ and the normal component of $\mathbf{B}_n^{(i)} = \mu_2 \mathbf{H}_n^{(i)}$ are continuous at the V_1, V_2 interface, therefore $\mathbf{H}_n^{(i)}$ must vanish also for $\mathbf{r} \notin V_i$. Because $\mathbf{E}_n^{(i)} \propto \nabla \times \mathbf{H}_n^{(i)}$ with a finite proportionality coefficient for $\mathbf{r} \notin V_i$, therefore $\mathbf{E}_n^{(i)}$ will also vanish there. Since the tangential component of $\mathbf{E}_n^{(i)}$ is continuous at the V_1, V_2 interface, therefore that component will vanish at that interface. Consequently each of the functions $\phi_n^{(i)}$ must have a constant value over any connected portion of the V_1, V_2 interface. Despite this restriction, a great deal of freedom exists in the construction of these functions. Therefore it should be possible to construct them so as to be a complete longitudinal set inside V_i. None of these functions will be divergence-free. That is because if it were divergence-free then we would have $\nabla^2 \phi_n^{(i)} = 0$ inside V_i and $\phi_n^{(i)}$ =const. over any connected portion of the V_1, V_2 interface. Therefore $\phi_n^{(i)}$ would have to equal that constant over all the interior of that connected portion. Consequently $\nabla \phi_n^{(i)}$ would vanish everywhere.

We will assume that the class 1 and class 2 functions comprise a complete set of divergence-free eigenfunctions inside V_i. Thus any square integrable, divergence-free field can be expanded in this basis inside V_i.

It should be emphasized that the classes 2 and 3 eigenstates are not quasistatic. All the quasistatic eigenstates, which are discussed in detail in Ref. [17] and have only real eigenvalues $u_n^{(i)} > 1$, are $\omega \to 0$ limits of the class 1 eigenstates. In that case, the scalar products of Eq. (15) become $\langle \psi|\phi\rangle_i \equiv \int dV \theta_i (\nabla \psi^* \cdot \nabla \phi)$.

If $\mathbf{J}_{ex} = 0$ for $\mathbf{r} \in V_i$ then the physical fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are divergence-free there. Therefore we only need the classes 1 and 2 eigenfunctions in order to expand the physical fields there. We shall show below how to represent those fields everywhere, i.e., in both V_1 and V_2 and also at the V_1, V_2 interface, without having to use any class 3 eigenfunctions. This is desirable because the class 3 eigenfunctions are more difficult to calculate numerically due to their huge degeneracy, i.e., they all have the same eigenvalue $s_n^{(i)} = u_n^{(i)} = 1$.

The simple example of an isolated spherical inclusion in an otherwise uniform host is discussed in detail in Appendix. In that case, all the eigenstates can be found essentially in closed form.

III. USING THE SCATTERING EIGENSTATES TO CALCULATE A PHYSICAL FIELD

We will usually assume that $\mathbf{J}_{ex} = 0$ in V_1, from which it follows, according to Eq. (2), that $\nabla \cdot \mathbf{E} = 0$ there. We will also assume that $\mathbf{J}_{ex}$ vanishes at sufficiently large distances. Assuming that the $\mathbf{E}_n^{(1)}$ are a complete set of divergence-free eigenstates in the Hilbert space of divergence-free vector functions in V_1 we first try to expand the local physical field $\mathbf{E}(\mathbf{r})$ inside V_1 as

$$\theta_1 \mathbf{E} = \theta_1 \sum_n A_n^{(1)} \mathbf{E}_n^{(1)}. \tag{16}$$

This expansion is valid if $\mathbf{E}(\mathbf{r})$ is square integrable over V_1. This is always the case if $\mathbf{E}(\mathbf{r})$ is the physical field produced by a source term $\mathbf{J}_{ex}(\mathbf{r})$ which vanishes inside V_1. In order to determine the expansion coefficients $A_n^{(1)}$, we consider the following two integrals:

$$\int dV \mathbf{E}_n^{(1)} \cdot \left[-\nabla \times (\nabla \times \mathbf{E}) + k_2^2 \mathbf{E}\right]$$
$$= u^{(1)} k_2^2 \langle \mathbf{E}_n^{(1)} | \mathbf{E} \rangle_1 - \frac{4\pi i \omega \mu_2}{c^2} \int dV \left(\mathbf{E}_n^{(1)} \cdot \mathbf{J}_{ex}\right)$$
$$= u^{(1)} k_2^2 \langle \mathbf{E}_n^{(1)} | \mathbf{E} \rangle_1 - \frac{4\pi i \omega \mu_2}{c^2} \langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2, \tag{17}$$

$$\int dV \mathbf{E} \cdot \left[-\nabla \times \left(\nabla \times \mathbf{E}_n^{(1)}\right) + k_2^2 \mathbf{E}_n^{(1)}\right]$$
$$= u_n^{(1)} k_2^2 \langle \mathbf{E} | \mathbf{E}_n^{(1)} \rangle_1, \tag{18}$$

where $\int dV (\mathbf{E}_n^{(1)} \cdot \mathbf{J}_{ex})$ was replaced by $\langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2$ because $\mathbf{J}_{ex}$ is nonzero only in V_2. The lhs's of these equations are equivalent, respectively, to

$$\int dV \{\nabla \cdot [\mathbf{E}_n^{(1)} \times (\nabla \times \mathbf{E})]$$
$$- \left(\nabla \times \mathbf{E}_n^{(1)}\right) \cdot (\nabla \times \mathbf{E}) + k_2^2 \left(\mathbf{E}_n^{(1)} \cdot \mathbf{E}\right)\},$$
$$\int dV \{\nabla \cdot [\mathbf{E} \times \left(\nabla \times \mathbf{E}_n^{(1)}\right)]$$
$$- (\nabla \times \mathbf{E}) \cdot \left(\nabla \times \mathbf{E}_n^{(1)}\right) + k_2^2 \left(\mathbf{E} \cdot \mathbf{E}_n^{(1)}\right)\}.$$

Only the first terms in these expressions differ. Those terms transform into surface integrals over the closed envelope of the system, which is assumed to be at large distances. As in the discussion of the eigenstates, both $\mathbf{E}$ and $\mathbf{E}_n^{(1)}$ have the form of Eqs. (7) and (8) there, since they both have the same asymptotic behavior of $e^{ik_i r}/r$. Therefore the two integrands under the surface integral tend to the same value as that surface is sent to ∞ and the integrals are finite and equal. Also, they both vanish under the regularization, as explained before Eq. (11). Consequently the rhs's of Eqs. (17) and (18) are equal and we get

$$\frac{4\pi i}{\omega \kappa_2} \langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2 = \left(u^{(1)} - u_n^{(1)}\right) \langle \mathbf{E}_n^{(1)} | \mathbf{E} \rangle_1$$
$$= \left(u^{(1)} - u_n^{(1)}\right) \langle \mathbf{E}_n^{(1)} | \mathbf{E}_n^{(1)} \rangle_1 A_n^{(1)}. \tag{19}$$

Therefore $\mathbf{E}(\mathbf{r})$ has the following expansion in terms of the eigenfunctions:

$$\theta_1 \mathbf{E} = \theta_1 \frac{4\pi i}{\omega \kappa_2} \sum_n \frac{\langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2}{\langle \mathbf{E}_n^{(1)} | \mathbf{E}_n^{(1)} \rangle_1} \frac{\mathbf{E}_n^{(1)}}{u^{(1)} - u_n^{(1)}}. \tag{20}$$

It would seem that this result is valid whenever the integrals $\int dV (\mathbf{E}_n^{(1)} \cdot \mathbf{J}_{ex})$ are finite. The last equation provides an expansion of $\mathbf{E}(\mathbf{r})$ for any $\mathbf{r} \in V_1$ in terms of a set of integrals which must be recalculated every time a different source term $\mathbf{J}_{ex}$ is used.

In order to extend this expansion to other values of $\mathbf{r}$ we use Eq. (20) to expand the term $u^{(1)} k_2^2 \theta_1 \mathbf{E}$ on the rhs of Eq. (1). We then replace the terms $u_n^{(1)} k_2^2 \theta_1 \mathbf{E}_n^{(1)}$ in the resulting sum using the lhs of Eq. (3). These manipulations result in the following equation:

$$\left(-\nabla \times (\nabla \times) + k_2^2\right) \left[\mathbf{E} - \sum_n A_n^{(1)} \frac{u^{(1)}}{u_n^{(1)}} \mathbf{E}_n^{(1)}\right] = -\frac{4\pi i \omega \mu_2}{c^2} \mathbf{J}_{ex}. \tag{21}$$

The expression in the square brackets is equal to what we shall call the "incident field" $\mathbf{E}_{01}$, which is the unique uniform medium scattering solution of

$$-\nabla \times (\nabla \times \mathbf{E}_{01}) + k_2^2 \mathbf{E}_{01} = -\frac{4\pi i \omega \mu_2}{c^2} \mathbf{J}_{ex}. \tag{22}$$

Using Eq. (19) to get an explicit expression for $A_n^{(1)}$, we obtain the following expansion for $\mathbf{E}(\mathbf{r})$, which is valid for any $\mathbf{r}$:

$$\mathbf{E} \equiv \mathbf{E}_{01} + \mathbf{E}_{sc1},$$
$$\mathbf{E}_{sc1} = \frac{4\pi i}{\omega \kappa_2} \sum_n \frac{u^{(1)}/u_n^{(1)}}{u^{(1)} - u_n^{(1)}} \frac{\langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2}{\langle \mathbf{E}_n^{(1)} | \mathbf{E}_n^{(1)} \rangle_1} \mathbf{E}_n^{(1)}, \tag{23}$$

where $\mathbf{E}_{sc1}$ is clearly the scattered field. This equation was already obtained in Ref. [14]. Comparing the last equation with Eq. (20), we also get the following expansion for $\mathbf{E}_{01}$ inside V_1:

$$\theta_1 \mathbf{E}_{01} = -\theta_1 \frac{4\pi i}{\omega \kappa_2} \sum_n \frac{\mathbf{E}_n^{(1)}}{u_n^{(1)}} \frac{\langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2}{\langle \mathbf{E}_n^{(1)} | \mathbf{E}_n^{(1)} \rangle_1}. \tag{24}$$

The last result is independent of $u^{(1)}$, as it should be. However, despite the appearance of $u_n^{(1)}$ and $\mathbf{E}_n^{(1)}$ in this expansion, it really must be independent of them too, since Eq. (22) is independent of the microstructure. Because of this we can use a different, simpler, microstructure to calculate $\mathbf{E}_{01}$ inside V_1 from Eq. (24). However, in order to exploit the expansion in Eq. (23) for $\mathbf{r} \notin V_1$, we need to know $\mathbf{E}_{01}$ there. In particular, we can use an artificial microstructure to calculate $\mathbf{E}_{01}$ using the last equation. In that microstructure, V_2 can be a sphere that encompasses all the field sources $\mathbf{J}_{ex}$ but is otherwise as small as possible. In that case, the eigenstates are known in essentially closed form, as shown in Appendix. Consequently Eq. (24) can be used to evaluate $\mathbf{E}_{01}$ in most of the two physical subvolumes V_1 and V_2, using the eigenfunctions and eigenvalues of the artificial microstructure. Equation (23) looks just like Eq. (20) minus Eq. (24), where these last two expansions are valid only inside V_1. However, we have now shown that the last equation is valid both inside and outside V_1! This equation is valid irrespective of how singular $\mathbf{J}_{ex}$ happens to be, as long as we are able to calculate $\mathbf{E}_{01}$ and the scalar products $\langle \mathbf{E}_n^{(1)} | \mathbf{E}_{01} \rangle_1$.

We can also express the product $\mathbf{E}_n^{(1)} \cdot \mathbf{J}_{ex}$ in terms of $\mathbf{E}_{01}$: by equating the rhs's of the following equivalent integrals

$$\int dV \mathbf{E}_{01} \cdot \left[-\nabla \times \left(\nabla \times \mathbf{E}_n^{(1)}\right) + k_2^2 \mathbf{E}_n^{(1)}\right]$$
$$= k_2^2 u_n^{(1)} \langle \mathbf{E}_{01} | \mathbf{E}_n^{(1)} \rangle_1,$$
$$\int dV \mathbf{E}_n^{(1)} \cdot \left[-\nabla \times (\nabla \times \mathbf{E}_{01}) + k_2^2 \mathbf{E}_{01}\right]$$
$$= -\frac{4\pi i \omega \mu_2}{c^2} \langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2,$$

we get

$$-\frac{4\pi i}{\omega \kappa_2 u_n^{(1)}} \langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2 = \langle \mathbf{E}_n^{(1)} | \mathbf{E}_{01} \rangle_1. \tag{25}$$

Substituting this result in Eq. (23), we finally get

$$\mathbf{E}_{sc1} = \sum_n \frac{u^{(1)}}{u_n^{(1)} - u^{(1)}} \frac{\langle \mathbf{E}_n^{(1)} | \mathbf{E}_{01} \rangle_1}{\langle \mathbf{E}_n^{(1)} | \mathbf{E}_n^{(1)} \rangle_1} \mathbf{E}_n^{(1)}$$
$$= \sum_n \frac{s_n^{(1)}}{s^{(1)} - s_n^{(1)}} \frac{\langle \mathbf{E}_n^{(1)} | \mathbf{E}_{01} \rangle_1}{\langle \mathbf{E}_n^{(1)} | \mathbf{E}_n^{(1)} \rangle_1} \mathbf{E}_n^{(1)}. \quad (26)$$

The same expansion was already obtained as Eq. (2.16) of Ref. [6]. This approach is more efficient than trying to solve Eq. (28) below numerically each time a different incident field is applied. We note that Eq. (26) can be used to expand the scattered field even if the incident field $\mathbf{E}_{01}$ is not produced by a given external current density $\mathbf{J}_{ex}$. This will be used in Appendix in order to calculate the scattered field when the incident field is a simple plane wave.

An alternative method for obtaining $\mathbf{E}_{sc1}$ is to note that Eq. (1) can be rewritten as

$$-\nabla \times [\nabla \times (\mathbf{E}_{01} + \mathbf{E}_{sc1})] + k_2^2(\mathbf{E}_{01} + \mathbf{E}_{sc1})$$
$$= u^{(1)} k_2^2 \theta_1 (\mathbf{E}_{01} + \mathbf{E}_{sc1}) - \frac{4\pi i\omega\mu_2}{c^2}\mathbf{J}_{ex}. \quad (27)$$

We consequently get the following PDE for $\mathbf{E}_{sc1}$:

$$-\nabla \times (\nabla \times \mathbf{E}_{sc1}) + k_2^2 \mathbf{E}_{sc1} = u^{(1)} k_2^2 \theta_1 (\mathbf{E}_{01} + \mathbf{E}_{sc1}). \quad (28)$$

We note that the eigenfunctions $\mathbf{E}_n^{(1)}(\mathbf{r})$ and eigenvalues $u_n^{(1)}$ do not depend upon $\mathbf{J}_{ex}$ but only upon k_2^2 and upon the microstructure which is characterized by $\theta_1(\mathbf{r})$. Therefore, if all the eigenstates are known then Eqs. (20), (23), and (24) are much simpler and quicker to apply for calculations of $\mathbf{E}(\mathbf{r})$ than brute force numerical solutions of the PDE's Eqs. (1), (22), and (28).

