Superexponential diffusion in nonlinear non-Hermitian systems

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We investigate the quantum diffusion of a periodically kicked particle subjected to both nonlinearity-induced self-interactions and $\mathcal{PT}$-symmetric potentials. We find that, due to the interplay between the nonlinearity and non-Hermiticity, the expectation value of the mean square of momentum scales with time in a superexponential form $\langle p^2(t)\rangle \propto \exp[\beta \exp(\alpha t)]$, which is faster than any known rates of quantum diffusion. In the $\mathcal{PT}$-symmetry-breaking phase, the intensity of a state increases exponentially with time, leading to the exponential growth of the interaction strength. The feedback of the intensity-dependent nonlinearity further turns the interaction energy into the kinetic energy, resulting in a superexponential growth of the mean energy. These theoretical predictions are in good agreement with numerical simulations in a $\mathcal{PT}$-symmetric nonlinear kicked particle. Our discovery establishes a mechanism of diffusion in interacting and dissipative quantum systems. Important implications and possible experimental observations are discussed.

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I. INTRODUCTION

Diffusion of particles is of fundamental importance in many disciplines of physics, e.g., statistical physics and condensed matter physics. Its mechanism governs the conductivity of electrons [1], the spin transport [2], the heat transfer [3], and the Fermi acceleration of cosmic ray particles [4–11], just to name a few. In the classical domain, a seminal result of the random motion of Brownian particles is normal diffusion [12], which is characterized by the linear growth ($\propto t$) of the second moment of the observable. Quantum mechanically, the random diffusion of microscopic particles in disordered potential is totally suppressed by quantum interference, leading to the well-known Anderson localization (AL) [1]. Its analog in momentum space is the dynamical localization (DL) [see the linear Hermitian (L-H) zone in Fig. 1], which occurs in chaotic systems periodically driven by impulsive fields [13–17].

In the Hermitian case, periodically driven systems exhibit interesting diffusion behaviors, such as power-law diffusion $\propto t^{\eta}$ [18] and exponential diffusion $\propto e^{\gamma t}$ [19,20] (see the L-H zone in Fig. 1), which originate from the quantum resonance. In the past two decades, extensive studies have been concentrated on the diffusion process in complex systems, where the disorder and nonlinearity may coexist [21–31]. Nonlinear effects appear in a broad range of systems, for instance, in the Bose-Einstein condensates [32] and in nonlinear optics [33]. Up to now, a wide spectrum of diffusion processes, from power-law diffusion [34–41] to exponential diffusion [42–45], has been found in nonlinear systems [see the nonlinear Hermitian (NL-H) zone in Fig. 1].

A common condition for the appearance of these diffusion processes is the assumption of Hermiticity of quantum mechanics. Even without Hermiticity, a new class of system with $\mathcal{PT}$ symmetry possesses the real eigenvalues as well [46–48]. The non-Hermitian Hamiltonian can be used to describe nonequilibrium relaxation problems [49], optical transport in lossy media [50,51], and open quantum systems [52]. Thus it has been a subject of extensive theoretical [53–59] and experimental studies [60–67]. The non-Hermitian extension of Floquet-driven systems stimulates fruitful studies on the quantum diffusion behavior, where the DL phenomenon and ballistic diffusion have been reported [68] [see the linear non-Hermitian (L-NH) zone in Fig. 1]. In this context, the quantum diffusion in a system with non-Hermiticity and nonlinearity deserves urgent investigation [69–77].

In this work, we investigate the wave-packet dynamics in a Floquet system, where both the $\mathcal{PT}$ symmetry and nonlinearity are periodically modulated in time. In the broken-$\mathcal{PT}$-symmetry phase, for which the quasienergies become

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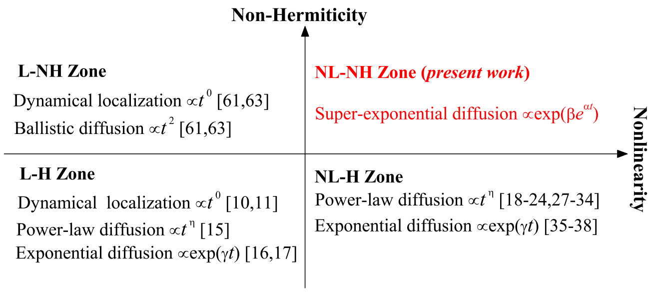

