Detectors interacting through quantum fields: Non-Markovian effects, nonperturbative generation of correlations, and apparent noncausality

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We study a system of two localized detectors (oscillators) interacting through a massless quantum field in a vacuum state via an Unruh-DeWitt coupling. This system admits an exact solution and provides a good model for addressing fundamental issues in particle-field interactions, causality, and locality in quantum field measurements that are relevant to proposed quantum experiments in space. Our analysis of the exact solution leads to the following results. (i) Common approximations used in the study of analogous open quantum systems fail when the distance between the detectors becomes of the order of the relaxation time. In particular, the creation of correlations between remote detectors is not well described by ordinary perturbation theory and the Markov approximation. (ii) There is a unique asymptotic state that is correlated; it is not entangled unless the detector separation is of the order of magnitude of the wavelength of the exchanged quanta. (iii) The evolution of seemingly localized observables is noncausal. The latter is a manifestation of Fermi's two-atom problem, albeit in an exactly solvable system. We argue that the problem of causality requires a reexamination of the notion of entanglement in relativistic systems, in particular, the physical relevance of its extraction from the quantum vacuum.

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I. INTRODUCTION

Understanding how spatially separated quantum systems interact via relativistic quantum fields becomes increasingly important. Many proposed quantum experiments in space lie in the regime where relativistic effects are important and may even provide tests of new physics [1]. Our ability to construct entangled states of atoms at large separations will reach a regime where the retarded propagation of photons will be a significant factor, thus allowing us to explore experimentally the relations between entanglement and relativistic causality. Furthermore, the interplay between localization and causality is a source of long-standing puzzles in the foundations of quantum field theory (QFT).

In this paper, we study an *integrable* model that allows us to address issues such as the above. The model consists of two harmonic oscillators interacting with a quantum field through the Unruh-DeWitt (UdW) coupling [2–4]. The field lies initially in the vacuum state. The harmonic oscillators can be viewed as particle detectors or as crude approximations to atoms (N-level systems). We find and analyze the exact solution to the system, to conclude the following.

(i) Common approximations that are employed in the treatment of analogous quantum systems (Markov approximation, Wigner-Weisskopf approximation, perturbative master equation) fail if the separation of the two detectors becomes of the order of the relaxation time. In particular, the above approximations break down completely in processes that involve the exchange of information between far separated detectors. While this result is derived in a specific model system, its context its quite generic for open quantum systems. In particular, it suggests that at least some entangled states for atoms at large separations decay nonexponentially.

(ii) There is a unique asymptotic state of the system. This state is correlated, however, correlations are suppressed at large separations between the two detectors. For distances of the order of the wavelength of the exchanged quanta, the asymptotic state is entangled. The generated entanglement evolves significantly at times of the order of the relaxation scale.

(iii) If we assume that the variables pertaining to detectors are localized quantum observables, then the reduced dynamics of the detector are noncausal. This is a manifestation of the famous Fermi two-atom problem (see below). Having an integrable solution allows us to show that this behavior is not an artifact of an approximation in the derivation of the dynamics. We believe that this noncausal behavior is fundamentally kinematical: we need to identify observables that also involve the field degrees of freedom in order to describe localized measurements. In other words, physical observables must be renormalized.

This conclusion implies that entanglement generated between the detectors may not be a physically meaningful quantum resource to harvest.

The context of our results is the following.

A. Non-Markovian dynamics

A localized quantum system, such as an atom, in an excited state decays to the vacuum through its interaction with a

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quantum field, even if the latter is in the vacuum state. Such decays are typically exponential. When the system is treated in the theory of open quantum systems [5,6], the exponential decay law arises as a consequence of Markovian open system dynamics.

Markovian dynamics are generic for weak coupling of the system to environment. The second-order Markovian master equation becomes exact at the van Hove limit [7]. In this limit, the system-environment coupling λ goes to 0, while the rescaled time $\lambda^2 t$ remains constant. This limit provides an excellent approximation for a large class of systems, especially in atom optics. However, comparison with exactly solvable models—as, for example, in quantum Brownian motion [8]—shows many regimes in which the second-order master equation fails. In particular, the van Hove limit may not be physically relevant when the open system dynamics are characterized by several long-time scales. This occurs, for example, if the environment has resonance frequencies or thresholds [9]. In this paper, we present another case of failure of the Markov approximation, when the time scale of retarded propagation is of the same order of magnitude as the decay or dissipation time.

The study of non-Markovian dynamics in open quantum systems has seen increased emphasis in recent years, because of the relevance of non-Markovian behavior to many different physical contexts, for example, condensed matter physics, quantum control, quantum biology, and quantum optics (see Ref. [10] and references therein). Our ability to prepare entangled states in multipartite systems provides technical and conceptual challenges to the theory of open quantum systems, because they go beyond the traditional paradigm of a central, localized system weakly interacting with an environment.

Consider, for example, two atoms prepared in an entangled state, separated by distance r and interacting with a quantum electromagnetic field. For small separations, this system is well described by the second-order master equation (see, for example, Ref. [11]). However, as the separation increases, approximations involved in the derivation of the master equation break down, for example, the rotating wave approximation [12,13]. When r becomes comparable to the decay time Γ^{-1}, the van Hove limit stops being a useful approximation, because it misrepresents strong effects due to retarded propagation. Simply by analyzing the mathematical assumptions involved in the Markov approximation, we expect the decay of an entangled pair of atoms to be strongly non-Markovian when Γr becomes of order unity or larger. This expectation is verified by our analysis.

Note that this breakdown of Markovian behavior is a nonperturbative effect: Γ is proportional to the coupling constant squared, but we can always find a distance r such that $\Gamma r \sim 1$. For atomic states relevant to entanglement experiments, the relevant length scale may be of the order of hundreds of meters or kilometers. Hence, the breakdown of Markovianity appears at scales relevant to macroscopic quantum phenomena.

B. Fermi's two-atom system

The two-atom system is a classic model for propagation of information through quantum fields. It was first studied by Fermi [14]. Fermi assumed that at time $t = 0$, atom A is in an excited state and atom B in the ground state. He asked when B will notice A and move from its ground state. In accordance with Einstein locality, he found that this happens only at times greater than r. It took about 30 years for Shirokov to point out that Fermi's result is an artifact of an approximation [15].

Several studies followed with conclusions depending on the approximations used [16]. It was believed that noncausality is due to the use of bare initial states and that it would not be present in a renormalized theory. However, Hegerfeldt showed that non-causality is generic [17,18], as it depends only on the assumption of energy positivity and on the existence of systems that are localized in disjoint space-time regions (see also the critique in [19]). The two-atom problem is a genuine problem of quantum theory that pertains to the definability of local observables and the meaning of locality in relation to quantum measurements.

C. Entanglement generation

It is well known that two systems that do not directly interact may become entangled through their interaction with a third system. This general result also applies to localized systems (detectors) interacting with the quantum field. The detectors may develop entanglement even if the field lies on its ground state [20]. This process is called *entanglement harvesting* and it has been extensively studied for different initial detector states, detector trajectories, or space-time geometries (see, for example, [21–23]). Interestingly, this process of entanglement creation may also take place between objects that remain spacelike separated, i.e., in some models, entanglement is seemingly generated outside the light cone [24–26].

However, it is far from obvious that the usual notion of entanglement, defined with reference to nonrelativistic physics, is an appropriate quantum resource for relativistic systems described by QFT. A proper quantum resource should be compatible with strong locality and causality constraints on acceptable physical observables that are required by QFT. Indeed, Fermi's problem is an indication that special care is needed in identifying acceptable local observables in a relativistic quantum system.

D. Our model

In this paper, we study the causal propagation of information between two separated Unruh-DeWitt detectors [2–4], rather than between two atoms. An Unruh-DeWitt detector is a pointlike quantum system that interacts with a quantum scalar field through a dipole coupling that mirrors the coupling of atoms to the electromagnetic field.[1] Here, we focus on static detectors; we do not consider effects due to the detectors' motion.

The main benefit of using the UdW detectors for studying information transfer in QFT is that they admit exact solutions. In particular, (i) if the self-Hamiltonian of each detector corresponds to a harmonic oscillator, and (ii) if the initial state

[1]Note that in many references the term "Unruh-DeWitt detector" is used only for pointlike *two-level* systems interacting with a field through a dipole coupling.

of the field is Gaussian, then the system of N detectors interacting with the quantum field is mathematically equivalent to a quantum Brownian motion (QBM) model [27] for N oscillators in a bath modeled by harmonic oscillators. This QBM model is exactly solvable [8,28,29]. Hence, we can compare the predictions of any approximation with those of the exact solution. The model considered here has also been studied by Lin and Hu [30] (see also [26] and [31]) for the same Hamiltonian but different detector trajectories. Reference [30] employs a very different approximation scheme and focuses on a different set of issues. Entanglement generation is a common issue, and there our results are compatible. However, we differ on the analysis of causality.

E. Structure of the paper

The structure of this paper is the following. In Sec. II, we present the general solution to the QBM model with N-system oscillators interacting with an environment, and we show that the system of two detectors interacting through a scalar field is a special solution. In Sec. III, we find the explicit solution to the two-detector system and prove that the Markov approximation breaks down completely for the transfer of information between remote detectors. In Sec. IV, we identify a unique asymptotic state that is correlated and show that it is entangled at small separations. In Sec. V, we show that this model manifests the same noncausal behavior with Fermi's two-atom system, and we discuss the implications and how causality can be restored. Section VI concludes the paper.

