Monogamy and polygamy for generalized W-class states using Rényi-α entropy

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Monogamy of entanglement is an indispensable feature in multipartite quantum systems. In this paper we investigate monogamy and polygamy relations with respect to any partition for generalized W-class states using Rényi-α entropy. First, we present analytical formulas of Rényi-α entanglement (RαE) and Rényi-α entanglement of assistance (RαEoA) for a reduced density matrix of an n-qudit pure state in a superposition of generalized W-class states and vacuum. Based on the analytical formulas, we show monogamy and polygamy relations in terms of RαE and RαEoA. Next a reciprocal relation of RαEoA in an arbitrary three-party quantum system is found. Later, we further develop tighter monogamy relations in terms of concurrence and convex-roof extended negativity than former ones. In order to show the usefulness of our results, two partition-dependent residual entanglements are established to get a comprehensive analysis of entanglement dynamics for generalized W-class states. Moreover, we apply our results to an interesting quantum game and find a bound of the difference between the quantum game and the classical game.

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I. INTRODUCTION

Quantum entanglement is a critical resource in quantum communication and quantum information processing. It has gradually become the subject of many studies over the years [1–6]. However, a very important feature of quantum entanglement is *monogamy of entanglement* (MoE) [1,7], i.e., a quantum system entangled with one of the parties cannot share its entanglement freely with the rest of the parties of the system. To ensure the security of quantum key distribution protocols [8], MoE plays a crucial role. Moreover MoE also has a significant influence when dealing with condensed-matter physics, including the N-representability problem in particle physics and the frustration effects of Heisenberg antiferromagnetic ground states [9–12].

By using squared concurrence, Coffman, Kundu, and Wootters first gave a mathematical expression of MoE, which is known as the CKW inequality [1]. Given a tripartite state ρ_{ABC}, the CKW inequality reads $\mathcal{C}_{AB}^2+\mathcal{C}_{AC}^2 \leqslant \mathcal{C}_{A(BC)}^2$ where $\mathcal{C}_{AB}$ and $\mathcal{C}_{AC}$ are the concurrence of $\rho_{AB}=\mathrm{Tr}_C(\rho_{ABC})$ and $\rho_{AC}=\mathrm{Tr}_B(\rho_{ABC})$. According to the CKW inequality, one can analyze the structure of multipartite entanglement and study the genuine multipartite entanglement in the dynamical evolution [13–15]. Based on a number of different entanglement measures besides concurrence, researchers have established variations of the CKW-type inequalities in multipartite quantum systems [16–27]. In these studies, however, it was found that monogamy relations using concurrence failed in the generalization of the CKW inequality for higher-dimensional quantum systems (qudit subsystem instead of qubit subsystem) [20]. Lancien *et al.* discovered that in some higher-dimensional quantum systems [21] no nontrivial monogamy relations are satisfied by a class of additive and normalized entanglement measures. Nevertheless, it was also detected that one entanglement measure, known as the squashed entanglement, does satisfy monogamy relations for arbitrary dimensional quantum systems [22]. This raises the question of studying MoE in terms of more efficient entanglement measures in higher-dimensional quantum systems.

There has already been some researches of MoE in higher-dimensional quantum systems. Kim and Sanders [23] first proposed the n-qubit *generalized W-class* (GW) states and further characterized the entanglement of these states by their partial entanglements with squared concurrence. Later, Kim studied these multiqubit GW states in Ref. [24] again. He analytically showed that the strong monogamy inequality of multiqubit entanglement was saturated by these GW states. In Ref. [25], Choi and Kim considered MoE of some other states: a superposition of multiqudit GW states and vacuum. They gave an analytical proof that strong monogamy inequality was saturated using squared *convex-roof extended negativity* (CREN) for these states. Later in Ref. [26], Kim then focused on a large class of mixed quantum states that were in a partially coherent superposition of an n-qudit GW state and vacuum. He found that this class of states obeyed a CKW-type monogamy inequality using squared CREN. Moreover, quite recently by using Tsallis-q entropy, Shi and Chen [27] have given monogamy relations for quantum states that were in a superposition of multiqudit GW states and vacuum. Inspired by these developments, we wish to further investigate MoE for the GW states with arbitrary partitions in higher-dimensional quantum systems.

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Choosing an efficient bipartite entanglement measure plays a strong part of characterizing MoE. In this paper we will use *Rényi-α entanglement* (RαE). RαE is based on Rényi-α entropy [28]. It is a generalization of *entanglement of formation* (EoF) [29] by using the Rényi-α entropy. For EoF, there is no CKW-type monogamy inequality to characterize MoE. For RαE, we do have a CKW-type monogamy inequality for certain α [30]. The entanglement measure RαE is also helpful in describing quantum phases with differing computational power [31], ground-state properties in multipartite quantum systems [32], and topologically ordered states [33]. Furthermore, RαE is also monotone as an entanglement measure, which means that it does not increase under local operations and classical communications. By using RαE, Kim and Sanders [30] provided monogamy inequalities for $\alpha \geqslant 2$ to quantify bipartite entanglement in multipartite quantum systems. Later, Song *et al.* [34] presented that a general monogamy inequality held using squared RαE for an arbitrary N-qubit mixed state. Also, lower and upper bounds for RαE were introduced in 2016 [35].

In this paper, we study the entanglement relations between the whole system and all possible partitions for an n-qudit pure state in superposition of generalized W-class states and vacuum using RαE. We also establish *partition-dependent residual entanglements* (PREs) using our results. PREs can help us get a full understanding of the entanglement dynamics for generalized W-class states with different levels and formats of partitions. We can even develop a possible comprehensive analysis of the entanglement dynamics in an infinite or finite time using PREs [36]. Interestingly, we also use a quantum game to show the application of our entanglement relations. Apart from their entertainment values, games among multiplayers can provide some intuitive means to understand complex problems. For example, a quantum game is an interesting and important tool for quantum cryptographic purposes. Tomamichel *et al.* [37] studied the probability that two players can simultaneously succeed in guessing the outcome exactly in a quantum game. From their results, we have that the optimal guessing probability can be achieved without using entanglement. The authors in Refs. [27,38] presented bounds on the difference between multiplayer quantum games and classical games using the monogamy of Tsallis-q entropy and squashed entanglement, respectively. We give a bound using our results which is independent of α and tighter than the bound in Ref. [38]. To some extent, the methods and applications in our paper can enrich the exploration of quantum entanglement.

This paper is organized in the following manner. In Secs. II and III, a few definitions of entanglement measures and GW states were briefly introduced, respectively. Section IV shows monogamy and polygamy relations for n-qudit GW states using RαE and Rényi-α entanglement of assistance (RαEoA). We also consider a class of generalized polygamy relations in terms of RαEoA in Sec. V, which reflects the reciprocal relation of RαEoA in a three-party quantum system. Section VI gives tighter monogamy relations using concurrence and CREN. The applications about PREs and quantum games are presented in Sec. VII. Finally discussion and conclusions are given in Sec. VIII.

II. DEFINITIONS

For a bipartite pure state $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|ii\rangle$, the concurrence $C(|\psi\rangle_{AB})$ is defined as [39]

$$\mathcal{C}(|\psi\rangle_{AB}) = \sqrt{2\left[1 - \mathrm{Tr}\left(\rho_A^2\right)\right]}, \tag{1}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|)$ (and analogously for ρ_B). For any mixed state ρ_{AB}, the concurrence is given via the so-called convex-roof extension

$$\mathcal{C}(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{C}(|\psi_i\rangle), \tag{2}$$

where the minimum is taken over all possible pure decompositions of $\rho_{AB} = \sum_i p_i |\psi_i\rangle_{AB}\langle\psi_i|$.

As the duality of concurrence, the concurrence of assistance (CoA) of any mixed state ρ_{AB} is defined as [40]

$$\mathcal{C}^a(\rho_{AB}) = \max_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{C}(|\psi_i\rangle), \tag{3}$$

where the maximum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle\}$ of ρ_{AB}.

For a bipartite state ρ_{AB} in a $d \otimes d' (d \leqslant d')$ quantum system, its negativity [41] is defined as

$$\mathcal{N}(\rho_{AB}) = \left\|\rho_{AB}^{T_A}\right\| - 1, \tag{4}$$

where $\rho_{AB}^{T_A}$ is the partial transpose with respect to the subsystem A and $\|X\|$ denotes the trace norm of X, $\|X\| = \mathrm{Tr}\sqrt{XX^\dagger}$.

In order to overcome the lack of separability criterion, negativity is modified as the CREN. For a bipartite mixed state ρ_{AB}, CREN is defined as

$$\widetilde{\mathcal{N}}(\rho_{AB}) = \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{N}(|\psi_i\rangle), \tag{5}$$

where the minimum is taken over all possible pure state decompositions $\{p_i, |\psi_i\rangle\}$ of ρ_{AB}.

One of the well-known measures of quantum entanglement is RαE [30]. For a bipartite pure state $|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}|ii\rangle$, the R$\alpha$E is defined as

$$E_\alpha(|\psi\rangle_{AB}) = S_\alpha(\rho_A) = \frac{1}{1-\alpha}\log_2\left(\mathrm{tr}\rho_A^\alpha\right), \tag{6}$$

where Rényi-α entropy is $S_\alpha(\rho_A) = [\log_2(\sum_i \lambda_i^\alpha)]/(1-\alpha)$. Here α is a non-negative real number and λ_i is the eigenvalue of reduced density matrix ρ_A. When the order α tends to 1, Rényi-α entropy $S_\alpha(\rho)$ converges to the von Neumann entropy.

For a bipartite mixed state ρ_{AB}, the RαE is defined via the convex-roof extension [30]

$$E_\alpha(\rho_{AB}) = \min \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}), \tag{7}$$

where the minimum is taken over all possible pure state decompositions of ρ_{AB}.

As a dual concept to Rényi-α entanglement, we define the RαEoA as

$$E_\alpha^a(\rho_{AB}) = \max \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}), \tag{8}$$

where the maximum is taken over all possible pure state decompositions of ρ_{AB}.

III. GENERALIZED W-CLASS STATES

Generalized W-class states are introduced by Kim and Sanders [23]. The structure of generalized W-class states is also investigated in their paper by considering arbitrary partitions of subsystems and they further proved the entanglements of generalized W-class states can be fully characterized by their partial entanglement.

A class of n-qubit W-class states and n-qudit GW states [23] are, respectively, expressed as

$$|\psi\rangle_{A_1A_2\dots A_n} = a_1|10\cdots0\rangle + a_2|01\cdots0\rangle + \dots + a_n|00\cdots1\rangle \tag{9}$$

and

$$|W_n^d\rangle_{A_1\cdots A_n} = \sum_{i=1}^{d-1}(a_{1i}|i0\cdots0\rangle + a_{2i}|0i\cdots0\rangle + \cdots + a_{ni}|00\cdots0i\rangle), \tag{10}$$

with the normalization condition $\sum_{i=1}^{n}|a_j|^2 = 1$ and $\sum_{s=1}^{n}\sum_{i=1}^{d-1}|a_{si}|^2 = 1$, respectively. The state in Eq. (10) with Hamming weight 1 is a coherent superposition of all n-qudit product states. Equation (10) includes n-qubit W-class states in Eq. (9) as a special case when $d = 2$.

For any partition $P = \{P_1, \dots, P_m\}$ of the set of subsystems $S = \{A_1, \dots, A_n\}$, $m \leqslant n$, the monogamy relation with respect to any partition P can be described by concurrence in the following lemma.

Lemma 1. [23] For any n-qudit generalized W-class states $|\psi\rangle_{A_1\cdots A_n}$ and a partition $P = \{P_1, \dots, P_m\}$ of the set of subsystems $S = \{A_1, \dots, A_n\}$, $m \leqslant n$

$$\mathcal{C}^2_{P_s(P_1\cdots\widehat{P_s}\cdots P_m)} = \sum_{k\neq s}\mathcal{C}^2_{P_sP_k} = \sum_{k\neq s}\left(\mathcal{C}^a_{P_sP_k}\right)^2 \tag{11}$$

and

$$\mathcal{C}_{P_sP_k} = \left(\mathcal{C}^a_{P_sP_k}\right), \tag{12}$$

for all $k \neq s$ and $(P_1\cdots\widehat{P_s}\cdots P_m) = (P_1\cdots P_s\cdots P_m) - (P_s)$.