Assuming first that V_1 is finite and only V_2 extends to infinity, then $\mathbf{E}(\mathbf{r})$, $\mathbf{E}_{01}(\mathbf{r})$, $\mathbf{E}_{sc1}(\mathbf{r})$, and $\mathbf{E}_n^{(1)}(\mathbf{r})$ will all behave asymptotically as $e^{ik_2 r}/r$.

In order to calculate the full physical field $\mathbf{E}(\mathbf{r}) \equiv \mathbf{E}_{01} + \mathbf{E}_{sc1}$, we still need to know the incident field $\mathbf{E}_{01}$. This can be found by solving the PDE of Eq. (22) in a uniform medium. An alternative method for calculating the incident field is to use Eq. (24), where the eigenfunctions $\mathbf{E}_n^{(1)}(\mathbf{r})$ are used to expand $\theta_1\mathbf{E}_{01}$ in V_1, and then use the eigenfunctions $\mathbf{E}_n^{(2)}(\mathbf{r})$ to expand $\theta_2\mathbf{E}_{01}$ in V_2. The problem with this approach is that $\mathbf{E}_{01}$ is not divergence-free in V_2. To overcome this, we define

$$\widetilde{\mathbf{E}}_{01} \equiv \mathbf{E}_{01} + \frac{4\pi i}{\omega\kappa_2}\mathbf{J}_{ex}. \quad (29)$$

From Eqs. (22) and (29), it follows that $\nabla \cdot \widetilde{\mathbf{E}}_{01} = 0$ both in V_1 (where $\mathbf{J}_{ex}$ vanishes by assumption) and in V_2 (where $\mathbf{J}_{ex}$ does not vanish everywhere).

We now first consider the following two equivalent integrals. (Note that these integrals are only equivalent if $\mathbf{E}_{01}$ is the incident field produced by a given source which is an external current density limited to a finite region in space. They are not equivalent if the incident field is an infinite plane wave. This will be relevant in Appendix when we consider the scattering of a plane wave).

$$\int dV \mathbf{E}_{01} \cdot \left[-\nabla \times \left(\nabla \times \mathbf{E}_m^{(1)}\right) + k_2^2 \mathbf{E}_m^{(1)}\right]$$
$$= k_2^2 u_m^{(1)} \langle \mathbf{E}_n^{(1)} | \mathbf{E}_{01} \rangle_1,$$
$$\int dV \mathbf{E}_m^{(1)} \cdot \left[-\nabla \times \left(\nabla \times \mathbf{E}_{01} + k_2^2 \mathbf{E}_{01}\right]\right.$$
$$= -\frac{4\pi i\omega\mu_2}{c^2} \langle \mathbf{E}_m^{(1)} | \mathbf{J}_{ex} \rangle_2,$$

from which it follows that

$$\langle \mathbf{E}_n^{(1)} | \mathbf{E}_{01} \rangle_1 = \langle \mathbf{E}_m^{(1)} | \widetilde{\mathbf{E}}_{01} \rangle_1 = -\frac{4\pi i\omega\mu_2}{c^2 k_2^2 u_m^{(1)}} \langle \mathbf{E}_n^{(1)} | \mathbf{J}_{ex} \rangle_2, \quad (30)$$

because $\mathbf{J}_{ex} = 0$ in V_1. Using this result, we can expand $\widetilde{\mathbf{E}}_{01}$ in V_1 as follows:

$$\theta_1 \widetilde{\mathbf{E}}_{01} = -\theta_1 \frac{4\pi i\omega\mu_2}{c^2 k_1^2} \sum_m \frac{\langle \mathbf{E}_n^{(2)} | \mathbf{J}_{ex} \rangle_2}{u_m^{(1)}} \frac{\mathbf{E}_m^{(1)}(\mathbf{r})}{\langle \mathbf{E}_m^{(1)} | \mathbf{E}_m^{(1)} \rangle_1}. \quad (31)$$

Another pair of equivalent integrals are

$$\int dV \mathbf{E}_{01} \cdot \left[-\nabla \times \left(\nabla \times \mathbf{E}_n^{(2)}\right) + k_1^2 \mathbf{E}_n^{(2)}\right]$$
$$= k_1^2 u_n^{(2)} \langle \mathbf{E}_n^{(2)} | \mathbf{E}_{01} \rangle_2,$$
$$\int dV \mathbf{E}_n^{(2)} \cdot \left[-\nabla \times (\nabla \times \mathbf{E}_{01} + k_1^2 \mathbf{E}_{01}\right]$$
$$= -\frac{4\pi i\omega\mu_2}{c^2} \langle \mathbf{E}_n^{(2)} | \mathbf{J}_{ex} \rangle_2$$
$$+ \left(k_1^2 - k_2^2\right)\left(\langle \mathbf{E}_n^{(2)} | \mathbf{E}_{01} \rangle_1 + \langle \mathbf{E}_n^{(2)} | \mathbf{E}_{01} \rangle_2\right),$$

from which we get

$$\langle \mathbf{E}_n^{(2)} | \mathbf{E}_{01} \rangle_2 = \frac{u^{(2)}}{u_n^{(2)} - u^{(2)}} \langle \mathbf{E}_n^{(2)} | \mathbf{E}_{01} \rangle_1$$
$$- \frac{4\pi i\omega\mu_2}{c^2 k_1^2 \left(u_n^{(2)} - u^{(2)}\right)} \langle \mathbf{E}_n^{(2)} | \mathbf{J}_{ex} \rangle_2.$$

Using the $\mathbf{E}_m^{(1)}$ eigenstates to expand the first term on the rhs of the last equation and then using Eq. (25) or Eq. (30) to replace the terms $\langle \mathbf{E}_m^{(1)} | \mathbf{E}_{01} \rangle_1$, we finally get

$$\langle \mathbf{E}_n^{(2)} | \widetilde{\mathbf{E}}_{01} \rangle_2 = \langle \mathbf{E}_n^{(2)} | \mathbf{E}_{01} \rangle_2 + \frac{4\pi i\omega\mu_2}{c^2 k_2^2} \langle \mathbf{E}_n^{(2)} | \mathbf{J}_{ex} \rangle_2$$
$$= -\frac{4\pi i\omega\mu_2}{c^2 k_2^2} \left[\langle \mathbf{E}_n^{(2)} | \mathbf{J}_{ex} \rangle_2 \frac{u_n^{(2)} - 1}{u_n^{(2)} - u^{(2)}} \right.$$
$$\left. + \sum_m \langle \mathbf{E}_m^{(1)} | \mathbf{J}_{ex} \rangle_2 \frac{\langle \mathbf{E}_m^{(1)} | \mathbf{E}_n^{(2)} \rangle_1}{\langle \mathbf{E}_m^{(1)} | \mathbf{E}_m^{(1)} \rangle_1} \frac{u^{(2)}}{u_m^{(1)}} \right]. \quad (32)$$

This result can now be used, together with the $\mathbf{E}_n^{(2)}$ eigenvectors, to expand $\widetilde{\mathbf{E}}_{01}$ in V_2, as we did above in Eq. (31):

$$\theta_2 \widetilde{\mathbf{E}}_{01} = -\theta_2 \frac{4\pi i\omega\mu_2}{c^2 k_1^2} \sum_n \frac{\mathbf{E}_n^{(2)}(\mathbf{r})}{\langle \mathbf{E}_n^{(2)} | \mathbf{E}_n^{(2)} \rangle_2} \left[\langle \mathbf{E}_n^{(2)} | \mathbf{J}_{ex} \rangle_2 \frac{1 - u^{(2)}}{u_n^{(2)} - u^{(2)}} \right.$$
$$\left. + \sum_m \langle \mathbf{E}_m^{(1)} | \mathbf{J}_{ex} \rangle_2 \frac{u^{(2)}}{u_m^{(1)}} \frac{\langle \mathbf{E}_m^{(1)} | \mathbf{E}_n^{(2)} \rangle_1}{\langle \mathbf{E}_m^{(1)} | \mathbf{E}_m^{(1)} \rangle_1} \right]. \quad (33)$$

If $\mathbf{J}_{ex}$ is nonzero only in V_1 then we cannot use the eigenfunctions $\mathbf{E}_n^{(1)}$ to expand the physical field $\mathbf{E}$ there, because $\nabla \cdot \mathbf{E}$ does not vanish there. We therefore define a new physical field, as we did in Eq. (29):

$$\widetilde{\mathbf{E}} \equiv \mathbf{E} + \frac{4\pi i}{\omega\kappa_2}\mathbf{J}_{ex}. \tag{34}$$

From Eqs. (1) and (2), it follows that $\nabla \cdot \widetilde{\mathbf{E}} = 0$ in both V_2 (where $\mathbf{J}_{ex}$ now vanishes) and in V_1 (where $\mathbf{J}_{ex}$ is not zero everywhere). This field satisfies the following PDE:

$$-\nabla \times (\nabla \times \widetilde{\mathbf{E}}) + k_2^2\widetilde{\mathbf{E}} = k_2^2 u^{(1)}\theta_1\widetilde{\mathbf{E}} - \frac{4\pi i}{\omega\kappa_1}\nabla \times (\nabla \times \mathbf{J}_{ex}). \tag{35}$$

The last term in this equation leads to

$$\begin{aligned}\langle \mathbf{E}_m^{(1)}|\nabla \times (\nabla \times \mathbf{J}_{ex})\rangle_1 &= \langle \nabla \times \mathbf{E}_m^{(1)}|\nabla \times \mathbf{J}_{ex}\rangle_1\\ &= k_2^2\big(1 - u_m^{(1)}\big)\langle \mathbf{E}_m^{(1)}|\mathbf{J}_{ex}\rangle_1,\end{aligned} \tag{36}$$

where integration by parts was applied to get the second term, and another integration by parts, along with the eigenvalue Eq. (3), was used to get the rhs. By considering the following two equivalent integrals:

$$\begin{aligned}&\int dV\, \mathbf{E}_m^{(1)} \cdot \big[-\nabla \times (\nabla \times \widetilde{\mathbf{E}}) + k_2^2\widetilde{\mathbf{E}}\big]\\ &= k_2^2 u^{(1)}\langle \mathbf{E}_m^{(1)}|\widetilde{\mathbf{E}}\rangle_1 - \frac{4\pi i}{\omega\kappa_1}\langle \nabla \times \mathbf{E}_m^{(1)}|\nabla \times (\nabla \times \mathbf{J}_{ex})\rangle_1,\\ &\int dV\, \widetilde{\mathbf{E}} \cdot \big[-\nabla \times \big(\nabla \times \mathbf{E}_m^{(1)}\big) + k_2^2\mathbf{E}_m^{(1)}\big]\\ &= k_2^2 u_m^{(1)}\langle \mathbf{E}_m^{(1)}|\widetilde{\mathbf{E}}\rangle_1,\end{aligned}$$

we obtain

$$\langle \mathbf{E}_m^{(1)}|\widetilde{\mathbf{E}}\rangle_1 = \frac{4\pi i\omega\mu_2}{c^2k_1^2}\frac{1 - u_m^{(1)}}{u^{(1)} - u_m^{(1)}}\langle \mathbf{E}_m^{(1)}|\mathbf{J}_{ex}\rangle_1. \tag{37}$$

Equations (36) and (37) lead to the following expansion for the rhs of Eq. (35) in V_1:

$$\begin{aligned}&k_2^2 u^{(1)}\theta_1\widetilde{\mathbf{E}} - \frac{4\pi i}{\omega\kappa_1}\nabla \times (\nabla \times \mathbf{J}_{ex})\\ &= \frac{4\pi i\omega\theta_1}{c^2(1 - u^{(1)})}\sum_n \frac{u_n^{(1)}\big(1 - u_n^{(1)}\big)}{u^{(1)} - u_n^{(1)}}\frac{\langle \mathbf{E}_n^{(1)}|\mathbf{J}_{ex}\rangle_1}{\langle \mathbf{E}_n^{(1)}|\mathbf{E}_n^{(1)}\rangle_1}\mathbf{E}_n^{(1)}(\mathbf{r}).\end{aligned}$$

In the last expression, we now replace $k_2^2 u_n^{(1)}\theta_1\mathbf{E}_n^{(1)}(\mathbf{r})$ by the lhs of the eigenvalue Eq. (3). This leads to the following PDE for $\widetilde{\mathbf{E}}$:

$$\begin{aligned}0 &= \big[-\nabla \times (\nabla \times\ ) + k_2^2\big]\\ &\quad\times\left[\widetilde{\mathbf{E}} - \frac{4\pi i}{\omega\kappa_1}\sum_n \frac{1 - u_n^{(1)}}{u^{(1)} - u_n^{(1)}}\frac{\langle \mathbf{E}_n^{(1)}|\mathbf{J}_{ex}\rangle_1}{\langle \mathbf{E}_n^{(1)}|\mathbf{E}_n^{(1)}\rangle_1}\mathbf{E}_n^{(1)}(\mathbf{r})\right].\end{aligned}$$

The expression in the large square brackets must vanish, therefore

$$\begin{aligned}\widetilde{\mathbf{E}} &\equiv \mathbf{E} + \frac{4\pi i}{\omega\kappa_1}\mathbf{J}_{ex}\\ &= \frac{4\pi i}{\omega\kappa_1}\sum_n \frac{1 - u_n^{(1)}}{u^{(1)} - u_n^{(1)}}\frac{\langle \mathbf{E}_n^{(1)}|\mathbf{J}_{ex}\rangle_1}{\langle \mathbf{E}_n^{(1)}|\mathbf{E}_n^{(1)}\rangle_1}\mathbf{E}_n^{(1)}(\mathbf{r}).\end{aligned} \tag{38}$$

We emphasize that, just as in the case of Eqs. (26) and (23), the last expression is valid for all values of $\mathbf{r}$, i.e., inside both V_1 and V_2. This is so despite the fact that only the V_1 eigenfunctions appear in the expansion.

If only V_1 extends to infinity we would have to expand the physical field using the $\mathbf{E}_n^{(2)}$ eigenfunctions, because only they have the correct asymptotic dependence on r at large distances, i.e., e^{ik_1r}/r. To that end we would have to repeat the entire discussion presented above using the $\mathbf{E}_n^{(2)}$ eigenfunctions instead of the $\mathbf{E}_m^{(1)}$ eigenfunctions. A better alternative is to switch the names of the two constituents. By doing that we would return to the kind of system treated in the previous discussions.

