Open quantum systems integrable by partial commutativity

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This article provides a framework to solve linear differential equations based on partial commutativity, which is introduced by means of the Fedorov theorem. The framework is applied to specific types of three-level and four-level quantum systems. The efficiency of the method is evaluated and discussed. The Fedorov theorem appears to answer the need for methods that allow us to study dynamical maps corresponding with time-dependent generators. By applying this method, one can investigate countless examples of dissipative systems such that the relaxation rates depend on time.

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I. INTRODUCTION

The problem of solving a differential equation belongs to most fundamental issues in the theory of open quantum systems. The ability to obtain a solution in closed form means that one can determine the trajectory of the system, which provides a complete characterization of how the quantum state changes in time. However, only particular types of differential equations allow solutions in closed forms. Additionally, a universal criterion for integrability does not exist. Therefore, there is a need for additional methods that can be applied to investigate different types of equations. In this article, we propose to implement the Fedorov theorem in the theory of open quantum systems.

The simplest dynamical map, which does not need any further comment at this point, can be obtained when the time-evolution is given by a master equation with the GKSL generator $\mathbb{L} : \mathbb{M}_N(\mathbb{C}) \rightarrow \mathbb{M}_N(\mathbb{C})$, where we assume that the space is finite-dimensional [1–3]. In such a case, the density matrix at any time instant can be computed by the semigroup

$$\rho(t) = \exp\left(\mathbb{L}t\right)[\rho(0)], \tag{1}$$

where $\rho(0)$ stands for the initial density matrix. A master equation governed by the GKSL generator is the most general type of Markovian and time-homogeneous evolution which preserves trace and positivity.

The closed-form solution of a master equation can be obtained straightforwardly as long as the generator is time-independent. The problem appears when the dynamics is governed by a master equation with a time-dependent linear generator:

$$\frac{d\rho(t)}{dt} = \mathbb{L}(t)[\rho(t)], \tag{2}$$

where the generator $\mathbb{L}(t)$ is defined on a time interval $\mathcal{I}$.

In 1949, Dyson published an article [4] in which he presented the formal solution of an explicitly time-dependent Schrödinger equation. The result was obtained by iteration and a time-ordering operator, which was later named the "Dyson series" after the author. Thus, the formal solution of Eq. (2) can be written by means of a superoperator $\Phi(t)$:

$$\rho(t) = \Phi(t)[\rho(0)] = \mathrm{T}\exp\left(\int_0^t \mathbb{L}(\tau)d\tau\right)[\rho(0)], \tag{3}$$

where T denotes the chronological product. The formula for the map $\Phi(t)$ can be expanded by applying the Dyson series [4]:

$$\Phi(t) = \mathbb{1}_N + \int_0^t dt_1 \mathbb{L}(t_1) + \int_0^t dt_1 \int_0^{t_1} dt_2 \mathbb{L}(t_1)\mathbb{L}(t_2) + \cdots, \tag{4}$$

provided it converges. One fundamental problem studied in the theory of open quantum systems relates to algebraic properties of $\mathbb{L}(t)$ which guarantee that the solution $\Phi(t)$ constitutes a legitimate dynamical map; see, e.g., Ref. [5]. Undoubtedly, such a question is relevant, but in this article we focus on the methods that provide solutions to time-dependent master equations of the form Eq. (2) without the necessity to utilize the infinite Dyson series.

In Sec. II, we revise the definitions and theorems connected with functionally commutative generators. Then, in Sec. III, we present the Fedorov theorem, which can be understood as a generalization of the Lappo-Danilevsky criterion. Along with the theorem, we propose a feasible framework for its application in concrete examples. Then, in Sec. IV, the framework is tested as we apply the Fedorov theorem to three-level and four-level open quantum systems with evolution governed by time-local generators. We study three particular types of three-level dynamics: *V-system*, *cascade*, and *Lambda*, as well as one example on four-level *cascade* systems, in order to prove that this technique can facilitate solving master equations with time-dependent generators.

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II. FUNCTIONAL AND INTEGRAL COMMUTATIVITY

To follow the trajectory of the systems, it is desirable to be able to write the solution of Eq. (2) in closed form:

$$\rho(t) = \exp\left(\int_0^t \mathbb{L}(\tau)d\tau\right)[\rho(0)], \tag{5}$$

which can be done only for specific generators $\mathbb{L}(t)$.

First, we shall analyze the sufficient conditions that, if satisfied by the generator $\mathbb{L}(t)$, guarantee that the solution can be written in closed form. We shall refer to the algebraic properties of the matrix representation of the generator $\mathbb{L}(t)$.

To begin with, let us recall a definition, assuming that $\mathbf{F}(t)$ stands for a matrix function and $\mathcal{I}$ denotes an interval within its domain.

Definition 1: Semiproper matrix function. A matrix function $\mathbf{F} : \mathcal{I} \to \mathbb{C}^{n\times n}$ is called *semiproper* on $\mathcal{I}$ if

$$\mathbf{F}(t)\mathbf{F}(\tau) = \mathbf{F}(\tau)\mathbf{F}(t) \ \ \forall\, t, \tau \in \mathcal{I}. \tag{6}$$

The definition of the semiproper function can be applied to time-dependent generators of evolution, which are a specific kind of complex-valued time-dependent function matrices. In other words, this property is called functional commutativity.

Definition 2: Functional commutativity. A time-dependent generator $\mathbb{L}(t)$ is functionally commutative (i.e., semiproper) iff

$$[\mathbb{L}(t), \mathbb{L}(s)] = 0 \ \ \forall\, t, s \in \mathcal{I}. \tag{7}$$

The notion of functional commutativity applied to generators of evolution allows one to formulate a theorem concerning the solvability of the dynamics Eq. (2) [6,7].

Theorem 1. If the generator of evolution $\mathbb{L}(t)$ satisfies the condition of functional commutativity Eq. (7), then the solution of Eq. (2) can be written in closed form according to Eq. (5).

The idea of semiproper matrix functions has received much attention in the second half of the 20th century. One noteworthy article was written by Martin in 1967 [8]. In one of the theorems, the author proved that the family of semiproper matrix functions can be completely characterized as commutative algebras generated by a basis of pairwise commutative constant matrices. Based on this result, we can say that $\mathbb{L}(t)$ is functionally commutative on $\mathcal{I}$ iff there exists a set of mutually commuting time-independent matrices $\{\mathbb{L}^{(k)}\}$ and piecewise continuous scalar functions $\{\alpha_k(t)\}$ such that

$$\mathbb{L}(t) = \sum_k \alpha_k(t)\mathbb{L}^{(k)}. \tag{8}$$

The decomposition of the generator of evolution Eq. (8) not only allows one to write the closed-form solution of Eq. (2), but it also simplifies the computing of the integral over time. However, finding such a decomposition of $\mathbb{L}(t)$ remains a challenge [9]. For this reason, Zhu proposed a different method to decompose a functionally commutative generator (called the spatial decomposition) [10], which was later developed by Kamizawa and applied to open quantum systems [11].

Another approach to the problem of solving the evolution equation of the form Eq. (2) is to apply to the notion of commutativity with the integral. It is another condition that, if satisfied by the generator $\mathbb{L}(t)$, implies that the solution of the evolution equation is given in closed form. Let us recall the definition.