II. THE MODEL

A. QBM models

The theory of open quantum systems studies the evolution of a quantum system S in interaction with an environment E. Typically, one assumes unitary evolution for the joint system S + E that includes the system and the environment and studies observables that pertain solely to S. In many problems, all information about such observables is contained in the *autonomous* time evolution of the *reduced density matrix* $\hat{\rho}_S(t)$ of the system, often expressed in terms of a master equation.

Quantum Brownian motion is one of the main paradigms in the theory of open quantum systems. In QBM, the system S consists of one or more particles. The environment is modeled as a thermal bath. It consists of a large number of harmonic oscillators initially in a thermal state. The number of environmental oscillators can be taken to infinity. The coupling term in the Hamiltonian is linear with respect to either the position or the momentum of the environmental oscillators. A key feature of QBM models is that the effects of the environment are contained in two generalized functions, the dissipation kernel and the noise kernel [27].

If the Hamiltonian of the system S is quadratic to positions and momenta of the system particles (harmonic oscillators and free particles) and the interaction Hamiltonian is linear with respect to the same variables, then the total system is integrable, and the reduced dynamics of the density matrix are *exactly solvable* [8]. The term "exactly solvable" is used in the sense of integrability: the exact time evolution of the reduced density matrix $\hat{\rho}_S(t)$ is expressed in terms of a *finite* number of functions of time that admit *explicit* definitions.[2] This is to be contrasted, for example, with perturbative evaluations of the dynamics, which typically involve an infinite number of functions of time that are defined recursively rather than explicitly. Exactly solvable QBM models essentially implement a full resummation of the perturbation series for the evolution operator.

Being exactly solvable the models above can be employed in order to test the validity of common approximations that are employed in the derivation of evolution equations for open quantum systems. In particular, they can be used to test the very common Markov assumption, i.e., the assumption that memory effects are negligible.

B. QBM in a multipartite system

1. The Hamiltonian

We consider a system of N harmonic oscillators of masses M_α and frequencies Ω_α interacting with a heat bath. The bath is modeled by a set of harmonic oscillators of masses m_i and frequencies ω_i. The Hamiltonian of the total system is

$$\hat{H} = \hat{H}_{\rm syst} + \hat{H}_{\rm env} + \hat{H}_{\rm int}, \tag{1}$$

where

$$\hat{H}_{\rm syst} = \sum_\alpha \left(\frac{1}{2M_\alpha}\hat{P}_\alpha^2 + \frac{M_\alpha \Omega_\alpha^2}{2}\hat{X}_\alpha^2 \right), \tag{2}$$

$$\hat{H}_{\rm env} = \sum_i \left(\frac{1}{2m_i}\hat{p}_i^2 + \frac{m_i \omega_i^2}{2}\hat{q}_i^2 \right), \tag{3}$$

$$\hat{H}_{\rm int} = \sum_i \sum_\alpha c_{i\alpha}\hat{X}_\alpha \hat{q}_i, \tag{4}$$

where $c_{i\alpha}$ are coupling constants.

Since the total Hamiltonian is quadratic with respect to all positions and momenta, the evolution operator $e^{-i\hat{H}t}$ can be explicitly constructed, and its position matrix elements are Gaussian.

We consider a factorized initial condition $\hat{\rho}_{\rm sys} \otimes \hat{\rho}_{\rm env}$ for the total system. If $\hat{\rho}_{\rm env}$ is Gaussian, then the reduced density matrix propagators can be computed explicitly. For $N = 1$, the reduced dynamics leads to the Hu-Paz-Zhang master equation [8].

In general, the assumption of a factorized initial condition between field and detectors is meaningful only as far as the field modes with energies of the order of the frequencies Ω_α are concerned. There is no preparation that can enforce separability for photons at the infrared and ultraviolet edges of the spectrum. However, a nonfactorized initial condition does not allow us to consider general initial states for the field [32], and in many model systems, including QBM, the effect of the nonfactorizing initial state dies out after a time scale of the order of a high-frequency cutoff [33].

[2]This does not mean that these functions can be evaluated exactly. Their evaluation may involve approximation schemes or be numerical.

2. The Wigner function propagator

In this paper, we employ the solution to the multipartite QBM model in the Wigner representation, first derived by Halliwell and Yu [34] (see also [35]). In particular, we employ the general solution of Ref. [28] for N system oscillators; for an alternative derivation of the dynamics for N system oscillators, see Ref. [29].

The key point in the derivation of [28] is that the propagator of the Wigner function for the reduced system is Gaussian, whose coefficients can be exactly evaluated. In particular, the propagator is described by two matrices: matrix $R(t)$, that corresponding to the classical dissipative equations of motion; and matrix $S(t)$, that containing the effect of environment-induced diffusion.

As we employ the results in Ref. [28], we do not consider the evolution of the environment degrees of freedom. Hence, all states and observables described in what follows refer solely to the system degrees of freedom. We therefore drop the subscript S in describing the reduced state of the system.

The Wigner function for a reduced density matrix $\hat{\rho}$ that describes N particles is given by

$$W(\mathbf{X},\mathbf{P}) = \frac{1}{(2\pi)^N}\int d\zeta\, e^{-\imath\,\mathbf{P}\cdot\zeta}\,\hat{\rho}\left(\mathbf{X}+\frac{1}{2}\zeta,\mathbf{X}-\frac{1}{2}\zeta\right). \quad (5)$$

We use the coordinates $\xi^a = (X_1, X_2, \ldots, X_N, P_1, P_2, \ldots, P_N)$ in phase space; the Wigner function is expressed as $W(\xi)$.

The dynamics in the Wigner picture is implemented by the Wigner function propagator $K_t(\xi_f, \xi_0)$, namely, a kernel that evolves the initial Wigner function W_0 to the Wigner function W_t at time t,

$$W_t(\xi_f) = \int \frac{d^{2N}\xi_0}{(2\pi)^N} K_t(\xi_f,\xi_0) W_0(\xi_0). \quad (6)$$

For QBM models, the Wigner function propagator is Gaussian. The most general form of a Gaussian propagator is

$$K_t(\xi_f,\xi_0) = \frac{\sqrt{\det S^{-1}}}{\pi^N}\,\exp\Big[-\frac{1}{2}\big[\xi_f^a-\xi_{cl}^a(t)\big] \times S^{-1}_{ab}(t)\big[\xi_f^b-\xi_{cl}^b(t)\big]\Big], \quad (7)$$

where S_{ab} is a positive definite matrix and

$$\xi_{cl}^a(t) = R_b^a(t)\xi_0^b. \quad (8)$$

Matrix R_b^a defines the solution to the classical equations of motion. Matrix S_{ab} determines the evolution of the environment-induced fluctuations. To see this, we consider the correlation matrix

$$V_{ab} := \tfrac{1}{2}\mathrm{Tr}[\hat{\rho}(\hat{\xi}_a\hat{\xi}_b+\hat{\xi}_b\hat{\xi}_a)] - \mathrm{Tr}(\hat{\rho}\hat{\xi}_a)\mathrm{Tr}(\hat{\rho}\hat{\xi}_b). \quad (9)$$

By Eq. (7),

$$V(t) = R(t)V(0)R^T(t) + S(t), \quad (10)$$

where V_0 is the correlation matrix of the initial state.

The explicit form of matrices R and S was derived in Ref. [28]. They depend on two kernels, the *dissipation kernel*,

$$\gamma_{\alpha\alpha'}(s) = -\sum_i \frac{c_{i\alpha}c_{i\alpha'}}{2m_i\omega_i^2}\sin(\omega_i s), \quad (11)$$

and the *noise kernel*,

$$\nu_{\alpha\alpha'}(s) = \sum_i \frac{c_{i\alpha}c_{i\alpha'}}{2m_i\omega_i^2}\coth\left(\frac{\omega_i}{2T}\right)\cos(\omega_i s), \quad (12)$$

which also characterize the path integral description of QBM [8,27]. The crucial step in the determination of matrices R and S is to find the solution to the homogeneous part of the linear integrodifferential equation [28]:

$$\ddot{\hat{X}}_\alpha(t) + \Omega_r^2\hat{X}_\alpha(t) + \frac{2}{M_\alpha}\sum_{\alpha'}\int_0^t ds\gamma_{\alpha\alpha'}(t-s)\hat{X}_{\alpha'}(s) = \sum_i \frac{c_{i\alpha}}{M_\alpha}\hat{q}_i^0(t). \quad (13)$$

The solution of Eq. (13) is

$$\hat{X}_a(t) = \sum_\alpha\left(\dot{u}_{\alpha\alpha'}(t)\hat{X}_{\alpha'} + \frac{1}{M_{\alpha'}}u_{\alpha\alpha'}(t)\hat{P}_{\alpha'}\right) + \sum_{\alpha'}\frac{1}{M_{\alpha'}}\int_0^t ds u_{\alpha\alpha'}(t-s)\sum_i c_{i\alpha'}\hat{q}_i^0(s), \quad (14)$$

where $u_{\alpha\alpha'}(t)$ is the solution of the homogeneous part of Eq. (13) with initial conditions $\dot{u}_{\alpha\alpha'}(0) = \delta_{\alpha\alpha'}$ and $u_{\alpha\alpha'}(0) = 0$. Equation (13) is essentially the classical equation of motion with a non-local-in-time dissipation term defined by the dissipation kernel.

Given the solution $u(t)$, we define matrix R as

$$R = \begin{pmatrix} \dot{u}(t) & u(t)M^{-1} \\ M\ddot{u}(t) & M\dot{u}(t)M^{-1}\end{pmatrix}, \quad (15)$$

where $M = \mathrm{diag}(M_1, \ldots, M_N)$ is the mass matrix for the system.