If both V_1 and V_2 extend out to infinity, then we will need to use the $\mathbf{E}_m^{(1)}$ eigenfunctions to expand $\mathbf{E}$ in the asymptotic regions of V_2 and the $\mathbf{E}_n^{(2)}$ eigenfunctions to expand $\mathbf{E}$ in the asymptotic regions of V_1. Here too we can switch the names of the two constituents in such a way that the relevant infinitely extending sub-volume is always V_2.

A case where only one of the subvolumes V_1 or V_2 extends to infinity is discussed in the Appendix. In that example, the microstructure is a single isolated spherical inclusion in an otherwise uniform background. The field source treated there is a current density localized at a single point $\mathbf{J}_{ex} = \mathbf{J}\delta^3(\mathbf{r} - \mathbf{r}_0)$, where $\mathbf{r}_0$ can be either inside or outside of the sphere.

If $\mathbf{J}_{ex}$ is nonzero in both constituents then we can write it as $\mathbf{J}_{ex} = \mathbf{J}_{ex}^{(1)} + \mathbf{J}_{ex}^{(2)}$, where $\mathbf{J}_{ex}^{(i)} = 0$ in V_i. We then calculate the physical fields $\mathbf{E}^{(i)}$ produced, separately, by $\mathbf{J}_{ex}^{(i)}$. The result for $\mathbf{E}(\mathbf{r})$ is then the sum of those two fields $\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)}$.

If $\mathbf{J}_{ex}(\mathbf{r})$ happens to be a two-dimensional (2D) surface current at the V_1, V_2 interface, where $\mathbf{E}_n^{(1)}(\mathbf{r})$ is discontinuous, then integrations like $\int dV\,(\mathbf{E}_n^{(1)} \cdot \mathbf{J}_{ex})$ can be carried out as follows: We first take $\mathbf{J}_{ex}(\mathbf{r})$ to have a smooth symmetric shape centered at the interface, and then let that shape tend to a one-dimensional (1D) Dirac delta function at the interface. The calculation of the physical field then proceeds by calculating $\mathbf{E}^{(1)}$ from $\mathbf{J}^{(1)}$, which is one half of the 1D δ function in V_2, and then $\mathbf{E}^{(2)}$ from $\mathbf{J}^{(2)}$, which is one half of the 1D δ function in V_1.

IV. EM EIGENSTATES AND THE SCATTERED FIELD OF A CLUSTER OF INCLUSIONS

As in the previous section, we assume that $\mu \equiv \mu_2$ everywhere and that only κ is heterogeneous. We also assume that $\mathbf{J}_{ex}$ is nonzero only in V_2.

If the subvolume V_1 of the κ_1 constituent is made of a cluster of nonoverlapping inclusions, then we can write

$$\theta_1(\mathbf{r}) = \sum_a \theta_a(\mathbf{r}), \tag{39}$$

where $\theta_a(\mathbf{r}) = 1$ if $\mathbf{r}$ is in the volume V_a of the inclusion a and vanishes elsewhere. We now try to expand the eigenstates $\mathbf{E}_n^{(1)}(\mathbf{r})$ of the V_1 subvolume in those of the individual isolated inclusions $\mathbf{E}_{a\alpha}(\mathbf{r})$:

$$\theta_a(\mathbf{r})\mathbf{E}_n^{(1)}(\mathbf{r}) = \theta_a(\mathbf{r})\sum_\alpha A_{a\alpha}^{(n)}\mathbf{E}_{a\alpha}(\mathbf{r}),\ A_{a\alpha}^{(n)} \equiv \frac{\langle \mathbf{E}_{a\alpha}|\mathbf{E}_n^{(1)}\rangle_a}{\langle \mathbf{E}_{a\alpha}|\mathbf{E}_{a\alpha}\rangle_a}. \tag{40}$$

Here we used an isolated inclusion scalar product defined by

$$\langle\mathbf{F}|\mathbf{E}\rangle_a \equiv \int dV\theta_a(\mathbf{r})[\mathbf{F}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r})] = \langle\mathbf{E}|\mathbf{F}\rangle_a. \tag{41}$$

We then calculate the following difference:

$$\int dV\mathbf{E}_{a\alpha}\cdot\left[-\nabla\times\left(\nabla\times\mathbf{E}_n^{(1)}\right)+k_2^2\mathbf{E}_n^{(1)}\right] - \int dV\mathbf{E}_n^{(1)}\cdot\left[-\nabla\times(\nabla\times\mathbf{E}_{a\alpha})+k_2^2\mathbf{E}_{a\alpha}\right].$$

As argued following Eq. (9), this difference vanishes. Using Eq. (3) and its isolated inclusion version

$$-\nabla\times(\nabla\times\mathbf{E}_{a\alpha})+k_2^2\mathbf{E}_{a\alpha} = u_{a\alpha}k_2^2\theta_a\mathbf{E}_{a\alpha}, \tag{42}$$

we then get

$$u_n^{(1)}\int dV\theta_1\left(\mathbf{E}_{a\alpha}\cdot\mathbf{E}_n^{(1)}\right) = u_{a\alpha}\int dV\theta_a\left(\mathbf{E}_n^{(1)}\cdot\mathbf{E}_{a\alpha}\right).$$

From Eqs. (39)–(41), we then get

$$u_n^{(1)}\sum_{b\beta}A_{b\beta}^{(n)}\int dV\theta_b(\mathbf{E}_{a\alpha}\cdot\mathbf{E}_{b\beta}) \equiv u_n^{(1)}\sum_{b\beta}A_{b\beta}^{(n)}\langle\mathbf{E}_{b\beta}|\mathbf{E}_{a\alpha}\rangle_b = u_{a\alpha}A_{a\alpha}^{(n)}\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{a\alpha}\rangle_a. \tag{43}$$

Here we used the fact that the eigenstates $\mathbf{E}_{a\alpha}(\mathbf{r})$ form a biorthogonal set in the volume V_a of the inclusion a according to the isolated inclusion scalar product defined in Eq. (41). Equation (43) is an infinite set of homogeneous linear algebraic equations for the expansion coefficients $A_{a\alpha}^{(n)}$. This only has nonzero solutions for special values of $u_n^{(1)}$. Thus this is a matrix eigenvalue problem with matrix $\widehat{M}$, eigenvalues $s_n^{(1)} \equiv 1/u_n^{(1)}$, and eigenvectors $A_{b\beta}^{(n)}$:

$$s_n^{(1)}A_{a\alpha}^{(n)} = \sum_{b\beta}M_{a\alpha,b\beta}A_{b\beta}^{(n)}, \quad M_{a\alpha,b\beta} \equiv \frac{1}{u_{a\alpha}}\frac{\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{b\beta}\rangle_b}{\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{a\alpha}\rangle_a}. \tag{44}$$

Note that when a and b are different inclusions the scalar product in the numerator of the last quotient is between eigenfunctions of two different isolated inclusions where the integration is over the volume of just one of those inclusions. These off-diagonal elements of $\widehat{M}$ represent the interactions between pairs of eigenstates of different inclusions. When $a = b$ then $M_{a\alpha,b\beta}$ vanishes unless also $\alpha = \beta$. This is then a diagonal element of $\widehat{M}$ which is just one of the isolated inclusion eigenvalues. The matrix eigenvalue equation must be solved numerically. That, however, is simpler than solving Eq. (3) numerically once the eigenstates of the isolated inclusions are known. This procedure becomes practical if the isolated inclusions all have similar simple shapes, such as spheres or circular cylinders or parallel slabs. In those cases, the isolated inclusion eigenstates, as well as the off-diagonal elements of the matrix $\widehat{M}$, can be calculated in closed analytical forms [6,7,14,15,18].

We now assert that the quotient $\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{b\beta}\rangle_b/u_{a\alpha}$ is symmetric under the transposition of $a\alpha$ and $b\beta$. Using Eq. (42) to substitute for $\theta_b\mathbf{E}_{b\beta}$, we get

$$\frac{1}{u_{a\alpha}}\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{b\beta}\rangle_b = \frac{1/k_2^2}{u_{a\alpha}u_{b\beta}}\int dV\mathbf{E}_{a\alpha}\cdot\left[-\nabla\times(\nabla\times\mathbf{E}_{b\beta})+k_2^2\mathbf{E}_{b\beta}\right] = \frac{1/k_2^2}{u_{a\alpha}u_{b\beta}}\int dV\{\nabla\cdot[\mathbf{E}_{a\alpha}\times(\nabla\times\mathbf{E}_{b\beta})] - (\nabla\times\mathbf{E}_{a\alpha})\cdot(\nabla\times\mathbf{E}_{b\beta})+k_2^2(\mathbf{E}_{a\alpha}\cdot\mathbf{E}_{b\beta})\}.$$

As shown in the discussion of Eq. (5), the last integral is symmetric in $\mathbf{E}_{a\alpha}$ and $\mathbf{E}_{b\beta}$, which proves our assertion. If the isolated inclusion eigenfunctions are all normalized to 1, i.e., $\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{a\alpha}\rangle_a = 1$, then $\widehat{M}$ is a symmetric matrix. Consequently its right and left eigenvalues are the same, as are its right and left eigenvectors. If we define the scalar product of two vectors $A_{a\alpha}, B_{a\alpha}$ by

$$\langle A|B\rangle \equiv \sum_{a\alpha}A_{a\alpha}B_{a\alpha} = \langle B|A\rangle,$$

then clearly two eigenvectors of $\widehat{M}$ that have different eigenvalues $u_n \neq u_m$ must be orthogonal. Thus these eigenvectors constitute a biorthogonal set.

If the need arises to find the $\mathbf{E}_n^{(1)}$ eigenstates outside V_1, when V_1 is a cluster of nonoverlapping inclusions, then we can proceed as follows: having found the V_1 eigenvectors $A_{a\alpha}^{(n)}$, we can use Eq. (40) to rewrite Eq. (3) as

$$0 = \left[-\nabla\times(\nabla\times\ )+k_2^2\right]\left[\mathbf{E}_n^{(1)} - \sum_{a\alpha}A_{a\alpha}^{(n)}\frac{u_n^{(1)}}{u_{a\alpha}}\mathbf{E}_{a\alpha}\right].$$

From this it follows that

$$\mathbf{E}_n^{(1)}(\mathbf{r}) = \sum_{a\alpha}A_{a\alpha}^{(n)}\frac{u_n^{(1)}}{u_{a\alpha}}\mathbf{E}_{a\alpha}(\mathbf{r}), \tag{45}$$

and that this is valid for all $\mathbf{r}$. This can now be used to represent $\mathbf{E}_n^{(1)}(\mathbf{r})$ in V_2.

If they are square integrable there, then the fields $\mathbf{E}_{sc1}$ and $\mathbf{E}_{01}$ can also be expanded in the isolated inclusion eigenfunctions for $\mathbf{r}\in V_a$ as

$$\theta_a\mathbf{E}_{sc1} = \theta_a\sum_\alpha\frac{\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{sc1}\rangle_a}{\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{a\alpha}\rangle_a}\mathbf{E}_{a\alpha}, \qquad \theta_a\mathbf{E}_{01} = \theta_a\sum_\alpha\frac{\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{01}\rangle_a}{\langle\mathbf{E}_{a\alpha}|\mathbf{E}_{a\alpha}\rangle_a}\mathbf{E}_{a\alpha}. \tag{46}$$

We now calculate the following difference:

$$\int dV\mathbf{E}_{a\alpha}\cdot\left[-\nabla\times(\nabla\times\mathbf{E}_{sc1})+k_2^2\mathbf{E}_{sc1}\right] - \int dV\mathbf{E}_{sc1}\cdot\left[-\nabla\times(\nabla\times\mathbf{E}_{a\alpha})+k_2^2\mathbf{E}_{a\alpha}\right].$$

As argued following Eq. (9), this difference vanishes. Using Eqs. (28) and (42), we compare the following equivalent integrals

$$\int dV \mathbf{E}_{a\alpha} \cdot \left[-\nabla \times (\nabla \times \mathbf{E}_{sc1}) + k_2^2 \mathbf{E}_{sc1}\right]$$
$$= \int dV \nabla \cdot [\mathbf{E}_{a\alpha} \times (\nabla \times \mathbf{E}_{sc1})] - \int dV (\nabla \times \mathbf{E}_{a\alpha}) \cdot (\nabla \times \mathbf{E}_{sc1}) + k_2^2 \int dV (\mathbf{E}_{a\alpha} \cdot \mathbf{E}_{sc1})$$
$$= u^{(1)} k_2^2 (\langle \mathbf{E}_{a\alpha} | \mathbf{E}_{01} \rangle_1 + \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{sc1} \rangle_1),$$
$$\int dV \mathbf{E}_{sc1} \cdot \left[- \nabla \times (\nabla \times \mathbf{E}_{a\alpha}) + k_2^2 \mathbf{E}_{a\alpha}\right]$$
$$= \int dV \nabla \cdot [\mathbf{E}_{sc1} \times (\nabla \times \mathbf{E}_{a\alpha})] - \int dV (\nabla \times \mathbf{E}_{sc1}) \cdot (\nabla \times \mathbf{E}_{a\alpha}) + k_2^2 \int dV (\mathbf{E}_{sc1} \cdot \mathbf{E}_{a\alpha})$$
$$= u_{a\alpha} k_2^2 \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{sc1} \rangle_a.$$

In this way, we get

$$u^{(1)} \sum_{b\beta} \frac{\langle \mathbf{E}_{a\alpha} | \mathbf{E}_{b\beta} \rangle_b}{\langle \mathbf{E}_{b\beta} | \mathbf{E}_{b\beta} \rangle_b} (\langle \mathbf{E}_{b\beta} | \mathbf{E}_{01} \rangle_b + \langle \mathbf{E}_{b\beta} | \mathbf{E}_{sc1} \rangle_b)$$
$$= u_{a\alpha} \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{sc1} \rangle_a.$$

If $\langle \mathbf{E}_{b\beta} | \mathbf{E}_{b\beta} \rangle_b = 1$, then this can be rewritten as

$$\frac{1}{u^{(1)}} \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{sc1} \rangle_a = \sum_{b\beta} M_{a\alpha,b\beta} (\langle \mathbf{E}_{b\beta} | \mathbf{E}_{sc1} \rangle_b + \langle \mathbf{E}_{b\beta} | \mathbf{E}_{01} \rangle_b). \tag{47}$$

This is an infinite set of inhomogeneous linear algebraic equations for the scalar products $\langle \mathbf{E}_{b\beta} | \mathbf{E}_{sc1} \rangle_b$ which can be used to expand $\mathbf{E}_{sc1}$ inside V_1.