In this paper, we select RαE as the entanglement measure. We note that there exists an analytical expression between the RαE and concurrence [30]. For any two-qubit mixed state,

$$E_\alpha(\rho_{AB}) = f_\alpha(\mathcal{C}^2(\rho_{AB})), \tag{13}$$

where the order $\alpha \geqslant 1$ and the function $f_\alpha(x)$ has the form

$$f_\alpha(x) = \frac{1}{1-\alpha}\log_2\left[\left(\frac{1-\sqrt{1-x}}{2}\right)^\alpha + \left(\frac{1+\sqrt{1-x}}{2}\right)^\alpha\right]. \tag{14}$$

Later, Wang *et al.* proved Eq. (13) still holds for $\alpha \geqslant (\sqrt{7}-1)/2$ [42]. So, making use of the analytic expression in Eq. (13) and monogamy relation in Lemma 1, we aim to show how the global entanglement for GW states can be characterized by partial entanglements using RαE with a different range of α.

Before that, we need several lemmas about the properties of Eq. (13) which are necessary in the proof of our main results.

Lemma 2. [34] The squared RαE $E_\alpha^2(\mathcal{C}^2)$ with $\alpha \geqslant (\sqrt{7}-1)/2$ in two-qubit mixed states is monotonically increasing and convex as a function of the squared concurrence $\mathcal{C}^2$.

Lemma 3. [34] The Rényi-α entanglement $E_\alpha(\mathcal{C}^2)$ with $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ is monotonically increasing and concave as a function of the squared concurrence $\mathcal{C}^2$.

Set $y = x^2$, denote $g_\alpha(y) = f_\alpha(x^2)$, and then for any two-qubit mixed state Eqs. (13) and (14) can be rephrased as

$$E_\alpha(\rho_{AB}) = g_\alpha(\mathcal{C}(\rho_{AB})), \tag{15}$$

where the order $\alpha \geqslant 1$ and the function $g_\alpha(y)$ has the form

$$g_\alpha(y) = \frac{1}{1-\alpha}\log_2\left[\left(\frac{1-\sqrt{1-y^2}}{2}\right)^\alpha + \left(\frac{1+\sqrt{1-y^2}}{2}\right)^\alpha\right]. \tag{16}$$

Lemma 4. [42] The function $g_\alpha(y)$ with $\alpha \geqslant (\sqrt{7}-1)/2$ is a monotonically increasing and convex function for $0 \leqslant y \leqslant 1$.

IV. MONOGAMY AND POLYGAMY RELATIONS FOR GENERALIZED W-CLASS STATES USING RéNYI-α ENTROPY

Consider an n-qudit pure state $|\psi\rangle_{A_1\cdots A_n}$ which is in a superposition of an n-qudit GW state in Eq. (10) and vacuum $|0\cdots0\rangle_{A_1\cdots A_n}$:

$$|\psi\rangle_{A_1A_2\cdots A_n} = \sqrt{p}|W_n^d\rangle_{A_1\cdots A_n} + \sqrt{1-p}|0\cdots0\rangle_{A_1\cdots A_n}, \tag{17}$$

with $0 \leqslant p \leqslant 1$. We first present two analytic formulas of RαE and RαEoA for $|\psi\rangle_{A_1\cdots A_n}$, which are helpful for us to investigate monogamy and polygamy relations later. Let us start with a structural property of n-qudit GW states.

Lemma 5. [25] Let $\rho_{A_{j_1}\cdots A_{j_m}}$ be a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ onto m-qudit subsystems $A_{j_1}\cdots A_{j_m}$ with $2 \leqslant m \leqslant n-1$. For any pure state decomposition of $\rho_{A_{j_1}\cdots A_{j_m}}$ such that

$$\rho_{A_{j_1}\cdots A_{j_m}} = \sum_k q_k|\phi_k\rangle_{A_{j_1}\cdots A_{j_m}}\langle\phi_k|, \tag{18}$$

$|\phi_k\rangle_{A_{j_1}\cdots A_{j_m}}$ is a superposition of a m-qudit generalized W-class state and vacuum.

Here the index vector $\overrightarrow{j} = (j_1, \cdots, j_m)$ with m distinct elements spans all the possible ordered subsets of the index set $\{1, 2, \cdots, n\}$. From the lemma above, we know that for any pure state decomposition $\{q_k, |\phi_k\rangle_{A_{j_1}\cdots A_{j_m}}\}$ of a reduced density matrix $\rho_{A_{j_1}\cdots A_{j_m}}$, $\rho_{A_{j_1}\cdots A_{j_m}}$ is a rank-2 operator.

Theorem 1. Assume $\rho_{A_1\cdots A_n}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); then we have

$$E_\alpha(\rho_{A_1|A_2\cdots A_n}) = f_\alpha(C^2(\rho_{A_1|A_2\cdots A_n})), \tag{19}$$

when $\alpha \geqslant (\sqrt{7}-1)/2$.

Proof. For convenience, we denote $\rho_{A_1|A_1\cdots A_n}$ as bipartite state ρ_{AB}. Assume $\{p_i, |\psi_i\rangle\}$ is the optimal decomposition for RαE of ρ_{AB}; then we have

$$\begin{aligned} E_\alpha(\rho_{AB}) &= \sum_i p_iE_\alpha(|\psi_i\rangle_{AB}) \\ &= \sum_i p_ig_\alpha(\mathcal{C}(|\psi_i\rangle_{AB})) \\ &\geqslant g_\alpha\left(\sum_i p_i\mathcal{C}(|\psi_i\rangle_{AB})\right) \\ &\geqslant g_\alpha(\mathcal{C}(\rho_{AB})), \end{aligned} \tag{20}$$

where the second equality is by the relation between RαE and concurrence for pure states in Eq. (15), the first inequality is due to convexity of g_α in Lemma 4, and the last inequality is by the monotonicity of g_α in Lemma 4 and the definition of $\mathcal{C}(\rho_{AB})$.

On the other hand, we can always assume $\rho_{AB} = \sum_h q_h|\phi_h\rangle_{AB}\langle\phi_h|$ is an optimal decomposition for $\mathcal{C}(\rho_{AB})$, then we have

$$\mathcal{C}(\rho_{AB}) = \sum_h q_h \mathcal{C}(|\phi_h\rangle_{AB}) \ .$$

Moreover, we will see the following relation is also true for all h:

$$\mathcal{C}(|\phi_h\rangle_{AB}) = \mathcal{C}(\rho_{AB}). \tag{21}$$

Since $\rho_{AB} = |\psi\rangle_{AB}\langle\psi|$ with $|\psi\rangle_{AB}$ in (17), then by the Hughston-Jozsa-Wootters theorem [25] the optimal decomposition $\rho_{AB} = \sum_h q_h|\phi_h\rangle_{AB}\langle\phi_h|$ for concurrence can be realized by a unitary matrix u_h with

$$|\phi_h\rangle_{AB} = u_h|\psi\rangle_{AB}.$$

So Eq. (21) is true due to $u_h u_h^* = I$ for all h.

Then

$$\begin{aligned} g_\alpha(\mathcal{C}(\rho_{AB})) &= g_\alpha\left(\sum_h q_h \mathcal{C}(|\phi_h\rangle_{AB})\right) \\ &= \sum_h q_h g_\alpha(\mathcal{C}(|\phi_h\rangle_{AB})) \\ &= \sum_h q_h E_\alpha(|\phi_h\rangle_{AB}) \\ &\geqslant E_\alpha(\rho_{AB}), \end{aligned} \tag{22}$$

where the second equality is by Eq. (21) and the inequality is by the definition of $E_\alpha(\rho_{AB})$.

Therefore, combining Eqs. (20) and (22), we finally get $E_\alpha(\rho_{AB}) = g_\alpha(C(\rho_{AB}))$, which is equivalent to $E_\alpha(\rho_{A_1|A_2\cdots A_n}) = f_\alpha(C^2(\rho_{A_1|A_2\cdots A_n}))$. ■

Lemma 5 leads to the fact that for any pure state decomposition of the reduced density matrix $\rho_{A_{j_1}\cdots A_{j_m}} = \sum_k q_k|\phi_k\rangle_{A_{j_1}\cdots A_{j_m}}\langle\phi_k|$, $|\phi_k\rangle_{A_{j_1}\cdots A_{j_m}}$ is also a pure state in a superposition of a m-qudit GW state and vacuum, so we naturally obtain $E_\alpha(\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}}) = f_\alpha(C^2(\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}}))$ from Theorem 1. Next we wish to establish a similar analytic formula for $|\psi\rangle_{A_1\cdots A_n}$ using RαEoA.

Theorem 2. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); then we have

$$E_\alpha^a(\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}}) = f_\alpha(\mathcal{C}^2(\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}})), \tag{23}$$

when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$.

Proof. For convenience, we denote $\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}}$ as ρ_{AB}. Since $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in Eq. (17), then $\mathcal{C}(\rho_{AB}) = \mathcal{C}^a(\rho_{AB})$ [23]. So it is enough for us to show $E_\alpha^a(\rho_{AB}) = f_\alpha([\mathcal{C}^a(\rho_{AB})]^2)$. First we prove $E_\alpha^a(\rho_{AB}) \leqslant f_\alpha([\mathcal{C}^a(\rho_{AB})]^2)$. Assume $\{p_i, |\psi_i\rangle\}$ is the optimal decomposition for RαEoA of ρ_{AB}; then we have

$$\begin{aligned} E_\alpha^a(\rho_{AB}) &= \sum_i p_i E_\alpha(|\psi_i\rangle_{AB}) \\ &= \sum_i p_i E_\alpha(\mathcal{C}^2(|\psi_i\rangle_{AB})) \\ &\leqslant E_\alpha\left(\sum_i p_i \mathcal{C}^2(|\psi_i\rangle_{AB})\right) \\ &\leqslant E_\alpha([\mathcal{C}^a(\rho_{AB})]^2) = f_\alpha([\mathcal{C}^a(\rho_{AB})]^2), \end{aligned} \tag{24}$$

where the first inequality is due to the concave property of $E_\alpha(\mathcal{C}^2)$ for $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ in Lemma 3 and the second inequality is due to the definition of $(\mathcal{C}^a(\rho_{AB}))^2$ and the increasing property of $E_\alpha(\mathcal{C}^2)$ in Lemma 3.

Next we show $E_\alpha^a(\rho_{AB}) \geqslant f_\alpha([\mathcal{C}^a(\rho_{AB})]^2)$. Assume $\{r_k, |\theta_k\rangle\}$ is the optimal decomposition for $\mathcal{C}^a(\rho_{AB})$. Then

$$\begin{aligned} f_\alpha\big([\mathcal{C}^a(\rho_{AB})]^2\big) &= g_\alpha(\mathcal{C}^a(\rho_{AB})) \\ &= g_\alpha\left(\sum_k r_k \mathcal{C}(|\theta_k\rangle_{AB})\right) \\ &\leqslant \sum_k r_k g_\alpha(\mathcal{C}(|\theta_k\rangle_{AB})) \\ &= \sum_k r_k E_\alpha(|\theta_k\rangle_{AB}) \\ &\leqslant E_\alpha^a(\rho_{AB}), \end{aligned} \tag{25}$$

where in the first inequality we have used the convex property of $g_\alpha(y)$ for $\alpha \geqslant (\sqrt{7}-1)/2$ in Lemma 4. The second inequality is due to the definition of $E_\alpha^a(\rho_{AB})$.

Thus combining (24) and (25), we have $E_\alpha^a(\rho_{AB}) = f_\alpha([\mathcal{C}^a(\rho_{AB})]^2)$ which completes the proof ■.