The matrix elements of S are given by

$$S_{X_\alpha X_{\alpha'}} = \sum_{\beta\beta'}\frac{1}{M_\beta M_{\beta'}}\int_0^t ds\int_0^t ds' u_{\alpha\beta}(s)\nu_{\beta\beta'}(s-s')u_{\beta'\alpha'}(s'), \quad (16)$$

$$S_{P_\alpha P_{\alpha'}} = M_\alpha M_{\alpha'}\sum_{\beta\beta'}\frac{1}{M_\beta M_{\beta'}}\int_0^t ds \times\int_0^t ds'\dot{u}_{\alpha\beta}(s)\nu_{\beta\beta'}(s-s')\dot{u}_{\beta'\alpha'}(s'), \quad (17)$$

$$S_{X_\alpha P_{\alpha'}} = M_{\alpha'}\sum_{\beta\beta'}\frac{1}{M_\beta M_{\beta'}}\int_0^t ds \times\int_0^t ds' u_{\alpha\beta}(s)\nu_{\beta\beta'}(s-s')\dot{u}_{\beta'\alpha'}(s'). \quad (18)$$

C. Two UdW detectors

We consider a system of two identical static harmonic oscillators of mass $M = 1$ and frequency Ω interacting with a scalar field through the UdW interaction Hamiltonian. The Hamiltonian of the total system form, where we assume that the detectors are localized at $x = x_1$ and $x = x_2$, is

$$\hat{H}_{\mathrm{int}} = \lambda\left(\int d^3x\hat{\phi}(x)\hat{q}_1\delta^3(x-x_1) + \int d^3x\hat{\phi}(x)\hat{q}_2\delta^3(x-x_2)\right), \quad (19)$$

where λ is a coupling constant.

For a free scalar field, the total Hamiltonian,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{\omega_k}} (\hat{a}(k)e^{ik\cdot x} + \hat{a}^\dagger(k)e^{-ik\cdot x}), \tag{20}$$

is a special case of the QBM Hamiltonian.

We compare the interaction term, (19), with the general form of the QBM interaction Hamiltonian,

$$\hat{H}_{\rm int} = \sum_i \sum_\alpha c_{i\alpha} \hat{X}_\alpha \hat{q}_i. \tag{21}$$

We note that the index i, which labels the environmental oscillators, corresponds to three momenta k, $m_i = 1$, $\omega_k = |k|$, and $c_{k\alpha} = \frac{\lambda}{\sqrt{2\omega_k}} e^{\imath k x_\alpha}$.

With the identifications above, the evaluation of the dissipation kernel is straightforward. By Eq. (11),

$$\gamma(s) = \gamma_0(s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \gamma_r(s) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{22}$$

where

$$\gamma_0(s) = -\frac{\lambda^2}{8\pi^2} \int_0^\infty dk \sin(ks), \tag{23}$$

$$\gamma_r(s) = -\frac{\lambda^2}{8\pi^2 r} \left[\int_0^\infty dk \frac{\sin(kr)\sin(ks)}{k} \right]. \tag{24}$$

The function $\gamma_0(s)$ is the dissipation kernel of the one-detector system [36].

In QBM models, expressions such as the above for the dissipation kernel are regularized through the introduction of an ultraviolet cutoff Λ. For example, for a single system oscillator the dissipation kernel is often expressed as $\gamma(s) = \int_0^\infty I(k)\sin(ks)$, where $I(k)$ is a function known as the *spectral density* of the environment [27]. The spectral density is typically assumed to decay rapidly for $k > \Lambda$, where Λ is a cutoff frequency.

For condensed-matter reservoirs, the cutoff frequency is a natural characteristic of the reservoir. If the environment corresponds to a quantum field, as in the present case or as in models of quantum optics, then Λ is introduced by hand, and it corresponds to a regime where the effective description of particles (system) interacting with a quantum field (environment) fails. For example, in quantum optics, Hamiltonians for particle-field interaction are derived subject to the dipole approximation. This asserts that the size r_0 of the particle is much smaller than the typical wavelengths of the field with which it interacts. Hence, particle-field interactions of this type do not make sense for frequencies of the order of r_0^{-1} and higher. For this reason, the introduction of a cutoff Λ of order r_0^{-1} is physically meaningful. It is also mathematically essential because it regulates divergences. In general, the introduction of a cutoff in QFT systems is rather arbitrary: physical predictions from such models are meaningful only if they are cutoff independent.

As it turns out, the introduction of an ultraviolet cutoff Λ is essential for $\gamma_0(s)$ in Eq. (23), in order to avoid divergences. In principle, we should introduce the same cutoff Λ to γ_r, however, γ_r is little affected unless r is of the order of Λ^{-1} or smaller. Alternatively, we can regularize γ_0 by equating it with γ_{r_0} for some $r_0 \ll r$.

By Eq. (12), the noise kernel is

$$\nu(s) = \nu_0(s) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \nu_r(s) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{25}$$

where

$$\nu_0(s) = \frac{\lambda^2}{8\pi} \delta(s), \tag{26}$$

$$\nu_r(s) = \nu_{21}(s) = \frac{\lambda^2}{32\pi r} [\mathrm{sgn}(r-s) + \mathrm{sgn}(r+s)]. \tag{27}$$

III. THE CLASSICAL EQUATIONS OF MOTION

A. The inverse Laplace transform

As explained in Sec. II B 2, the reduced dynamics of the system are expressed in terms of the Wigner function propagator. The latter is constructed from the knowledge of matrices $R(t)$ and $S(t)$. Matrix $R(t)$ corresponds to the classical equation of motion of the system variables, and it is constructed from the knowledge of the function $u(t)$, which is a solution to the classical equation of motion

$$\ddot{u}_\alpha(t) + \Omega_r^2 u_\alpha(t) + 2\sum_{\alpha'} \int_0^t ds \gamma_{\alpha\alpha'}(t-s) u_{\alpha'}(s) = 0, \tag{28}$$

subject to the initial conditions $\dot{u}_{\alpha\alpha'}(0) = \delta_{\alpha\alpha'}$ and $u_{\alpha\alpha'}(0) = 0$.

Next, we evaluate the solutions $u_{\alpha\alpha'}(t)$ of the classical equation of motion, (13). Since Eq. (13) is linear, it can be solved by a Laplace transform. It is straightforward to evaluate the Laplace transform $\tilde{u}(z)$ of $u(t)$ as $A^{-1}(z)$, where $A(z)$ is the 2×2 matrix with elements

$$A_{\alpha\alpha'}(z) = (z^2 + \Omega_\alpha^2)\delta_{\alpha\alpha'} + 2\tilde{\gamma}_{\alpha\alpha'}(z), \tag{29}$$

where $\tilde{\gamma}_{\alpha\alpha'}(z)$ is the Laplace transform of the dissipation kernel. The Laplace transforms of γ_0 and γ_r are

$$\tilde{\gamma}_0(z) = -\frac{\lambda^2}{16\pi^2} \ln\left(1 + \frac{\Lambda^2}{z^2}\right) \simeq -\frac{\lambda^2}{8\pi^2} \ln\left(\frac{\Lambda}{z}\right), \tag{30}$$

$$\tilde{\gamma}_r(z) = -\frac{\lambda^2}{16\pi r z} [e^{-rz}\bar{\mathrm{Ei}}(rz) - e^{rz}\mathrm{Ei}(-rz)], \tag{31}$$

where we have simplified $\gamma_0(z)$ by assuming that the relevant values of z satisfy $|z| \ll \Lambda$; Ei stands for the exponential integral function, defined by [37]

$$\mathrm{Ei}(z) = \gamma + \ln|z| + \sum_{z=1}^\infty \frac{z^n}{n!n}, \tag{32}$$

where γ is the Euler-Mascheroni constant and $\bar{\mathrm{Ei}}(z) = \mathrm{Ei}(\bar{z})$.

It follows that

$$\tilde{u}(z) = \frac{1}{2}\left[\frac{1}{z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) + 2\tilde{\gamma}_r(z)} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) - 2\tilde{\gamma}_r(z)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]. \tag{33}$$

Hence, $u(t)$ takes the form

$$u(t) = \frac{1}{2}\left[f_+(t) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + f_-(t) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right] \tag{34}$$

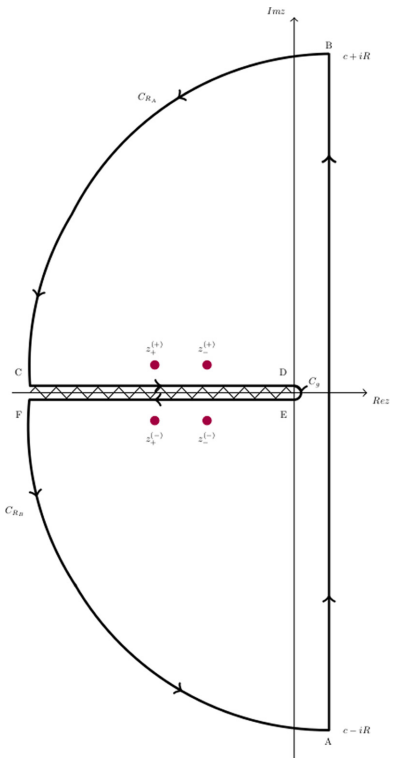

FIG. 1. Bromwich contour, branch cut, and poles related to Eq. (35). Integration is along a straight line from $c - i\infty$ to $c + i\infty$, where c is a real constant larger than the real part of the poles of the integrand. The contour is closed by a semicircle of radius $R \to \infty$.

in terms of functions $f_\pm(t)$, which are defined by the Bromwich integrals

$$f_\pm(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \frac{e^{zt}}{z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) \pm 2\tilde{\gamma}_r(z)}, \tag{35}$$

where c is a real constant larger than the real part of any pole in the integrand.