FIG. 1. Schematic diagram for quantum diffusion in different situations: left, linear (L) zone; right, nonlinear (NL) zone; bottom, Hermitian (H) zone; top, non-Hermitian (NH) zone.

complex, the wave packets diffuse in a way that the mean square of momentum is the exponent of time, i.e., $\langle p^2(t)\rangle \propto \exp[\beta e^{\alpha t}]$. To the best of our knowledge, this is the first report of a superexponential diffusion (SED). The underlying physics is due to the coexistence of two facts: (i) the exponentially fast growth of the intensity of the wave function due to the non-Hermiticity and (ii) the positive feedback mechanism of the intensity-dependent nonlinearity. Our theoretical prediction of the law of the SED is consistent with numerical results. We note in passing that exponential acceleration of particles has received extensive investigation. It has been reported that a sequence of highly correlated motions of a random walk model leads to the exponential Fermi acceleration in the time-dependent billiard systems [9–11], which allows for exploring the ultrafast acceleration in different systems. Our finding of SED sheds light on the ultrafast acceleration of the Fermi-Ulam model.

II. MODEL AND RESULTS

The system we consider is a quantum kicked particle in an infinite square well [78,79]. We make an extension to the $\mathcal{PT}$-symmetric kicking potential which, in contrast to Hermitian kicking, induces exotic transport behaviors [70,71,80]. The interatomic interaction is described by a mean-field nonlinear term, which is temporally modulated by delta kicks. The Hamiltonian in dimensionless units reads

$$\mathrm{H} = \frac{p^2}{2} + V(x) + V_{\mathrm{K}}(x)\sum_n \delta(t-n) + \mathrm{H}_{\mathrm{I}}\sum_n \delta(t-n), \tag{1}$$

where

$$V(x) = \begin{cases} 0 & \text{if } |x| < \frac{L}{2} \\ +\infty & \text{otherwise}, \end{cases} \tag{2}$$

$$V_{\mathrm{K}}(x) = K[\cos(x) + i\lambda\sin(x)], \tag{3}$$

and

$$\mathrm{H}_{\mathrm{I}} = g_0|\psi(x,t)|^2. \tag{4}$$

Here $p = -i\hbar_{\mathrm{eff}}\partial/\partial x$ is the momentum operator, x is the coordinate, and $\hbar_{\mathrm{eff}}$ denotes the effective Planck constant with the commutation relation $[x, p] = i\hbar_{\mathrm{eff}}$. The parameter L controls the width of the infinite square well. In the kicking potential $V_{\mathrm{K}}(x)$, the parameter K indicates the strength of its real part, and λ is the strength of its imaginary part. The parameter g_0 controls the nonlinear interaction strength. It is worth noting that the delta-kick nonlinearity induces rich physics, such as exponential instability [42–45] and dispersionless dynamics of wave packets [81], which are absent in systems with static nonlinear interactions [82]. In addition, this kind of system with rigid boundary conditions has served as a prototype for investigating the Fermi acceleration of particles in a cosmic ray in astrophysical plasmas [4–8]. Therefore, the $\mathcal{PT}$-symmetric extension of this system is of broad interest.

Let $|\varphi_j\rangle$ be the eigenstate of the unperturbed Hamiltonian $\mathrm{H}_0 = p^2/2 + V(x)$, with $\mathrm{H}_0|\varphi_j\rangle = E_j|\varphi_j\rangle$. In the representation of $|\varphi_j\rangle$, an arbitrary state can be expressed as $|\psi(t)\rangle = \sum_{j=0}^{+\infty}\psi_j(t)|\varphi_j\rangle$, with $\psi_j(t)$ being the component of the eigenstate $|\varphi_j\rangle$. The initial state is taken as the ground state, i.e., $\psi(x,0) = \sqrt{2/L}\cos(\pi x/L)$. The time evolution of the quantum state over a period is governed by $|\psi(t+1)\rangle = U|\psi(t)\rangle$. Due to the periodic kicking, the Floquet operator has the expression $U = U_{\mathrm{f}}U_{\mathrm{K}}$, where the kicking evolution operator is $U_{\mathrm{K}} = \exp[-iV_{\mathrm{K}}(x)/\hbar_{\mathrm{eff}} - i\mathrm{H}_{\mathrm{I}}(x,t)/\hbar_{\mathrm{eff}}]$, and the free evolution operator is $U_{\mathrm{f}} = \exp(-ip^2/2\hbar_{\mathrm{eff}})$. Specifically, one-period evolution contains four steps: (i) the kicking evolution in x space, $\tilde{\psi}(x,t) = U_{\mathrm{K}}(x,t)\psi(x,t)$; (ii) the sine transformation of the state $|\tilde{\psi}(t)\rangle$ from coordinate space to momentum space; (iii) the free evolution in p space, $\psi(p,t+1) = U_{\mathrm{f}}(p)\tilde{\psi}(p,t)$; and (iv) the inverse sine transformation of the state $|\psi(t+1)\rangle$ from p space to x space for the next period evolution. Without loss of generality, we consider the case with $L = 2\pi$, for which the particle can experience the kicking potential of a full period of 2π.