Theorems 1 and 2 have shown the analytic expression between Rényi-α entropy and concurrence for a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$ with a different range of α. Taking the intersection of the different range leads us to the following theorem.

Theorem 3. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); then we have

$$\begin{aligned} E_\alpha^a(\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}}) &= E_\alpha(\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}}) \\ &= f_\alpha(\mathcal{C}^2(\rho_{A_{j_1}|A_{j_2}\cdots A_{j_m}})), \end{aligned} \tag{26}$$

when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$.

With the analytical expressions obtained above, we then begin to investigate the monogamy relations with respect to a partition $P = \{P_1, P_2, \cdots, P_k\}$ of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$, $k \leqslant m < n$ using Rényi-α entropy. Before that, we need a lemma to ensure the analytical expressions obtained above are all available for $|\psi\rangle_{P_1,\cdots,P_k}$.

Lemma 6. [26] Let $|\psi\rangle_{A_1,\cdots,A_n}$ be an n-qudit pure state in Eq. (17), then for any partition $P = \{P_1, \ldots, P_k\}$ of the set of subsystems $S = \{A_1, \ldots, A_n\}$, $k \leqslant n$, the state $|\psi\rangle_{P_1,\cdots,P_k}$ is also a superposition of a k-party generalized W-class state in Eq. (10) and vacuum in higher-dimensional quantum systems. Here $P_s \cap P_t = \emptyset$ for $s \neq t$, and $\bigcup_s P_s = S$.

Theorem 4. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, \cdots, P_k\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$, $k \leqslant m < n$; when $\alpha \geqslant$

$(\sqrt{7}-1)/2$, we have the following monogamy inequality:

$$E_\alpha^2(\rho_{P_1|P_2\cdots P_k}) \geqslant \sum_{i=2}^{k} E_\alpha^2(\rho_{P_1P_i}). \tag{27}$$

Proof. For $\alpha \geqslant (\sqrt{7}-1)/2$, we have

$$\begin{aligned}E_\alpha^2(\rho_{P_1|P_2\cdots P_k}) &= f_\alpha^2(\mathcal{C}^2(\rho_{P_1|P_2\cdots P_k}))\\ &= f_\alpha^2\left(\sum_{i=2}^{k}\mathcal{C}^2(\rho_{P_1P_i})\right)\\ &\geqslant \sum_{i=2}^{k} f_\alpha^2(\mathcal{C}^2(\rho_{P_1P_i}))\\ &= \sum_{i=2}^{k} E_\alpha^2(\rho_{P_1P_i}),\end{aligned} \tag{28}$$

where in the first equality we use Theorem 1, the second equality is due to the monogamy relation in Lemma 1, the first inequality is due to the convex property of squared Rényi-α entanglement in Lemma 2, and the last equality is obtained by Theorem 1 again. ■

Theorem 4 deals with the monogamy inequality of the squared RαE in a partition of $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$. However, we can also analogously show the monogamy inequality for the μth power of RαE for GW states when $\mu \geqslant 2$.

Corollary 1. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, \cdots, P_k\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$, $k \leqslant m < n$; when $\alpha \geqslant 2$, we have the following monogamy inequality:

$$E_\alpha^\mu(\rho_{P_1|P_2\cdots P_k}) \geqslant \sum_{i=2}^{k} E_\alpha^\mu(\rho_{P_1P_i}), \tag{29}$$

for $\mu \geqslant 2$.

Proof. In Theorem 3 of Ref. [23], the authors presented that

$$f_\alpha(x^2+y^2) \geqslant f_\alpha(x^2) + f_\alpha(y^2), \tag{30}$$

with $\alpha \geqslant 2$ and $0 \leqslant x, y \leqslant 1, 0 \leqslant x^2+y^2 \leqslant 1$.

Then when $\mu \geqslant 2$, we can get

$$\begin{aligned}f_\alpha^\mu(x^2+y^2) &\geqslant \left[f_\alpha^2(x^2)+f_\alpha^2(y^2)\right]^{\frac{\mu}{2}}\\ &= f_\alpha^\mu(x^2)\left(1+\frac{f_\alpha^2(y^2)}{f_\alpha^2(x^2)}\right)^{\frac{\mu}{2}}\\ &\geqslant f_\alpha^\mu(x^2)+f_\alpha^\mu(y^2).\end{aligned} \tag{31}$$

Here the second inequality is obtained by $(1+t)^x \geqslant 1+t^x$ for any real number x and t, $0 \leqslant t \leqslant 1, x \in [1, \infty]$.

Denote $Q = P_3 \cdots P_k$, then

$$\begin{aligned}E_\alpha^\mu(\rho_{P_1|P_2\cdots P_k}) &= f_\alpha^\mu(\mathcal{C}^2(\rho_{P_1|P_2\cdots P_k}))\\ &= f_\alpha^\mu[\mathcal{C}^2(\rho_{P_1P_2})+\mathcal{C}^2(\rho_{P_1Q})]\\ &\geqslant f_\alpha^\mu(\mathcal{C}^2(\rho_{P_1P_2}))+f_\alpha^\mu(\mathcal{C}^2(\rho_{P_1Q}))\\ &= E_\alpha^\mu(\rho_{P_1P_2})+E_\alpha^\mu(\rho_{P_1Q}),\end{aligned} \tag{32}$$

where the first equality is due to Theorem 1, and the second equality is obtained from Lemma 1. If $\min\{\mathcal{C}(\rho_{P_1P_2}), \mathcal{C}(\rho_{P_1Q})\} = 0$, obviously the inequality holds. If $\min\{\mathcal{C}(\rho_{P_1P_2}), \mathcal{C}(\rho_{P_1Q})\} > 0$, assuming $\mathcal{C}(\rho_{P_1P_2}) \geqslant \mathcal{C}(\rho_{P_1Q})$, then the inequality holds owing to (31).

By partitioning the qudit system Q into two subsystems, P_3 and a $2^{|p_4|+\cdots+|p_k|}$-dimensional quantum system, and using the inequality (31) repeatedly, one gets (29). Here $|p_i|$ denotes the number of qubits in party P_i for $i = 1, 2, \cdots, k$. ■

As a duality of monogamy relations, polygamy relations using RαEoA for GW states can also be developed.

Theorem 5. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, \cdots, P_k\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$, $k \leqslant m < n$; when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, we have the following polygamy inequality:

$$E_\alpha^a(\rho_{P_1|P_2\cdots P_k}) \leqslant \sum_{i=2}^{k} E_\alpha^a(\rho_{P_1P_i}). \tag{33}$$

Proof. When $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, we have

$$\begin{aligned}E_\alpha^a(\rho_{P_1|P_2\cdots P_k}) &= f_\alpha(\mathcal{C}^2(\rho_{P_1|P_2\cdots P_k}))\\ &= f_\alpha\left(\sum_{i=2}^{k}\mathcal{C}^2(\rho_{P_1P_i})\right)\\ &\leqslant \sum_{i=2}^{k} f_\alpha(\mathcal{C}^2(\rho_{P_1P_i}))\\ &= \sum_{i=2}^{k} E_\alpha^a(\rho_{P_1P_i}),\end{aligned} \tag{34}$$

where the first equality is obtained by Theorem 2, the second equality is due to the monogamy relation in Lemma 1, the first inequality is due to the concave property of Rényi-α entanglement in Lemma 3, and the last equality is obtained by Theorem 2. ■

Taking similar consideration in Corollary 1, we can have a polygamy inequality of μth power of RαEoA.

Corollary 2. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, \cdots, P_k\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$, $k \leqslant m < n$; when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, we have the following polygamy inequality:

$$(E_\alpha^a)^\mu(\rho_{P_1|P_2\cdots P_k}) \leqslant \sum_{i=2}^{k} \left(E_\alpha^a\right)^\mu(\rho_{P_1P_i}), \tag{35}$$

for $0 < \mu \leqslant 1$.

Proof. In Lemma 2 of Ref. [43], the authors showed that

$$f_\alpha(x^2+y^2) \leqslant f_\alpha(x^2) + f_\alpha(y^2), \tag{36}$$

with $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$ and $0 \leqslant x, y \leqslant 1, 0 \leqslant x^2+y^2 \leqslant 1$.

Then when $0 < \mu \leqslant 1$, we have

$$\begin{aligned}f_\alpha^\mu(x^2+y^2) &\leqslant [f_\alpha(x^2)+f_\alpha(y^2)]^\mu\\ &\leqslant f_\alpha^\mu(x^2)+f_\alpha^\mu(y^2).\end{aligned} \tag{37}$$

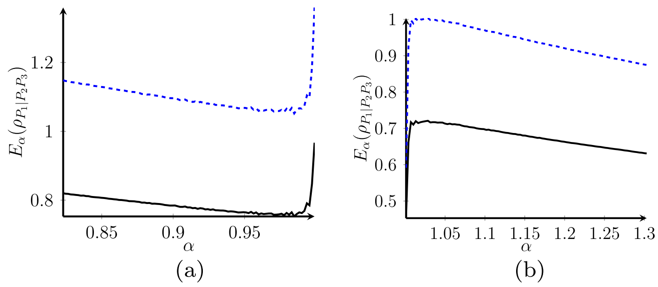

FIG. 1. Lower and upper bounds are shown for $E_\alpha(\rho_{P_1|P_2P_3})$ when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, $\alpha \neq 1$. The solid black line is the function $\sqrt{E_\alpha^2(\rho_{P_1P_2}) + E_\alpha^2(\rho_{P_1P_3})}$ of variable α, which is a lower bound for $E_\alpha(\rho_{P_1|P_2P_3})$. The dashed blue line is the function $E_\alpha^a(\rho_{P_1P_2}) + E_\alpha^a(\rho_{P_1P_3})$ of variable α, which is an upper bound for $E_\alpha(\rho_{P_1|P_2P_3})$. (a) $(\sqrt{7}-1)/2 \leqslant \alpha < 1$; (b) $1 < \alpha \leqslant (\sqrt{13}-1)/2$.

Here the second inequality is obtained by $(1+t)^x \leqslant 1 + t^x$ with $t \geqslant 0$, $x \in (0, 1]$ and a similar consideration in the proof of inequality (31).

Denote $Q = P_3 \cdots P_k$, then

$$\begin{aligned}\left(E_\alpha^a\right)^\mu(\rho_{P_1|P_2\cdots P_k}) &= f_\alpha^\mu(\mathcal{C}^2(\rho_{P_1|P_2\cdots P_k}))\\ &\leqslant \sum_{i=2}^{k} f_\alpha^\mu(\rho_{P_1P_i})\\ &= \sum_{i=2}^{k} \left(E_\alpha^a\right)^\mu(\rho_{P_1P_i}).\end{aligned} \tag{38}$$

Here the first equality is from Theorem 2, and the inequality is obtained from the iterative use of inequality (37) and a similar consideration in the proof of inequality (32). ■

As an example to show the application of the above results, we consider the four-qubit GW state

$$\begin{aligned}|\psi\rangle_{A_1A_2A_3A_4} = 0.3|0001\rangle + 0.4|0010\rangle + 0.5|0100\rangle\\ + \sqrt{0.5}|1000\rangle.\end{aligned} \tag{39}$$

Here we choose $\rho_{A_1A_2A_3}$ as the reduced density matrix of $|\psi\rangle_{A_1A_2A_3A_4}$ and $P = \{P_1, P_2, P_3\}$ with $P_1 = A_1, P_2 = A_2, P_3 = A_3$. First, we have

$$\rho_{A_1A_2A_3} = 0.09|000\rangle\langle 000| + |\phi\rangle\langle\phi|, \tag{40}$$

where $|\phi\rangle = 0.4|001\rangle + 0.5|010\rangle + \sqrt{0.5}|100\rangle$. Using the calculation method in Ref. [44], we get $\mathcal{C}(\rho_{P_1P_2}) = \frac{\sqrt{2}}{2}$, $\mathcal{C}(\rho_{P_1P_3}) = \frac{2\sqrt{2}}{5}$. Then when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, $\alpha \neq 1$, using Theorem 3, we have

$$\begin{aligned}E_\alpha(\rho_{P_1P_2}) &= E_\alpha^a(\rho_{P_1P_2}) = f_\alpha\left[\left(\frac{\sqrt{2}}{2}\right)^2\right],\\ E_\alpha(\rho_{P_1P_3}) &= E_\alpha^a(\rho_{P_1P_3}) = f_\alpha\left[\left(\frac{2\sqrt{2}}{5}\right)^2\right].\end{aligned} \tag{41}$$

Since Theorem 4 leads to $\sqrt{E_\alpha^2(\rho_{P_1P_2}) + E_\alpha^2(\rho_{P_1P_3})} \leqslant E_\alpha(\rho_{P_1|P_2P_3})$, and Theorem 5 leads to $E_\alpha^a(\rho_{P_1|P_2P_3}) \leqslant E_\alpha^a(\rho_{P_1P_2}) + E_\alpha^a(\rho_{P_1P_3})$, and Theorem 3 ensures the equality $E_\alpha(\rho_{P_1|P_2P_3}) = E_\alpha^a(\rho_{P_1|P_2P_3})$, then we have lower and upper bounds for $E_\alpha(\rho_{P_1|P_2P_3})$; see Fig. 1.