The integrand in Eq. (35) has a branch cut at $z = 0$. For this reason, we consider the integration contour in Fig. 1, which circles around the branch cut. Using Cauchy's theorem, we find that the functions $f_\pm(t)$ consist of two parts,

$$f_\pm(t) = f_\pm^0(t) + I_\pm(t). \tag{36}$$

The part $f_\pm^0(t)$ contains the contribution from the poles in the region enclosed by the contour, as in Fig. 1; we refer to it as the *pole term*. The part $I_\pm(t)$ includes the contribution from the negative imaginary axis; we refer to this as the *branch-cut term*.

B. The pole term

For sufficiently small λ, the poles can be identified perturbatively. To this end, we set $z_+^\pm = \pm i\Omega + \lambda^2 x$, and we solve the equation

$$z^2 + \Omega^2 + 2\tilde{\gamma}_0(z) \pm 2\tilde{\gamma}_r(z) = 0 \tag{37}$$

to leading order in λ^2. We find that the poles associated with f_+ are at $z_+^\pm = \pm i\Omega + i\delta\Omega_+ - \Gamma_+$ and the poles associated with f_- at $z_-^\pm = \pm i\Omega + i\delta\Omega_- - \Gamma_-$, where

$$\delta\Omega_\pm = -\frac{\lambda^2}{8\pi^2\Omega}\left(\ln\left(\frac{\Lambda}{\Omega}\right) \pm \frac{\cos(r\Omega)}{r\Omega}\mathrm{Si}(r\Omega) \mp \frac{\sin(r\Omega)}{r\Omega}\mathrm{Ci}(r\Omega)\right), \tag{38}$$

$$\Gamma_\pm = \Gamma_0\left(1 \pm \frac{\sin(r\Omega)}{r\Omega}\right), \tag{39}$$

$$\Gamma_0 = \frac{\lambda^2}{16\pi\Omega}. \tag{40}$$

$\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are the trigonometric integrals [37]. The constant Γ_0 is the decay rate of a single oscillator interacting with a scalar field.

Besides the two poles above, there exists a pole that is not accessible by perturbation theory. This solution corresponds to the regime $|z| \ll \Omega$. For example, consider the case where $r \to \infty$, so that the contribution of the $\tilde{\gamma}_r(z)$ term is negligible, and Eq. (37) has a root for $\mathrm{Re}(z) \simeq \Lambda e^{-\frac{\pi\Omega}{2\Gamma_0}}$. For finite r the solution acquires an imaginary part. Since the real part of the root is positive, its contribution to $u(t)$ blows up exponentially as $t \to \infty$. This term is unphysical, because it is incompatible with the dissipative nature of the open system evolution.

An analogous term appears in the Abraham-Lorentz classical treatment of the radiation reaction, which leads to a third-order equation for a particle's position [38]. In fact, the exponentially runaway solution in such systems was first found by Planck [39]. For the role of these solutions in QBM models of particle-field interaction, see Ref. [36].

These runaway solutions originate from the inadequacy of the particle-field coupling to account for soft photons. In the present context, runaway solutions can be avoided by an infrared regularization. For example, we can regularize by assuming a finite mass μ for the scalar field. This is equivalent to shifting the argument of $\gamma_0(z)$ by μ, so that we redefine

$$\gamma_0(z) = -\frac{\lambda^2}{16\pi^2}\ln\left(1 + \frac{\Lambda^2}{(z+\mu)^2}\right). \tag{41}$$

For $\mu > \Lambda e^{-\frac{\pi\Omega}{2\Gamma_0}}$, the third pole has a negative real part and does not lead to runaway solutions. This regularization results in the integrand manifesting branch cuts at $z = -\mu \pm i\Lambda$, which have to be taken into account by an appropriate modification of the contour integral. In the weak-coupling limit $(\Gamma_0/\Omega \ll 1)$, μ^{-1} is much larger and Λ^{-1} is much smaller

than physically relevant time scales, so we can simply ignore the contribution of this pole at physically relevant time scales. In contrast, for strong coupling, the runaway solutions cannot be regularized away. The system of the two UdW detectors coupled with the scalar field is physically meaningful only in the weak-coupling limit.

We conclude that in the weak-coupling limit, except at very early times ($t \sim O(\lambda^4)$), the pole term is well approximated by

$$f_{\pm}^{(0)}(t) = \frac{\sin \tilde{\Omega}_{\pm} t}{\tilde{\Omega}_{\pm}} e^{-\Gamma_{\pm} t}. \tag{42}$$

C. The branch-cut term

To evaluate the integral along the negative near axis, we use the identities

$$\tilde{\gamma}_0(-s \pm i\epsilon) = F(s) \mp i\frac{\lambda^2}{16\pi}, \tag{43}$$

$$\tilde{\gamma}_r(s \pm i\epsilon) = G(s) \mp i\frac{\lambda^2}{16\pi sr}\sinh(rs) \tag{44}$$

for positive $\epsilon \to 0$. The functions $F(s)$ and $G(s)$ are

$$F(s) = -\frac{\lambda^2}{8\pi^2}\ln\left(\frac{\Lambda}{s}\right), \tag{45}$$

$$G(s) = -\frac{\lambda^2}{8\pi^2 rs}[\cosh(rs)\mathrm{Shi}(rs) - \sinh(rs)\mathrm{Chi}(rs)] \tag{46}$$

where Shi is the hyperbolic sine integral function and Chi the hyperbolic cosine integral function, defined as

$$\mathrm{Shi}(z) = \int_0^t \frac{\sinh(t)}{t} dt,$$

$$\mathrm{Chi}(z) = \gamma + \ln z + \int_0^z \frac{\cosh(t) - 1}{t} dt. \tag{47}$$

Then

$$I_{\pm}(t) = -\frac{\lambda^2}{8\pi^2}\int_0^{\infty} ds e^{-st} \times \frac{1 \pm \frac{\sinh(rs)}{rs}}{(s^2 + \Omega^2 + 2F(s) + 2G(s))^2 + \left(\frac{\lambda^2}{8\pi}\right)^2\left(1 \pm \frac{\sinh(rs)}{rs}\right)^2}. \tag{48}$$

The function $I_{\pm}(t)$ cannot be evaluated analytically. A good approximation that is valid for $t > r$ is to ignore the terms of order λ^2 in the denominator, so that

$$I_{\pm}(t) = -\frac{\lambda^2}{8\pi^2}\int_0^{\infty} ds e^{-st}\frac{1 \pm \frac{\sinh(rs)}{rs}}{(s^2 + \Omega^2)^2}. \tag{49}$$

For $t < r$, the approximation above does not hold, because dropping the terms of order λ^2 in the denominator renders the integral divergent.

For $\Omega t \gg 1$, Eq. (49) becomes

$$I_{\pm}(t) = -\frac{\lambda^2}{8\pi^2\Omega^4}\left[\frac{1}{t} \pm \frac{1}{r}\tanh^{-1}(r/t)\right]. \tag{50}$$

In Fig. 2 we plot $I_{\pm}$ as a function of $\Gamma_0 t$ for different values of Ωr. It is negative valued and increases asymptotically to 0. It is unlike the pole term, in that it does not involve any oscillations.

D. The Markov approximation

Equation (35) is similar to the equation for the persistence amplitude of an unstable quantum state in the random phase approximation [9]. In fact, the two kernels $\tilde{\gamma}_0$ and $\tilde{\gamma}_r$ are similar to the ones that appear in the evolution of a pair of atomic qubits interacting with the electromagnetic field [40]. The difference is that the dominant term contains a quadratic rather than a linear term with respect to z, reflecting that in a harmonic oscillator we consider both positive-frequency and negative-frequency solutions.

The split, (36), into a pole term and a branch-cut term is generic whenever the kernels describing the effect of the environment contain branch cuts. A common approximation in the study of unstable systems is the Wigner-Weisskopf approximation (WWA), in which (i) the branch-cut term is neglected, and (ii) the poles are calculated to leading order in perturbation theory [9]. The WWA leads to exponential decay. It coincides with the van Hove limit, namely, taking the limit $\lambda \to 0$, with $\lambda^2 t$ kept constant. In the open quantum system context, the van Hove limit leads to the second-order master equation that describes Markovian dynamics [5].

It is straightforward to evaluate the van Hove limit of Eq. (35). A function of the form

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dz \frac{e^{zt}}{z^2 + \Omega^2 + \lambda^2 a(z)}, \tag{51}$$

for some kernel $\lambda^2 a(z)$, can be written as

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{dz}{i\sqrt{\Omega^2 + 2\lambda^2 a(z)}}\left[\frac{1}{z - i\sqrt{\Omega^2 + 2\lambda^2 a(z)}} - \frac{1}{z + i\sqrt{\Omega^2 + 2\lambda^2 a(z)}}\right]. \tag{52}$$

We set $z = i\Omega + \lambda^2 x$ in the first term and $z = -i\Omega + \lambda^2 x$ in the second. Then we take the limit $\lambda \to 0$, with $\lambda^2 t$ constant, to obtain

$$f(t) = \frac{1}{\Omega}\left(e^{-i\Omega t - \frac{\lambda^2 a(i\Omega)}{\Omega} t} - e^{i\Omega t - \frac{\lambda^2 a(-i\Omega)}{\Omega} t}\right), \tag{53}$$

i.e., the pole term with a perturbative evaluation of the poles.