A commonly used quantity to characterize the quantum diffusion in momentum space is the expectation value of kinetic energy,

$$\langle p^2(t)\rangle = \sum_j p_j^2|\psi_j(t)|^2/\mathcal{N}, \tag{5}$$

where the norm of the quantum state is $\mathcal{N}(t) = \sum_j|\psi_j(t)|^2$. This quantity coincides with the expectation value of kinetic energy up to a factor of 1/2. Note that in the phase that breaks the $\mathcal{PT}$ symmetry, the norm $\mathcal{N}(t)$ could increase exponentially with time. Thus, the above definition of the expectation value drops the contribution of the norm $\mathcal{N}(t)$.

In the present work, we investigate both numerically and theoretically the time dependence of the mean kinetic energy $\langle p^2(t)\rangle$. We consider the case that the system is in the $\mathcal{PT}$-symmetry-breaking phase, which is guaranteed by setting the value of the imaginary part of the kicking potential to be sufficiently large. Figure 2(a) shows that, for $g_0 = 0$, the mean kinetic energy is suppressed during the time evolution, which is just the phenomenon of DL. Interestingly, for a specific value of the nonlinear strength (e.g., $g_0 = 0.1$), the mean kinetic energy follows that of the linear case $g_0 = 0$ for time smaller than a threshold value t_c, beyond which it increases in a superexponential way. Such an intrinsic phenomenon occurs even if the nonlinear strength is very small, e.g., $g_0 = 10^{-7}$. More importantly, we theoretically find the law of the superexponential growth of mean kinetic energy,

$$\langle p^2(t)\rangle \approx \exp(\beta e^{\alpha t}), \tag{6}$$

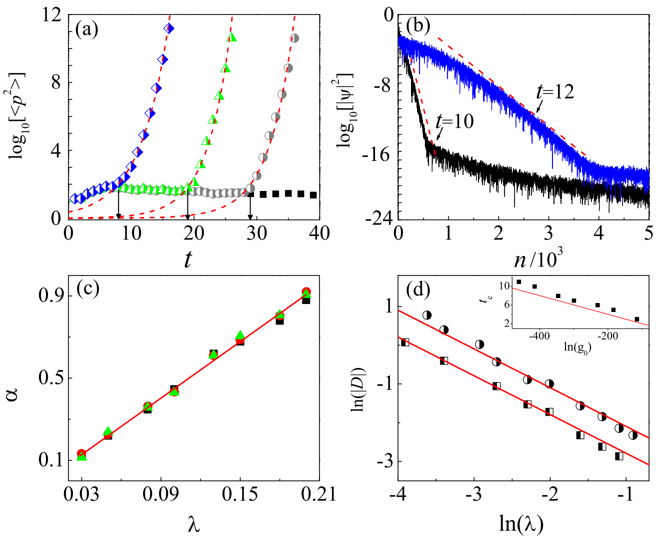

FIG. 2. (a) Time dependence of the mean energy $\langle p^2\rangle$ for $\lambda = 0.05$ with $g_0 = 0$ (squares), 10^{-7} (circles), 10^{-4} (triangles), and 0.1 (diamonds). Dashed lines (in red) denote the fitting function of the form $\langle p^2\rangle \approx \exp(\beta e^{\alpha t})$. Arrows mark the threshold time t_c for the appearance of the superexponential diffusion. (b) Probability density distribution in eigenstate space with $\lambda = 0.05$ and $g_0 = 0.1$ at times $t = 10$ (black curve) and 12 (blue curve). Dashed lines (in red) indicate the exponential profile of the form $|\psi_n|^2 \propto \exp(-n/\xi)$ with ξ being the localization length. (c) The growth rate α of the mean energy versus λ for $g_0 = 0.1$ (squares), 10^{-3} (circles), and 10^{-5} (triangles). The red line indicates our theoretical prediction $\alpha \propto K\lambda/\hbar_{\rm eff}$ in Eq. (14). Other parameters are $K = 5$, $\hbar_{\rm eff} = 0.5$, and $L = 2\pi$. (d) The growth rate $|D|$ versus λ with $K = 5$ for $\hbar_{\rm eff} = 0.5$ (circles) and 0.25 (squares). Red lines indicate the theoretical prediction $|D| = \hbar_{\rm eff}/(2K\lambda)$ in Eq. (10). Inset: The threshold time t_c versus $\ln(g_0)$ for $K = 5$, $\lambda = 0.5$, and $\hbar_{\rm eff} = 0.1$. The solid line indicates our theoretical prediction in Eq. (10).