Definition 3: Integral commutativity. A time-dependent generator $\mathbb{L}(t)$ is said to commute with its integral iff:

$$\mathbb{L}(t)\int \mathbb{L}(t)dt = \int \mathbb{L}(t)dt\ \mathbb{L}(t) \iff \left[\mathbb{L}(t), \int \mathbb{L}(t)dt\right] = 0. \tag{9}$$

A thorough study of time-dependent matrices that commute with their integrals was published by Bogdanov and Cheboratev in 1959 [12]. Necessary and sufficient conditions for integral commutativity can be given in relation to the properties of the Jordan canonical form of $\mathbb{L}(t)$ [13,14]. Based on the notion of integral commutativity, one can formulate a theorem concerning the solvability of evolution equations [15].

Theorem 2. If the generator of evolution $\mathbb{L}(t)$ satisfies the condition of integral commutativity Eq. (9), then the fundamental solution of Eq. (2) has the closed form Eq. (5).

It is worth noting that $\mathbb{L}(t)$ is said to be analytic in a neighborhood of $t = t_0$ when each element of $\mathbb{L}(t)$ [and thus $\mathbb{L}(t)$ itself] can be represented as a Taylor series centered at t_0 which converges in some neighborhood of t_0. If the time-dependent generator $\mathbb{L}(t)$ is an analytic complex-valued matrix function, then $\mathbb{L}(t)$ satisfies the condition of functional commutativity if and only if it commutes with its integral, which means that in such a case both criteria are compatible [16].

Theorem 2 could be equivalently formulated in terms of the generator that commutes with its derivative, which is a common way to express and study this criterion; see, e.g., Refs. [17–19]. Nonetheless, for the sake of the content of this article, we stay with the notion of integral commutativity, originally introduced by Lappo-Danilevsky, which is a starting point for further analysis.

III. PARTIAL COMMUTATIVITY AND A FRAMEWORK FOR ITS APPLICATION

Either functional or integral commutativity is sufficient to write the solution of Eq. (2) in closed form according to Eq. (5). However, these conditions are not necessary. It may happen that a generator of evolution satisfies neither of the two conditions, but one is still able to write the solution of the dynamics equation in closed form. To be more specific, in this article we shall investigate the Fedorov theorem, which demonstrates that a closed-form solution can be obtained under the condition of partial commutativity [20] (for English, refer to pp. 39–44 in [6]).

First, one should be reminded that every time-dependent linear generator $\mathbb{L}(t)$ can always be represented as a matrix, which makes it possible to study the algebraic properties of the generator. On the other hand, the evolution equation given by Eq. (2) can always be transformed into a differential equation with the generator $\mathbb{L}(t)$ in its matrix form multiplying the vectorized density matrix $\mathrm{vec}\{\rho(t)\}$, which is simpler from the computational point of view. The operator $\mathrm{vec}\{\rho(t)\}$ should be understood as a vector constructed by stacking the columns of $\rho(t)$ one underneath the other, and such an operation shall be referred to as the "vec operator." Thus, let us consider the master equation in the vectorized form, i.e.,

$$\text{vec}\{\dot{\rho}(t)\} = \mathbb{L}(t)\ \text{vec}\{\rho(t)\}, \tag{10}$$

and for such dynamics we shall formulate the Fedorov theorem [20].

Theorem 3: Fedorov theorem. If the generator of evolution $\mathbb{L}(t)$ (its matrix representation) satisfies the condition

$$[\,\mathbb{L}(t), B^n(t)\,]\,\alpha = 0 \quad \forall\, n = 1, 2, 3, \dots, \tag{11}$$

where $B(t) = \int_0^t \mathbb{L}(\tau)d\tau$ and α is a constant vector, then the solution of Eq. (10) can be written in closed form:

$$\text{vec}\{\rho(t)\} = \exp[B(t)]\,\alpha. \tag{12}$$

Proof. There exists an obvious decomposition of $\exp[B(t)]$, i.e.,

$$\exp[B(t)] = \sum_{m=0}^{\infty} \frac{1}{m!} B^m(t), \tag{13}$$

which allows one to write a formula for the first derivative of $\exp[B(t)]$:

$$\begin{aligned}\frac{d \exp[B(t)]}{dt} &\\ = \mathbb{L}(t) &+ \frac{1}{2!}\{\mathbb{L}(t)B(t) + B(t)\mathbb{L}(t)\}\\ &+ \frac{1}{3!}\left\{\mathbb{L}(t)B^2(t) + B(t)\mathbb{L}(t)B(t) + B^2(t)\mathbb{L}(t)\right\} + \cdots\\ = \sum_{m=1}^{\infty} &\frac{1}{m!}\sum_{k=1}^{m} B^{k-1}(t)\mathbb{L}(t)B^{m-k}(t).\end{aligned} \tag{14}$$

On the other hand, one can notice that the assumption [see Eq. (11)] can be transformed in the following way (for any $m, n \in \mathbb{N}$):

$$\begin{aligned}\mathbb{L}(t)B^n(t)\,\alpha = B^n(t)\mathbb{L}(t)\,\alpha \quad &\Leftrightarrow\\ B^m(t)\mathbb{L}(t)B^n(t)\,\alpha = B^{n+m}(t)\mathbb{L}(t)\,\alpha.&\end{aligned} \tag{15}$$

Keeping in mind Eqs. (14) and (15), one can check whether $\text{vec}\{\rho(t)\} = \exp[B(t)]\alpha$ satisfies the evolution equation given by Eq. (10):

$$\begin{aligned}\frac{d\ \text{vec}\{\rho(t)\}}{dt} &= \frac{d\ \exp[B(t)]\,\alpha}{dt}\\ &= \sum_{m=1}^{\infty}\frac{1}{m!}\sum_{k=1}^{m} B^{k-1}(t)\mathbb{L}(t)B^{m-k}(t)\\ &= \mathbb{L}(t)\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{k=1}^{m} B^{m-1}(t)\,\alpha = \mathbb{L}(t)\sum_{m=1}^{\infty}\frac{1}{m!} m B^{m-1}(t)\,\alpha\\ &= \mathbb{L}(t)\sum_{m=1}^{\infty}\frac{1}{(m-1)!} B^{m-1}(t)\,\alpha = \mathbb{L}(t)\sum_{m=0}^{\infty}\frac{1}{m!} B^{m}(t)\,\alpha\\ &= \mathbb{L}(t)\ \exp[B(t)]\,\alpha = \mathbb{L}(t)\ \text{vec}\{\rho(t)\}.\end{aligned} \tag{16}$$

It means that $\text{vec}\{\rho(t)\}$ defined by the formula Eq. (12) satisfies the dynamics given by Eq. (10), which completes the proof.

There are three issues that one should be aware of in connection with the Fedorov theorem.

First, the Fedorov theorem enables us to write the solution of the evolution equation in closed form. However, there is a significant limitation—as the initial vectors one can use only the vector (or vectors) α which satisfy the condition of partial commutativity introduced by Eq. (11). Naturally, if one has two linearly independent vectors α_1 and α_2 and both of them satisfy Eq. (11), then the linear combination of them, $c_1\alpha_1 + c_2\alpha_2$, also satisfies the condition from the Fedorov theorem. Therefore, all vectors α that satisfy Eq. (11) constitute a subspace in the vector space. The subspace that contains all vectors α shall be denoted by $\mathcal{M}(\mathbb{L}(t))$.

Second, from a physical point of view, it is important to be able to determine the trajectory of the state on the basis of the solution of the evolution equation. However, it may happen that when one determines α satisfying the condition Eq. (11) for a specific generator of evolution, it turns out that after devectorization α is not a proper density matrix. In such a case, the solution with α as the initial vector is not a legitimate state trajectory. For this reason, from a physical point of view, it is required to use as the initial vectors only such α that belongs to the intersection $\mathcal{M}(\mathbb{L}(t)) \cap \text{vec}\{S(\mathcal{H})\}$, where $\text{vec}\{S(\mathcal{H})\}$ refers to the state set of all vectorized density matrices associated with the Hilbert space $\mathcal{H}$.