V. GENERALIZED POLYGAMY RELATIONS FOR GENERALIZED W-CLASS STATES USING RéNYI-α ENTROPY

In this section, we develop generalized polygamy relations of multipartite systems in terms of RαEoA. Now we first investigate a generalized polygamy relation which shows the reciprocal relation of RαEoA in an arbitrary three-party quantum system.

Theorem 6. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, P_3\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$; when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, we have the following polygamy inequality:

$$E_\alpha^a(\rho_{P_1|P_2P_3}) \leqslant E_\alpha^a(\rho_{P_2|P_1P_3}) + E_\alpha^a(\rho_{P_3|P_1P_2}). \tag{42}$$

Proof. Assume $\{p_i, |\psi_i\rangle\}$ is the optimal decomposition for RαEoA of $\rho_{P_1|P_2P_3}$ such that $E_\alpha^a(\rho_{P_1|P_2P_3}) = \sum_i p_i E_\alpha(|\psi_i\rangle_{P_1|P_2P_3})$. Let $T(\rho) = 2[1 - \mathrm{Tr}(\rho^2)]$ be the linear entropy which is related to the concurrence [45]. For each pure state $|\psi_i\rangle_{P_1|P_2P_3}$ in this optimal decomposition, one has $\rho^i_{P_2P_3} = \mathrm{Tr}_{P_1}(|\psi_i\rangle_{P_1P_2P_3}\langle\psi_i|)$, $\rho^i_{P_2} = \mathrm{Tr}_{P_1P_3}(|\psi_i\rangle_{P_1P_2P_3}\langle\psi_i|)$, $\rho^i_{P_3} = \mathrm{Tr}_{P_1P_2}(|\psi_i\rangle_{P_1P_2P_3}\langle\psi_i|)$.

When $[\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3}) + \mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2})] \leqslant 1$, we have

$$\begin{aligned}E_\alpha(|\psi_i\rangle_{P_1|P_2P_3}) &= f_\alpha(\mathcal{C}^2(|\psi_i\rangle_{P_1|P_2P_3}))\\ &= f_\alpha\left(T\left(\rho^i_{P_2P_3}\right)\right)\\ &\leqslant f_\alpha\left[T\left(\rho^i_{P_2}\right) + T\left(\rho^i_{P_3}\right)\right]\\ &= f_\alpha[\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3}) + \mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2})]\\ &\leqslant f_\alpha(\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3})) + f_\alpha(\mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2}))\\ &= E_\alpha(|\psi_i\rangle_{P_2|P_1P_3}) + E_\alpha(|\psi_i\rangle_{P_3|P_1P_2}),\end{aligned} \tag{43}$$

where the first equality is from Theorem 1, and the first inequality is due to the subadditivity of the linear entropy [45] and the monotonically increasing property of $f_\alpha(x)$ in Lemma 3. The second inequality is due to inequality (36).

When $[\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3}) + \mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2})] \geqslant 1$, defining $S = [\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3}) + \mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2})] - 1 > 0$, we have

$$\begin{aligned}E_\alpha(|\psi_i\rangle_{P_1|P_2P_3}) &= f_\alpha(\mathcal{C}^2(|\psi_i\rangle_{P_1|P_2P_3}))\\ &\leqslant f_\alpha(1)\\ &= f_\alpha[\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3}) + \mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2}) - S]\\ &\leqslant f_\alpha(\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3})) + f_\alpha[\mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2}) - S]\\ &\leqslant f_\alpha(\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3})) + f_\alpha(\mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2}))\\ &= E_\alpha(|\psi_i\rangle_{P_2|P_1P_3}) + E_\alpha(|\psi_i\rangle_{P_3|P_1P_2}).\end{aligned} \tag{44}$$

Here the second inequality is by using inequality (36) with respect to $\mathcal{C}^2(|\psi_i\rangle_{P_2|P_1P_3})$ and $\mathcal{C}^2(|\psi_i\rangle_{P_3|P_1P_2})) - S$, and the third inequality is due to the increasing property of f_α on the second term.

Therefore, we have

$$\begin{aligned}E_\alpha^a(\rho_{P_1|P_2P_3}) &= \sum_i p_i E_\alpha(|\psi_i\rangle_{P_1|P_2P_3})\\ &\leqslant \sum_i p_i E_\alpha(|\psi_i\rangle_{P_2|P_1P_3})\end{aligned}$$

$$+\sum_i p_i E_\alpha(|\psi_i\rangle_{P_3|P_1P_2})$$
$$\leqslant E_\alpha^a(\rho_{P_2|P_1P_3}) + E_\alpha^a(\rho_{P_3|P_1P_2}), \quad (45)$$

where the first inequality is from inequality (43) and the second inequality is due to the definition of RαEoA. ■

Theorem 6 tells that the sum of two RαEoA's with respect to two possible bipartitions $(P_2 - P_1P_3, P_3 - P_1P_2)$ always bounds the RαEoA with respect to the remaining bipartition $(P_1 - P_2P_3)$. Moreover, iterative use of Eq. (42) leads us to the generalization of Theorem 6.

Corollary 3. Assume $\rho_{A_{j_1}\cdots A_{jm}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, \cdots, P_k\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$, $k \leqslant m < n$; when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, we have the following polygamy inequality:

$$E_\alpha^a(\rho_{P_1|P_2,\cdots,P_k}) \leqslant E_\alpha^a(\rho_{P_2|P_1P_3,\cdots,P_k}) + \cdots$$
$$+ E_\alpha^a(\rho_{P_k|P_1P_2,\cdots,P_{k-1}}). \quad (46)$$

According to the relations between RαE and RαEoA in Theorem 3, we can have another corollary of Theorem 6 for RαE.

Corollary 4. Assume $\rho_{A_{j_1}\cdots A_{jm}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, P_3\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$; when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, we have the following monogamy inequality:

$$E_\alpha(\rho_{P_1|P_2P_3}) \leqslant E_\alpha(\rho_{P_2|P_1P_3}) + E_\alpha(\rho_{P_3|P_1P_2}). \quad (47)$$

Next making use of Theorem 6, we can obtain a polygamy-type upper bound of multiqubit entanglement for RαEoA between the two-party subsystem and the remaining k-party subsystem.

Theorem 7. Assume $\rho_{A_{j_1}\cdots A_{jm}}$ is a reduced density matrix of an n-qudit pure state $|\psi\rangle_{A_1\cdots A_n}$ in (17); here we denote by $\{P_1, P_2, Q_1, Q_2, \cdots, Q_k\}$ a partition of the set $\{A_{j_1}, A_{j_2}, \cdots, A_{j_m}\}$; when $\alpha \in [(\sqrt{7}-1)/2, (\sqrt{13}-1)/2]$, we have

$$E_\alpha^a(\rho_{P_1P_2|Q_1\cdots Q_k}) \leqslant 2E_\alpha^a(\rho_{P_1P_2}) + \sum_{i=1}^k E_\alpha^a(\rho_{P_1Q_i})$$
$$+ \sum_{i=1}^k E_\alpha^a(\rho_{P_2Q_i}). \quad (48)$$

Proof. Consider $P_1P_2Q_1\cdots Q_k$ as a three-party quantum state with $Q = Q_1\cdots Q_k$. Then Theorem 6 leads us to

$$E_\alpha^a(\rho_{P_1P_2|Q}) \leqslant E_\alpha^a(\rho_{P_1|P_2Q}) + E_\alpha^a(\rho_{P_2|P_1Q}). \quad (49)$$

From Theorem 5, we have

$$E_\alpha^a(\rho_{P_1|P_2Q}) \leqslant E_\alpha^a(\rho_{P_2|P_1}) + \sum_{i=1}^k E_\alpha^a(\rho_{P_1Q_i}), \quad (50)$$

and

$$E_\alpha^a(\rho_{P_2|P_1Q}) \leqslant E_\alpha^a(\rho_{P_1|P_2}) + \sum_{i=1}^k E_\alpha^a(\rho_{P_2Q_i}). \quad (51)$$

Combining (49)–(51), we obtain (48). ■

Theorem 7 implies that the entanglement (RαEoA) between the parties P_1P_2 and the other parties cannot be more than the sum of the individual entanglement between P_1 and the $k+1$ remaining parties and the individual entanglement between P_2 and the $k+1$ remaining parties.

For the case when $\xi_{P_1P_2|Q_1\cdots Q_k} = \rho_{P_1|Q_1\cdots Q_k} \otimes |0\rangle_{P_2}\langle 0|$, inequality (48) in Theorem 7 reduced to the polygamy relation (33) in terms of RαEoA in Theorem 5. So that is why we call the result in Theorem 7 a generalized polygamy relation for the GW state using RαEoA.

VI. TIGHTER MONOGAMY RELATIONS FOR GENERALIZED *W*-CLASS STATES

This section gives tighter monogamy relations for GW states in terms of concurrence and negativity. A general monogamy relation is also developed using Rényi-α entropy. First, we prove a pivotal lemma.

Lemma 7. For real numbers $t \in [0, 1]$, $x \geqslant k \geqslant 1$, we have

$$(1+x)^t \geqslant 1 + \left(\frac{(1+k)^t - 1}{k^t}\right)x^t. \quad (52)$$

Proof. Consider the function $f_t(x) = \frac{(1+x)^t-1}{x^t}$, since

$$\frac{df_t(x)}{dx} = tx^{-(t+1)}[1-(1+x)^{t-1}] \geqslant 0$$

for $t \in [0, 1]$ and $x \geqslant 1$.

In other words, the function $f_t(x)$ is an increasing function with $x \geqslant 1$. Since $x \geqslant k \geqslant 1$, then $f_t(x) \geqslant f_t(k)$. ■

Now if we set the partition $\{P_1, P_2, P_3\}$ as a subset of the set $\{A_1, A_2, ..., A_n\}$, then the monogamy relation in Lemma 1 can be rewritten as

$$\mathcal{C}^2_{P_1|P_2P_3} = \mathcal{C}^2_{P_1P_2} + \mathcal{C}^2_{P_1P_3}, \quad (53)$$

$$\mathcal{C}_{P_1P_2} = \mathcal{C}^a_{P_1P_2}. \quad (54)$$

Theorem 8. Assume $|\psi\rangle_{A_1A_2\cdots A_n}$ is an n-qudit generalized W-class state and set the partition $\{P_1, P_2, P_3\}$ as a subset of the set $\{A_1, A_2, ..., A_n\}$; if $\mathcal{C}^\alpha_{P_1P_3} \geqslant k\mathcal{C}^\alpha_{P_1P_2}$, we have

$$\left(\mathcal{C}^a_{P_1|P_2P_3}\right)^\beta \geqslant h\left(\mathcal{C}^a_{P_1P_3}\right)^\beta + \left(\mathcal{C}^a_{P_1P_2}\right)^\beta, \quad (55)$$

with $\beta \in [0, \alpha]$, $\alpha \geqslant 2$, $h = \frac{(1+k)^{\frac{\beta}{\alpha}}-1}{k^{\frac{\beta}{\alpha}}}$, $k \geqslant 1$.