Another way to calculate the infinite dimensional vector $A^{(sc1)}$ of expansion coefficients $A^{(sc1)}_{a\alpha} \equiv \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{sc1} \rangle_a$, knowing the infinite dimensional vector $A^{(01)}$ of expansion coefficients $A^{(01)}_{a\alpha} \equiv \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{01} \rangle_a$, is to first rewrite the previous equation and Eq. (44) in symbolic forms as (recall that $s^{(1)} \equiv 1/u^{(1)}$)

$$s^{(1)} A^{(sc1)} = \widehat{M} (A^{(sc1)} + A^{(01)}), \qquad s_n^{(1)} A^{(n)} = \widehat{M} A^{(n)}.$$

The equation for $A^{(sc1)}$ can be solved as

$$A^{(sc1)} = \frac{\widehat{M}}{s^{(1)} - \widehat{M}} A^{(01)} = \sum_n \frac{s_n^{(1)}}{s^{(1)} - s_n^{(1)}} \frac{\langle A^{(n)} | A^{(01)} \rangle}{\langle A^{(n)} | A^{(n)} \rangle} A^{(n)}.$$

Thus, in order to calculate the expansion coefficients $A^{(sc1)}_{a\alpha}$ we only need to sum over the eigenvalues and eigenvectors of $\widehat{M}$, using the appropriate physical value of $s^{(1)} \equiv 1/u^{(1)} \equiv \kappa_2/(\kappa_2 - \kappa_1)$.

In order to expand $\mathbf{E}_{sc1}$ outside V_1, we substitute the $\mathbf{r} \in V_1$ expansions of $\mathbf{E}_{sc1}(\mathbf{r})$ and $\mathbf{E}_{01}(\mathbf{r})$ on the rhs of Eq. (28) to get

$$0 = \left(-\nabla \times (\nabla \times) + k_2^2\right) \times \left[\mathbf{E}_{sc1} - \sum_{a\alpha} \frac{u^{(1)}}{u_{a\alpha}} \frac{\langle \mathbf{E}_{a\alpha} | \mathbf{E}_{sc1} \rangle_a + \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{01} \rangle_a}{\langle \mathbf{E}_{a\alpha} | \mathbf{E}_{a\alpha} \rangle_a} \mathbf{E}_{a\alpha}\right]. \tag{48}$$

Because the differential equation $-\nabla \times (\nabla \times \mathbf{E}) + k_2^2 \mathbf{E} = 0$ in all space has no nonzero solutions that decay to 0 when $\mathbf{r} \to \infty$, therefore the square brackets in Eq. (48) must vanish. We thus get the following expansion for $\mathbf{E}_{sc1}(\mathbf{r})$:

$$\mathbf{E}_{sc1} = \sum_{a\alpha} \frac{u^{(1)}}{u_{a\alpha}} \frac{\langle \mathbf{E}_{a\alpha} | \mathbf{E}_{sc1} \rangle_a + \langle \mathbf{E}_{a\alpha} | \mathbf{E}_{01} \rangle_a}{\langle \mathbf{E}_{a\alpha} | \mathbf{E}_{a\alpha} \rangle_a} \mathbf{E}_{a\alpha}. \tag{49}$$

This is valid for all $\mathbf{r}$, both inside and outside V_1. The form of the last expansion differs from that in Eq. (46), which is valid only inside V_1. This is due to the fact that in the latter equation each eigenfunction $\mathbf{E}_{a\alpha}$ is used only inside its inclusion a. By contrast, in Eq. (49) each of those eigenfunctions is used for all values of $\mathbf{r}$. Thus every isolated inclusion eigenfunction $\mathbf{E}_{a\alpha}(\mathbf{r})$ is used also for $\mathbf{r}$'s that lie inside other inclusions or outside any inclusion.

The above discussion is valid when the field source lies in V_2, i.e., not inside any of the inclusions a. However, when the field source lies in V_1, i.e., inside one or more of those inclusions, then we need to switch the indices 1 and 2 and proceed as explained near the end of Sec. III. In that case, there is usually no simple way to calculate the V_1 eigenstates, where V_1 is now the complement of the inclusions volume. However, we can still use the approach described in Eqs. (34)–(38) to expand the physical field in the V_1 eigenfunctions even though it is not divergence-free there.

V. EM EIGENSTATES WHEN BOTH κ AND μ ARE HETEROGENEOUS AND THEIR EXPLOITATION FOR EXPANDING A PHYSICAL FIELD

When the values of μ as well as those of κ differ in the two constituents, the computation of the scattering eigenstates of Maxwell's equations in a two-constituent composite and their exploitation for expanding a physical field become more complicated. Those equations are first reduced to the following partial differential equations (PDE's) for the local electromagnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$:

$$-\nabla \times (\nabla \times \mathbf{E}) + k_2^2 \mathbf{E} + v \nabla \times [\theta_1 (\nabla \times \mathbf{E})] = u k_2^2 \theta_1 \mathbf{E} - \frac{4\pi i \omega \mu_2}{c^2} \mathbf{J}_{ex}, \tag{50}$$

$$-\nabla \times (\nabla \times \mathbf{H}) + k_2^2 \mathbf{H} + t \nabla \times [\theta_1 (\nabla \times \mathbf{H})] = w k_2^2 \theta_1 \mathbf{H} - \frac{4\pi}{c\kappa_2} \nabla \times [(1 + t\theta_1) \mathbf{J}_{ex}], \tag{51}$$

where

$$\kappa_i \equiv \epsilon_i + \frac{4\pi i\sigma_i}{\omega}, \quad k_i^2 \equiv \frac{\omega^2}{c^2}\kappa_i\mu_i, \quad u \equiv 1 - \frac{\kappa_1}{\kappa_2}, \quad t \equiv 1 - \frac{\kappa_2}{\kappa_1} = \frac{u}{u-1}, \quad v \equiv 1 - \frac{\mu_2}{\mu_1}, \quad w \equiv 1 - \frac{\mu_1}{\mu_2} = \frac{v}{v-1}.$$

Here $\theta_1(\mathbf{r}) = 1$ for $\mathbf{r}$ inside the volume V_1 of the No. 1 constituent and $\theta_1(\mathbf{r}) = 0$ elsewhere, while $\mathbf{J}_{ex}$ is a monochromatic external current density; ϵ_i, σ_i, and μ_i, $i = 1, 2$, are the electric permittivity, electric conductivity, and magnetic permeability of the two constituents. Note that in this section we omit the upper index of $u^{(1)}$, $s^{(1)}$, etc. [see Eqs. (1) and (2)], since we only present expressions for the case where $\mathbf{J}_{ex} = 0$ in V_1.

When the solution of Eq. (50) is known then the solution of Eq. (51) can be obtained by using one of Maxwell's equations. namely,

$$\nabla \times \mathbf{E}(\mathbf{r}) = \frac{i\omega\mu(\mathbf{r})}{c}\mathbf{H}(\mathbf{r}). \tag{52}$$

Likewise, the solution of Eq. (50) can be obtained from that of Eq. (51) by

$$\nabla \times \mathbf{H}(\mathbf{r}) = -\frac{i\omega\kappa(\mathbf{r})}{c}\mathbf{E}(\mathbf{r}) + \frac{4\pi}{c}\mathbf{J}_{ex}(\mathbf{r}). \tag{53}$$

When both κ_i and μ_i can have complex values, the asymptotic behavior of $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ at large distances needs to be reconsidered. In any electromagnetically passive material EM energy can only be dissipated, never created. Nevertheless, materials where EM energy gain occurs can be synthesized by embedding atoms or molecules or quantum dots which are in excited states and can emit EM energy by jumping to a lower quantum state. In a homogeneous medium, a plane wave characterized by a wave vector $\mathbf{k}$ is always an eigenstate. As was shown many years ago by Victor Veselago, the direction of energy propagation can then be either along $\mathbf{k}$ or along $-\mathbf{k}$ [19].

When the physical EM fields are produced by a source $\mathbf{J}_{ex}(\mathbf{r})$ with a finite spatial extent then one expects that energy must propagate away from that source, at least at large distances. However, the most important property of those fields is that they must decay to 0 when $|\mathbf{r}| \to \infty$. Therefore the asymptotic behavior of those fields will be

$$\mathbf{E} = \frac{e^{ik_i r}}{r}\mathbf{a}, \quad \mathrm{Im}k_i > 0, \quad \mathbf{a} \perp \mathbf{n},$$
$$k_i^2 = \kappa_i\mu_i\frac{\omega^2}{c^2}, \quad r \in V_i, \quad |k_i r| \gg 1.$$

This will sometimes result in $\mathrm{Re}k_i < 0$, e.g., when both κ_i and μ_i have a large negative real part and a small positive imaginary part. That is the case when such a material is "left handed" and therefore exhibits "negative refraction" [20].

Eigenstates of Eqs. (50) and (51) will have to exhibit the same kind of asymptotic behavior, namely, decay to 0 when $|\mathbf{r}| \to \infty$. Therefore they too will sometimes have $\mathrm{Re}k_i < 0$. In that case, they would be incoming rather than outgoing waves.

We now define two special sets of scattering eigenstates. These are solutions of Eq. (50) or Eq. (51) when there are no sources, i.e., when $\mathbf{J}_{ex} \equiv 0$, which are exponentially decaying waves at large distances. Such nonzero solutions exist only for special values, i.e, eigenvalues, of the various moduli u, v, t, w. Because such solutions do not exist when all of those moduli have permissible physical values, the eigenvalues will necessarily be unphysical. This means that the special values of some of the parameters κ_i and μ_i which correspond to those eigenvalues will have negative imaginary parts. The energy gains resulting from these values will exactly compensate for the energy losses from the other values, and also from radiation losses due to outgoing waves, so that the total EM energy of any eigenstate is conserved in time.

The two sets of eigenstates are solutions of the following equations:

$$-\nabla \times \left(\nabla \times \mathbf{E}_n^{(u)}\right) + k_2^2\mathbf{E}_n^{(u)} = u_n k_2^2\theta_1\mathbf{E}_n^{(u)}, \tag{54}$$

$$-\nabla \times \left(\nabla \times \mathbf{E}_m^{(v)}\right) + k_2^2\mathbf{E}_m^{(v)} = -v_m\nabla \times \left[\theta_1\left(\nabla \times \mathbf{E}_m^{(v)}\right)\right]. \tag{55}$$

As in the preceding sections, these sets of eigenfunctions will be used to expand $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$.

The solutions of Eq. (54) were already discussed in Sec. II—see Eqs. (3)–(12). In order to identify the solutions of Eq. (55), we could follow a procedure similar to that of those equations. Instead of that we note that it follows from Eq. (53) that, for $\mathbf{r} \in V_1$, we can write

$$-\frac{i\omega\kappa_{1m}}{c}\mathbf{E}_m^{(v)} = \nabla \times \mathbf{H}_m^{(w)},$$

where $\mathbf{H}_m^{(w)}$ is an eigensolution of Eq. (51) when $\mathbf{J}_{ex} \equiv 0$ and $t = 0$, and where w_m is the special value of w, i.e., the eigenvalue. That equation has the same form as Eq. (54), therefore $\mathbf{H}_m^{(w)}(\mathbf{r}) \equiv \mathbf{E}_m^{(u)}(\mathbf{r})$ and $w_m = u_m$. Note that $\kappa_{1m} = \kappa_2(1 - u_m)$. Consequently, for $\mathbf{r} \in V_1$, we get

$$\mathbf{E}_m^{(v)} = -\frac{c}{i\omega\kappa_{1m}}\nabla \times \mathbf{E}_m^{(u)}, \quad v_m = \frac{w_m}{w_m - 1} = \frac{u_m}{u_m - 1}.$$

We note that, in contrast with the states $\mathbf{E}_n^{(u)}$, the states $\mathbf{E}_m^{(v)}$ are not an orthogonal set in terms of the usual scalar product as defined in Eq. (15). However, from Eq. (52), it follows that, for $\mathbf{r} \in V_1$, we have

$$\nabla \times \mathbf{E}_m^{(v)} = \frac{i\omega\mu_{1m}}{c}\mathbf{H}_m^{(w)} = \frac{i\omega\mu_{1m}}{c}\mathbf{E}_m^{(u)}, \tag{56}$$

where $\mu_{1m} = \mu_2(1 - w_m)$. Therefore we get

$$\left\langle \nabla \times \mathbf{E}_m^{(v)} \middle| \nabla \times \mathbf{E}_n^{(v)} \right\rangle_1 = -\frac{\omega^2\mu_{1m}^2}{c^2}\left\langle \mathbf{E}_m^{(u)} \middle| \mathbf{E}_n^{(u)} \right\rangle_1,$$

and this vanishes if $u_n \neq u_m$, i.e., if $v_n \neq v_m$. Thus the set of eigenfunctions $\mathbf{E}_m^{(v)}$ is an orthogonal set if $\langle \nabla \times \mathbf{E} | \nabla \times \mathbf{F} \rangle_1$ is taken to be the scalar product.

We now assume that $\mathbf{J}_{ex} = 0$ inside V_1. The physical field is then divergence-free there. We therefore expand it there in a series of the eigenfunctions, $\mathbf{E}_n^{(u)}$ or $\mathbf{E}_m^{(v)}$, which are also divergence-free there:

$$\theta_1 \mathbf{E} = \theta_1 \sum_n A_n \mathbf{E}_n^{(u)}, \quad \theta_1 \mathbf{E} = \theta_1 \sum_m B_m \mathbf{E}_m^{(v)}. \tag{57}$$

In order to calculate the expansion coefficients A_n, we proceed as in the discussion following Eq. (16):

$$\begin{aligned} 0 &= \int dV \{\mathbf{E}_n^{(u)} \cdot [-\nabla \times (\nabla \times \mathbf{E}) + k_2^2 \mathbf{E}] - \mathbf{E} \cdot [-\nabla \times (\nabla \times \mathbf{E}_n^{(u)}) + k_2^2 \mathbf{E}_n^{(u)}]\} \\ &= \int dV \mathbf{E}_n^{(u)} \cdot \left\{ -v \nabla \times [\theta_1 (\nabla \times \mathbf{E})] + u k_2^2 \theta_1 \mathbf{E} - \frac{4\pi i \omega \mu_2}{c^2} \mathbf{J}_{ex} \right\} - u_n k_2^2 \int dV \theta_1 (\mathbf{E} \cdot \mathbf{E}_n^{(u)}) \\ &= (u - u_n) k_2^2 \langle \mathbf{E}_n^{(u)} | \mathbf{E} \rangle_1 - v \langle \nabla \times \mathbf{E}_n^{(u)} | \nabla \times \mathbf{E} \rangle_1 - \frac{4\pi i \omega \mu_2}{c^2} \int dV (\mathbf{E}_n^{(u)} \cdot \mathbf{J}_{ex}) \\ &= (u - u_n) k_2^2 \langle \mathbf{E}_n^{(u)} | \mathbf{E}_n^{(u)} \rangle_1 A_n - v \sum_p A_p \langle \nabla \times \mathbf{E}_n^{(u)} | \nabla \times \mathbf{E}_p^{(u)} \rangle_1 - \frac{4\pi i \omega \mu_2}{c^2} \int dV (\mathbf{E}_n^{(u)} \cdot \mathbf{J}_{ex}). \end{aligned} \tag{58}$$

This is a system of inhomogeneous linear algebraic equations for A_n with a matrix that is symmetric.