Third, in practice, there is no need to take into account in Eq. (11) all powers of $B^n(t)$ up to infinity because one can always use the Cayley-Hamilton theorem [21–23], which states that every matrix satisfies its characteristic polynomial. Therefore, if $B(t)$ is a $\mu \times \mu$ matrix, the μth power of $B(t)$ linearly depends on the lower powers. Thus, in general, it is sufficient to consider the powers of $B(t)$ up to $\mu - 1$. The number of necessary powers may be additionally reduced provided one can determine the degree of the minimal polynomial of $B(t)$, which can be done numerically for some generators $\mathbb{L}(t)$.

In the context of the Fedorov theorem, it is important to explain how the vectors α satisfying the condition Eq. (11) can be obtained. One should notice that we are searching for the subspace which can be expressed by the following formula:

$$\mathcal{M}(\mathbb{L}(t)) := \bigcap_{n=1}^{\mu-1} \text{Ker}[\,\mathbb{L}(t), B^n(t)\,]. \tag{17}$$

The formula Eq. (17) cannot be easily calculated, however one might notice a significant similarity between this issue and the problem of finding common eigenvectors of two matrices [24,25]. Therefore, in the context of the Fedorov theorem, one can use the approach introduced by Shemesh in order to transform the formula for the subspace $\mathcal{M}(\mathbb{L}(t))$ into an expression, which will be straightforward in computing. Let us first prove a lemma.

Lemma 1. For any set of linear operators $\{R_1, \dots, R_\kappa\}$, the following relation holds true:

$$\bigcap_{i=1}^{\kappa} \text{Ker}\ R_i = \text{Ker} \sum_{i=1}^{\kappa} R_i^\dagger R_i, \tag{18}$$

where $R_i^\dagger$ denotes the operator dual to R_i.

Proof. Let us prove the lemma for two operators R_1 and R_2 since one can easily generalize the reasoning for a higher number of operators. Then, on the left-hand side of Eq. (18), we have $\mathrm{Ker}R_1 \cap \mathrm{Ker}R_2$. Next, we observe

$$
\begin{aligned}
&x \in \mathrm{Ker}R_1 \cap \mathrm{Ker}R_2 \;\Leftrightarrow\; x \in \mathrm{Ker}R_1 \wedge x \in \mathrm{Ker}R_2,\\
&R_1\, x = 0 \wedge R_2\, x = 0,\\
&R_1^{\dagger}R_1\, x = 0 \wedge R_2^{\dagger}R_2\, x = 0,\\
&(R_1^{\dagger}R_1 + R_2^{\dagger}R_2)\, x = 0,\\
&x \in \mathrm{Ker}(R_1^{\dagger}R_1 + R_2^{\dagger}R_2),
\end{aligned} \tag{19}
$$

and the last part finishes the proof.

Based on Lemma 1, we can conclude that the closed-form solution according to Eq. (12) can be obtained for the initial vectors α which belong to the subspace $\mathcal{M}(\mathbb{L}(t))$ such that

$$\mathcal{M}(\mathbb{L}(t)) = \mathrm{Ker} \sum_{n=1}^{\mu-1} [\,\mathbb{L}(t), B^n(t)\,]^{\dagger}[\,\mathbb{L}(t), B^n(t)\,]. \tag{20}$$

To sum up, if one wants to apply the Fedorov theorem in order to obtain a closed-form solution of a differential equation with a time-dependent generator $\mathbb{L}(t)$, one needs to prove that the subspace $\mathcal{M}(\mathbb{L}(t))$ defined by Eq. (20) is nonempty, which can be done effectively thanks to the Shemesh criterion. Then, one can write a closed-form solution of the evolution equation: $\mathrm{vec}\{\rho(t)\} = \exp[B(t)]\alpha$. This solution generates a legitimate trajectory from the physical point of view only if the initial vector α can be considered as a vectorized density matrix, i.e., $\alpha \in \mathcal{M}(\mathbb{L}(t)) \cap \mathrm{vec}\{S(\mathcal{H})\}$. Generators $\mathbb{L}(t)$ such that the corresponding subspace $\mathcal{M}(\mathbb{L}(t))$ is nonempty can be called *partially commutative*.

IV. FEDOROV THEOREM IN DYNAMICS OF OPEN QUANTUM SYSTEMS

A. Preliminaries

In this article, we shall consider the evolution generator $\mathbb{L}(t)$ of d-level quantum systems in the form [26,27]

$$\mathbb{L}(t)[\rho] = -i[H,\rho] + \sum_k \gamma_k(t)\left(V_k\rho V_k^{\dagger} - \frac{1}{2}\{V_k^{\dagger}V_k, \rho\}\right), \tag{21}$$

which can be regarded as a specific type of time-dependent GKSL generator [1,2], such that the jump operators V_k are represented by constant matrices while the relaxation rates $\gamma_k(t)$ are time-dependent. The operator H is Hermitian, i.e., $H^{\dagger} = H$, and it can be interpreted as the effective Hamiltonian that accounts for the unitary evolution. This generator preserves the Hermiticity and trace of the density matrix, but for negative relaxation rates in some time intervals the evolution features non-Markovian effects [28]. For this reason, we shall restrict our analysis only to the relaxation rates such that $\gamma_k(t) \geqslant 0$ for all $t \geqslant 0$ and for any k, which means that the evolution may be called time-dependent Markovian (though the corresponding dynamical map is not a semigroup).

One of the algebraic methods used in the analysis is the technique to obtain a matrix representation of the generator of evolution. Such a procedure is feasible if we apply the property connected with the vec operator. For any three matrices A, B, C such that their product ABC is computable, we have the following relation [29]:

$$\mathrm{vec}\,(ABC) = (C^T \otimes A)\;\mathrm{vec}B, \tag{22}$$

which shall be called the Roth's column lemma. This property has been excessively studied within the field of pure mathematics [30–32] as well as applied to physics in order to search for matrix representations of given GKSL generators of evolution [33–35]. Taking into account the Roth's column lemma Eq. (22), one transforms the generator of evolution given originally by Eq. (21) into the matrix form

$$
\begin{aligned}
\mathbb{L}(t)
&= i\big(H^T \otimes \mathbb{1}_d - \mathbb{1}_d \otimes H\big)\\
&\quad + \sum_k \gamma_k(t)\left(\overline{V}_k \otimes V_k - \frac{1}{2}\mathbb{1}_d \otimes V_k^{\dagger}V_k - \frac{1}{2}V_k^{T}\overline{V}_k \otimes \mathbb{1}_d\right),
\end{aligned} \tag{23}
$$

where $\overline{V}_k$ denotes the complex conjugate of the jump operator V_k.

In our analysis, we consider three specific generators of evolution which govern the dynamics of three-level systems: V-system, *cascade*, and Λ-system [36]. For years such types of dynamics have been an important field of research since they are connected to optimal control of quantum dissipative systems in the context of laser cooling [37–39]. Therefore, we assume that $d = 3$, and the vectors $\{|1\rangle, |2\rangle, |3\rangle\}$ stand for the standard basis in the Hilbert space $\mathcal{H}$. A jump operator V_k, which corresponds to the transition from the jth level to the ith level, shall be defined as $V_k := |i\rangle\langle j| \equiv E_{ij}$.