Proof. Since $\mathcal{C}^\alpha_{P_1P_3} \geqslant k\mathcal{C}^\alpha_{P_1P_2}$, we have

$$\begin{aligned}
\left(\mathcal{C}^a_{P_1|P_2P_3}\right)^\beta &= (\mathcal{C}_{P_1|P_2P_3})^\beta \\
&\geqslant \left(\mathcal{C}^\alpha_{P_1P_2} + \mathcal{C}^\alpha_{P_1P_3}\right)^{\frac{\beta}{\alpha}} \\
&= \mathcal{C}^\beta_{P_1P_2}\left(1 + \frac{\mathcal{C}^\alpha_{P_1P_3}}{\mathcal{C}^\alpha_{P_1P_2}}\right)^{\frac{\beta}{\alpha}} \\
&\geqslant \mathcal{C}^\beta_{P_1P_2}\left[1 + \frac{(1+k)^{\frac{\beta}{\alpha}}-1}{k^{\frac{\beta}{\alpha}}}\left(\frac{\mathcal{C}^\alpha_{P_1P_3}}{\mathcal{C}^\alpha_{P_1P_2}}\right)^{\frac{\beta}{\alpha}}\right] \\
&= \mathcal{C}^\beta_{P_1P_2} + \frac{(1+k)^{\frac{\beta}{\alpha}}-1}{k^{\frac{\beta}{\alpha}}}\mathcal{C}^\beta_{P_1P_3} \\
&= \left(\mathcal{C}^a_{P_1P_2}\right)^\beta + \frac{(1+k)^{\frac{\beta}{\alpha}}-1}{k^{\frac{\beta}{\alpha}}}\left(\mathcal{C}^a_{P_1P_3}\right)^\beta. \quad (56)
\end{aligned}$$

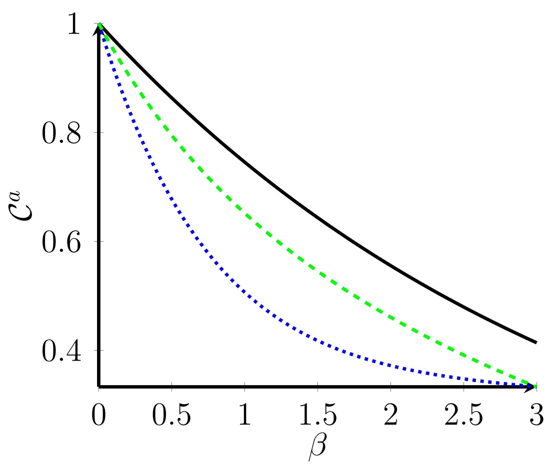

FIG. 2. Concurrence of assistance and lower bounds for three-qubit GW states given in Eq. (57). The solid black line is the function $\mathcal{C}^a_{P_1|P_2P_3}$ of variable β. The dotted blue line is the lower bound of $\mathcal{C}^a_{P_1|P_2P_3}$ obtained from [27]. The dashed green line is the lower bound of $\mathcal{C}^a_{P_1|P_2P_3}$ obtained from our result.

Here the first inequality is due to (53). The second inequality is obtained from Lemma 7 and the last equality is due to (54). ■

Theorem 8 shows a general monogamy inequality for GW states using CoA which is tighter than the result in Ref. [27]. Here we present an example to illustrate this. We consider a three-qubit GW state:

$$|\psi\rangle_{A_1A_2A_3} = \frac{1}{\sqrt{6}}|100\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{2}{\sqrt{6}}|001\rangle. \quad (57)$$

Set the partition $P_1 = A_1$, $P_2 = A_2$, $P_3 = A_3$. After computation, we get $\mathcal{C}(\rho_{P_1|P_2P_3}) = \mathcal{C}^a(\rho_{P_1|P_2P_3}) = \frac{\sqrt{5}}{3}$, $\mathcal{C}(\rho_{P_1P_2}) = \mathcal{C}^a(\rho_{P_1P_2}) = \frac{1}{3}$, $\mathcal{C}(\rho_{P_1P_3}) = \mathcal{C}^a(\rho_{P_1P_3}) = \frac{2}{3}$. Choose $\alpha = 3$, then $1 \leqslant k \leqslant 8$. Assuming $k = 5$, we get the lower bound for $\mathcal{C}^a_{P_1|P_2P_3}$ from the monogamy inequality in Theorem 8. One can also get a lower bound for $\mathcal{C}^a_{P_1|P_2P_3}$ from Ref. [27]. We plot these two lower bounds for $\mathcal{C}^a_{P_1|P_2P_3}$ in Fig. 2. From Fig. 2, we can easily see that our bound is tighter than the one in Ref. [27].

Next we generalize the results to the multipartite GW state in terms of CoA under some restricted conditions.

Theorem 9. Assume $\rho_{P_1\cdots P_m}$ is a reduced density matrix of an n-qudit pure GW state $|\psi\rangle_{A_1\cdots A_n}$; if we have $k\mathcal{C}^\alpha_{P_1P_i} \leqslant \mathcal{C}^\alpha_{P_1|P_{i+1}\cdots P_{m-1}}$ for $i = 2, 3, \cdots, t$, and $\mathcal{C}^\alpha_{P_1P_j} \geqslant k\mathcal{C}^\alpha_{P_1|P_{j+1}\cdots P_m}$ for $j = t+1, \cdots, m-1$, then we have

$$\left(\mathcal{C}^a_{P_1|P_2,\cdots,P_m}\right)^\beta \geqslant \sum_{i=2}^{t} h^{i-2}\left(\mathcal{C}^a_{P_1P_i}\right)^\beta + h^t \sum_{i=t+1}^{m-1}\left(\mathcal{C}^a_{P_1P_i}\right)^\beta + h^{t-1}\left(\mathcal{C}^a_{P_1P_m}\right)^\beta, \quad (58)$$

with $\beta \in [0, \alpha]$, $\alpha \geqslant 2$, $h = \frac{(1+k)^{\frac{\beta}{\alpha}}-1}{k^{\frac{\beta}{\alpha}}}$, $k \geqslant 1$.

Proof. Since $k\mathcal{C}^\alpha_{P_1P_i} \leqslant \mathcal{C}^\alpha_{P_1|P_{i+1}\cdots P_{m-1}}$ for $i = 2, 3, \cdots, n$, then using Theorem 8, we have

$$\begin{aligned}\left(\mathcal{C}^a_{P_1|P_2\ldots P_m}\right)^\beta &\geqslant \left(\mathcal{C}^a_{P_1P_2}\right)^\beta + h\left(\mathcal{C}^a_{P_1|P_3\ldots P_m}\right)^\beta \\ &\geqslant \cdots \\ &\geqslant \sum_{i=2}^{t} h^{i-2}\left(\mathcal{C}^a_{P_1P_i}\right)^\beta + h^{t-1}\left(\mathcal{C}^a_{P_1|P_{t+1}\ldots P_m}\right)^\beta. \end{aligned} \quad (59)$$

Since $\mathcal{C}^\alpha_{P_1P_j} \geqslant k\mathcal{C}^\alpha_{P_1|P_{j+1}\cdots P_m}$ for $j = t+1, \cdots, m-1$, using Theorem 8 again, we get

$$\begin{aligned}\left(\mathcal{C}^a_{P_1|P_{t+1}\ldots P_m}\right)^\beta &\geqslant h\left(\mathcal{C}^a_{P_1P_{t+1}}\right)^\beta + \left(\mathcal{C}^a_{P_1|P_{t+2}\ldots P_m}\right)^\beta \\ &\geqslant \cdots \\ &\geqslant h\sum_{i=t+1}^{m-1}\left(\mathcal{C}^a_{P_1P_i}\right)^\beta + \left(\mathcal{C}^a_{P_1P_m}\right)^\beta. \end{aligned} \quad (60)$$

Combining (59) and (60), we finally obtain

$$\begin{aligned}\left(\mathcal{C}^a_{P_1|P_2\ldots P_m}\right)^\beta &\geqslant \sum_{i=2}^{t} h^{i-2}\left(\mathcal{C}^a_{P_1P_i}\right)^\beta + h^{t-1}\left(\mathcal{C}^a_{P_1|P_{t+1}\ldots P_m}\right)^\beta \\ &\geqslant \sum_{i=2}^{t} h^{i-2}\left(\mathcal{C}^a_{P_1P_i}\right)^\beta + h^t\sum_{i=t+1}^{m-1}\left(\mathcal{C}^a_{P_1P_i}\right)^\beta \\ &\quad + h^{t-1}\left(\mathcal{C}^a_{P_1P_m}\right)^\beta. \end{aligned} \quad (61)$$

■

When $k = 1$, inequality (58) reduces to the inequality (44) in Ref. [27]. Note that Theorems 8 and 9 are presented for the n-qudit GW state. Furthermore, we claim that Theorems 8 and 9 are also valid for the mixed state

$$\begin{aligned}\rho_{A_1\cdots A_n} &= p|W_n^d\rangle_{A_1\cdots A_n}\langle W_n^d| \\ &\quad + (1-p)|0\cdots 0\rangle_{A_1\ldots A_n}\langle 0\cdots 0|. \end{aligned} \quad (62)$$

Since $\rho_{A_1\cdots A_n}$ is an operator of rank 2, we can always have a purification of $\rho_{A_1\cdots A_n}$ such that

$$\begin{aligned}|\psi\rangle_{A_1\cdots A_nA_{n+1}} &= \sqrt{p}|W_n^d\rangle_{A_1\cdots A_n} \otimes |0\rangle_{A_{n+1}} \\ &\quad + \sqrt{1-p}|0\cdots 0\rangle_{A_1\cdots A_n} \otimes |x\rangle_{A_{n+1}}, \end{aligned} \quad (63)$$

with $|x\rangle_{A_{n+1}} = \sum_{1=i}^{d-1} a_{n+1i}|i\rangle_{A_{n+1}}$ a one-qudit quantum state of A_{n+1}; then using (10), Eq. (63) can be rewritten as

$$\begin{aligned}|\psi\rangle_{A_1\cdots A_{n+1}} =& \sum_{i=1}^{d-1}[\sqrt{p}(a_{1i}|i\cdots 00\rangle_{A_1\cdots A_{n+1}} + \cdots \\ &+ a_{ni}|0\cdots i0\rangle)_{A_1\cdots A_{n+1}} \\ &+ \sqrt{1-p}a_{n+1i}|0\cdots 0i\rangle_{A_1\cdots A_{n+1}}]. \end{aligned} \quad (64)$$

One can find that it is an $(n+1)$-qudit GW state.

Moreover, we note that CREN is equivalent to concurrence for any pure state with Schmidt rank 2 [12]. Choi and Kim [25] proved that for an n-qudit pure state $|\psi\rangle_{A_1\cdots A_n}$ in Eq. (17), we have

$$\widetilde{\mathcal{N}}^2(|\psi\rangle_{A_1\cdots A_n}) = \sum_{j=2}^{n}\widetilde{\mathcal{N}}^2(\rho_{A_1A_j}). \quad (65)$$

Later Kim [26] presented that for the reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$, we have

$$\widetilde{\mathcal{N}}^2(\rho_{A_1\cdots A_n}) = \sum_{j=2}^{n} \widetilde{\mathcal{N}}^2(\rho_{A_1A_j}). \tag{66}$$

The inequalities (65) and (66) are still true for the counterexamples violating the CKW inequality using squared concurrence. We can use the similar method in the proof of Corollary 1 to generalize inequality (66) into μth power for $\mu \geqslant 2$:

$$\widetilde{\mathcal{N}}^\mu(\rho_{A_1\cdots A_n}) \geqslant \sum_{j=2}^{n} \widetilde{\mathcal{N}}^\mu(\rho_{A_1A_j}). \tag{67}$$

Therefore, playing a similar trick in Theorems 8 and 9, we can obtain the general monogamy relations for CREN which are tighter than the corresponding results in Ref. [27].