A similar procedure is implemented to calculate B_m:

$$\begin{aligned} 0 &= \int dV \{\mathbf{E}_m^{(v)} \cdot [-\nabla \times (\nabla \times \mathbf{E}) + k_2^2 \mathbf{E}] - \mathbf{E} \cdot [-\nabla \times (\nabla \times \mathbf{E}_m^{(v)}) + k_2^2 \mathbf{E}_m^{(v)}]\} \\ &= -v \langle \nabla \times \mathbf{E}_m^{(v)} | \nabla \times \mathbf{E} \rangle_1 + u k_2^2 \langle \mathbf{E}_m^{(v)} | \mathbf{E} \rangle_1 + v_m \langle \nabla \times \mathbf{E}_m^{(v)} | \nabla \times \mathbf{E} \rangle_1 - \frac{4\pi i \omega \mu_2}{c^2} \int dV (\mathbf{E}_m^{(v)} \cdot \mathbf{J}_{ex}) \\ &= -(v - v_m) B_m \langle \nabla \times \mathbf{E}_m^{(v)} | \nabla \times \mathbf{E}_m^{(v)} \rangle_1 + u k_2^2 \sum_p B_p \langle \mathbf{E}_m^{(v)} | \mathbf{E}_p^{(v)} \rangle_1 - \frac{4\pi i \omega \mu_2}{c^2} \int dV (\mathbf{E}_m^{(v)} \cdot \mathbf{J}_{ex}). \end{aligned} \tag{59}$$

This too is a system of inhomogeneous linear algebraic equations for B_m with a matrix that is symmetric.

Note that in deriving the last two equations, we applied the following algebraic transformation:

$$\int dV\, \mathbf{F} \cdot \{\nabla \times [\theta_1 (\nabla \times \mathbf{E})]\} = -\int dV\, \nabla \cdot \{\mathbf{F} \times [\theta_1 (\nabla \times \mathbf{E})]\} + \int dV\, \theta_1 (\nabla \times \mathbf{F}) \cdot (\nabla \times \mathbf{E}).$$

On the rhs the first integral transforms to a surface integral over the system envelope which vanishes, while the second integral can be written as $\langle \nabla \times \mathbf{F})|(\nabla \times \mathbf{E}\rangle_1$.

We note that Eqs. (58) and (59) are solvable even if the (biorthogonal) normalization terms $\langle \mathbf{E}_n^{(u)} | \mathbf{E}_n^{(u)} \rangle_1$ or $\langle \nabla \times \mathbf{E}_m^{(v)} | \nabla \times \mathbf{E}_m^{(v)} \rangle_1$ happen to vanish!

We now possess two expansions for $\mathbf{E}(\mathbf{r})$ which are valid inside V_1. In order to get an expansion that is valid everywhere we process Eq. (50) as follows. The term $u k_2^2 \theta_1 \mathbf{E}$ is represented using the first expansion in Eq. (57). The term $v \nabla \times [\theta_1 (\nabla \times \mathbf{E})]$ is represented using the second expansion in Eq. (57). In the resulting sums over eigenstates, the terms $u_n k_2^2 \theta_1 \mathbf{E}_n^{(u)}$ and $-v_m \nabla \times [\theta_1 (\nabla \times \mathbf{E}_m^{(v)})]$ are then replaced by the expressions on the lhs's of Eqs. (54) and (55), respectively. This leads to the following PDE for $\mathbf{E}(\mathbf{r})$:

$$[-\nabla \times (\nabla \times \ ) + k_2^2] \left[\mathbf{E} - \sum_m \frac{v}{v_m} B_m \mathbf{E}_m^{(v)} - \sum_n \frac{u}{u_n} A_n \mathbf{E}_n^{(u)} \right] = -\frac{4\pi i \omega \mu_2}{c^2} \mathbf{J}_{ex}.$$

Defining $\mathbf{E}_{01}(\mathbf{r})$ as the uniform medium scattering solution of Eq. (22), namely,

$$-\nabla \times (\nabla \times \mathbf{E}_{01}) + k_2^2 \mathbf{E}_{01} = -\frac{4\pi i \omega \mu_2}{c^2} \mathbf{J}_{ex},$$

we then get

$$\mathbf{E} = \mathbf{E}_{01} + \sum_m \frac{v}{v_m} B_m \mathbf{E}_m^{(v)} + \sum_n \frac{u}{u_n} A_n \mathbf{E}_n^{(u)}. \tag{60}$$

In this way, we get the final result for the physical field $\mathbf{E}$ when both κ and μ are heterogeneous and $\mathbf{J}_{ex} \neq 0$ only in V_2.

If $\mathbf{J}_{ex} \neq 0$ only in V_1 then the entire discussion of this section can be repeated switching the roles of the two constituents.

If $\mathbf{J}_{ex}$ is nonzero in both constituents then we can write it as a sum $\mathbf{J}_{ex} = \mathbf{J}_{ex}^{(1)} + \mathbf{J}_{ex}^{(2)}$, where $\mathbf{J}_{ex}^{(i)} \neq 0$ only in V_i, and solve separately for $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, as explained in the paragraph before last of Sec. III. The result for $\mathbf{E}(\mathbf{r})$ is then the sum of those two fields $\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)}$.

VI. SUMMARY AND DISCUSSION

We have presented a detailed general theory of the EM eigenstates of the full Maxwell equations in a two-constituent composite medium where both the local electric permittivity $\kappa(\mathbf{r})$ and the local magnetic permeability $\mu(\mathbf{r})$ have different uniform values in the two constituents. These eigenstates were shown to allow an expansion of any physical local field solution $\mathbf{E}(\mathbf{r})$ of Maxwell's equations in such a system. Those fields are always square integrable in any finite volume that does not include the field sources. In the case where $\mu(\mathbf{r})$ is uniform, Eq. (26) provides such an expansion for the scattered field $\mathbf{E}_{sc1}(\mathbf{r}) \equiv \mathbf{E}(\mathbf{r}) - \mathbf{E}_{01}(\mathbf{r})$ which is valid everywhere when the field $\mathbf{E}(\mathbf{r})$ results from an incident field $\mathbf{E}_{01}(\mathbf{r})$. If the physical field is produced by a source current density $\mathbf{J}_{ex}(\mathbf{r})$, then Eq. (23) gives the scattered field $\mathbf{E}_{sc1}(\mathbf{r})$ for any $\mathbf{r}$ without having to first calculate $\mathbf{E}_{01}(\mathbf{r})$.

These expressions allow calculation of the physical field $\mathbf{E}(\mathbf{r})$ and the scattered field $\mathbf{E}_{sc1}$ without having to solve Maxwell's equations anew every time a different source current density $\mathbf{J}_{ex}(\mathbf{r})$ or incident field $\mathbf{E}_{01}(\mathbf{r})$ is introduced or every time the physical value of κ_1 is changed. What needs to be done is to calculate all the EM eigenstates for the given microstructure and the given values of κ_2 and k_2^2. That can be time consuming for complex microstructures, but the eigenfunctions and eigenvalues only depend upon κ_2, k_2^2, and the microstructure. Moreover, as shown in Sec. IV, when the microstructure is made of many identical inclusions the computation of the eigenstates can be greatly simplified.

In contrast with some previous published derivations [6,15,18], the calculations described here did not require the introduction of Green's tensor for solving Eqs. (1) and (3). This enabled us to avoid having to deal with difficulties arising from the singularity of that tensor when its two position vectors coincide [8].

For an isolated inclusion that is simply shaped, i.e., a sphere [6,7] or circular cylinder [15] or flat slab [14], all the eigenstates can be found in essentially closed analytical form. The interactions between pairs of eigenstates of such different similarly shaped inclusions can also be obtained in closed form [6,14,15]. The eigenstates of a cluster of such inclusions can then be found by finding the eigenvectors of an appropriate infinite dimensional discrete matrix with diagonal elements that are the isolated inclusion eigenvalues and off-diagonal elements that are the interactions.

In practice, only a small number of eigenstates are usually required in order to get accurate results from the expansions for $\mathbf{E}(\mathbf{r})$, especially when the subvolume V_1, where $\mathbf{J}_{ex} \equiv 0$, does not extend to infinity. This is so because those expansions are not finite-convergence-radius series in powers of a volume fraction V_i/V, but are uniformly convergent series in a hopefully complete set of bi-orthogonal eigenstates of Maxwell's equations.

In the case where both $\kappa(\mathbf{r})$ and $\mu(\mathbf{r})$ are nonuniform a briefer theory was presented in Sec. V. Expansions for $\mathbf{E}(\mathbf{r})$ then appear in Eqs. (57) and (60).

We would like to stress that there is no need to invoke the concept of "quasinormal modes" [11] in order to get useful expansions for the physical field $\mathbf{E}(\mathbf{r})$. As we have shown, true normalizable eigenstates can be defined rigorously and calculated either analytically, for isolated simply shaped inclusions, or numerically for clusters of identical inclusions or for more complicated microstructures.

For spatially periodic microstructures, the eigenfunctions are Bloch-Floquet functions. Those too can be calculated using efficient, robust computational techniques. This will be described elsewhere.

ACKNOWLEDGMENT

P.Y.C. was supported, in part, by a Tel-Aviv University Center for NanoScience and Nanotechnology Post-Doc Scholarship.

APPENDIX: EIGENFUNCTIONS OF AN ISOLATED SPHERE AND THEIR APPLICATION FOR CALCULATING A PHYSICAL FIELD

The sphere is assumed to be centered at the origin. The classes 1 and 2 eigenfunctions are then the vector spherical harmonics (VSH) where the radial part is constructed from a spherical Bessel function of $k_i r$, as described in Ref. [6]. The microstructure under discussion is invariant under rotations about the origin. This means that Maxwell's equations are also invariant under these rotations. Therefore the eigenfunctions of those equations can be chosen to also be eigenfunctions of the operators of total angular momentum for the EM field [21]:

$$\mathbf{J} \equiv \mathbf{L} + \mathbf{S}, \quad \mathbf{L} \equiv -i(\mathbf{r} \times \nabla), \quad \mathbf{S} \equiv i\mathbf{e}\times,$$

where $\mathbf{e}$ is a unit vector. Therefore these eigenfunctions are constructed from vector multipole fields (VMF) of two types (see Eq. (A13) below and Ref. [21] for a definition of the vector spherical harmonics (VSH) $\mathbf{Y}_{JlM}(\Omega)$):

$$\mathbf{E}^{(iE)}_{nlm}(\mathbf{r}) = f^{(iE)}_{nl}(r)\mathbf{X}_{lm}(\Omega), \tag{A1}$$

$$\mathbf{H}^{(iM)}_{nlm}(\mathbf{r}) = f^{(iM)}_{nl}(r)\mathbf{X}_{lm}(\Omega), \tag{A2}$$

$$\mathbf{X}_{lm}(\Omega) \equiv \mathbf{Y}_{llm}(\Omega) \equiv \frac{\hat{\mathbf{L}}Y_{lm}}{[l(l+1)]^{1/2}} \equiv -\frac{i(\mathbf{r}\times\nabla Y_{lm})}{\sqrt{l(l+1)}}, \tag{A3}$$

where the E superscript signifies that this is a transverse electric field, i.e., perpendicular to $\mathbf{r}$, while the M superscript signifies that this is a transverse magnetic field, and where $i = 1$ or $i = 2$ signifies that these are V_1 or V_2 eigenfunctions. The function $Y_{lm}(\Omega)$ is the regular spherical harmonic.

The outside of the sphere is defined to be the subvolume V_2, while the inside of the sphere is defined to be V_1. The radial functions then have the following forms:

$$f^{(F)}_{nl}(r) = \begin{cases} A^{(F)}_{nl} j_l(k_1 r) & \text{for } r < a, \\ B^{(F)}_{nl} h^{(1)}_l(k_2 r) & \text{for } r > a, \end{cases} \tag{A4}$$

$$\frac{k_1}{k_2} = \left(\frac{\kappa_1}{\kappa_2}\right)^{\frac{1}{2}}, \tag{A5}$$

where F stands for either $1E$ or $2E$ or $1M$ or $2M$, j_l and $h^{(1)}_l$ are the spherical Bessel functions, and k_i is the wave number in the subvolume V_i. Because the eigenvalues are special values of $u^{(1)}$ or $u^{(2)}$, therefore κ_1 and k_1 will have special values. Because the physical field $\mathbf{E}(\mathbf{r})$ will behave asymptotically as $e^{ik_2 r}/r$, therefore we will only use the V_1 eigenfunctions $\mathbf{E}^{(1M)}_{nlm}$ and $\mathbf{E}^{(1E)}_{nlm}$, which also exhibit the same asymptotic behavior, when trying to expand $\mathbf{E}(\mathbf{r})$.

The continuity condition on the tangential components of $\mathbf{E}^{(1E)}_n(\mathbf{r})$ is easily obtained using the fact that $\mathbf{X}_{lm} \perp \mathbf{r}$. In this way, we get

$$A_{nl}^{(1E)} j_l(k_{1nl}a) = B_{nl}^{(1E)} h_l^{(1)}(k_2 a). \tag{A6}$$

The continuity condition on the tangential components of $\mathbf{H}_n^{(1E)}(\mathbf{r}) = c/[i\omega\mu(\mathbf{r})][\nabla \times \mathbf{E}_n^{(1E)}(\mathbf{r})]$ is most easily obtained by noting that

$$\mathbf{r} \times [\nabla \times \phi(r)\mathbf{X}_{lm}] = -\frac{\partial}{\partial r}[r\phi(r)]\mathbf{X}_{lm}. \tag{A7}$$

This leads to

$$A_{nl}^{(1E)}\left[j_l(k_{1nl}a) + k_{1nl}a j_l'(k_{1nl}a)\right] = B_{nl}^{(1E)}\left[h_l^{(1)}(k_2 a) + k_2 a h_l^{(1)\prime}(k_2 a)\right].$$

When this equation is divided by Eq. (A6) and the quotient is simplified, we get the following equation for the eigenvalues $u_{nl}^{(1E)}$:

$$\left.\frac{y j_l'(y)}{j_l(y)}\right|_{y=k_{1nl}a\equiv k_2 a\left[1-u_{nl}^{(1E)}\right]^{1/2}} = \left.\frac{x h_l^{(1)\prime}(x)}{h_l^{(1)}(x)}\right|_{x=k_2 a}. \tag{A8}$$

The continuity condition on the tangential components of $\mathbf{H}_n^{(1M)}(\mathbf{r})$ is easily obtained using the fact that $\mathbf{X}_{lm} \perp \mathbf{r}$. In this way, we get

$$A_{nl}^{(1M)} j_l(k_{1nl}a) = B_{nl}^{(1M)} h_l^{(1)}(k_2 a). \tag{A9}$$

The continuity condition on the tangential components of $\mathbf{E}_n^{(1M)}(\mathbf{r}) = ic/[\omega\kappa(\mathbf{r})][\nabla \times \mathbf{H}_n^{(1M)}(\mathbf{r})]$ is most easily obtained by using Eq. (A7) once again to get

$$\frac{A_{nl}^{(1M)}}{\kappa_{1nl}}\left[j_l(k_{1nl}a) + k_{1nl}a j_l'(k_{1nl}a)\right] = \frac{B_{nl}^{(1M)}}{\kappa_2}\left[h_l^{(1)}(k_2 a) + k_2 a h_l^{(1)\prime}(k_2 a)\right].$$

When this equation is divided by Eq. (A9) and the quotient is simplified we get the following equation for the eigenvalues $u_{nl}^{(1M)}$:

$$\left(\frac{1}{y^2} + \frac{j_l'(y)}{y j_l(y)}\right)_{y=k_{1nl}a\equiv k_2 a\left[1-u_{nl}^{(1M)}\right]^{1/2}} = \left(\frac{1}{x^2} + \frac{h_l^{(1)\prime}(x)}{x h_l^{(1)}(x)}\right)_{x=k_2 a}. \tag{A10}$$

Note that we allowed κ_2 and k_2 to retain their physical values and let κ_{1nl} and k_{1nl} represent the eigenvalues. We note that the eigenstates of the isolated sphere are known spherical Bessel functions and only the eigenvalues usually need to be calculated by numerical solution of one of Eqs. (A8) or (A10).