As far as four-level systems are concerned ($d = 4$), the standard basis is denoted by $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$. We demonstrate that one can define *cascade*-type evolution with three jump operators accompanied by time-dependent decoherence rates, and then apply the Fedorov theorem to search for the dynamical map.

B. Three-level V-system

The three-level V-system relates to a physical scenario when an atom has two excited levels denoted by $|1\rangle$ and $|3\rangle$, but one ground state $|2\rangle$. The dynamics describes a decay from one of the excited levels into the ground state. Thus, we have two jump operators: $E_{21} := |2\rangle\langle 1|$ and $E_{23} := |2\rangle\langle 3|$. We assume that the corresponding decoherence rates are given by the functions $\gamma_{21}(t) := \sin^2\omega t$ and $\gamma_{23}(t) := \cos^2\omega t$. Then, based on the Roth's column lemma, the matrix form of the generator can be found according to Eq. (23):

$$
\begin{aligned}
\mathbb{L}_V(t) &= i\big(H_V^T \otimes \mathbb{1}_3 - \mathbb{1}_3 \otimes H_V\big)\\
&\quad + \sin^2\omega t\left(E_{21} \otimes E_{21} - \frac{1}{2}\mathbb{1}_3 \otimes E_{11} - \frac{1}{2}E_{11} \otimes \mathbb{1}_3\right)\\
&\quad + \cos^2\omega t\left(E_{23} \otimes E_{23} - \frac{1}{2}\mathbb{1}_3 \otimes E_{33} - \frac{1}{2}E_{33} \otimes \mathbb{1}_3\right),
\end{aligned} \tag{24}
$$

where H_V denotes the unperturbed Hamiltonian which describes three energy levels of the V-system, i.e.,

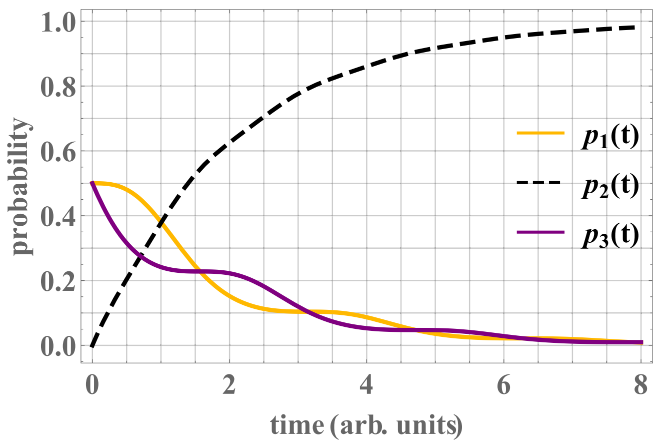

FIG. 1. The probability of finding the three-level V-system in one of the possible states.

$H_V = \mathrm{diag}(\mathcal{E}_1, 0, \mathcal{E}_3)$ (the energy of the ground level is normalized to zero, i.e., $\mathcal{E}_2 = 0$).

One can check that the generator for the V-system satisfies the following relations:

$$[\mathbb{L}_V(t),\ \mathbb{L}_V(\tau)] = 0 \qquad \forall\, t, \tau \geqslant 0, \\ \left[\mathbb{L}_V(t),\ \int \mathbb{L}_V(t)\, dt\right] = 0, \tag{25}$$

which implies that the closed-form solution of the evolution equation can be obtained based on the Lappo-Danilevsky criterion (without the Fedorov generalization):

$$\rho(t) = \exp\left(\int_0^t \mathbb{L}_V(\tau) d\tau\right)[\rho(0)]. \tag{26}$$

Let us investigate, as a specific example, the trajectory of the initial state: $\rho(0) = 1/2\,|1\rangle\langle 1| + 1/2\,|3\rangle\langle 3|$, which corresponds to a statistical mixture of two excited states with equal probabilities. The trajectory of this state can be described by the following dynamical map:

$$\rho(t) \\ = \begin{pmatrix} \frac{1}{2}e^{\frac{-2\omega t+\sin(2\omega t)}{4\omega}} & 0 & 0 \\ 0 & 1 - e^{-\frac{t}{2}}\cosh\left[\frac{\sin(2\omega t)}{4\omega}\right] & 0 \\ 0 & 0 & \frac{1}{2}e^{-\frac{2\omega t+\sin(2\omega t)}{4\omega}} \end{pmatrix}. \tag{27}$$

To study in detail the dynamics governed by the generator Eq. (24), let us consider the probability of finding the quantum system in each of the possible states as a function of time. By $p_i(t)$ we denote the probability of finding the system in the ith state at time instant t. One can find the plots in Fig. 1.

One can observe that the probability of finding the system in the state $|2\rangle$ is an increasing function, the value of which converges asymptotically to 1. It is not an unexpected result since the V-model describes a three-level system that decays into the ground state in time. Nonetheless, it is worth noting that the probabilities $p_1(t)$ and $p_3(t)$ present specific shapes due to the fact that we introduced the oscillating functions (i.e., $\sin\omega t$ and $\cos\omega t$) into the decoherence rates. One could exchange the relaxation rates of the generator Eq. (24) into different time-dependent functions and then explore other time characteristics of the probabilities.

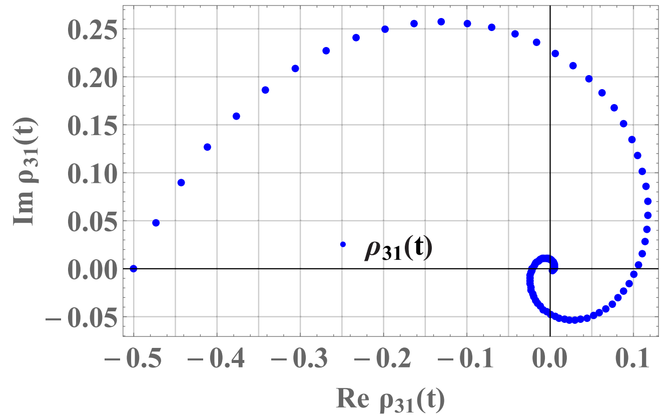

FIG. 2. The trajectory of $\rho_{31}(t)$ on the complex plane, assuming that the initial value of the relative phase equals π.

To investigate more effects, one can add phase factors into the off-diagonal elements of the initial density matrix, i.e., $\rho_{13}(0) = 1/2\, e^{-i\phi}$ and $\rho_{31}(0) = 1/2\, e^{i\phi}$, where ϕ stands for the relative phase between the states $|1\rangle$ and $|3\rangle$. Such a generalization does not affect the formulas for probabilities as presented in Fig. 1, but it allows one to additionally study how the phase factors change in time. Then, by applying the dynamics Eq. (26), one would obtain

$$\rho_{13}(t) = \tfrac{1}{2}\, e^{[-1/2+i(\mathcal{E}_3-\mathcal{E}_1)]t}\, e^{-i\phi} \quad \text{and} \quad \rho_{31}(t) = \overline{\rho_{13}(t)}, \tag{28}$$

which means that the relative phase ϕ between the energy states $|1\rangle$ and $|3\rangle$ vanishes while the initial state decays into the ground level $|2\rangle$. The phase-damping effect is caused by the factor $e^{-1/2t}$, whereas the other coefficient emerging from the evolution, i.e., $e^{i(\mathcal{E}_3-\mathcal{E}_1)t}$, makes the phase factor rotate on the complex plane. For arbitrary $\mathcal{E}_3$ and $\mathcal{E}_1$, the time evolution of the phase factor $\rho_{31}(t)$ is presented in Fig. 2.