Theorem 10. Assume $\rho_{A_1A_2A_3}$ is the reduced density matrix of a three-qudit pure state $|\psi\rangle_{A_1A_2A_3}$ in a superposition of a three-qudit generalized W-class state and vacuum; if $\widetilde{\mathcal{N}}^\alpha_{A_1A_3} \geqslant k\widetilde{\mathcal{N}}^\alpha_{A_1A_2}$, we have

$$\big(\widetilde{\mathcal{N}}_{A_1|A_2A_3}\big)^\beta \geqslant h\big(\widetilde{\mathcal{N}}_{A_1A_3}\big)^\beta + \big(\widetilde{\mathcal{N}}_{A_1A_2}\big)^\beta, \tag{68}$$

with $\beta \in [0, \alpha]$, $\alpha \geqslant 2$, $h = \frac{(1+k)^{\frac{\beta}{\alpha}}-1}{k^{\frac{\beta}{\alpha}}}$, $k \geqslant 1$.

Proof.

$$\begin{aligned}(\widetilde{\mathcal{N}}_{A_1|A_2A_3})^\beta &\geqslant \big(\widetilde{\mathcal{N}}^\alpha_{A_1A_2} + \widetilde{\mathcal{N}}^\alpha_{A_1A_3}\big)^{\frac{\beta}{\alpha}}\\ &= \widetilde{\mathcal{N}}^\beta_{A_1A_2}\left(1 + \frac{\widetilde{\mathcal{N}}^\alpha_{A_1A_3}}{\widetilde{\mathcal{N}}^\alpha_{A_1A_2}}\right)^{\frac{\beta}{\alpha}}\\ &\geqslant \big(\widetilde{\mathcal{N}}_{A_1A_2}\big)^\beta + h\big(\widetilde{\mathcal{N}}_{A_1A_3}\big)^\beta.\end{aligned} \tag{69}$$

Here the first inequality is due to (67). The second inequality is obtained from Lemma 7 and $\widetilde{\mathcal{N}}^\alpha_{A_1A_3} \geqslant k\widetilde{\mathcal{N}}^\alpha_{A_1A_2}$. ■

Theorem 11. Assume $\rho_{A_1\cdots A_n}$ is the reduced density matrix of an n-qudit pure state $|\psi\rangle_{A_1\cdots A_n}$ in a superposition of an n-qudit generalized W-class state in (10) and vacuum; if $k\widetilde{\mathcal{N}}^\alpha_{A_1A_i} \leqslant \widetilde{\mathcal{N}}^\alpha_{A_1|A_{i+1}\cdots A_n}$ for $i = 2, 3, \cdots, m$, and $\widetilde{\mathcal{N}}^\alpha_{A_1A_j} \geqslant k\widetilde{\mathcal{N}}^\alpha_{A_1|A_{j+1}\cdots A_n}$ for $j = m+1, \cdots, n-1$, then we have

$$\begin{aligned}\big(\widetilde{\mathcal{N}}_{A_1|A_2\ldots A_n}\big)^\beta \geqslant &\sum_{i=2}^{m} h^{i-2}\big(\widetilde{\mathcal{N}}_{A_1A_i}\big)^\beta\\ &+ h^m \sum_{i=m+1}^{n-1} \big(\widetilde{\mathcal{N}}_{A_1A_i}\big)^\beta\\ &+ h^{m-1}\big(\widetilde{\mathcal{N}}^a_{A_1A_n}\big)^\beta,\end{aligned} \tag{70}$$

with $\beta \in [0, \alpha]$, $\alpha \geqslant 2$, $h = \frac{(1+k)^{\frac{\beta}{\alpha}}-1}{k^{\frac{\beta}{\alpha}}}$, $k \geqslant 1$.

Finally, we explore the general monogamy relations for RαE.

Theorem 12. Assume $\rho_{A_{j_1}\cdots A_{j_m}}$ is a reduced density matrix of $|\psi\rangle_{A_1\cdots A_n}$ in a superposition of an n-qudit generalized W-class state in (10) and vacuum, and set the partition $\{P_1, P_2, P_3\}$ as a subset of the set $\{A_{j_1}\cdots A_{j_m}\}$; if $(E_\alpha(\rho_{P_1P_3}))^\mu \geqslant k(E_\alpha(\rho_{P_1P_2}))^\mu$, when $\alpha \geqslant 2$, we have

$$\big(E_\alpha(\rho_{P_1|P_2P_3})\big)^\beta \geqslant h\big(E_\alpha(\rho_{P_1P_3})\big)^\beta + \big(E_\alpha(\rho_{P_1P_2})\big)^\beta, \tag{71}$$

with $\beta \in [0, \mu]$, $\mu \geqslant 2$, $h = \frac{(1+k)^{\frac{\beta}{\mu}}-1}{k^{\frac{\beta}{\mu}}}$, $k \geqslant 1$.

Proof. From Theorem 1, we have

$$\begin{aligned}(E_\alpha(\rho_{P_1|P_2P_3}))^\beta &= \big(f_\alpha\big(\mathcal{C}^2_{P_1|P_2P_3}\big)\big)^\beta\\ &= \big[f_\alpha\big(\mathcal{C}^2_{P_1P_2} + \mathcal{C}^2_{P_1P_3}\big)\big]^\beta\\ &\geqslant \big[f^\mu_\alpha\big(\mathcal{C}^2_{P_1P_2}\big) + f^\mu_\alpha\big(\mathcal{C}^2_{P_1P_3}\big)\big]^{\frac{\beta}{\mu}}\\ &\geqslant \big(f_\alpha\big(\mathcal{C}^2_{P_1P_2}\big)\big)^\beta + h\big(f_\alpha\big(\mathcal{C}^2_{P_1P_3}\big)\big)^\beta\\ &= \big(E_\alpha(\rho_{P_1P_2})\big)^\beta + h(E_\alpha(\rho_{P_1P_3}))^\beta.\end{aligned} \tag{72}$$

Here the first inequality is due to inequality (30) and $(a+b)^x \geqslant a^x + b^x$ for $a \geqslant 0, b \geqslant 0, x \geqslant 2$. The second inequality is obtained from Lemma 7 and the condition $(E_\alpha(\rho_{P_1P_3}))^\mu \geqslant k(E_\alpha(\rho_{P_1P_2}))^\mu$. ■

The iterative use of inequality (71) naturally leads us to the generalization of Theorem 12 into multiparty quantum systems.

Theorem 13. Assume $\rho_{A_{j_1}\cdots A_{jm}}$ is a reduced density matrix of an n-qudit pure state $|\psi\rangle_{A_1\cdots A_n}$ in a superposition of an n-qudit generalized W-class state in (10) and vacuum $|0\cdots 0\rangle_{A_1\cdots A_n}$, and set the partition $\{P_1, P_2, \cdots, P_s\}$ as a subset of the set $\{A_{j_1}\cdots A_{j_m}\}$; if $k(E_\alpha(\rho_{P_1P_i}))^\mu \leqslant (E_\alpha(\rho_{P_1|P_{i+1}\cdots P_s}))^\mu$ for $i = 2, 3, \cdots, t$, and $(E_\alpha(\rho_{P_1P_j}))^\mu \geqslant k(E_\alpha(\rho_{P_1|P_{j+1}\cdots P_m}))^\mu$ for $j = t+1, \cdots, s-1$, when $\alpha \geqslant 2$, then we have

$$\begin{aligned}(E_\alpha(\rho_{P_1|P_2\ldots P_m}))^\beta \geqslant &\sum_{i=2}^{t} h^{i-2}(E_\alpha(\rho_{P_1P_i}))^\beta\\ &+ h^t \sum_{i=t+1}^{s-1} (E_\alpha(\rho_{P_1P_i}))^\beta\\ &+ h^{t-1}(E_\alpha(\rho_{P_1P_m}))^\beta,\end{aligned} \tag{73}$$

with $\beta \in [0, \mu]$, $\mu \geqslant 2$, $h = \frac{(1+k)^{\frac{\beta}{\mu}}-1}{k^{\frac{\beta}{\mu}}}$, $k \geqslant 1$.

VII. APPLICATION

A. Partition-dependent residual entanglement

Monogamy and polygamy relations can offer an insight into multipartite entanglement. The relations we have obtained involve the exploration of both bipartite and multipartite entanglements with respect to all the possible partitions for GW states of the whole system. In this subsection, we begin with deriving some monogamylike inequalities of partition-dependent entanglement for GW states.

We can always design the partition in several steps; we first divide the whole system into two parties, then divide each of them into another two, and next keep dividing in this way until we obtain the desired partition of the whole system or a specific subsystem. Here, for convenience, we first choose a partition $P = \{P_1, P_2\}$ of the set $\{A_1, A_2, \cdots, A_n\}$ for a GW state $|\psi\rangle_{A_1A_2\ldots A_n}$ in Eq. (17) with $P_1 = \{A_1, A_2, \cdots, A_s\}$ and $P_2 = \{A_{s+1}, \cdots, A_n\}$; then we continue dividing P_1 and P_2 into $P_1 = \{P_{11}, P_{12}\}$ and $P_2 = \{P_{21}, P_{22}\}$, in which $P_{11} = \{A_1, A_2, \cdots, A_g\}$, $P_{12} = \{A_{g+1}, \cdots, A_s\}$, $P_{21} = \{A_{s+1}, \cdots, A_h\}$, and $P_{22} = \{A_{h+1}, \cdots, A_n\}$. Now, under this partition, we can have

$$\begin{aligned} E_\alpha^2(\rho_{P_1|P_{21}P_{22}}) &\geqslant E_\alpha^2(\rho_{P_1P_{21}}) + E_\alpha^2(\rho_{P_1P_{22}}) \\ &\geqslant E_\alpha^2(\rho_{P_{11}P_{21}}) + E_\alpha^2(\rho_{P_{12}P_{21}}) \\ &\quad + E_\alpha^2(\rho_{P_{11}P_{22}}) + E_\alpha^2(\rho_{P_{12}P_{22}}) \\ &\geqslant \sum_{i=1}^{s}\sum_{j=s+1}^{n} E_\alpha^2(\rho_{A_iA_j}), \end{aligned} \tag{74}$$

where the first inequality is from Theorem 4 with the partition P_1 and $\{P_{21}, P_{22}\}$, and the second inequality can be obtained from Theorem 4 with the partition $\{P_{11}, P_{12}\}$ and P_{21} (P_{22}). When reducing the partition into a single-qubit subsystem, we get the third inequality from Theorem 4. Here $1 \leqslant g \leqslant s < h \leqslant n$.

Based on the monogamylike inequalities in (74), we then put forward the following PREs in terms of Rényi entropy for GW states:

$$\begin{aligned} \Gamma_{P_{11}P_{12}|P_{21}P_{22}} &= E_\alpha^2(\rho_{P_{11}P_{12}|P_{21}P_{22}}) \\ &\quad - E_\alpha^2(\rho_{P_{11}P_{21}}) - E_\alpha^2(\rho_{P_{12}P_{21}}) \\ &\quad - E_\alpha^2(\rho_{P_{11}P_{22}}) - E_\alpha^2(\rho_{P_{12}P_{22}}) \end{aligned} \tag{75}$$

and

$$\Gamma'_{P_1|P_2} = E_\alpha^2(\rho_{P_1|P_2}) - \sum_{i=1}^{s}\sum_{j=s+1}^{n} E_\alpha^2(\rho_{A_iA_j}). \tag{76}$$

With concrete partitions of the system in term of s, g, and h, PREs can reflect how the global entanglement for GW states can be characterized by partial entanglement using RαE with different formats.