with $\alpha \propto K\lambda/\hbar_{\rm eff}$ and $\beta \propto g_0^2$. As a further step, we numerically calculate the growth rate α for different λ, as shown in Fig. 2(c). One can see that the growth rate α increases linearly with λ. Moreover, its change with respect to g_0 is negligible, coinciding with our theoretical prediction that $\alpha \propto K\lambda/\hbar_{\rm eff}$. The corresponding probability distribution in eigenstate space is shown in Fig. 2(b), which demonstrates the exponentially decayed profile, i.e., $|\psi_n|^2 \propto \exp(-n/\xi)$ with ξ being the localization length. Taking into account $\langle p^2(t)\rangle \propto \hbar_{\rm eff}^2\xi^2$, the localization length will increase in the superexponential way, which is dramatically different from the phenomenon of the DL.

From Fig. 2(a), one can also see that the threshold time t_c for the appearance of the SED decreases with the increase of nonlinear strength g_0. Numerical results of t_c for different g_0 are depicted in the inset of Fig. 2(d), which demonstrates the good agreement with the analytical formula in Eq. (10). To further confirm our analytical analysis, we numerically investigate the growth rate D of t_c for different λ. The numerical results are in good agreement with the theoretical prediction, i.e., $D = -\hbar_{\rm eff}/(2K\lambda)$ [see Fig. 2(d)]. We want to stress that we have also numerically investigated the system with the periodic boundary condition, which is just the $\mathcal{PT}$-symmetric extension of the kicked rotor model. This system exhibits the same SED, which can be predicted by our theory as well. We believe that the superexponential diffusion is a general phenomenon that goes beyond the simple model systems studied in this work. The universality of superexponential diffusion in other non-Hermitian nonlinear systems is left for our future studies.

III. THEORETICAL ANALYSIS

We concentrate on the case of the $\mathcal{PT}$-symmetry-breaking phase, i.e., $K\lambda/\hbar_{\rm eff} \gg 1$, for which the norm exponentially increases with time: $\mathcal{N} \approx \exp(K\lambda t/\hbar_{\rm eff})$. As an estimation, we analyze the time evolution of the quantum state at the point $x_0 = \pi/2$, which corresponds to the maximal value of the imaginary part of the kicking potential, i.e., $V_{\rm i}(x_0) = K\lambda\sin(x_0) = K\lambda$. After several kicking periods, the quantum state is extremely centered at x_0, since the action of the imaginary kicking term of the Floquet operator $U_{\rm K}^{\rm i}(x_0) = \exp(K\lambda/\hbar_{\rm eff})$ on a quantum state can greatly amplify the probability amplitude of the state in x_0 if $K\lambda/\hbar_{\rm eff} \gg 1$. Accordingly, the time evolution of the probability amplitude for $x_0 = \pi/2$ is approximately given by

$$|\psi(x_0,t)|^2 \propto \mathcal{N}(t) \propto \exp\left(2\frac{K\lambda}{\hbar_{\rm eff}}t\right)|\psi(x_0,0)|^2. \quad (7)$$

As a consequence, the nonlinear interaction strength increases exponentially as

$$g_0|\psi(x_0,t)|^2 \propto g_0|\psi(x_0,0)|^2 \exp\left(2\frac{K\lambda}{\hbar_{\rm eff}}t\right). \quad (8)$$

Our previous investigations on a $\mathcal{PT}$-symmetric kicked rotor model demonstrate that, in the $\mathcal{PT}$-symmetry-breaking phase, the imaginary part of the kicking potential leads to the formation of the localized modes in both real and momentum space [70,71]. Interestingly, such localized modes can be approximately described by Gaussian wave packets [70]. That is to say, the $\mathcal{PT}$-symmetric kicking potential induces the localization in x_0; meanwhile, the quantum state is localized in a specific position p_0 in momentum space. It is worth noting that the effects of nonlinear interaction lead to the spreading of the quantum state in different momentum sites, that is, the delocalization in momentum space. Based on this, we propose a competing mechanism to explain the appearance of SED.