Because of the spherical symmetry, the eigenvalues $u_{nl}^{(F)}$ have much degeneracy—they are independent of m. We can therefore reorganize the biorthogonality properties of the eigenstates as follows [6]. First of all we now define the scalar product in the more standard way as

$$\langle \mathbf{E}|\mathbf{F}\rangle_i \equiv \int dV \theta_i (\mathbf{E}^* \cdot \mathbf{F}) = \langle \mathbf{F}|\mathbf{E}\rangle_i^*.$$

Second, the left eigenfunction that is conjugate to any of the right eigenfunctions is defined by

$$\mathcal{C}[f_{nl}(r)\mathbf{X}_{lm}(\Omega)] \equiv f_{nl}^*(r)\mathbf{X}_{lm}(\Omega),$$
$$\mathcal{C}[\nabla \times f_{nl}(r)\mathbf{X}_{lm}(\Omega)] \equiv \nabla \times f_{nl}^*(r)\mathbf{X}_{lm}(\Omega),$$

i.e., only the radial part is complex-conjugated. As shown in Eq. (A4) the functions $f_{nl}(r)$ are the spherical Bessel functions $j_l(kr)$ or $h_l^{(1)}(kr)$. Therefore [see Eq. (5.9.19) in Ref. [21]]

$$\begin{aligned}\nabla \times f_l(kr)\mathbf{X}_{lm} &= i\left(\frac{l}{2l+1}\right)^{1/2}\left(\frac{d}{dr} - \frac{l}{r}\right) f_l(kr)\mathbf{Y}_{l\,l+1\,m} + i\left(\frac{l+1}{2l+1}\right)^{1/2}\left(\frac{d}{dr} + \frac{l+1}{r}\right) f_l(kr)\mathbf{Y}_{l\,l-1\,m}\\ &= -ik\left[\left(\frac{l}{2l+1}\right)^{1/2} f_{l+1}(kr)\mathbf{Y}_{l\,l+1\,m} - \left(\frac{l+1}{2l+1}\right)^{1/2} f_{l-1}(kr)\mathbf{Y}_{l\,l-1\,m}\right],\end{aligned} \tag{A11}$$

where f_l can be any spherical Bessel function. From this, it follows that

$$\{\mathcal{C}[\nabla \times f_l(kr)\mathbf{X}_{lm}(\Omega)]\}^* = ik\left[\left(\frac{l}{2l+1}\right)^{1/2} f_{l+1}(kr)\mathbf{Y}_{l\,l+1\,m}^* - \left(\frac{l+1}{2l+1}\right)^{1/2} f_{l-1}(kr)\mathbf{Y}_{l\,l-1\,m}^*\right]. \tag{A12}$$

The functions $\mathbf{Y}_{ll\pm 1m}(\Omega)$ are special cases of the general VSH (see Eqs. (5.9.18)–(5.9.20) and (5.9.10) in Ref. [21])

$$\mathbf{Y}_{JlM}(\Omega) = \sum_{mq} Y_{lm}(\Omega)\mathbf{e}_q (lm1q|l1JM), \tag{A13}$$

where $\mathbf{e}_q$ are the complex unit vectors

$$\mathbf{e}_0 \equiv \mathbf{e}_z, \quad \mathbf{e}_{\pm 1} \equiv \mp\frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y)$$

and $(lm1q|l1JM)$ is a Clebsch-Gordan coefficient (CG). It follows from these definitions that, when J, l, M are integers, then $\mathbf{Y}_{JlM}^*(\Omega) = (-1)^{l+1-J-M}\mathbf{Y}_{Jl\,-M}(\Omega)$. Therefore $\mathbf{Y}_{ll\pm 1\,m}^*(\Omega) = (-1)^m \mathbf{Y}_{ll\pm 1\,-m}(\Omega)$ and $\mathbf{X}_{lm}^*(\Omega) = (-1)^{1-m}\mathbf{X}_{l\,-m}(\Omega)$.

Three special cases of Eq. (A13) that will interest us later are $\mathbf{Y}_{llm}$, which was already defined earlier in Eq. (A3), and

$\mathbf{Y}_{l\,l\pm1\,m}$. In particular, when $l=1$ then

$$\mathbf{Y}_{10m}(\Omega)=\frac{\mathbf{e}_m}{\sqrt{4\pi}},$$

$$\mathbf{Y}_{11m}(\Omega)\equiv\mathbf{X}_{1m}(\Omega)=\frac{\mathbf{e}_m\times\mathbf{n}}{\sqrt{6\pi}}-\frac{1}{\sqrt{60}}[(\mathbf{e}_0\times\mathbf{n})Y_{2m}(\Omega)-(\mathbf{e}_1\times\mathbf{n})Y_{2m-1}(\Omega)-(\mathbf{e}_{-1}\times\mathbf{n})Y_{2m+1}(\Omega)].$$

The VSH satisfy the following general orthogonality relations [21]:

$$\int d\Omega[\mathbf{X}^*_{lM}(\Omega)\cdot\mathbf{X}_{l'M'}(\Omega)]=\delta_{ll'}\delta_{MM'},$$

$$\int d\Omega[\mathbf{Y}^*_{JlM}(\Omega)\cdot\mathbf{Y}_{J'l'M'}(\Omega)]=\delta_{JJ'}\delta_{ll'}\delta_{MM'}.$$

Thus only the radial functions are relevant for ensuring the nontrivial biorthogonalty of the eigenfunctions.

Equation (A8) can be rewritten as

$$\frac{y j_{l-1}(y)}{j_l(y)}=\frac{x h^{(1)}_{l-1}(x)}{h^{(1)}_l(x)}. \tag{A14}$$

We now restrict our discussion to the case where $x\equiv k_2a\ll 1$. In that case,

$$\frac{x h^{(1)}_{l-1}(x)}{h^{(1)}_l(x)}\cong\frac{x^2}{2l-1}\ll 1.$$

If also $y\ll 1$ then

$$\frac{y j_{l-1}(y)}{j_l(y)}\cong 2l+1=O(1).$$

There is therefore no solution to Eq. (A14) with $y\ll 1$. Solutions do exist whenever y is close to a zero of $j_{l-1}(y)$, denoted by $y_{n\,l-1}$. This requires $|y|=\mathcal{O}(1)\gg|x|$, therefore

$$1-u^{(1E)}_{nl}\equiv\left(\frac{k^{(1E)}_{nl}}{k_2}\right)^2\approx\frac{y^2_{n\,l-1}}{x^2}=O\left(\frac{1}{|k_2|^2a^2}\right)\gg 1,$$

$$j_{l-1}\left(k^{(1E)}_{nl}a\right)\approx\frac{(k_2a)^2}{(2l-1)y_{n\,l-1}}j_l(y_{nl-1})\ll 1.$$

Similarly, Eq. (A10) can be rewritten as

$$\left(\frac{j_{l-1}(y)}{y j_l(y)}-\frac{l}{y^2}\right)_{y=k_2a(1-u^{(1M)}_{nl})^{1/2}}=\left(\frac{h^{(1)}_{l-1}(x)}{x h^{(1)}_l(x)}-\frac{l}{x^2}\right)_{x=k_2a}. \tag{A15}$$

We again restrict our discussion to the case where $x\equiv k_2a\ll 1$. The rhs of the last equation then becomes

$$\frac{1}{2l-1}-\frac{l}{x^2}\gg 1.$$

If also $y\ll 1$ then Eq. (A15) becomes

$$\frac{l+1}{y^2}\approx\frac{1}{2l-1}-\frac{l}{x^2},$$

which has the unique solution

$$\frac{y^2}{x^2}=1-u^{(1M)}_{nl}=-\frac{l+1}{l}+O(|k_2a|^2)$$
$$\Rightarrow u^{(1M)}_{nl}\cong\frac{2l+1}{l}=O(1).$$

These eigenvalues are the same as the quasistatic eigenvalues of Refs. [1,2,17]. They will be assigned the value of $n=0$. All the other solutions of Eq. (A15) will have y near a zero of $j_l(y)$, denoted by $y_{n\,l}$, therefore $|y|=O(1)\gg|x|$. This leads to

$$u^{(1M)}_{nl}=1-\frac{y^2}{x^2}=O\left(\frac{1}{|k_2|^2a^2}\right)\gg 1,$$

$$j_l\left(k^{(1M)}_{nl}a\right)=-\frac{(k_2a)^2}{y_{nl}}j_{l-1}(y_{nl})\ll 1.$$

These large eigenvalues will be assigned values of $n\neq 0$.

While the zeros of $j_l(y)$ are usually only available numerically (with the exception of $y_{n0}=n\pi$), some of the sums over them can be found in closed form by exploiting different expansions of $j_l(y)$—see, e.g., Eqs. (9.5.10) and (10.1.2) of Ref. [22]:

$$\sum_n\frac{1}{y^2_{nl}}=\frac{1}{2(2l+3)}, \tag{A16}$$

$$\sum_n\frac{1}{y^4_{nl}}=\frac{2l+7}{8(2l+3)^2(2l+5)}. \tag{A17}$$

The $1E$ eigenfunctions are

$$\mathbf{E}^{(1E)}_{nlm}=A^{(1E)}_{nl}j_l\left(k^{(1E)}_{nl}r\right)\mathbf{X}_{lm}(\Omega),\quad r<a$$
$$\mathbf{E}^{(1E)}_{nlm}=B^{(1E)}_{nl}h^{(1)}_l(k_2r)\mathbf{X}_{lm}(\Omega),\quad r>a,$$

where $\frac{(k^{(1E)}_{nl})^2}{k_2^2}\equiv\frac{\kappa^{(1E)}_{nl}}{\kappa_2}=1-u^{(1E)}_{nl}\approx\frac{y^2_{n\,l-1}}{k_2^2a^2}$.

Using Eq. (A11) with $f_l(kr)=j_l(k^{(1M)}_{nl}r)$ when $r<a$ and $f_l(kr)=h^{(1)}_l(k_2r)$ when $r>a$ we now get expressions for the $1M$ eigenfunctions $\mathbf{E}^{(1M)}_{nlm}$:

$$\mathbf{E}^{(1M)}_{nlm}\equiv\frac{ic}{\omega\kappa(\mathbf{r})}\left(\nabla\times\mathbf{H}^{(1M)}_{nlm}\right)$$
$$=\frac{ck^{(1M)}_{nl}}{\omega\kappa^{(1M)}_{nl}}A^{(1M)}_{nl}\left[\left(\frac{l}{2l+1}\right)^{1/2}j_{l+1}\left(k^{(1M)}_{nl}r\right)\mathbf{Y}_{l\,l+1\,m}(\Omega)-\left(\frac{l+1}{2l+1}\right)^{1/2}j_{l-1}\left(k^{(1M)}_{nl}r\right)\mathbf{Y}_{l\,l-1\,m}(\Omega)\right],\quad r<a,$$

$$\mathbf{E}^{(1M)}_{nlm}=\frac{ck_2}{\omega\kappa_2}B^{(1M)}_{nl}\left[\left(\frac{l}{2l+1}\right)^{1/2}h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m}(\Omega)-\left(\frac{l+1}{2l+1}\right)^{1/2}h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{l\,l-1\,m}(\Omega)\right],\quad r>a,$$

where $\frac{(k^{(1M)}_{nl})^2}{k_2^2}\equiv\frac{\kappa^{(1M)}_{nl}}{\kappa_2}=1-u^{(1M)}_{nl}\approx\frac{y^2_{nl}}{k_2^2a^2}$ when $n\neq 0$, while $\frac{(k^{(1M)}_{0l})^2}{k_2^2}\equiv\frac{\kappa^{(1M)}_{0l}}{\kappa_2}=1-u^{(1M)}_{0l}\approx-\frac{l+1}{l}$.

The normalization properties of the V_1 eigenfunctions become

$$1=\int_{r<a} dV\,\big[\mathcal{C}\mathbf{E}^{(1E)}_{nlm}\big]^*\cdot\mathbf{E}^{(1E)}_{nlm}=\big[A^{(1E)}_{nl}\big]^2\int_{r<a} dV\,[j_l(ry_{nl-1}/a)]^2|\mathbf{X}_{lm}|^2=\big[A^{(1E)}_{nl}\big]^2\int_{r<a} r^2\,dr\,[j_l(ry_{nl-1}/a)]^2$$
$$=\big[A^{(1E)}_{nl}\big]^2\left\{\frac{r^3}{2}([j_l(ry_{nl-1}/a)]^2-j_{l-1}(ry_{nl-1}/a)j_{l+1}(ry_{nl-1}/a))\right\}_0^a, \tag{A18}$$

$$1=\int_{r<a} dV\,\big[\mathcal{C}\mathbf{E}^{(1M)}_{nlm}\big]^*\cdot\mathbf{E}^{(1M)}_{nlm}$$
$$=\left(\frac{ck_{1n}}{\omega\kappa_{1n}}\right)^2\int_{r<a} dV\big[A^{(1M)}_{nl}\big]^2\left[j_{l+1}\big(k^{(1M)}_{nl}r\big)\left(\frac{l}{2l+1}\right)^{1/2}\mathbf{Y}^*_{l\,l+1\,m}(\Omega)-j_{l-1}\big(k^{(1M)}_{nl}r\big)\left(\frac{l+1}{2l+1}\right)^{1/2}\mathbf{Y}^*_{l\,l-1\,m}(\Omega)\right]$$
$$\times\left[j_{l+1}\big(k^{(1M)}_{nl}r\big)\left(\frac{l}{2l+1}\right)^{1/2}\mathbf{Y}_{l\,l+1\,m}(\Omega)-j_{l-1}\big(k^{(1M)}_{nl}r\big)\left(\frac{l+1}{2l+1}\right)^{1/2}\mathbf{Y}_{l\,l-1\,m}(\Omega)\right]$$
$$=\frac{(k_2a)^2\mu_2}{y_{nl}^2\kappa_2}\big[A^{(1M)}_{nl}\big]^2\int_{r<a} r^2dr\left\{\frac{l}{2l+1}\big[j_{l+1}\big(k^{(1M)}_{nl}r\big)\big]^2+\frac{l+1}{2l+1}\big[j_{l-1}\big(k^{(1M)}_{nl}r\big)\big]^2\right\}$$
$$=\frac{(k_2a)^2\mu_2}{y_{nl}^2\kappa_2}\big[A^{(1M)}_{nl}\big]^2\frac{a^3}{2}\left[\frac{l}{2l+1}\big(j^2_{l+1}(y_{nl})-j_l(y_{nl})j_{l+2}(y_{nl})\big)+\frac{l+1}{2l+1}\big(j^2_{l-1}(y_{nl})-j_{l-2}(y_{nl})j_l(y_{nl})\big)\right],\quad n\neq 0, \tag{A19}$$

$$1=\int_{r<a} dV\,\big[\mathcal{C}\mathbf{E}^{(1M)}_{0lm}\big]^*\cdot\mathbf{E}^{(1M)}_{0lm}$$
$$=-\frac{l\mu_2}{(l+1)\kappa_2}\frac{\big(A^{(1M)}_{0l}\big)^2}{2l+1}\int_{r<a} r^2dr\big[l\,j^2_{l-2}(ik_2r\sqrt{(l+1)/l})+(l+1)j^2_{l-1}(ik_2r\sqrt{(l+1)/l})\big]$$
$$\approx(-1)^{l-1}\frac{\mu_2a^3}{\kappa_2}\left(\frac{l+1}{l}\right)^{l-2}\frac{\big[A^{(1M)}_{0l}\big]^2(l+1)}{[(2l+1)!!]^2}(k_2a)^{2l-2}. \tag{A20}$$

These expressions determine the normalization coefficients $A^{(1E)}_{nl}$ and $A^{(1M)}_{nl}$. The other V_1 normalization coefficients $B^{(1E)}_{nl}$ and $B^{(1M)}_{nl}$ are obtained by using Eqs. (A6) and (A9), respectively. These eigenfunctions will be used to expand the physical field when that field is divergence-free.