We select an n-qubit W state to show the application of PREs:

$$|\psi\rangle_{A_1A_2\ldots A_n} = \frac{1}{\sqrt{n}}(|10\cdots0\rangle + |01\cdots0\rangle + \ldots + |00\cdots1\rangle). \tag{77}$$

Then, when $\alpha \geqslant (\sqrt{7}-1)/2$, Theorem 1 leads us to

$$\begin{aligned} \Gamma_{P_{11}P_{12}|P_{21}P_{22}} &= f_\alpha^2(\mathcal{C}^2_{P_{11}P_{12}|P_{21}P_{22}}) \\ &\quad - f_\alpha^2(\mathcal{C}^2_{P_{11}P_{21}}) - f_\alpha^2(\mathcal{C}^2_{P_{12}P_{21}}) \\ &\quad - f_\alpha^2(\mathcal{C}^2_{P_{11}P_{22}}) - f_\alpha^2(\mathcal{C}^2_{P_{12}P_{22}}) \end{aligned} \tag{78}$$

and

$$\Gamma'_{P_1|P_2} = f_\alpha^2(\mathcal{C}^2_{P_1|P_2}) - \sum_{i=1}^{s}\sum_{j=s+1}^{n} f_\alpha^2(\mathcal{C}^2_{A_iA_j}). \tag{79}$$

After calculation, we have $\mathcal{C}^2_{P_{11}P_{12}|P_{21}P_{22}} = \frac{4s(n-s)}{n^2}$, and $\mathcal{C}^2_{P_{11}P_{21}} = \frac{1}{n^2}[\sqrt{(n-s)^2 + 4g(h-s)} - (n-s)]^2$, $\mathcal{C}^2_{P_{12}P_{21}} = \frac{1}{n^2}[\sqrt{(n-s)^2 + 4(s-g)(h-s)} - (n-s)]^2$, $\mathcal{C}^2_{P_{11}P_{22}} = \frac{1}{n^2}[\sqrt{(n-s)^2 + 4g(n-h)} - (n-s)]^2$, $\mathcal{C}^2_{P_{12}P_{22}} = \frac{1}{n^2}[\sqrt{(n-s)^2 + 4(s-g)(n-h)} - (n-s)]^2$, and $\mathcal{C}^2_{A_iA_j} = \frac{1}{n^2}[\sqrt{(n-2)^2 + 4} - (n-2)]^2$.

TABLE I. The values of PRE $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ for the different entanglement orders α and all the possible values for g labeled as (s, g, h) when $n = 6$, $s = 4$, $h = 5$.

	(4,1,5)	(4,2,5)	(4,3,5)	(4,4,5)
$\alpha = 0.95$	0.7652	0.7898	0.7652	0.6966
$\alpha = 1.05$	0.7715	0.7927	0.7715	0.7128
$\alpha = 1.15$	0.7720	0.7901	0.7720	0.7217
$\alpha = 1.25$	0.7686	0.7841	0.7686	0.7253

As shown in Eqs. (78) and (79), $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ and $\Gamma'_{P_1|P_2}$ are related to the bipartition in terms of s, while the former relies on the further partitions in terms of g and h. Choose α as 0.95, 1.05, 1.15, and 1.25 randomly, and set $n = 6$ and $s = 4$. Since $1 \leqslant g \leqslant s$ and $s < h \leqslant n$, then g can be the value of 1, 2, 3, and 4 and h equals 5 or 6. We calculate the first indicator for the six-qubit W state with all the possible values of g and h which are labeled as (s, g, h) in Tables I and II. When $h = 5$ in Table I, PRE $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ has the same value for (4,1,5) and (4,3,5), and the maximum value is at (4,2,5) while the minimum is at (4,4,5). When $h = 6$ in Table II, PRE $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ has the same value for (4,1,6) and (4,3,6), and the maximum value is at (4,2,6) while the minimum is at (4,4,6). In other words, the value of PRE $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ for partition $A_1|A_2A_3A_4|A_5|A_6$ ($A_1|A_2A_3A_4|A_5A_6$) equals to the value for partition $A_1A_2A_3|A_4|A_5|A_6$ ($A_1A_2A_3|A_4|A_5A_6$) for the six-qubit W state in (77). Comparing Tables I and II, the values when $h = 5$ in Table I are all bigger than the corresponding ones when $h = 6$ in Table II, and the values at (4,4,5) and (4,2,6) are equal. It is also interesting to find that PRE $\Gamma'_{P_1|P_2}$ has the same value when $s = 1$ and 5 ($s = 2$ and 4) in Table III. In fact, the values of PREs in Tables I– III are not only related to the values of s, g, h, and n, but also the product of the difference value between s, g, h, and n, such as $s(n-s)$, $g(h-s)$, $(s-g)(h-s)$, $g(n-h)$, and $(s-g)(n-h)$. These significant results are mainly due to the special structure of the six-qubit W state in (77).

In order to explore the relationship between PREs and α, we also plot the function PREs of variable α when s, g, and h are fixed for $\alpha \geqslant (\sqrt{7}-1)/2$ in Figs. 3 and 4. In Fig. 3, the values of PRE $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ tend to increase first and then decrease. Figure 3(a) [Fig. 3(b)] corresponds to the case in Table I (Table II), which gives very good agreement with the analysis for Table I (Table II). In Fig. 4, the values of PRE $\Gamma'_{P_1|P_2}$ tend to decrease when $s = 1, 2, 4, 5$. When $s = 3$, the function of PRE $\Gamma'_{P_1|P_2}$ monotonically increases and ap-

TABLE II. The values of PRE $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ for the different entanglement orders α and all the possible values of g labeled as (s, g, h) when $n = 6$, $s = 4$, $h = 6$.

	(4,1,6)	(4,2,6)	(4,3,6)	(4,4,6)
$\alpha = 0.95$	0.6543	0.6966	0.6543	0.5309
$\alpha = 1.05$	0.6704	0.7128	0.6704	0.5494
$\alpha = 1.15$	0.6804	0.7217	0.6804	0.5638
$\alpha = 1.25$	0.6859	0.7253	0.6859	0.5745

TABLE III. The values of PRE $\Gamma'_{P_1|P_2}$ for the different entanglement orders α and all the possible values for s when $n = 6$.

	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
$\alpha = 0.95$	0.4380	0.8485	0.9981	0.8485	0.4380
$\alpha = 1.05$	0.4051	0.8352	0.9989	0.8352	0.4051
$\alpha = 1.15$	0.3753	0.8218	0.9993	0.8218	0.3753
$\alpha = 1.25$	0.3486	0.8085	0.9996	0.8085	0.3486

proaches to 1. The reason is that the first term of $\Gamma'_{P_1|P_2}$ equals 1 when $n = 6$ and $s = 3$, which leads to a difference. These results in Fig. 4 fit with the ones in Table III.

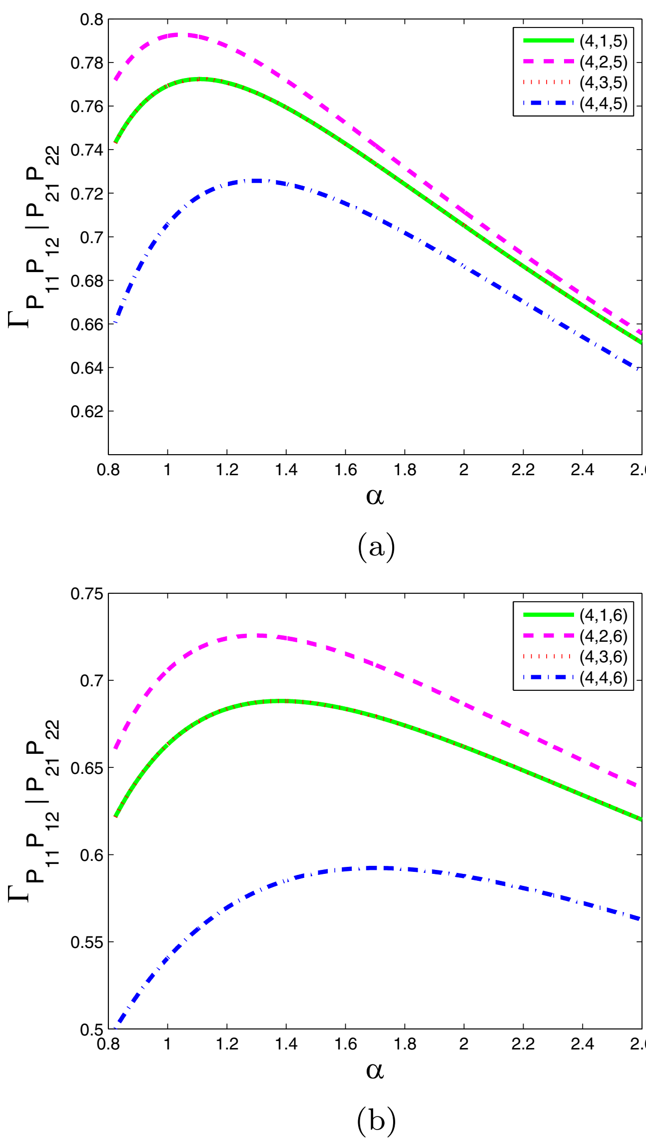

FIG. 3. The function PRE $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ of variable α is shown for $\alpha \geqslant (\sqrt{7} - 1)/2$. $\Gamma_{P_{11}P_{12}|P_{21}P_{22}}$ gets the maximum (minimum) value when $g = 2$ ($g = 4$) for some fixed α. (a) $n = 6$, $s = 4$, $h = 5$ and $1 \leqslant g \leqslant s$. The curves coincide at (4; 1; 5) and (4; 3; 5); (b) $n = 6$, $s = 4$, $h = 6$ and $1 \leqslant g \leqslant s$. The curves coincide at (4; 1; 6) and (4; 3; 6).

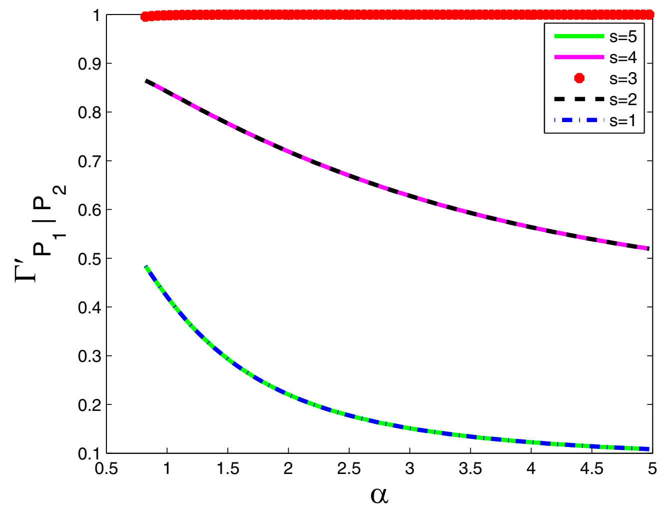

FIG. 4. The function PRE $\Gamma'_{P_1|P_2}$ of variable α for $\alpha \geqslant (\sqrt{7} - 1)/2$ when $n = 6$ and $1 \leqslant s < n$. The curves coincide at $s = 1 (s = 2)$ and $s = 5 (s = 4)$. The value of $\Gamma'_{P_1|P_2}$ at $s = 3$ gets the maximum value, which is close to 1.

Therefore, we have done a comprehensive analysis of the whole system using the PREs to get a full understanding of entanglement dynamics for the six-qubit W state in (77). By virtue of PREs, we can develop a possible comprehensive analysis of the entanglement dynamics in an infinite or finite time for future study.