In this system, there is a competition between the non-Hermitian kicking potential and the nonlinear interaction. The non-Hermitian kicking potential leads to localization. Meanwhile, the nonlinear interaction destroys the localization. At the initial time, the nonlinear interaction at x_0, i.e., $g_0|\psi(x_0,0)|^2$, is much smaller than the imaginary part of the non-Hermitian kicking potential. Thus, during the short-time evolution, the dynamics of this system is governed by the non-Hermitian kicking potential. When the nonlinear interaction strength exceeds the imaginary kicking strength, the effects of nonlinear interaction dominate the dynamical behavior of the system, consequently causing the appearance of the SED. It is hence straightforward to get the threshold time t_c by

$$g_0|\psi(x_0,t)|^2 = K\lambda. \quad (9)$$

Combining Eqs. (8) and (9) yields the relation

$$t_c \propto -\frac{\hbar_{\text{eff}}}{2K\lambda}\ln(g_0) + \frac{\hbar_{\text{eff}}}{2K\lambda}\ln\left(\frac{K\lambda}{|\psi(x_0,0)|^2}\right), \tag{10}$$

which is confirmed by our numerical results [see the inset of Fig. 2(d)].

Next, we evaluate the time dependence of the mean kinetic energy. Previously, we developed a hybrid quantum-classical (HQC) theory to explain the exponential diffusion induced by temporally modulated nonlinear interactions [44,45]. Our mathematical analysis is based on the investigation of a periodically modulated rotor model whose dynamics is governed by the nonlinear Schrödinger equation. We find the mathematical equivalence between this system and a generalized kicked rotor (GKR) model. Our HQC theory gets the iterative equation of mean energy from the classical mapping equations, which yields the law of the exponential diffusion [44,45]. A detailed derivation of the HQC theory can be found in Refs. [44,45]. Our diffusion-based theory is verified by the another method with integration of the Schrödinger equation [43].

Here, we make an extension of the theory to systems with non-Hermiticity. Our HQC theory predicts an iterative equation of energy,

$$\langle p^2(t+1)\rangle \approx \langle p^2(t)\rangle + Cg^2(t)\langle p^2(t)\rangle, \tag{11}$$

where the time-dependent nonlinear interaction strength is defined as $g(t) = g_0|\psi(x,t)|^2$, and C is an unimportant constant (see Refs. [44,45] for the derivation details of the above equation). As an estimation, we use $|\psi(x_0,t)|^2$ to replace $|\psi(x,t)|^2$, which is reasonable since $|\psi(x_0,t)|^2$ accounts the maximal contribution. Substituting Eq. (8) into Eq. (11) yields

$$\langle p^2(t+1)\rangle \approx \langle p^2(t)\rangle + Cg_0^2\exp\left(\frac{4K\lambda t}{\hbar_{\text{eff}}}\right)\langle p^2(t)\rangle. \tag{12}$$

In the continuous-time limit, the above equation yields

$$\frac{d\ln(\langle p^2\rangle)}{dt} \approx Cg_0^2\exp\left(\frac{4K\lambda t}{\hbar_{\text{eff}}}\right). \tag{13}$$

Therefore, the time dependence of the mean kinetic energy takes the form

$$\langle p^2(t)\rangle \propto \exp\left[g_0^2\exp\left(\frac{4K\lambda}{\hbar_{\text{eff}}}t\right)\right]. \tag{14}$$

The validity of our theoretical prediction is confirmed by the numerical results of the growth rate of mean energy, i.e., $\alpha \propto K\lambda/\hbar_{\text{eff}}$ [see Fig. 2(c)]. We want to stress that our study focuses on the regime away from quantum resonance, i.e., $\hbar_{\text{eff}} \neq 4\pi r/s$ with r and s coprime integers. Although in the quantum main-resonance case $\hbar_{\text{eff}} = 4\pi$ the intensity of the quantum state increases in the way of Eq. (8), the mean square of momentum does not obey the iterative equation in Eq. (11). As a consequence, the quantum diffusion is not superexponentially fast. We leave the quantum diffusion in quantum resonance situation for further investigation.