From Eqs. (A18)–(A20), (A6), and (A9), we get the following results for the normalization coefficients of the $\mathbf{E}^{(1E)}_{nlm}$ and $\mathbf{E}^{(1M)}_{nlm}$ eigenfunctions when $|k_2|a\ll 1$ [recall that y_{nl} is the n-th zero of $j_l(y)$]:

$$A^{(1E)}_{nl}\approx\left(\frac{2}{a^3}\right)^{1/2}\frac{1}{j_l(y_{n\,l-1})}, \tag{A21}$$

$$B^{(1E)}_{nl}\approx i\left(\frac{2}{a^3}\right)^{1/2}\frac{(k_2a)^{l+1}}{(2l-1)!!}, \tag{A22}$$

$$A^{(1M)}_{n\neq0\,l}\approx\left[\frac{2\kappa_2}{a^3\mu_2}\right]^{1/2}\frac{y_{nl}}{k_2a}\frac{1}{j_{l-1}(y_{nl})}, \tag{A23}$$

$$B^{(1M)}_{n\neq0\,l}\approx -i\left[\frac{2\kappa_2}{a^3\mu_2}\right]^{1/2}\frac{(k_2a)^{l+2}}{(2l-1)!!}, \tag{A24}$$

$$A^{(1M)}_{0l}\approx(-i)^{l-1}\left[\frac{\kappa_2}{(l+1)\mu_2a^3}\right]^{1/2}\frac{(2l+1)!!}{(k_2a)^{l-1}}\left(\frac{l}{l+1}\right)^{\frac{l-2}{2}}, \tag{A25}$$

$$B^{(1M)}_{0l}\approx-\left(\frac{\kappa_2}{(l+1)\mu_2a^3}\right)^{1/2}\frac{l+1}{l}\frac{(k_2a)^{l+2}}{(2l-1)!!}. \tag{A26}$$

Note that the sign used when calculating the square root of $(A^{(F)}_{nl})^2$ to get $A^{(F)}_{nl}$ was chosen arbitrarily. This does not affect the final results for the physical fields.

From these results, we can write the following closed form expressions for the eigenfunctions when $k_2 a \ll 1$:

$$\mathbf{E}^{(1E)}_{nlm} \approx \left(\frac{2}{a^3}\right)^{1/2} \frac{j_l(ry_{n\,l-1}/a)}{j_l(y_{n\,l-1})}\mathbf{X}_{lm}, \quad r < a, \tag{A27}$$

$$\mathbf{E}^{(1E)}_{nlm} \approx i\left(\frac{2}{a^3}\right)^{1/2} \frac{(k_2a)^{l+1}}{(2l-1)!!} h^{(1)}_l(k_2r)\mathbf{X}_{lm}, \quad r > a, \tag{A28}$$

$$\mathbf{E}^{(1M)}_{n\neq 0\,lm} \approx \left(\frac{2}{(2l+1)a^3}\right)^{1/2} \frac{1}{j_{l-1}(y_{nl})}[l^{1/2} j_{l+1}(ry_{nl}/a)\mathbf{Y}_{l\,l+1\,m}(\Omega) - (l+1)^{1/2} j_{l-1}(ry_{nl}/a)\mathbf{Y}_{l\,l-1\,m}(\Omega)], \ r < a, \tag{A29}$$

$$\mathbf{E}^{(1M)}_{n\neq 0\,lm} \approx -i\left(\frac{2(2l+1)}{a^3}\right)^{1/2} \frac{(k_2a)^{l+2}}{(2l+1)!!}\left[l^{1/2} h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m}(\Omega) - (l+1)^{1/2} h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{l\,l-1\,m}(\Omega)\right], \ r > a, \tag{A30}$$

$$\mathbf{E}^{(1M)}_{0lm} \approx (-i)^l \left(\frac{2l+1}{(l+1)a^3}\right)^{1/2} \left(\frac{l}{l+1}\right)^{\frac{l-1}{2}} \frac{(2l-1)!!}{(k_2a)^{l-1}} [l^{1/2} j_{l+1}[ik_2r\sqrt{(l+1)/l}]\mathbf{Y}_{l\,l+1\,m}(\Omega)$$
$$- (l+1)^{1/2} j_{l-1}[ik_2r\sqrt{(l+1)/l}]\mathbf{Y}_{l\,l-1\,m}(\Omega)], \ r < a, \tag{A31}$$

$$\mathbf{E}^{(1M)}_{0lm} \approx -\left(\frac{2l+1}{(l+1)a^3}\right)^{1/2} \frac{l+1}{l} \frac{(k_2a)^{l+2}}{(2l+1)!!}\left[l^{1/2} h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m}(\Omega) - (l+1)^{1/2} h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{l\,l-1\,m}(\Omega)\right], \ r > a. \tag{A32}$$

We now turn to the problem of calculating the local physical field $\mathbf{E}(\mathbf{r})$ when the incident field $\mathbf{E}_{01}(\mathbf{r})$ is a plane wave traveling along z axis in the No. 2 constituent with right/left-handed circular polarization when $m = 1$ or $m = -1$, namely (see Eq. (16.139) in Ref. [13]):

$$\mathbf{E}_{01}(\mathbf{r}) \equiv -m\sqrt{2}\,\mathbf{e}_m e^{ik_2z} = \sum_{l=1}^{\infty} i^l \sqrt{4\pi(2l+1)}$$
$$\times \left\{ j_l(k_2r)\mathbf{X}_{l\,m} - im\left[\left(\frac{l}{2l+1}\right)^{1/2} j_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m} - \left(\frac{l+1}{2l+1}\right)^{1/2} j_{l-1}(k_2r)\mathbf{Y}_{l\,l-1\,m}\right]\right\}. \tag{A33}$$

The scalar products between this field and the various eigenfunctions become

$$\left\langle \mathbf{E}^{(1E)}_{nlm} \middle| \mathbf{E}_{01}\right\rangle_1 = i^l\sqrt{8\pi(2l+1)a^3}\frac{(k_2a)^l}{y^2_{n\,l-1}(2l-1)!!}, \tag{A34}$$

$$\left\langle \mathbf{E}^{(1M)}_{n\neq 0\,lm} \middle| \mathbf{E}_{01}\right\rangle_1 = mi^{l-1}\sqrt{8\pi(2l+1)a^3}\frac{(k_2a)^{l+1}}{y^2_{nl}\,(2l-1)!!}, \tag{A35}$$

$$\left\langle \mathbf{E}^{(1M)}_{0lm} \middle| \mathbf{E}_{01}\right\rangle_1 = mi^l\left(\frac{4\pi(l+1)a^3}{2l+1}\right)^{1/2}\frac{(k_2a)^{l-1}}{(2l-1)!!}. \tag{A36}$$

When these results are used in Eq. (26), along with the explicit forms of the eigenfunctions from Eqs. (A27)–(A32) and the various eigenvalues we find that these eigenfunctions contribute to $\mathbf{E}_{sc1}$ outside the sphere as follows. When κ_1, and therefore $u^{(1)} \equiv 1 - \kappa_1/\kappa_2$, are finite, the eigenstates $\mathbf{E}^{(1E)}_{nlm}$ and $\mathbf{E}^{(1M)}_{n\neq 0\,lm}$ contribute terms of order $(k_2a)^{2l+3}$ and $(k_2a)^{2l+4}$, respectively, to $\mathbf{E}_{sc1}$, while $\mathbf{E}^{(1M)}_{n=0\,lm}$ contribute terms of order $(k_2a)^{2l+1}$. However, when κ_1 and $u^{(1)}$ are infinite then $\mathbf{E}^{(1E)}_{nlm} \propto (k_2a)^{2l+1}$, $\mathbf{E}^{(1M)}_{n\neq 0\,lm} \propto (k_2a)^{2l+2}$, and $\mathbf{E}^{(1M)}_{n=0\,lm} \propto (k_2a)^{2l+1}$. Therefore, when $u^{(1)}$ is finite only $\mathbf{E}^{(1M)}_{n=0\,l=1\,m}$ makes an important contribution to the series in Eq. (26). However, when $u^{(1)}$ is infinite then $\mathbf{E}^{(1M)}_{n=0\,l=1\,m}$ and $\mathbf{E}^{(1E)}_{nl=1\,m}$ for all values of n make important contributions to that series. This leads to the following closed form approximation for the scattered field when $r > a$:

$$\mathbf{E}_{sc1} \approx m\sum_l i^l \frac{\kappa_1-\kappa_2}{l\kappa_1+(l+1)\kappa_2}\frac{\sqrt{4\pi}(l+1)\,(k_2a)^{2l+1}}{(2l+1)!!\,(2l-1)!!}\left[\sqrt{l}\,h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m}(\Omega) - \sqrt{l+1}\,h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{l\,l-1\,m}(\Omega)\right]$$
$$+ m\sum_{ln\neq 0}\frac{i^l}{y^2_{nl}}\frac{\kappa_1-\kappa_2}{\kappa_1(k_2a)^2-\kappa_2y^2_{nl}}\frac{4\sqrt{\pi}(k_2a)^{2l+5}}{(2l+1)\,[(2l-1)!!]^2}\left[\sqrt{l}\,h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m}(\Omega) - \sqrt{l+1}\,h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{l\,l-1\,m}(\Omega)\right]$$
$$+ \sum_{ln} i^{l-1}\frac{\kappa_1-\kappa_2}{\kappa_1(k_2a)^2-\kappa_2y^2_{n\,l-1}}\frac{4\sqrt{\pi(2l+1)}}{[(2l-1)!!]^2}\frac{(k_2a)^{2l+3}}{y^2_{nl-1}}h^{(1)}_l(k_2r)\mathbf{X}_{lm}(\Omega), \quad m = \pm 1. \tag{A37}$$

In this result, the first sum is over the $\mathbf{E}^{(1M)}_{n=0\,lm}$ eigenstates, the second sum is over the $\mathbf{E}^{(1M)}_{n\neq 0\,lm}$ eigenstates, and the third sum is over the $\mathbf{E}^{(1E)}_{nlm}$ eigenstates. Clearly, if $(k_2a)^2 \ll \kappa_2/\kappa_1$ then at any value of l only the terms in the first sum are important. However, if $(k_2a)^2 \gg \kappa_2/\kappa_1$ then those terms in the first and third sums have similar orders of magnitude, namely they are $\propto (k_2a)^{2l+1}$, while the terms in the second sum are much smaller, namely they are $\propto (k_2a)^{2l+3}$. In the case of a perfectly conducting sphere, when $\kappa_1 \to \infty$, the first and third sums in this equation have the same form as the classic expression for the scattered electric field when a plane wave impinges upon such a sphere which is much smaller than the wavelength $2\pi/k_2$—see Eq. (16.141) of Ref. [13]. The second sum does not appear in the latter equation, probably because every l-term contributes negligibly compared to the corresponding terms in the other sums. Note that the value of $m = +1$ or $m = -1$ signifies that the incident plane wave $\mathbf{E}_{01}$ was right/left-hand circularly polarized. Note also that when $1 - u^{(1)} \equiv \kappa_1/\kappa_2$ is equal to $1 - u_{0l}^{(1M)} \equiv -(l+1)/l$ then the scattered field due to the $\mathbf{E}_{0lm}^{(1M)}$ eigenstate, i.e., the l term in the first sum on the rhs of Eq. (A37) diverges. This occurs when $u^{(1)}$ equals the eigenvalue $u_{0l}^{(1M)} \equiv (2l+1)/l$. That is the quasistatic order-l electric multipole resonance of the isolated sphere at $s = s_{0l}^{(1M)} \equiv l/(2l+1)$.