B. Quantum game

A referee and two isolated players, Alice and Bob, are playing a quantum game $G = (A, B, X, Y, \pi, v)$, in which two players only communicate with the referee and not between themselves [37]. A, B, X, Y are finite sets. π is a probability distribution: $X \times Y \longrightarrow [0, 1]$. v is a verification function: $X \times Y \times A \times B \longrightarrow [0, 1]$. Based on some probability distribution π, the referee chooses a question pair (x, y) on the question alphabets $X \times Y$. Then he sends x to Alice and y to Bob. Later the two players give their answers a and b from the set A and set B. If $v(x, y, a, b) = 1$ for the verification function, then they win. The classical value of the game

$$cv(G) = \sup_{a_x, b_y} \sum_{x,y,a,b} \pi(x, y)v(a, b, x, y) \int_{\Omega} a_x(\omega)b_y(\omega)d\mathbb{P}(\omega)$$

is the maximum winning probability when two players can use optimal deterministic strategies $\sum_a a_x(\omega) = \sum_b b_y(\omega) = 1$ based on some classical correlation $\mathbb{P}(\omega)$. The quantum value for a bipartite entangled state ρ_{AB} of the game is

$$qv(G) = \sup_{\rho, E_x^a, F_y^b} \sum_{x,y,a,b} \pi(x, y)v(a, b, x, y)\mathrm{tr}\big(\rho E_x^a \otimes F_y^b\big)$$

where the maximum takes over all the positive operator-valued measures (POVMs) E_x^a and F_y^b, $\sum_a E_x^a = 1$, $\sum_b F_y^b = 1$. It is clear that for all games $cv(G) \leqslant qv(G)$.

In Ref. [38], the authors assume that Alice has a d-dimensional system A. She can simultaneously share quantum or classical correlation with an arbitrary number of players $B_1, B_2, ..., B_n$. So the referee chooses a player B_i randomly and plays the game $G_i = (A, B_i, X_i, Y_i, \pi_i, v_i)$ with Alice and B_i. For $\{G_i\}_{1\leqslant i\leqslant n}$, they define the average entangled value:

$$Aqv(\{G_i\}) = \sup_{\rho,E_x^a,F_{1,y}^b,\cdots,F_{n,y}^b} \frac{1}{n}\sum_{i=1}^{n}\sum_{a,b,x,y}\pi_i(x,y)\, v_i(a,b,x,y)\mathrm{tr}\big(\rho^{AB_i}E_x^a\otimes F_{i,y}^b\big), \tag{80}$$

here $E_x^a, F_{1,y}^b, \cdots, F_{n,y}^b$ are POVMs on $A, B_1, \cdots, B_n$, respectively. $\rho^{AB_1\cdots B_n}$ is a multipartite state and the dimension of A is at most d. Since the classical correlation used for different G_i can be combined, then the average classical value was given by

$$Acv(\{G_i\}) = \frac{1}{n}\sum_{i=1}^{n} cv(G_i). \tag{81}$$

In Ref. [27], the authors reconsider the bound of the difference between the quantum games and the classical games restricting to GW states using Tsallis-q entropy for $q\in(1,2]$. Here we get a new bound of the difference between the quantum games and the classical games restricting to GW states using Rényi-α entropy for $\alpha\geqslant 1$. Let $G = (A, B, X, Y, \pi, v)$ be a quantum game. For fixed auxiliary systems A and B and POVMs E_x^a and F_y^b, the value function becomes a positive linear function:

$$\mathrm{lin}_G(\rho_{AB}) = \sum_{x,y,a,b}\pi(x,y)v(a,b,x,y)\mathrm{tr}\big(\rho E_x^a\otimes F_y^b\big).$$

Note that lin_G is of norm at most 1; then, for a separable σ_{AB} and an arbitrary ρ_{AB},

$$\begin{aligned}\mathrm{lin}_G(\rho_{AB}) &\leqslant \mathrm{lin}_G(\rho_{AB}-\sigma_{AB}) + \mathrm{lin}_G(\sigma_{AB})\\ &\leqslant \|\rho_{AB}-\sigma_{AB}\|_1 + cv(G).\end{aligned} \tag{82}$$

For a bipartite pure state $|\psi\rangle_{AB} = \sum_{i=0}^{d-1}\sqrt{\lambda_i}|ii\rangle$, we show there exists a separable state σ_{AB} such that

$$\||\psi\rangle_{AB}\langle\psi| - \sigma_{AB}\|_1 \leqslant 2\sqrt{2E_\alpha(\rho_{AB})}. \tag{83}$$

for $\alpha\geqslant 1$. First we select $\sigma_{AB} = |00\rangle\langle 00|$; then we compute the trace norm $\||\psi\rangle_{AB}\langle\psi| - |00\rangle\langle 00|\|_1 = 2\sqrt{1-\lambda_0}$. So according to (83) we need to show $2\sqrt{1-\lambda_0}\leqslant 2\sqrt{\frac{-2\log_2\sum_{i=0}^{d-1}\lambda_i^\alpha}{\alpha-1}}$ in the following.

When $\alpha\geqslant 1$, $\lambda\in[0,1]$, one has $\sum_{i=0}^{d-1}\lambda_i^\alpha \geqslant \lambda_0^\alpha + (1-\lambda_0)^\alpha$; then it is enough for us to prove $2\sqrt{1-\lambda_0}\leqslant 2\sqrt{\frac{-2\log_2[\lambda_0^\alpha+(1-\lambda_0)^\alpha]}{\alpha-1}}$.

Let $h_\alpha(\lambda_0) = -2\log_2[\lambda_0^\alpha + (1-\lambda_0)^\alpha] - (1-\lambda_0)(\alpha-1)$; we need to show $h_\alpha(\lambda_0)\geqslant 0$. Since

$$h'_\alpha(\lambda_0) = \frac{-2\alpha\lambda_0^{\alpha-1} + 2\alpha(1-\lambda_0)^{\alpha-1}}{[\lambda_0^\alpha + (1-\lambda_0)^\alpha]\ln 2} + (\alpha-1), \tag{84}$$

and after analysis, we find $h'_\alpha(\lambda_0)$ in (84) has only one zero $\epsilon\in[0,1]$. $h_\alpha(\lambda_0)$ is monotonically increasing for $\lambda_0\in[0,\epsilon]$ and monotonically decreasing for $\lambda_0\in[\epsilon,1]$. However, $\lambda_0\geqslant\frac{1}{d}$, so it is enough to show $h_\alpha(0)\geqslant 0$ and $h_\alpha(\frac{1}{d})\geqslant 0$. $h_\alpha(0)\geqslant 0$ is clear. After computation, we obtain $h_\alpha(\frac{1}{d}) = -\log_2[1+(d-1)^\alpha] + \alpha\log_2 d - \frac{(d-1)(\alpha-1)}{d}$. One can easily get $h_\alpha(\frac{1}{d})\geqslant 0$ for any pure state with Schmidt rank equal to or less than 2 when $\alpha\geqslant 1$.

When ρ is a mixed state, assume $\{p_i, |\psi_i\rangle\}$ is the optimal decomposition of ρ in terms of $E_\alpha(\rho_{AB})$; then

$$\begin{aligned}\|\rho_{AB}-\sigma_{AB}\|_1 &= \|\sum_i p_i|\psi_i\rangle_{AB}\langle\psi_i| - \sum_i p_i|\theta_i\rangle_{AB}\langle\theta_i|\|_1\\ &\leqslant \sum_i p_i\||\psi_i\rangle_{AB}\langle\psi_i| - |\theta_i\rangle_{AB}\langle\theta_i|\|_1\\ &\leqslant 2\sqrt{2}\sum_i\sqrt{p_i}\sqrt{p_iE_\alpha(|\psi_i\rangle_{AB})}\\ &\leqslant 2\sqrt{2}\sqrt{E_\alpha(\rho_{AB})}.\end{aligned} \tag{85}$$

Here we use the subadditivity of the one-norm. The second inequality is due to (83) and the last inequality is due to the definition of RαEoA and Theorem 3.

Using the monogamy inequality in Theorem 4, we have

$$\begin{aligned}\sum_{i=1}^{n} E_\alpha^2(\rho_{AB_i}) &\leqslant E_\alpha^2(\rho_{A|B_1\ldots B_n})\\ &\leqslant \left(\frac{\log_2 d^{1-\alpha}}{1-\alpha}\right)^2 = (\log_2 d)^2.\end{aligned} \tag{86}$$

Combining (82) and (85), we find that

$$\begin{aligned}Aqv(G) &\leqslant \frac{2\sqrt{2}}{n}\sum_{i=1}^{n}\sqrt{E_\alpha(\rho_{A|B_1\ldots B_n})} + Acv(G)\\ &\leqslant \frac{2\sqrt{2}}{n^{\frac{1}{4}}}\sqrt{E_\alpha(\rho_{AB_i})} + Acv(G)\\ &\leqslant \frac{2\sqrt{2}}{n^{\frac{1}{4}}}(\log_2 d)^{\frac{1}{2}} + Acv(G).\end{aligned} \tag{87}$$

Here the second inequality comes from Hölder's inequality. The last inequality is due to (86).

Finally, we get a bound of the difference between the quantum games and the classical games using Rényi-α entropy:

$$Aqv(G) - Acv(G) \leqslant \frac{2\sqrt{2}}{n^{\frac{1}{4}}}(\log_2 d)^{\frac{1}{2}}. \tag{88}$$

It is interesting that this bound is independent of α. In Ref. [27], the bound obtained by Tsallis-q entropy is $Aqv(G) - Acv(G)\leqslant\frac{2\sqrt{2}}{n^{\frac{1}{4}}}\frac{1}{\sqrt{q-1}}$. When $q=2$, this bound is the same as our bound for $d=2$. Compared with the result in Ref. [38], $Aqv(G) - Acv(G)\leqslant\frac{3.1}{n^{\frac{1}{4}}}d(\log_2 d)^{\frac{1}{4}}$, our bound is tighter due to $d\geqslant(\log_2 d)^{\frac{1}{4}}$.

VIII. DISCUSSION AND CONCLUSION

By using Rényi-α entropy, we have provided a class of monogamy and polygamy inequalities of multipartite entanglement for GW states with respect to different partitions. At the beginning, we have shown analytical formulas of RαE and RαEoA for a reduced density matrix of GW states when $\alpha\geqslant(\sqrt{7}-1)/2$ and $\alpha\in[(\sqrt{7}-1)/2,(\sqrt{13}-1)/2]$, respectively. According to the two analytical formulas, we have obtained monogamy and polygamy inequalities. We further generalize them into μth power with a specific range of μ. Moreover, we have shown generalized polygamy relations for

GW states using RαEoA, which shows a reciprocal relation of RαEoA. Last but not the least, we provide tighter monogamy relations in terms of concurrence and CREN than existing ones and also obtain general monogamy relations for GW states using RαE.

To show the application of our main results, we first establish two partition-independent residual entanglements using our main results and get a full cognition of entanglement dynamics for GW states. Our defined PREs can have unique usefulness when one deals with the entanglement-changing process in arbitrary partitions of multiqubit systems. Just to make it more interesting, we consider a quantum game to show the application of our results. It is important to find that the bound of the difference between the quantum games and the classical games using Rényi-α entropy for $\alpha \geqslant 1$ is independent of α. When $d = 2$, our result is the same as the result obtained by Tsallis-q entropy for $q = 2$ in Ref. [27]. Our bound is also tighter than the result in Ref. [38].

Our paper is similar but actually different from the results in Ref. [27] which considered the monogamy relations for GW states using Tsallis-q entropy. Tsallis-q entropy is a generalization of von Neumann entropy, and it is relevant to the study of separability of compound quantum systems [46] and global quantum discord [47] while Rényi-α entropy is a generalization of Shannon entropy with applications in the study of nonlinear properties of quantum states [48] and quantum correlations in fermionic systems [49]. We can use PREs to explore more in the applications mentioned above of Rényi-α entropy for future study. Additionally, the two entanglement measures also have different properties. For example, Rényi-α entropy for $\alpha \neq 1$ is not in general subadditive [4] while Tsallis-q entropy is subadditive for $q \geqslant 1$ [50]. This leads to the methods of the proofs in Sec. V being quite different from the ones in Ref. [27]. Furthermore, we obtain tighter monogamy relations than the ones in Ref. [27] which can imply finer characterizations of entanglement distributions. Therefore, we believe our results can be useful for future research in higher-dimensional multipartite quantum systems.

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