We would like to mention that the phenomenon of SED also occurs in cases with a non-Hermitian kicking potential, i.e., $V_{\text{K}}(x) = (K + i\lambda)\cos(x)\sum_n \delta(t-n)$, where K and λ indicate the strength of the real and imaginary parts of the kicking potential. Our consideration of the $\mathcal{PT}$-symmetric potential in the present work is inspired by the fact that this kind of system could display fruitful physics due to the $\mathcal{PT}$-symmetry-breaking transition. Another reason for us to investigate the $\mathcal{PT}$-symmetric kicking potential is based on the experimental interest, i.e., the realization of this kind of kicking potential by using the optical setting of Fabry-Pérot optical resonator with intracavity phase and loss gratings, which was first proposed in Ref. [68]. Based on this proposal, we provide an experimental setup of a $\mathcal{PT}$-symmetric optical system with periodically placed Kerr media and phase gratings. This system has the advantage of the controllability of the kicking times by engineering the numbers of the layers of both the Kerr media and phase grating.

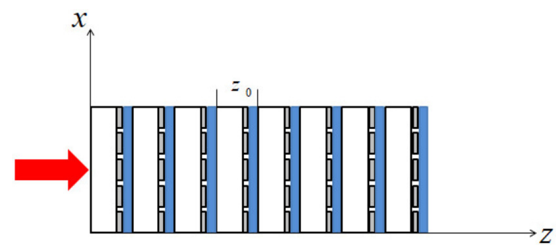

FIG. 3. Schematic illustration of our proposed optical system with multilayer media, where the blue layers denote Kerr media and the gray layers represent the phase gratings.

As a further step, we propose an optical setup to emulate the wave dynamics described by the Hamiltonian in Eq. (1). Optical waveguides provide an ideal platform for the observation of the wave-packet transport in $\mathcal{PT}$-symmetric systems [60,62,69,73,83–86]. Under the paraxial approximation, the propagation of light is governed by an equation mathematically equivalent to the Schrödinger equation [87,88], where the longitude dimension of light mimics the time variable. We consider an optical system consisting of a periodic sequence of multilayers of phase gratings and Kerr media in the propagation direction (see Fig. 3), which is a modification of the realization of the kicked rotor model using optical settings [68,89–91]. It was proposed that the effect of sinusoidal and quarter-wave-shifted gratings is described by the $\mathcal{PT}$-symmetric potential, which means that these gratings introduce the "gain-or-loss" mechanisms to the system [68]. The Kerr effects of media induce an intensity-dependent nonlinear term in Eq. (1). To realize the delta kicks in time, both the sizes of the phase grating and Kerr media in the propagation direction z should be much smaller than the period of the optical sequence. The light is trapped in the transverse dimension by waveguides, which resembles the reflective boundary condition of an infinite square well in Eq. (1). The propagation of light in such an optical system is governed the Hamiltonian in Eq. (1). Therefore, our finding of the SED is within reach of current experiments and may shed light on the fundamental problems of quantum diffusion.

IV. SUMMARY

We investigate, both numerically and analytically, the SED in a $\mathcal{PT}$-symmetric kicking system. The underlying physics of such an intrinsic phenomenon is the positive feedback mechanism of the nonlinearity, which turns the exponential growth of the intensity of wave packets in the $\mathcal{PT}$-symmetry-breaking phase into the kinetic energy. Our theoretical prediction of the threshold time for the appearance of the SED and the law of SED are in good agreement with numerical results. This behavior is of particular importance in the field of Fermi acceleration, where the process for accelerating cosmic ray particles to large energy scales is still an open question [5,92].

The effect of nonlinear interaction on the AL and DL is a long-standing problem. Most of the investigations concentrate on the case with static nonlinear interactions; namely, it is not time dependent. Although there are no strict theoretical conclusions on this issue, extensive numerical experiments have demonstrated the power-law diffusion of particles due to the destruction of the localization by nonlinear effects [21–27]. Remarkably, analytical predictions of the Anderson localization of Bogoliubov quasiparticles under the interatomic interaction has been shown in Refs. [28–31]. The unprecedented control of nonlinear interactions by the Feshbach resonance in ultracold atoms and by the femtosecond laser writing technique in nonlinear optics opens the opportunity for investigating the quantum diffusion under periodically modulated nonlinearity. There are only a few studies on the fate of AL and DL in the presence of temporally modulated nonlinear interactions [41–45]. Therefore, our finding of the SED is of significance, especially for the engineering of diffusion in recent experiments.

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