When $|k_2/k_1| \ll |k_2a|^2 \ll 1$ we can write the following approximate expression for $\mathbf{E}_{sc1}$, where only the first and third sums and only the $l = 1$ terms are kept and the well known result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

was used to sum over $1/y_{n0}^2 = 1/(n\pi)^2$:

$$\mathbf{E}_{sc1} \approx \left(\frac{4\pi}{3}\right)^{1/2} (k_2a)^3\, h_1^{(1)}(k_2r)\mathbf{X}_{1m}(\Omega) + i\,m\frac{\sqrt{8}}{3}(k_2a)^3\left[\sqrt{2\pi}h_2^{(1)}(k_2r)\mathbf{Y}_{12m}(\Omega) - h_0^{(1)}(k_2r)\mathbf{e}_m\right], \quad m = \pm 1. \tag{A38}$$

When the incident field is produced by an externally given current density which is nonzero only outside the sphere, then we can use the eigenfunctions of Eqs. (A27)–(A32) to calculate their overlap integrals with $\mathbf{J}_{ex}(\mathbf{r})$. We will assume that these source currents are a point source, namely, $\mathbf{J}_{ex}(\mathbf{r}) = \mathbf{J}_0\delta^3(\mathbf{r} - \mathbf{r}_0)$ where $\mathbf{r}_0$ is outside the sphere. In that case, we find

$$\langle \mathbf{E}_{nlm}^{(1E)} | \mathbf{J}_{ex} \rangle_2 = i\left(\frac{2}{a^3}\right)^{1/2} \frac{(k_2a)^{l+1}}{(2l-1)!!} h_l^{(1)}(k_2r_0)(\mathbf{X}_{lm}^*(\Omega_0)\cdot\mathbf{J}_0), \tag{A39}$$

$$\langle \mathbf{E}_{n\neq 0\,lm}^{(1M)} | \mathbf{J}_{ex} \rangle_2 = i\left(\frac{2(2l+1)}{a^3}\right)^{1/2} \frac{(k_2a)^{l+2}}{(2l+1)!!}\left(\left[\sqrt{l}h_{l+1}^{(1)}(k_2r_0)\mathbf{Y}_{ll+1m}^*(\Omega_0) - \sqrt{l+1}h_{l-1}^{(1)}(k_2r_0)\mathbf{Y}_{ll-1m}^*(\Omega_0)\right]\cdot\mathbf{J}_0\right), \tag{A40}$$

$$\langle \mathbf{E}_{n=0\,lm}^{(1M)} | \mathbf{J}_{ex} \rangle_2 = \left(\frac{(2l+1)}{(l+1)a^3}\right)^{1/2} \frac{l+1}{l} \frac{(k_2a)^{l+2}}{(2l+1)!!}\left(\left[\sqrt{l}h_{l+1}^{(1)}(k_2r_0)\mathbf{Y}_{ll+1m}^*(\Omega_0) - \sqrt{l+1}h_{l-1}^{(1)}(k_2r_0)\mathbf{Y}_{ll-1m}^*(\Omega_0)\right]\cdot\mathbf{J}_0\right). \tag{A41}$$

Using these results in Eq. (23) leads to the following approximate expression for $\mathbf{E}_{sc1}$:

$$\begin{aligned}
\mathbf{E}_{sc1} = {} & \frac{8\pi i}{\omega\kappa_2 a^3} \sum_{nlm} \frac{\kappa_1 - \kappa_2}{\kappa_1(k_2a)^2 + \kappa_2 y_{nl-1}^2} \frac{(k_2a)^{2l+6}}{[(2l-1)!!]^2} \frac{1}{y_{nl-1}^2 - (k_2a)^2} h_l^{(1)}(k_2r_0)(\mathbf{X}_{lm}^*(\Omega_0)\cdot\mathbf{J}_0)h_{lm}^{(1)}(k_2r)\mathbf{X}lm(\Omega) \\
& - \frac{8\pi i}{\omega\kappa a^3} \sum_{n\neq 0\,lm} \left\{ \frac{(k_2a)^{2l+8}}{(2l-1)!!\,(2l+1)!!} \frac{\kappa_1 - \kappa_2}{\kappa_1(k_2a)^2 - \kappa_2 y_{nl}^2} \frac{1}{y_{nl}^2 - (k_2a)^2} \right. \\
& \times \left[\left(l^{1/2}h_{l+1}^{(1)}(k_2r_0)\mathbf{Y}_{ll+1\,m}^*(\Omega_0) - (l+1)^{1/2}h_{l-1}^{(1)}(k_2r_0)\mathbf{Y}_{ll-1\,m}^*(\Omega_0)\right)\cdot\mathbf{J}_0\right] \\
& \left. \times \left[l^{1/2}h_{l+1}^{(1)}(k_2r)\mathbf{Y}_{ll+1\,m}(\Omega) - (l+1)^{1/2}h_{l-1}^{(1)}(k_2r)\mathbf{Y}_{ll-1\,m}(\Omega)\right]\right\} \\
& - \frac{4\pi i}{\omega\kappa_2 a^3} \sum_{lm} \frac{(k_2a)^{2l+4}(l+1)}{[(2l+1)!!]^2} \frac{\kappa_1 - \kappa_2}{l\kappa_1 + (l+1)\kappa_2} \\
& \times \left[\left(l^{1/2}h_{l+1}^{(1)}(k_2r_0)\mathbf{Y}_{ll+1\,m}^*(\Omega_0) - (l+1)^{1/2}h_{l-1}^{(1)}(k_2r_0)\mathbf{Y}_{ll-1\,m}^*(\Omega_0)\right)\cdot\mathbf{J}_0\right] \\
& \times \left[l^{1/2}h_{l+1}^{(1)}(k_2r)\mathbf{Y}_{ll+1\,m}(\Omega) - (l+1)^{1/2}h_{l-1}^{(1)}(k_2r)\mathbf{Y}_{ll-1\,m}(\Omega)\right].
\end{aligned} \tag{A42}$$

Here too, when $|k_2a|^2 \ll |\kappa_2/\kappa_1| \ll 1$ then only the third sum, which represents the contributions of the $\mathbf{E}_{n=0\,lm}^{(1M)}$ eigenstates, is important. However, when $|\kappa_2/\kappa_1| \ll |k_2a|^2 \ll 1$ then that sum as well as the first sum, which represents the contributions of the $\mathbf{E}_{nlm}^{(1E)}$ eigenstates, are important.

Another case worth discussing is when the physical field is produced by a given external current density $\mathbf{J}_{ex}(\mathbf{r})$ that is nonzero only inside the sphere, i.e., for $r < a$. This case was not treated previously using the EM eigenstates. In that case, we use Eq. (38) to expand the divergence-free physical field $\widetilde{\mathbf{E}} \equiv \mathbf{E} + [4\pi i/(\omega\kappa_2)]\mathbf{J}_{ex}$ in the divergence-free $\mathbf{E}_n^{(1)}$ eigenfunctions. In order to do that we need the following overlap integrals of $\mathbf{E}_n^{(1)}$ and $\mathbf{J}_{ex}$.

Again we assume that $\mathbf{J}_{ex}(\mathbf{r}) = \delta^3(\mathbf{r}-\mathbf{r}_0)$, where now $r_0 < a$:

$$\langle \mathbf{E}^{(1E)}_{nlm}|\mathbf{J}_{ex}\rangle_1 = \left(\frac{2}{a^3}\right)^{1/2}\frac{j_l(r_0 y_{n\,l-1}/a)}{j_l(y_{n\,l-1})}(\mathbf{X}^*_{lm}(\Omega_0)\cdot\mathbf{J}_0), \tag{A43}$$

$$\langle \mathbf{E}^{(1M)}_{n\neq 0\,lm}|\mathbf{J}_{ex}\rangle_1 = -\left(\frac{2}{(2l+1)a^3}\right)^{1/2}\frac{1}{j_{l-1}(y_{n\,l})}([l^{1/2}j_{l+1}(r_0 y_{n\,l}/a)\mathbf{Y}^*_{l\,l+1\,m}(\Omega_0) - (l+1)^{1/2}j_{l-1}(r_0 y_{n\,l}/a)\mathbf{Y}^*_{l\,l-1\,m}(\Omega_0)]\cdot\mathbf{J}_0), \tag{A44}$$

$$\langle \mathbf{E}^{(1M)}_{n=0\,lm}|\mathbf{J}_{ex}\rangle_1 = -i\left(\frac{2l+1}{a^3}\right)^{1/2}\left(\frac{r_0}{a}\right)^{l-1}(\mathbf{Y}^*_{l\,l-1\,m}(\Omega_0)\cdot\mathbf{J}_0). \tag{A45}$$

Using these results, along with Eqs. (A27)–(A32), in Eq. (38), we get

$$\begin{aligned}\widetilde{\mathbf{E}}(\mathbf{r}) \approx{}& \frac{8\pi k_2}{\kappa_1\omega a^3}\sum_{n\neq 0\,lm}\frac{y^2_{nl}}{\kappa_1(k_2a)^2-\kappa_2 y^2_{nl}}\frac{1}{j_{l-1}(y_{nl})}\frac{(k_2a)^{l+2}}{(2l+1)!!}([j_{l+1}(r_0y_{nl}/a)\mathbf{Y}^*_{l\,l+1\,m}(\Omega_0) - j_{l-1}(r_0y_{nl}/a)\mathbf{Y}^*_{l\,l+1\,m}(\Omega_0)]\cdot\mathbf{J}_0)\\ &\times\left[h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m}\Omega) - h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{l\,l+1\,m}\Omega)\right]\\ &-\frac{4\pi k_2^4}{\omega\kappa_1}\sum_{lm}\frac{(l+1)^{3/2}}{l\kappa_1+(l+1)\kappa_2}\frac{(k_2r_0)^{l-1}}{l\,(2l-1)!!}(\mathbf{Y}^*_{l\,l-1\,m}(\Omega_0)\cdot\mathbf{J}_0)\left[h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{l\,l+1\,m}(\Omega) - h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{l\,l+1\,m}(\Omega)\right]\\ &+\frac{8\pi k_2}{\kappa_1\omega a^3}\sum_{nlm}\frac{y^2_{nl-1}}{\kappa_1(k_2a)^2-\kappa_2y^2_{nl-1}}\frac{(k_2a)^{l+1}}{(2l-1)!!}\frac{j_l(r_0y_{nl-1}/a)}{j_l(y_{nl-1})}(\mathbf{X}^*_{lm}(\Omega_0)\cdot\mathbf{J}_0)h^{(1)}_l(ry_{nl-1}/a)\mathbf{X}_{lm}(\Omega).\end{aligned} \tag{A46}$$

Here only the second sum, which represents the contribution of the $\mathbf{E}^{(1M)}_{n=0\,lm}$ eigenstates and is independent of the sphere radius a, is important, irrespective of the values of κ_1 and κ_2. Because $r_0 < a$ and $|k_2a|^2 \ll 1$, the most important contribution comes from the $l = 1$ term in that sum, and it is independent of the precise location of the point source $\mathbf{r}_0$. To see this note that $\mathbf{Y}_{10m}(\Omega_0) = \mathbf{e}_m/\sqrt{4\pi}$ is independent of Ω_0.

For the sake of completeness we note that the class 3 eigenfunctions are now $\nabla\phi_{nlm}$, where $\phi_{nlm}(\mathbf{r})$ is a scalar spherical harmonic Y_{lm} multiplied by a radial function $f_{nlm}(r)$ that vanishes at $r = a$. Returning to the notation where V_1 is the inside of the sphere while V_2 is its outside, we find that the V_1 normalized radial function $f_{nlm}(r)$ is proportional to the spherical Bessel function $j_l(y_{nl}r/a)$ as follows:

$$f_{nlm}(r) = \pm\left(\frac{2}{a}\right)^{1/2}\frac{j_l(y_{nl}r/a)}{y_{nl}\,j_{l-1}(y_{nl})}, \quad r < a. \tag{A47}$$

Similarly, the class 3 normalized V_2 radial eigenfunctions are

$$g_{nlm}(r) = \pm i\left(\frac{2}{a}\right)^{1/2}\frac{f_l(x_{nl}r/a)}{x_{nl}\,f_{l-1}(x_{nl})}, \quad r > a, \tag{A48}$$

where $f_l(z)$ can be either $j_l(z)$ or the spherical Bessel function of the second kind $n_l(z)$, which is real for real z and tends to 0 when $z\to\infty$. x_{nl} is the n-th zero of $f_l(z)$. Integrands where any of these eigenfunctions appears at $r > a$ require a multiplicative factor $e^{-\delta r}$, $\delta > 0$ for convergence of the integral. At the end of the calculation the limit $\delta\to 0^+$ needs to be taken.

[1] D. J. Bergman, The dielectric constant of a two-component granular composite—a practical scheme for calculating the pole spectrum, Phys. Rev. B **19**, 2359 (1979).

[2] D. J. Bergman, The Dielectric constant of a simple - cubic array of identical spheres, J. Phys. C **12**, 4947 (1979).

[3] D. J. Bergman and J. Ben-Yakar, Static electric field of a point charge in a two-constituent composite medium: Implications for perfect imaging, Proc. SPIE **8809**, 880923 (2013).

[4] D. J. Bergman, Perfect imaging of a point charge in the quasistatic regime, Phys. Rev. A **89**, 015801 (2014).

[5] A. Farhi and D. J. Bergman, Analysis of a Veselago lens in the quasistatic regime, Phys. Rev. A **90**, 013806 (2014).

[6] D. J. Bergman and D. Stroud, Theory of resonances in the electromagnetic scattering by macroscopic bodies, Phys. Rev. B **22**, 3527 (1980).

[7] D. J. Bergman, U. Evra, and X. Li, Radiative lifetimes of nanoplasmonic states, Proc. SPIE **5927**, 59270I (2005).

[8] O. J. F. Martin and Nicolas B. Piller, Electromagnetic scattering in polarizable backgrounds, Phys. Rev. E **58**, 3909 (1998).

[9] G. T. Papadakis, D. Fleischman, A. Davoyan, P. Yeh, and H. A. Atwater, Optical Magnetism in planar metamaterial heterostructures, Nat. Commun. **9**, 296 (2018).

[10] D. R. Smith, W. J. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, A Composite Medium with Simultaneously Neg-

ative Permeability and Permittivity, Phys. Rev. Lett. **84**, 4184 (2000).

[11] C. Sauvan, J. P. Hugonin, I. S. Maksymov, and P. Lalanne, Theory of the Spontaneous Optical Emission of Nanosize Photonic and Plasmon Resonators, Phys. Rev. Lett. **110**, 237401 (2013).

[12] P. Lalanne, W. Yan, K. Vynck, C. Sauvan, and J.-P. Hugonin, Light interaction with photonic and plasmonic resonances, Laser Photonics Rev. **12**, 1700113 (2018).

[13] J. D. Jackson, *Classical Electrodynamics* 2nd ed. (Wiley and Sons, New York, 1962).

[14] A. Farhi and D. J. Bergman, Electromagnetic eigenstates and the field of an oscillating point electric dipole in a flat-slab composite structure, Phys. Rev. A **93**, 063844 (2016).

[15] D. J. Bergman, Electromagnetic eigenstates of finite cylinders and cylinder-clusters: Application to macroscopic response of meta-materials, Proc. SPIE **7032**, 70321A (2008).

[16] D. J. Bergman and K. J. Dunn, Bulk Effective Dielectric Constant of a Composite with a Periodic Microgeometry, Phys. Rev. B **45**, 13262 (1992).

[17] D. J. Bergman and A. Farhi, Spectral method for the static electric potential in a composite medium, Phys. Rev. A **97**, 043855 (2018).

[18] D. J. Bergman and Y. Ben-Yakar, Electromagnetic eigenstates applied to the theoretical discussion of meta-materials with negative refraction, Proc. SPIE **7394**, 73941Z (2009).

[19] V. G. Veselago, The electrodynamics of substances with simultaneously negative values of ϵ and μ, Usp. Fiz. Nauk **92**, 517 (1964) [Sov. Phys. Usp. **10**, 509 (1968)].

[20] J. B. Pendry, Negative Refraction Makes a Perfect Lens, Phys. Rev. Lett. **85**, 3966 (2000).

[21] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[22] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).