The van Hove limit essentially substitutes the classical equation of motion with non-local-in-time dissipation with an equation that is local in time. Hence, it removes memory effects from the evolution equation. A local-in-time equation for dissipation is a necessary—but usually not a sufficient—condition for Markovian dynamics. This can be seen in path integral derivations of the QBM master equation [8,27]; Markovian behavior requires that the noise kernel also becomes local.

To summarize, the Markov approximation to the system under study presupposes the validity of the WWA. Hence, the violation of the latter is a definite sign of the existence of non-Markovian dynamics.

E. Non-Markovian dynamics

The WWA, and consequently the exponential decay law, cannot be valid at all times (see the reviews [9,41], and [42]). Exponential decay fails at very early times due to quantum

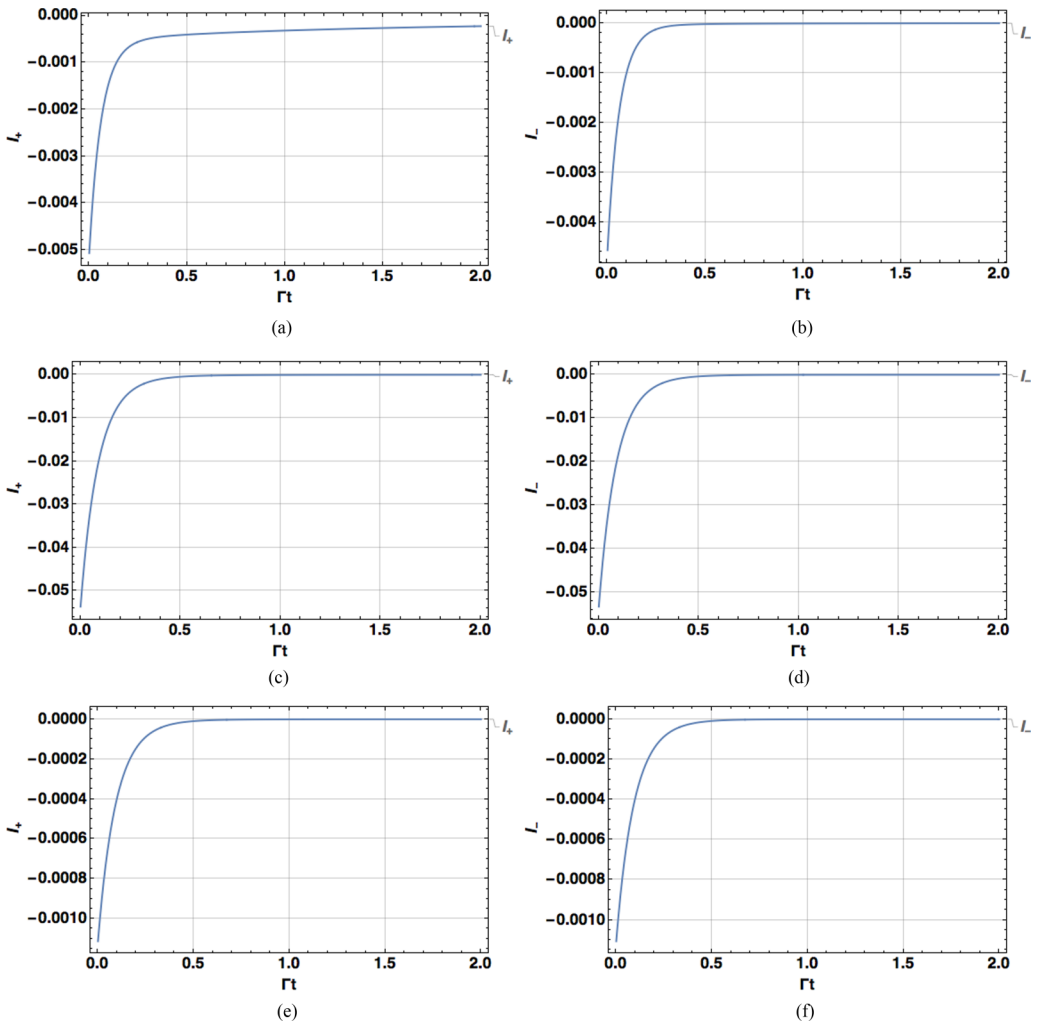

FIG. 2. Evolution of $\Omega I_{\pm}$ as a function of $\Gamma_0 t$ for different values of the dimensionless Ωr, where $\Gamma_0/\Omega = 10^{-3}$ and $\Omega = 1$. In figures (a), (b) $\Omega r = 1$, (c), (d) $\Omega r = 10$ and (e), (f) $\Omega r = 1000$.

Zeno dynamics. It also fails at very late times: the branch-cut term typically falls off as an inverse power of t and eventually becomes larger than the pole term, which decays exponentially. However, the time scale for this decay is much larger than the relaxation time. For example, in optical systems even for Γ_0/Ω as large as 10^{-3}, the breakdown of the exponential decay takes place at $\Gamma_0 t \sim 30$ when less than $1 : 10^{26}$ of the initial systems remains in the excited state.

A violation of the WWA is physically meaningful only if it takes place at time scales compatible with the dissipation time, i.e., if it happens when $\Gamma_0 t$ is a small number. We show that this takes place in the system studied here when the detectors are separated by a large distance r.

Equation (34) implies that $u_{11} = u_{22} = \frac{1}{2}(f_+ + f_-)$ and that $u_{12} = u_{21} = \frac{1}{2}(f_+ - f_-)$. The terms u_{11} and u_{12} describe the dependence of the variables of one detector on the initial conditions of the second detector, while u_{12} and u_{21} essentially describe the correlations developed between the two detectors.

Equations (38) and (39) imply that as $r \to \infty$, $\Gamma_+ = \Gamma_-$ and $\delta\Omega_+ = \delta\Omega_-$. By Eq. (42), $f_+^{(0)}(t) = f_-^{(0)}(t)$ as $r \to \infty$, for all t. Hence, the pole part of $u_{12}(t)$ vanishes for all t as $r \to \infty$. In contrast, the branch-cut term remains finite. By continuity, for any given t there is a finite distance r, at which the branch-cut term dominates over the pole term, and hence, the WWA fails.

We have verified this behavior numerically as shown in Fig. 3. There, we present a semilogarithmic plot of the pole term of u_{12} divided by the full u_{12}, as a function of time. We chose $\Gamma_0/\Omega = 10^{-3}$, i.e., we work well within the weak-coupling regime. By construction, this ratio is very close to 0 if the WWA holds, and it differs significantly from 0 if the WWA fails. The plots show that the behavior of this function changes when r becomes of the order of Γ_0^{-1}. At this scale, we see significant violations of the WWA at the scale of $\Gamma_0 t \sim 1$ and a complete breakdown as $\Gamma_0 t$ becomes about 5. Note that both violations and the breakdown of the WWA occur early, when a significant fraction of the energy remains in the system.

The WWA is well preserved for u_{11} and u_{22} in the regime where it fails for u_{12}. Nonetheless, the WWA also fails for u_{11} and u_{22} at sufficiently large times. This is to be expected, because—as mentioned earlier—the WWA is guaranteed to

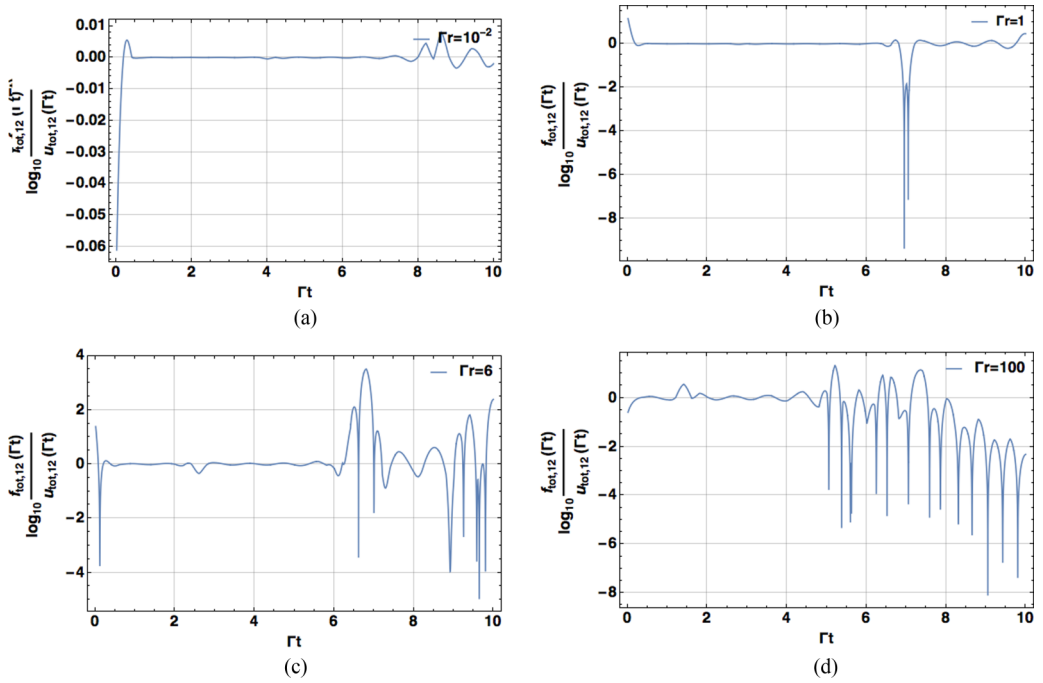

FIG. 3. Evolution of the quantity $\frac{u_{12}^{(0)}}{u_{12}}$, where $u_{12}^{(0)}$ stands for the Markovian part of u_{12}, as a function of $\Gamma_0 t$ and for different values of the dimensionless Γr. In this plot, $\Gamma_0/\Omega = 10^{-3}$.

fail in the long-time limit. What is rather unexpected is that for sufficiently large r, the WWA breaks down at relatively early times also for u_{11} and u_{22}. We found that for $\Gamma_0 r < 10$, the breakdown of the WWA occurs at $\Gamma_0 t \simeq 15$, i.e., at a time where a negligible amount of energy remains in the system. However, for $\Gamma_0 r > 50$, the WWA breaks down much earlier, when $\Gamma_0 t \simeq 5$.