C. Three-level *cascade* system

The three-level model called *cascade* describes a situation when the system can relax from the state $|3\rangle$ into the middle level $|2\rangle$ and then into the ground state denoted by $|1\rangle$. Since two kinds of transition are admissible, we have two jump operators: $E_{23} := |2\rangle\langle 3|$ and $E_{12} := |1\rangle\langle 2|$. We assume that the corresponding relaxation rates are again given by the functions $\gamma_{23}(t) := \sin^2\omega t$ and $\gamma_{12}(t) := \cos^2\omega t$. This leads to the generator of evolution in the following representation:

$$\mathbb{L}_C(t) = i\big(H_C^T \otimes \mathbb{1}_3 - \mathbb{1}_3 \otimes H_C\big) \\ + \sin^2\omega t\left(E_{23} \otimes E_{23} - \frac{1}{2}\mathbb{1}_3 \otimes E_{33} - \frac{1}{2}E_{33} \otimes \mathbb{1}_3\right) \\ + \cos^2\omega t\left(E_{12} \otimes E_{12} - \frac{1}{2}\mathbb{1}_3 \otimes E_{22} - \frac{1}{2}E_{22} \otimes \mathbb{1}_3\right), \tag{29}$$

where H_C denotes the unperturbed Hamiltonian that describes three symmetric energy levels, i.e., $H_C = \mathrm{diag}(-\mathcal{E}, 0, \mathcal{E})$ (the energy of the intermediate level is normalized to zero).

One can check that for the generator $\mathbb{L}_C(t)$, we obtain

$$[\,\mathbb{L}_C(t),\ \mathbb{L}_C(\tau)\,] \neq 0, \qquad \left[\mathbb{L}_C(t),\ \int \mathbb{L}_C(t)\,dt\right] \neq 0, \tag{30}$$

which implies that the sufficient conditions for the closed-form solution are not satisfied. Therefore, there is a need for a more general approach. One can consider the Fedorov theorem as a possible technique to solve the evolution equation with the generator Eq. (29).

To effectively apply the Fedorov theorem, we first need to numerically determine the minimal polynomial of $\int \mathbb{L}_C(t)dt$. The specific coefficients of the polynomial are of little interest since we focus on its degree, which equals 6. This means that for any $t \geqslant 0$, the operator $[\int \mathbb{L}_C(t)dt]^6$ can be expressed by means of the lower powers of $\int \mathbb{L}_C(t)dt$. Combining this observation with the earlier result Eq. (20), we need to investigate the kernel of the operator

$$\Gamma^{(C)} \equiv \sum_{n=1}^{5} \left[\mathbb{L}_C(t), \left(\int \mathbb{L}_C(t)\,dt\right)^n\right]^\dagger \left[\mathbb{L}_C(t), \left(\int \mathbb{L}_C(t)\,dt\right)^n\right]. \tag{31}$$

The matrix representation of $\Gamma^{(C)}$ can be found numerically. One can obtain that $\Gamma^{(C)}_{99} = g(t) \neq 0$ and all the other elements are equal to zero. This means the intersection of vec $\mathcal{S}(\mathcal{H})$ and $\mathcal{M}(\mathbb{L}_C(t)) = \text{Ker}\,\Gamma^{(C)}$ can be written as

$$\text{vec}\,\rho \in \text{vec}\,\mathcal{S}(\mathcal{H}) \cap \mathcal{M}(\mathbb{L}_C(t)) \ \Leftrightarrow\ \rho \in \mathcal{S}(\mathcal{H}) \wedge \rho_{33} = 0, \tag{32}$$

which implies that the evolution equation with the generator Eq. (29) has a closed-form solution only for the initial states which assume zero probability for the level $|3\rangle$. Thus, the dynamical map can be written as

$$\rho(t) = \exp\left(\int_0^t \mathbb{L}_C(\tau)d\tau\right)[\rho(0)], \tag{33}$$

where $\rho(0) = p\,|1\rangle\langle 1| + (1-p)\,|2\rangle\langle 2|$ and $0 \leqslant p \leqslant 1$ (one may add phase factors on the off-diagonal elements). The explicit form of $\rho(t)$ can be computed as

$$\rho(t) = \begin{pmatrix} 1-\xi(t) & 0 & 0 \\ 0 & \xi(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{34}$$

where

$$\xi(t) := (1-p)\exp\left(-\frac{2\omega t + \sin(2\omega t)}{4\omega}\right). \tag{35}$$

To illustrate the results of the method, let us assume that $p = 0$, i.e., the initial density matrix $\rho(0) = |2\rangle\langle 2|$. The plots in Fig. 3 present the probabilities $p_1(t)$ and $p_2(t)$ (naturally $p_3(t) = 0$ for all $t \geqslant 0$).

The results demonstrate the decay from the middle state $|2\rangle$ into the ground state $|1\rangle$ in the time domain. The character of the probability graphs could by changed by modifying the functions that define the time-dependent relaxation rates: $\gamma_{23}(t)$ and $\gamma_{12}(t)$.

The process of relaxation within the *cascade* model can also be analyzed by means of the time evolution of the purity and the von Neumann entropy. For a system described by a density matrix $\rho(t)$, the purity, which shall be denoted by $\pi(t)$, is defined as $\pi(t) := \text{Tr}\{\rho^2(t)\}$. The von Neumann entropy has the standard definition $S(t) := \text{Tr}\{\rho(t)\ln\rho^2(t)\}$. Note that usually these figures are computed for a given state, whereas we treat them as the functions of time since we wish to follow the dynamics of entropy and purity for the initial state $\rho(0) = p\,|1\rangle\langle 1| + (1-p)\,|2\rangle\langle 2|$. We obtain the formulas

$$\pi(t) = 2\xi^2(t) - 2\xi(t) + 1, \qquad S(t) = -[1-\xi(t)]\ln\{1-\xi(t)\} - \xi(t)\ln\{\xi(t)\}. \tag{36}$$

To be more specific, let us again assume that $p = 0$. And for the initial state $\rho(0) = |2\rangle\langle 2|$ we can plot the functions $\pi(t)$ and $S(t)$ (see Fig. 4).

Since the input was a pure state, we have $\pi(0) = 1$ and $S(0) = 0$. Then, the state is getting more mixed with time. At some point, we have equal probabilities for $|2\rangle$ and $|1\rangle$, which means that the purity drops down to its minimal value, i.e., $\pi(t') = 1/2$, whereas the von Neumann entropy reaches its maximum value $S(t') = \ln 2 \approx 0.693\,15$. In time, both functions are approaching their initial values since the final state

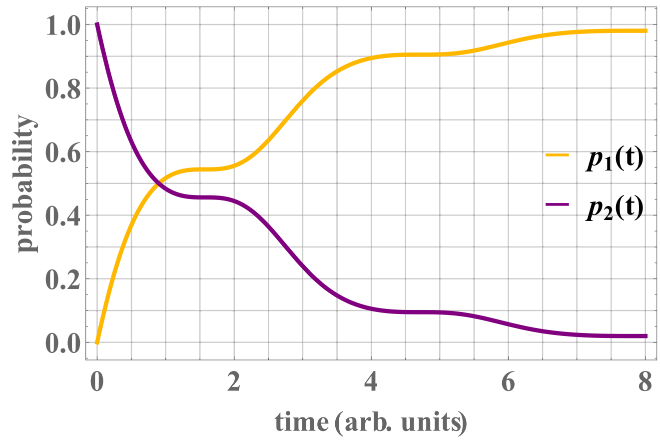

FIG. 3. The probability of finding the three-level *cascade* system in one of the possible states: $|1\rangle$ or $|2\rangle$.