In all regimes that we have studied, the WWA breaks down at the u_{12} term both earlier and more strongly than it does at the u_{11} and u_{22} terms. Therefore, the WWA fails primarily for terms that describe the creation of correlations between distant detectors. For these terms, the branch-cut contribution dominates. This result strongly suggests that the creation of correlations over large distances is a nonperturbative effect. It cannot be described correctly by perturbative approximation schemes, such as the von Hove limit or the second-order master equation.

The conclusion above is unquestionable for the present model, because we have an integrable system and, consequently, full control over all approximation schemes. We conjecture that this behavior may characterize other open systems with similar Hamiltonians, for example, N-level atoms coupled to the electromagnetic field, found at separations r of order of the relaxation time. However, to see such effects, we have to treat them using methods beyond the second-order master equation, which provides the most widely used treatment of such systems.

The system also exhibits non-Markovian behavior at the opposite regime $r \to 0$, as $\gamma_r \to \gamma_0$, and f_- becomes simply $\frac{1}{\Omega}\sin\Omega t$. This behavior has been extensively studied in multipartite QBM models (see, for example, [30] and [43]). We are not concerned with this regime here, because the limit $\Omega r \ll 1$ is not compatible with the identification of the oscillators either with atoms or with particle detectors.

IV. ASYMPTOTIC STATES AND GENERATION OF ENTANGLEMENT

In this section, we show that the open system dynamics of the detectors lead to a unique asymptotic state. This state is correlated, and it is entangled for small separations.

A. Asymptotic state

In Sec. II, we showed that the reduced density matrix propagator for this model is fully determined by matrices $R(t)$ and $S(t)$. In Sec. III, we evaluated $R(t)$ and showed its non-Markovian behavior for $\Gamma_0 r \gtrsim 1$. Matrix $S(t)$ is determined by Eqs. (16)–(18).

When evaluating the matrix elements $S_{ab}(t)$, we find that even for the nondiagonal elements the dominant contribution comes from the functions $u_{11}(t)$ and $u_{22}(t)$ and their derivatives. These functions are well described by the pole term except for very long times. Hence, we expect that the WWA is accurate for $S_{ab}(t)$. Numerically, we find that the difference between the S_{ab} calculated via the WWA and the exact expression is of the order of $\Gamma_0/\Omega \ll 1$. If we substitute solely the pole term for $u(t)$ in Eqs. (16)–(18), integrations can be carried out analytically. They lead to an analytic expression for $S_{ab}(t)$ that is accurate to order Γ_0/Ω.

The functions $u_{\alpha\alpha'}(t)$ vanish as $t \to \infty$, hence, so does the matrix $R_{ab}(t)$. Equation (7) implies that as $t \to 0$, the Wigner function propagator becomes independent of ξ_0. Numerical evaluation of $S_{ab}(t)$ shows that it asymptotes to a constant

matrix for large t; we denote this matrix $S(\infty)$. Hence, asymptotically the system is described by the Wigner function,

$$W_\infty(\xi) = \frac{1}{\pi\sqrt{\det[S(\infty)]}} \exp\left[-\frac{1}{2}S^{-1}_{ab}(\infty)\xi^a\xi^b\right]. \quad (54)$$

By Eq. (10), the correlation matrix at infinity $V_{ab}(\infty)$ coincides with $S_{ab}(\infty)$.

Interestingly, the matrix $S(\infty)$ involves correlations between the two detectors: the matrix elements $S_{X_1X_2}(\infty)$, $S_{P_1P_2}(\infty)$, and $S_{X_1P_2}(\infty)$ that describe such correlations are nonzero. To see this, we use the fact that the dominant contribution to $S_{ab}(\infty)$ is well approximated by the WWA. Substituting Eq. (42) into Eqs. (16)–(18), taking the limit $t \to \infty$, and keeping terms to leading order in Γ_0/Ω, we obtain

$$S_{X_1X_1}(\infty) = S_{X_2X_2}(\infty) = \frac{\Gamma_0}{\Omega}\left[\frac{1}{\Gamma_+} + \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} - \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right], \quad (55)$$

$$S_{P_1P_1}(\infty) = S_{P_2P_2}(\infty) = \Gamma_0\Omega\left[\frac{1}{\Gamma_+} + \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} - \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right], \quad (56)$$

$$S_{X_1P_1}(\infty) = S_{X_2P_2}(\infty) = \frac{2\Gamma_0}{\Omega}\left(\frac{\delta\Omega}{\Omega} + \frac{\sin(\Omega_+ r) - \sin(\Omega_- r)}{4\Omega r}\right), \quad (57)$$

$$S_{X_1X_2}(\infty) = S_{X_2X_1}(\infty) = \frac{\Gamma_0}{\Omega}\left[\frac{1}{\Gamma_+} - \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} + \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right], \quad (58)$$

$$S_{P_1P_2}(\infty) = S_{P_2P_1}(\infty) = \Gamma_0\Omega\left[\frac{1}{\Gamma_+} - \frac{1}{\Gamma_-} - \frac{1}{2\Omega r}\left(\frac{\sin(\Omega_+ r)}{\Gamma_+} + \frac{\sin(\Omega_- r)}{\Gamma_-}\right)\right], \quad (59)$$

$$S_{X_1P_2}(\infty) = S_{X_2P_1}(\infty) = \frac{\Gamma_0}{\Omega}\left(-1 + \frac{\sin(\Omega_+ r) + \sin(\Omega_- r)}{2\Omega r}\right). \quad (60)$$

Remarkably, the correlation terms $S_{X_1X_2}$ and $S_{P_1P_2}$ turn out to be of order $(\Gamma_0/\Omega)^0$, i.e., of the same order as the diagonal terms. However, unlike the diagonal terms, the correlation terms are suppressed as Ωr becomes significantly larger than unity. For $\Omega r \simeq 20$ or less, there is a significant residual correlation between the detectors. This may appear surprising, but we note that the destruction of correlations at late times may be a common feature of either high-temperature baths or systems of qubits, but it is *not* a generic property of open quantum systems. The existence of asymptotic correlations appears more intuitive when viewing the oscillators as actual particle detectors. We would expect the detectors to develop correlations if they dominantly interact with particles with a de Broglie wavelength of the order of their distance.[3]

Next, we examine whether the asymptotic state is entangled. To this end, we employ the positive partial transpose separability criterion of Peres and Horodecki [44,45]. In the present context, the positive partial transpose criterion is applied to the correlation matrix V. A correlation matrix on $L^2(R) \otimes L^2(R)$ is separable if it satisfies

$$V \geqslant -\frac{i}{2}\tilde{J}, \quad \tilde{J} = \Lambda J \Lambda, \quad (61)$$

where J is the symplectic form in the four-dimensional phase space of two particles and Λ is the matrix of the positive partial transpose operation $\Lambda = \mathrm{diag}(1, 1, 1, -1)$ [46]. Hence, if the matrix $V \geqslant +\frac{i}{2}\tilde{J}\Omega$ has a negative eigenvalue, the associated state is entangled.

In Fig. 4, we plot the minimal eigenvalue λ_- of $S(\infty) + \frac{i}{2}J$ as a function of Ωr. A negative value of λ_- indicates an entangled Gaussian state; a positive value, a separable Gaussian state. If $V \geqslant -\frac{i}{2}\tilde{J}$, then $2iJ\tilde{V} \geqslant I$, where $\tilde{V} = \Lambda V \Lambda$. Hence, all symplectic eigenvalues of $2\tilde{V}$ are greater than 1. If this inequality does not hold, then (at least) the smallest symplectic eigenvalue ν_- of $2\tilde{V}$ is smaller than unity. It follows that $\lambda_- < 0$ if $\nu_- < 1$, and vice versa. Hence, the criterion for entanglement employed here is equivalent to the one provided by the negativity $\log \nu_-$ of the quantum state [47].

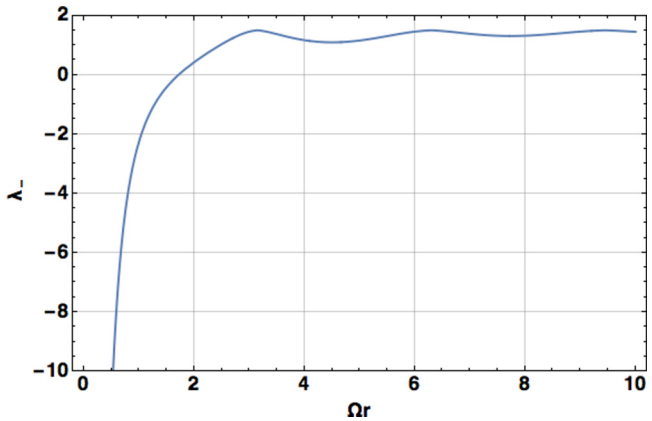

FIG. 4. The minimal eigenvalue λ_- of the matrix $S(\infty) + \frac{i}{2}\tilde{J}$ as a function of Ωr.