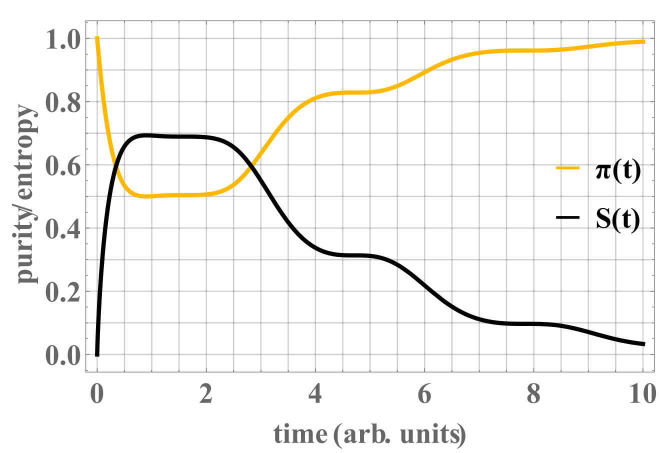

FIG. 4. The purity $\pi(t)$ and the von Neumann entropy $S(t)$ of a dissipative system subject to the *cascade* decoherence model.

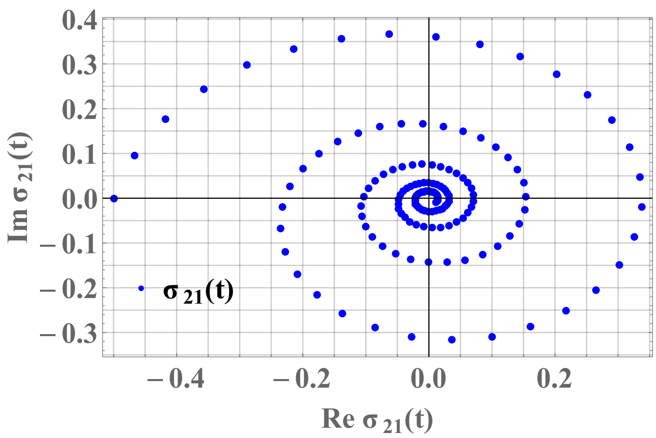

FIG. 5. The trajectory of $\sigma_{21}(t)$ on the complex plane, assuming that the initial value of the relative phase equals π.

is also pure. The shape of the functions reflects the definitions of the relaxation rates.

One can also consider the time evolution of off-diagonal elements of the density matrix by imposing a relative phase ϕ between the states $|1\rangle$ and $|2\rangle$. Then, the initial density matrix $\sigma(0)$ can be introduced in the form

$$\sigma(0) = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\phi} & 0 \\ e^{i\phi} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (37)$$

Such a change in the initial density matrix allows one to study the dynamics of the phase factors. Based on the dynamical map Eq. (33), we obtain

$$\sigma_{12}(t) = \frac{1}{2}\exp\left[\left(-\frac{1}{4} + \mathcal{E}i\right)t - \frac{\sin 2\omega t}{8\omega}\right]e^{-i\phi} \quad (38)$$

and $\sigma_{21}(t) = \overline{\sigma_{12}(t)}$, which gives the trajectory of the phase factor as presented in Fig. 5 (for arbitrary ω and $\mathcal{E}$).

D. Three-level Λ-system

The quantum Λ-system with three energy levels belongs to very useful models studied in different areas of modern physics; see, e.g., Refs. [40–42]. It is assumed that the system decays from the excited level $|2\rangle$ into one of two lower-energy states: $|1\rangle$ or $|3\rangle$. Thus, we have two jump operators: $E_{12} := |1\rangle\langle 2|$ and $E_{32} := |3\rangle\langle 2|$. We shall consider the following generator of evolution:

$$\begin{aligned}\mathbb{L}_\Lambda(t) = {} & i\big(H_\Lambda^T \otimes \mathbb{1}_3 - \mathbb{1}_3 \otimes H_\Lambda\big) \\ & + f_1(t)\big(E_{12}\otimes E_{12} - \tfrac{1}{2}\mathbb{1}_3 \otimes E_{22} - \tfrac{1}{2}E_{22}\otimes \mathbb{1}_3\big) \\ & + f_2(t)\big(E_{32}\otimes E_{32} - \tfrac{1}{2}\mathbb{1}_3 \otimes E_{22} - \tfrac{1}{2}E_{22}\otimes \mathbb{1}_3\big),\end{aligned} \quad (39)$$

where the functions $f_i(t) : \mathcal{I} \to \mathbb{R}_+$ are assumed to be linearly independent, and H_Λ stands for the Hamiltonian that describes the energy levels, i.e., $H_\Lambda = \mathrm{diag}(-\mathcal{E}_1, 0, -\mathcal{E}_3)$ for $\mathcal{E}_1, \mathcal{E}_3 > 0$. One can notice that this generator is not functionally commutative, nor does it commute with its integral. The minimal polynomial of Eq. (39) cannot be easily determined without any assumptions concerning the functions $f_1(t), f_2(t)$ and the energies $\mathcal{E}_1, \mathcal{E}_3$, which means that in order to consider the Fedorov theorem in the context of Λ-systems, we need to search for the kernel of

$$\Gamma^{(\Lambda)} \equiv \sum_{n=1}^{8}\left[\mathbb{L}_\Lambda(t), \left(\int \mathbb{L}_\Lambda(t)\,dt\right)^n\right]^\dagger\left[\mathbb{L}_\Lambda(t), \left(\int \mathbb{L}_\Lambda(t)\,dt\right)^n\right]. \quad (40)$$

Interestingly, regardless of the functions $f_1(t), f_2(t)$ and the energies $\mathcal{E}_1, \mathcal{E}_3$, it can be checked numerically that $\Gamma^{(\Lambda)}_{55} \neq 0$ and all the other elements are zeros. For this reason, we can write

$$\mathrm{vec}\,\rho \in \mathrm{vec}\,\mathcal{S}(\mathcal{H}) \cap \mathcal{M}(\mathbb{L}_\Lambda(t)) \;\Leftrightarrow\; \rho \in \mathcal{S}(\mathcal{H}) \wedge \rho_{22} = 0, \quad (41)$$

which means that the differential equation with the generator Eq. (39) has a closed-form solution, for example when the initial state is given by $\rho_S(0) = p\,|1\rangle\langle 1| + (1-p)\,|3\rangle\langle 3|$. However, such a state, which is a statistical mixture of two lower-energy states, is stationary because the dynamics does not allow any transitions between the levels $|1\rangle$ and $|3\rangle$. Thus, for any functions $f_1(t)$ and $f_2(t)$, we have

$$\rho(t) = \exp\left(\int_0^t \mathbb{L}_\Lambda(\tau)d\tau\right)[\rho_S(0)] = \rho_S(0). \quad (42)$$

Alternatively, one can impose a relative phase between the states $|1\rangle$ and $|3\rangle$ and consider how the dynamics influence the off-diagonal elements. If we introduce the initial state in the form

$$\rho(0) = \frac{1}{2}\begin{pmatrix} 1 & 0 & e^{-i\phi} \\ 0 & 0 & 0 \\ e^{i\phi} & 0 & 1 \end{pmatrix}, \quad (43)$$

where ϕ stands for the relative phase, then one can observe that such an initial state also satisfies the condition of partial commutativity. If we impose the dynamical map $\exp\left(\int_0^t \mathbb{L}_\Lambda(\tau)d\tau\right)$ on the state Eq. (43), we obtain

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1 & 0 & e^{(\mathcal{E}_3-\mathcal{E}_1)t\,i}e^{-i\phi} \\ 0 & 0 & 0 \\ e^{(\mathcal{E}_1-\mathcal{E}_3)t\,i}e^{i\phi} & 0 & 1 \end{pmatrix}, \quad (44)$$

which means that the phase factor rotates on the complex plane in time. The oscillations of the phase factor are attributed solely to the unitary evolution. If the energy levels were degenerate, i.e., $\mathcal{E}_3 = \mathcal{E}_1 = 0$, then the input state Eq. (43) would be stationary.

E. Four-level *cascade* system

The four-level *cascade* model describes a physical situation when the system can relax from the highest state $|4\rangle$ into the lower level $|3\rangle$, then into the state $|2\rangle$, and finally into the ground state denoted by $|1\rangle$. Since three kinds of transition are admissible, we have three jump operators: $E_{34} := |3\rangle\langle 4|$, $E_{23} := |2\rangle\langle 3|$, and $E_{12} := |1\rangle\langle 2|$. There are plenty of possible time-dependent decoherence rates that might be analyzed in the context of such dynamics. We shall assume that the corresponding relaxation rates are given by the functions $\gamma_{34}(t) := e^{-\omega t}$ and $\gamma_{23}(t) = \gamma_{12}(t) = \sin^2(3\,\omega t)$. This leads to the generator of evolution in the following representation:

$$\begin{aligned}\mathbb{L}_{\text{FC}}(t) &= i\left(H_{\text{FC}}^T \otimes \mathbb{1}_4 - \mathbb{1}_4 \otimes H_{\text{FC}}\right) \\ &+ e^{-\omega t}\left(E_{34} \otimes E_{34} - \frac{1}{2}\mathbb{1}_4 \otimes E_{44} - \frac{1}{2}E_{44} \otimes \mathbb{1}_4\right) \\ &+ \sin^2(3\,\omega\,t)\left(E_{23} \otimes E_{23} - \frac{1}{2}\mathbb{1}_4 \otimes E_{33} - \frac{1}{2}E_{33} \otimes \mathbb{1}_4\right) \\ &+ \sin^2(3\,\omega\,t)\left(E_{12} \otimes E_{12} - \frac{1}{2}\mathbb{1}_4 \otimes E_{22} - \frac{1}{2}E_{22} \otimes \mathbb{1}_4\right),\end{aligned} \tag{45}$$

where H_{FC} denotes a four-level *cascade* Hamiltonian. The energy levels are assumed to be symmetric, i.e., $H_{\text{FC}} = \text{diag}(-\mathcal{E}_2, -\mathcal{E}_1, \mathcal{E}_1, \mathcal{E}_2)$ for $\mathcal{E}_1, \mathcal{E}_2 > 0$. One can verify that the generator $\mathbb{L}_{\text{FC}}(t)$ satisfies neither the condition of functional commutativity nor commutativity with its integral. Therefore, it is desirable to search for other methods that can be used to solve the evolution equation governed by the generator Eq. (45).

We investigate the kernel of the operator $\Gamma^{(\text{FC})}$ [cf. Eq. (31)]. The matrix representation of this operator can be determined numerically. One can then observe that $\Gamma^{(\text{FC})}_{16\,16} = g(t)$, whereas the other elements are zeros. This means the intersection of $\text{vec}\,\mathcal{S}(\mathcal{H})$ and $\mathcal{M}(\mathbb{L}_{\text{FC}}(t)) = \text{Ker}\,\Gamma^{(\text{FC})}$ can be written as

$$\text{vec}\,\rho \in \text{vec}\,\mathcal{S}(\mathcal{H}) \cap \mathcal{M}(\mathbb{L}_{\text{FC}}(t)) \;\Leftrightarrow\; \rho \in \mathcal{S}(\mathcal{H}) \wedge \rho_{44} = 0, \tag{46}$$

which implies that the evolution equation with the generator Eq. (45) has a closed-form solution only for the initial states that assume zero probability for the level $|4\rangle$. In other words, we are able to follow the dynamics in closed form only if we reduce the dimension of the system by 1. Then, the dynamical map can be written as

$$\rho(t) = \exp\left(\int_0^t \mathbb{L}_{\text{FC}}(\tau)d\tau\right)[\rho(0)], \tag{47}$$

where $\rho(0)$ denotes an initial state satisfying Eq. (46), e.g., $\rho(0) = q_1\ |1\rangle\langle 1| + q_2\ |2\rangle\langle 2| + q_3\ |3\rangle\langle 3|$, and $\{q_1, q_2, q_3\}$ stands for a probability distribution (one may add phase factors on the off-diagonal elements).

Let us study a specific example of this kind of dynamics by assuming that the initial state has a form $\rho(0) = 1/3\ |2\rangle\langle 2| + 2/3\ |3\rangle\langle 3|$. Based on the closed-form solution Eq. (47) one can compute

$$\begin{aligned}p_1(t) &= 1 + \frac{1}{18\omega}\left(e^{\frac{-6\omega t+\sin(6\,\omega\,t)}{12\omega}}[-6(3+t)\omega + \sin(6\,\omega\,t)]\right), \\ p_2(t) &= \frac{1}{18\omega}\left(e^{\frac{-6\omega t+\sin(6\,\omega\,t)}{12\omega}}[6(1+t)\omega - \sin(6\,\omega\,t)]\right), \\ p_3(t) &= \frac{2}{3}e^{\frac{-6\omega t+\sin(6\,\omega\,t)}{12\omega}},\end{aligned} \tag{48}$$

where $p_k(t)$, like before, stands for the probability of finding the system in the kth state. To track the changes that occur in the system during the evolution, the functions $p_k(t)$ are presented in Fig. 6.

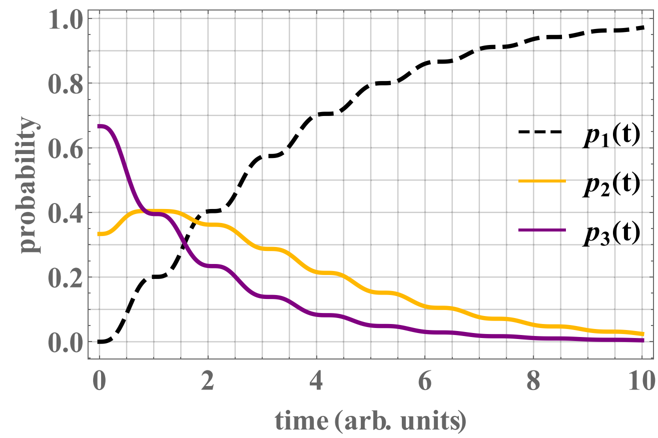

FIG. 6. The probability of finding the four-level *cascade* system in one of the possible states: $|1\rangle$, $|2\rangle$, or $|3\rangle$.

Similarly as before, one can follow other characteristics of a quantum system, such as the purity, denoted by $\pi(t)$, and the von Neumann entropy, denoted by $S(t)$. In Fig. 7 one can observe the plots of these functions.