[3]There is no lower limit to Ω in our model—except for the infrared cutoff—so the detectors could be correlated even if they are separated by macroscopically large distances. Of course, actual particle detectors are macroscopic systems, and the variables $\hat{X}_\alpha$ are highly coarse-grained. The inclusion of additional degrees of freedom to the detector would introduce decoherence effects that would suppress such correlations beyond some length scale L.

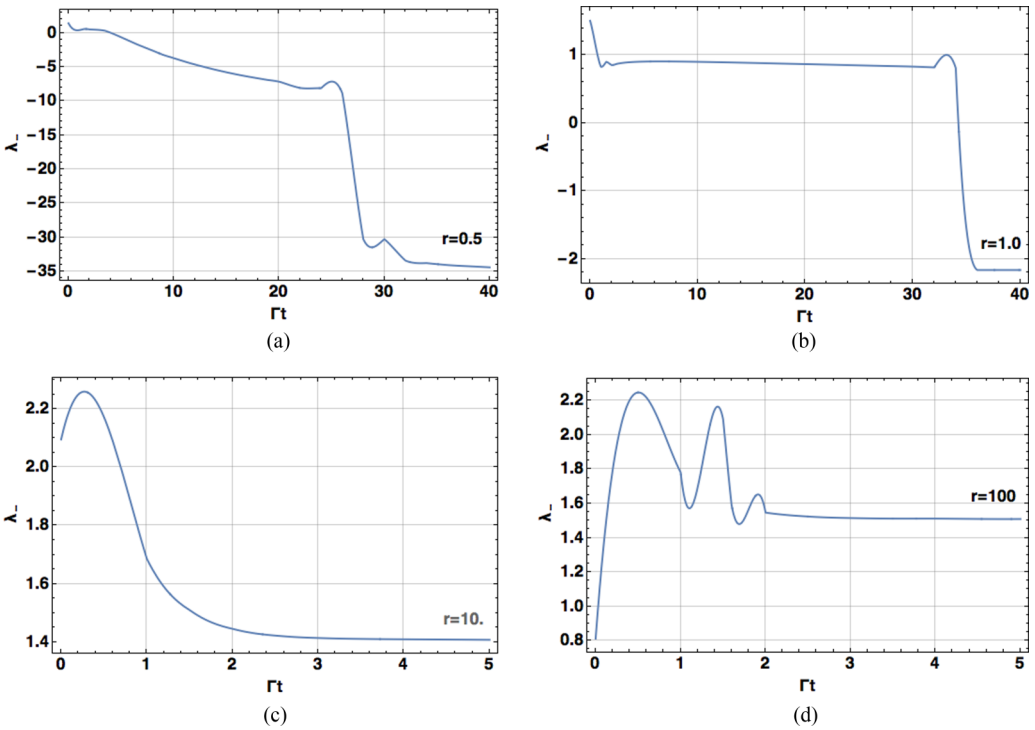

FIG. 5. The evolution of minimal eigenvalue λ_- of $V_t + \frac{i}{2}\tilde{J}$ for initial factorized state $|z\rangle \otimes |z'\rangle$ and for different values of Ωr. We see that entanglement is generated only for small r. In these plots, $\Omega = 1$.

We see that the asymptotic state is entangled for $\Omega r \lessapprox 1.79$ and that the entanglement is stronger as $r \to 0$. The results are qualitatively compatible with the analysis in Ref. [20] (which ignores backreaction) and with the analysis in Ref. [30] (which employs an expansion scheme). We note that Eqs. (55)–(60) provide the exact asymptotic expression of S in the weak-coupling limit.

B. Entanglement generation

Having established the asymptotic behavior of the two-detector system and identified the asymptotic behavior of entanglement, we examine how entanglement is generated in time. Again, we employ the separability criterion, (61). We consider an initial factorized state $|z\rangle \otimes |z'\rangle$ that is a product of coherent states. In Fig. 5(a), we plot the lowest eigenvalue of $V_t + \frac{i}{2}\tilde{J}$ as a function of $\Gamma_0 t$, where V_t is given by Eq. (10).

We note the following. First, the relaxation time for distances such that Ωr is of order unity is significantly larger than Γ_0^{-1}, because the small decay constant Γ_- is much smaller than Γ_0. For example, for $\Omega r = 0.5$, $\Gamma_-^{-1} = 25\Gamma_0^{-1}$. For small values of Ωr, entanglement is generated at early times, as shown in Fig. 5(a). For a large value of Ωr, there is no generation of entanglement. However, for Ωr close to unity (but still smaller than the bounding value, 1.79), entanglement is generated only at late times, when the system converges to its asymptotic entangled state.

The behavior of entanglement depends crucially on the parameter Ωr. The choice of the initial state $|z\rangle \otimes |z'\rangle$ does not significantly affect the creation of entanglement. Other factorized initial states exhibit the same behavior.

For $z = z' = 0$, the initial state is $|0, 0\rangle$, i.e., the ground state of the system of two oscillators. However, this state is not the lowest-energy state for the full field detector Hamiltonian. For this reason, the energy of the detector degrees of freedom momentarily increases as a result of the interaction with the environment, which would be paradoxical if $|0, 0\rangle$ were a true ground state.

The state $|0, 0\rangle$ may be viewed as a ground state of the system if we can assume a setup in which the field detector coupling switches on at $t = 0$. As long as the switching-on takes place at time scales much smaller than Γ^{-1}, the solutions to the reduced dynamics derived here are applicable.

In this context, the creation of entanglement from an initial vacuum state is referred to as *harvesting* of the QFT vacuum. Most research on harvesting focuses on the evaluation of the effect at the lowest order of time-dependent perturbation theory. This is a good approximation as long as the interaction is switched on for a time interval much smaller than the relaxation time. For longer times, perturbation theory is not reliable. An open quantum system treatment that takes backreaction into account is essential; otherwise the effects of relaxation cannot be incorporated into the description. For example, the asymptotic creation of entanglement as in Fig. 5(b) occurs at time scales of the relaxation time.

We emphasize that most research on harvesting refers to moving detectors, while here we restrict our consideration to static ones. We find that there is no significant generation of entanglement outside the light cone for static detectors.

V. THE CHALLENGE OF CAUSALITY

An important motivation of this work is to understand how causality is implemented in the communication of

separated localized quantum objects through a quantum field. The present model, being exactly solvable, provides an explicit demonstration of Fermi's two-atom problem, in which the fundamental physical issues are not obscured by questions about the validity of approximations.

It is straightforward to verify that the classical equations of motion, (13), are not causal: $\hat{X}_2(t)$ depends on the value of $\hat{X}_1(0)$, even for times $t < r$. This result is not surprising. Equation (13) describes the interaction between the oscillators in terms of direct coupling in position—even if it is nonlocal in time—and it is well known that direct particle coupling cannot lead to causal dynamics in relativistic systems. The problem is that Eq. (13) describes the evolution of the expectation values of the observables $\hat{X}_{1,2}$, hence, its noncausal behavior seemingly implies superluminal signals.

Having an exactly solvable model allows us to demonstrate explicitly that this noncausal behavior is not an artifact of common approximations employed in such systems; for a treatment of causality violation in interactions between oscillator UdW detectors, see [48] and [49]. In particular, noncausality is not due to the choice of a factorizing initial condition, which was employed in the derivation of the density matrix propagator. Such a condition cannot hold exactly, because any preparation of the system cannot affect arbitrarily high energies of the field. Factorizability holds at most up to a cutoff energy scale. However, existing models in the theory of quantum open systems strongly suggest that such correlations are mostly significant at early times and that their effects becomes negligible as correlations are established between system and environment due to dynamical interaction.

More importantly, we can derive an exact evolution equation for the expectation value $\langle \hat{X}_r \rangle$ [28],

$$\frac{d^2}{dt^2}\langle \hat{X}_\alpha(t)\rangle + \Omega_\alpha^2 \langle \hat{X}_\alpha(t)\rangle + 2\sum_{\alpha'} \int_0^t \gamma_{\alpha\alpha'}(t-s)\langle \hat{X}_{\alpha'}(s)\rangle = \sum_i \frac{c_{i\alpha}}{M_\alpha}\langle \hat{q}_i^0(t)\rangle, \tag{62}$$

where $\hat{q}_i$ is the field operator associated with the ith mode, evolving according to the free equations of motion for the field. We can also choose the initial state to satisfy $\langle \hat{\phi}(x)\rangle = \langle \hat{\pi}(x)\rangle = 0$, where $\hat{\pi}(x)$ is the field conjugate momentum.[4] This condition implies that $\langle \hat{q}_i^0(t)\rangle = 0$, hence, $\langle \hat{X}_\alpha(t)\rangle$ satisfies Eq. (13). Mean values evolve noncausally, irrespective of the initial condition.

The situation is analogous to that of Fermi's two-level atom mentioned in Sec. I. In this sense, it is generic to all relativistic systems when we attempt to describe their subsystems as completely localized in space. Hegerfeldt proved with minimal assumptions that for any systems A and B, in disjoint regions, interacting through a quantum field, the excitation probability of B is nonzero immediately after $t = 0$ [17]. The present model exemplifies Hegerfeldt's theorem in an exactly solvable system.

[4]This is a natural condition for a state that behaves like the field vacuum. In any case, the mean value of the field and its conjugate momentum can be shifted to any value by a unitary action of the Weyl group, which is generated by the field canonical algebra.