It is worth noting that one can choose any specific state satisfying the condition Eq. (46) (e.g., with phase factors) and track its characteristics in time, assuming that the evolution is governed by the generator Eq. (45). For instance, we may consider a state in the form

$$\sigma(0) = \frac{1}{3}\begin{pmatrix} 1 & e^{-i\,\phi_{12}} & e^{-i\,\phi_{13}} & 0 \\ e^{i\,\phi_{12}} & 1 & e^{i\,(\phi_{12}-\phi_{13})} & 0 \\ e^{i\,\phi_{13}} & e^{i\,(\phi_{13}-\phi_{12})} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{49}$$

where ϕ_{12} denotes the relative phase between the states $|1\rangle$ and $|2\rangle$ (and analogously for ϕ_{13}). By applying the dynamical map Eq. (47) to the state Eq. (49), we can determine the dynamics

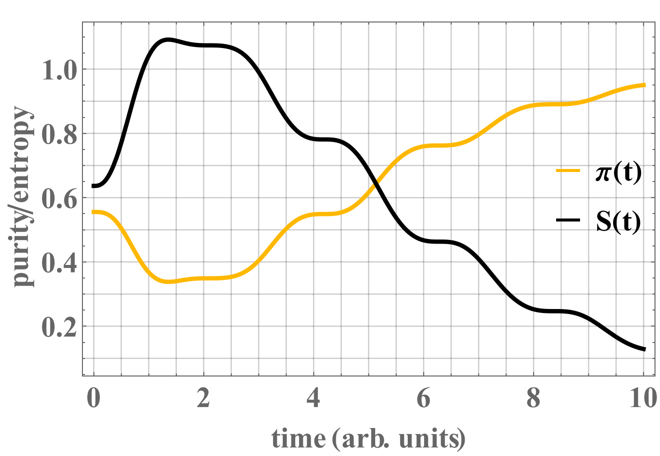

FIG. 7. The purity and the von Neumann entropy of a dissipative four-level system subject to the *cascade* decoherence model.

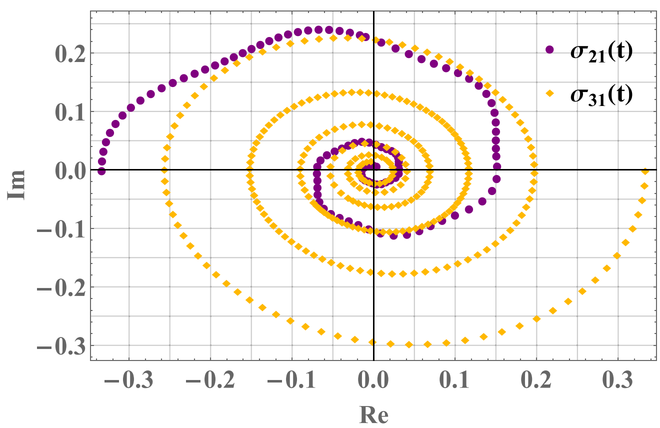

FIG. 8. The trajectories of $\sigma_{21}(t)$ and $\sigma_{31}(t)$ on the complex plane with the initial values of the relative phases: $\phi_{12} = \pi$ and $\phi_{13} = 0$.

of the off-diagonal elements:

$$\begin{aligned} \sigma_{21}(t) &= \frac{1}{3}\exp\left(-\frac{1}{4}t + i(\mathcal{E}_1 - \mathcal{E}_2)t + \frac{\sin(6\omega t)}{24\omega}\right)e^{i\phi_{12}}, \\ \sigma_{31}(t) &= \frac{1}{3}\exp\left(-\frac{1}{4}t - i(\mathcal{E}_1 + \mathcal{E}_2)t + \frac{\sin(6\omega t)}{24\omega}\right)e^{i\phi_{13}}, \quad (50) \\ \sigma_{32}(t) &= \frac{1}{3}\exp\left(-\frac{1}{2}t - 2\mathcal{E}_1 it + \frac{\sin(6\omega t)}{12\omega}\right)e^{i(\phi_{13}-\phi_{12})}, \end{aligned}$$

and from $\sigma_{ij}(t) = \overline{\sigma_{ji}(t)}$ we can get the other half. The trajectories can be presented graphically on the complex plane if we assume some arbitrary values of the parameters characterizing the evolution, i.e., $\omega, \mathcal{E}_1, \mathcal{E}_2$. For two exemplary phase factors, this is done in Fig. 8.

F. Discussion and analysis

The Fedorov theorem provides a useful generalization of the Lappo-Danilevsky criterion. This method was originally introduced by Fedorov in a two-page article in Russian [20] and later included in the book by Erugin [6]. For a long time, the theorem was unnoticed in the field of linear differential equations. However, in 2018 it was rediscovered by Kamizawa [43], who proposed an effective analytical method for studying partial commutativity, although with no reference to physics.

This article contributes to the field of open quantum system dynamics by demonstrating that the Fedorov theorem can be applied to search for dynamical maps if the corresponding generator depends on time. We considered three particular types of three-level dynamics: *V-system*, *cascade*, and *Lambda* along with one example on four-level systems. Such evolution models are commonly studied in laser physics.

In the case of the V-system, it turns out that the generator of evolution Eq. (24) is functionally commutative (even if the relaxation rates are substituted with different time-dependent functions). This allows us to follow the trajectory for any initial state by the closed-form solution. For specific examples, we obtained plots that show how the probabilities of a system being in basis states change in time. Interestingly, if one imposes a relative phase factor in the off-diagonal elements of the density matrix, we shall observe phase-damping effects that can be presented by trajectories of the phase factor on the complex plane.

The results for the *cascade* model demonstrate that the Fedorov theorem can be useful but limited at the same time. The closed-form solution can be obtained only if there is zero probability for the initial state to be in the highest energy level. This means that we can study only the dynamics of a reduced, two-level subsystem. In spite of this limitation, one can determine the solution for a spectrum of density matrices and study time characteristics of the corresponding probabilities. The analysis can be further extended by analyzing the dynamics of the purity and the von Neumann entropy. In addition, one can analyze the dynamics of the off-diagonal elements of the density matrix by following the trajectories of phase factors on the complex plane.

Third, in the case of the famous *Lambda*-system, the Fedorov theorem allows one to write the solution only for such states that are stationary in terms of the probabilities. The system, given as a statistical mixture of the two lower states, remains unchanged subject to the generator of evolution. However, if we impose nonzero off-diagonal elements of the initial density matrix, we can observe oscillations of the phase factor, which is attributed to the unitary part of the generator.

Finally, an example of four-level systems with *cascade* dynamics was studied. Based on the Fedorov theorem, we could obtain a closed-form solution for a three-level subset of initial states. The dynamics of such states can be investigated by following the probabilities, purity, von Neumann entropy, as well as the trajectories of phase factors.

The examples studied in the article show that the applicability of the Fedorov theorem depends on the algebraic structure of the generator $\mathbb{L}(t)$. For some types of dynamics, the Fedorov theorem may allow one to obtain a closed-form solution and track the time changes in quantum systems. This problem requires further research. More kinds of time-dependent generators should be tested in connection with the Fedorov theorem. Multilevel quantum systems subject to relaxation (e.g., laser cooling) are an area of intensive research, both theoretical and experimental, e.g., [44–46]. The Fedorov theorem can provide an effective framework to study the dynamics of such systems.

V. SUMMARY AND OUTLOOK

In the article, we have proposed the Fedorov theorem as a technique to solve differential equations that describe the dynamics of open quantum systems. The method was applied to specific types of three-level and four-level systems. The generators studied in the article are in line with evolution models considered within laser physics. Thus, the results provide valuable insight into the dynamics of relaxation systems. Various characteristics of dissipative systems, such as the purity or the von Neumann entropy, can be investigated in the time domain based on the Fedorov theorem.

In the future, the Fedorov theorem shall be applied to other multilevel quantum systems, which may bring significant advancement in understanding the dynamics of dissipative systems composed of atoms interacting with light. When a high-dimensional Hilbert space is concerned, we expect that by partial commutativity one can study closed-form solutions of evolution equations within the admissible subset of initial quantum states. Further research into the Fedorov theorem seems relevant for pure mathematics as well as in the context of physical applications.

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