The only known theory of relativistic interactions is QFT, and this is subject to constraints about localization of observables [50–52]. The noncausal behavior of models of particle-field interaction is therefore no surprise at the fundamental level. However, there is an important result that supports causality in UdW detectors: Cliche and Kempf [53] considered a pair of pointlike two-level UdW detectors, prepared initially in a factorized state. The first detector is in a general state; the second detector is in the ground state. They showed that even if the reduced matrix of the second detector becomes excited immediately, it does not depend on the initial state of the second detector, as long as the two detectors are spacelike separated.

The result above does not contradict our analysis, as it refers to a different system (two-level vs oscillator detectors). However, it demonstrates a different behavior of detectors, and it is important to understand the origins of the difference. One possibility is that two-level detectors inherently behave better than oscillator ones. Strictly speaking, the interaction Hamiltonian (both in this paper and in [53]) is ill defined as a self-adjoint operator, as it involves the value of a quantum field (an operator-valued distribution) at specific points without smearing. It is conceivable that the problems from this ill definition are magnified by the dimension of the detector's Hilbert space, leading eventually to noncausal behavior. Note that even for two-level detectors the use of smeared fields in the Hamiltonian leads to noncausal behavior [48].

Another possibility is that the result in Ref. [53] holds only for the specific initial state of vacuum for one of the two detectors. To the best of our knowledge, there is no analysis that shows causal behavior for expectation values of general observables in two-level detectors, for example, showing that in two-level systems and for spacelike separation the expectation value of $\hat{\sigma}_x$ for one detector is independent of the expectation value of any $\hat{\sigma}_i$ from the other detector. This is a difficult property to guarantee, given that the two detectors develop entanglement instantaneously [20].

To better understand causality in UdW detectors, it is important to adapt the analysis in Ref. [53] to higher-dimensional Hilbert spaces and to the behavior of general observables. The causality result in Ref. [53] follows from the analysis of the Dyson series for the total system of detectors and field. For the system studied here, the Dyson series can be fully resummed, so, in principle, we can have a precise identification of the terms responsible for noncausal behavior.

We believe that the type of noncausality identified here is not an artifact of unphysical dynamics, for example, due to the limited validity of the field-particle coupling of this model. One way to see this is the following. Field-particle couplings can be derived for the dynamics of an N-level atom coupled to the electromagnetic field [54]. The harmonic oscillators considered here can be viewed as atoms with equal spacing in the levels and $N \to \infty$. The starting point in such derivations is the full quantum electrodynamics. The crucial condition that leads to couplings of the form of (19) is the *dipole approximation*. This asserts that the size of the localized systems is much smaller than the wavelength of the

emitted radiation. Since the size of these systems defines the cutoff frequency Λ, the dipole approximation is expected to hold with an accuracy of the order of Ω/Λ. Hence, corrections to the dipole approximation (and, hence, to the field-particle coupling) are expected to increase with Ω and to be sensitive to the cutoff Λ. This is the case for the runaway solutions that are regularized away (see Sec. III B). In contrast, the noncausal behavior that characterizes Eq. (13) is insensitive to Ω or to Λ.

For this reason, we believe that the problem of causality in detector-field interactions is fundamentally *kinematical* and not dynamical. This is supported by several theorems on the impossibility of defining localization observables in relativistic quantum systems [50–52]. Existing definitions of localized observables conflict with the requirement for relativistic causality. Observables that appear to be local and causal in classical theory or in on-relativistic quantum theory (e.g., a particle's position) fail to be so in relativistic quantum theory.

In order to clarify this point, we must clarify the sense in which we use the word "local," because this term has different meanings in quantum information theory and in QFT. In quantum information theory, locality is related to factorizability with respect to the tensor product. For example, in a system described by a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, operators of the form $\hat{A} \otimes \hat{I}$ or $\hat{I} \otimes \hat{B}$ correspond to local observables. In QFT, the locality of an observable refers to the space-time (or spatial region) of support for this observable. For example, the observable $\hat{\phi}(f) := \int d^3x \hat{\phi}(x) f(x)$ is localized in a region C, if the smearing function f has support only in C.

Consider the observables $\hat{X}_\alpha$ and $\hat{P}_\alpha$, which describe the degrees of freedom of the oscillator detectors in the present model. Certainly, they are local observables in the former sense. It would be natural to assert that they are local also in the second sense, i.e., that they correspond to observables localized in a small neighborhood around the points $x = x_1$ and $x = x_2$. For example, one may assume that the detector degrees of freedom correspond to another field $\hat{\psi}$, so that $\hat{X}_1$ and $\hat{P}_1$ correspond to local observables for this field around x_1 and $\hat{X}_2$ and $\hat{P}_2$ correspond to local observables for this field around x_2.

This interpretation might make sense for a free field $\hat{\psi}$, because the Fock space of the field can factorize as $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_{\rm rest}$, where $\mathcal{H}_1$ involves field variables localized in a region C_1, $\mathcal{H}_2$ involves field variables localized in a region C_2, and $\mathcal{H}_{\rm rest}$ involves field variables localized outside $C_1 \cup C_2$. Then we could identify $\hat{X}_1$ and $\hat{P}_1$ with operators in $\mathcal{H}_1$ and $\hat{X}_2$ and $\hat{P}_2$ with operators in $\mathcal{H}_2$. Note that such an identification might not be compatible with a local QFT, as one would have to show that the evolution of $\hat{X}_\alpha$ and $\hat{P}_\alpha$ is generated by a local Hamiltonian, which is the integral of a Hamiltonian density.

The point is that $\hat{X}_\alpha$ and $\hat{P}_\alpha$ might be interpreted as local observables for $\hat{\psi}$, only as long as $\hat{\psi}$ and $\hat{\phi}$ do not interact. When the interaction is present, $\hat{X}_\alpha$ and $\hat{P}_\alpha$ can no longer be viewed as localized observables pertaining to a single detector. But then, if they are nonlocal observables, the noncausal interaction identified earlier is not conceptually problematic.

This also means that a causal description of relativistic transmission of information requires a consistent definition of localized observables. The Hilbert space of the total system is $\mathcal{H}_{\rm tot} = \mathcal{H}_{d1} \otimes \mathcal{H}_{d2} \otimes \mathcal{H}_{\rm field}$, where $\mathcal{H}_{d\alpha}$ is a Hilbert space associated with the α detector and $\mathcal{H}_{\rm field}$ the field Hilbert space. An operator that corresponds to a measurement in a region around $\mathbf{x}_1$ through the detector should not be of the form $\hat{A} \otimes \hat{I} \otimes \hat{I}$ (i.e., local in the quantum information sense), but rather it should be a nonfactorized operator on $\mathcal{H}_{\rm tot}$ that reduces to the factorizing form for $\lambda \to 0$. Heuristically, a local observable should be dressed by "virtual photons"[5] in order to be compatible with causality. This is in accordance with Hegerfeld's proposal, about reconciling QFT with his non-go theorems [18].

Still, it is doubtful that self-adjoint operators that generalize $\hat{X}_\alpha$ and $\hat{P}_\alpha$ for the interacting system can be defined in a way that is compatible with causality. There are strong arguments that ideal measurements—i.e., measurements corresponding to self-adjoint operators—are incompatible with causality in QFT [56]. These arguments are completely independent of the analysis of Fermi's two-atom problem, and they involve a QFT analysis of measurement. They strongly suggest that *all* QFT measurements must be nonprojective. However, if $\hat{X}_\alpha$ and $\hat{P}_\alpha$ exist as self-adjoint operators in a QFT underlying the present models, they would define projective measurements, in contradiction to the above result.

Finally, we note that one of us has proposed the use of time-extended observables for the description of particle localization [57]. Such observables correspond to positive operator-valued measures that partly depend upon the dynamics of the quantum system [58]. Hence, a model with exactly solvable dynamics, such as the one analyzed here, is important for the explicit construction of such observables and for the testing of their causal behavior.

VI. CONCLUSIONS

Future quantum experiments will allow us to test important issues at the foundations of QFT and of quantum information, pertaining to the principles of causality and locality and their relation to nonclassical correlations like entanglement. Exactly solvable models, like the one analyzed here, allow us to explore regimes that will be experimentally accessible, but they are not adequately describedby the usual approximation

[5]Note a key difference between QFT and quantum mechanics. Consider a composite quantum system with Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. In quantum mechanics, it is possible to define both a noninteracting Hamiltonian of the form $\hat{h}_1 \otimes \hat{I} + \hat{I} \otimes \hat{h}_2$ and an interacting Hamiltonian in this Hilbert space. This is not possible in QFT. According to a crucial theorem by Haag, an interacting Hamiltonian does not exist in the same Hilbert space with the noninteracting Hamiltonian [55]. Indeed, this is the reason for the infinities of QFT perturbation theory. It follows that a bipartite splitting of the Hilbert space of the form $\mathcal{H}_1 \otimes \mathcal{H}_2$ is not possible in an interacting theory. Observables by necessity are dressed, i.e., they involve both components. For example, in QED, fermionic observables necessarily involve photonic degrees of freedom, in quantum optics, atomic observables involve photonic degrees of freedom, and so on.

schemes, such as the Markov approximation or the perturbative analysis of master equations. Our conclusion that the generation of correlations between subsystems at large separations is a nonperturbative process is particularly important in relation to this context.

We believe that the model presented here provides an important tool for addressing foundational issues in QFT, because it has a formal exact solution and provides full mathematical control of all approximation schemes. It may be used for constructing localized observables to address the Fermi problem, for understanding causal propagation of signals and information in QFT, and for generalizing existing quantum information concepts to relativistic systems.

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