# Canonical forms of two-qubit states under local operations

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Canonical forms of two-qubits under the action of stochastic local operations and classical communications (SLOCC) offer great insight for understanding nonlocality and entanglement shared by them. They also enable geometric picture of two-qubit states within the Bloch ball. It has been shown [Phys. Rev. A **64**, 010101(R) (2001)] that an arbitrary two-qubit state gets transformed under SLOCC into one of the *two* different canonical forms. One of these happens to be the Bell diagonal form of two-qubit states and the other a nondiagonal canonical form is obtained for a family of rank deficient two-qubit states. The method employed by Verstraete *et al.* [Phys. Rev. A **64**, 010101(R) (2001)] required highly nontrivial results on matrix decompositions in *n*-dimensional spaces with an indefinite metric. Here we employ an entirely different approach—inspired by the methods developed by Rao *et al.* [J. Mod. Opt. **45**, 955 (1998)] in classical polarization optics—which leads naturally towards the identification of two inequivalent SLOCC invariant canonical forms for two-qubit states. In addition, our approach results in a simple geometric visualization of two-qubit states in terms of their SLOCC canonical forms.

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## I. INTRODUCTION

Geometric intuition inscribed in the Bloch ball picture of qubits serves as a powerful tool in the field of quantum information processing. Simplicity of this geometric representation of qubits inspired its generalization to quantum single party systems in higher dimensions [1–3]. However, these attempts resulted in complicated geometric features. On the other hand, physically relevant visualization of the simplest bipartite quantum system viz., joint state of two-qubits, has been developed by several groups [4–13]. Geometric representation of two-qubit states inside the Bloch ball provides a natural picture to understand correlation properties, such as entanglement [4,6,12–15], quantum discord [10–12], and nonlocal steering [6,12,14–19].

Previously, Verstraete *et al.* [5] and Verstraete [6] highlighted that SLOCC on a two-qubit density matrix $\rho_{AB}$ correspond to Lorentz transformations on the $4 \times 4$ real matrix parametrization $\Lambda$ of $\rho_{AB}$ and they arrived at two different types of canonical forms for the real matrix $\Lambda$. The canonical forms of $\Lambda$ correspond to its Lorentz singular value decompositions, offering a natural classification of the set of all two-qubit density matrices into *two* different SLOCC families. The canonical SLOCC transformations also paved way to visualization of two-qubit state—as an ellipsoid inscribed inside the Bloch ball [5,6,10–12]. However, the mathematical recipe used in Refs. [5,6] to arrive at the SLOCC canonical forms is highly technical and depended on nontrivial results on matrix decompositions in spaces with indefinite metric [20]. Moreover, it was pointed out [12] that this approach fails to reveal the geometric features in an unambiguous fashion. A more detailed investigation by Jevtic *et al.* [12] focused towards an elegant geometric representation, mapping a two-qubit state to an ellipsoid lying inside the Bloch ball in a complete manner with the help of suitable SLOCC transformations. However, this paper did not address the relevant issue of identifying canonical forms of two-qubit density matrix $\rho_{AB}$, based on the Lorentz singular value decomposition of the associated $4 \times 4$ real matrix $\Lambda$. A straightforward method to identify Lorentz singular value decomposition, which, in turn, gets connected with the SLOCC canonical forms of two-qubit states, is still lacking. In this paper we address this issue, using the methods developed in classical polarization optics by some of us [21,22]. Our method leads to the identification of two different types of SLOCC canonical forms for two-qubit states. The canonical forms identified by our approach are shown to be Lorentz equivalent to the ones obtained in Ref. [5]. Our detailed analysis gives a fresh perspective on the geometric representation of two-qubit states in terms of their SLOCC inequivalent canonical forms.

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Contents of the paper are organized as follows: In Sec. II, we obtain the $4 \times 4$ real matrix parametrization $\Lambda$ of a two-qubit density matrix $\rho_{AB}$ shared between Alice and Bob. After giving a brief outline on Minkowski space notions of positive, neutral, negative four-vectors, and orthochronous proper Lorentz group (OPLG), we show that: (i) the real $4 \times 4$ matrix $\Lambda$ gets pre- and postmultiplied by the $4 \times 4$ Lorentz matrices $L_A$ and $L_B^T$ (the superscript "$T$" denotes matrix transposition) under the action of SLOCC transformation—implemented by Alice and Bob, respectively, on their individual qubits; (ii) the $4 \times 4$ real matrix $\Lambda$ maps the set of all four-vectors with non-negative Minkowski norm into itself. Section III gives details on finding the canonical forms of real symmetric matrices $\Omega_A = \Lambda G \Lambda^T$ and $\Omega_B = \Lambda^T G \Lambda$, [which are constructed from the real matrix $\Lambda$ and the Minkowski metric $G = \mathrm{diag}(1, -1, -1, -1)$] using Lorentz congruent transformations $L_A \Omega_A L_A^T$ and $L_B \Omega_B L_B^T$, respectively. Lorentz singular value decompositions $\Lambda^c = L_A \Lambda L_B^T$ of two inequivalent canonical forms $\Lambda^{\mathrm{I_c}}$, $\Lambda^{\mathrm{II_c}}$ of the $4 \times 4$ real matrix $\Lambda$ and the corresponding two-qubit density matrices $\rho_{AB}^{\mathrm{I_c}}$, $\rho_{AB}^{\mathrm{II_c}}$, and $\rho_{BA}^{\mathrm{II_c}}$ are also given here. Furthermore, equivalence between the canonical forms obtained earlier by Verstraete *et al.* [5] and Verstraete [6] with the ones realized based on our approach, is established in Sec. III. Geometric representation to aid visualization of SLOCC canonical forms of the two-qubit states is discussed in Sec. IV. A concise summary of our results is presented in Sec. V.

## II. REAL PARAMETRIZATION OF TWO-QUBIT DENSITY MATRIX AND SLOCC TRANSFORMATIONS

Consider a two-qubit state $\rho_{AB}$ belonging to the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbb{C}_2 \otimes \mathbb{C}_2$, shared between two parties Alice and Bob. It can be expressed in the Hilbert-Schmidt basis $\{\sigma_\mu \otimes \sigma_\nu, \mu, \nu = 0\text{–}3\}$ as

$$\rho_{AB} = \frac{1}{4} \sum_{\mu,\nu=0}^{3} \Lambda_{\mu\nu} (\sigma_\mu \otimes \sigma_\nu), \tag{2.1}$$

where

$$\Lambda_{\mu\nu} = \mathrm{Tr}[\rho_{AB}(\sigma_\mu \otimes \sigma_\nu)]. \tag{2.2}$$

Here $\sigma_0 = \mathbb{1}_2$, denotes a $2 \times 2$ identity matrix, and $\sigma_1$, $\sigma$, and $\sigma_3$ are the Pauli matrices.

Expressed in the $2 \times 2$ block form, the $4 \times 4$ real matrix $\Lambda$ defined in (2.2) takes the following compact form:

$$\Lambda = \begin{pmatrix} 1 & \mathbf{b}^T \\ \mathbf{a} & T \end{pmatrix}, \tag{2.3}$$

with the superscript $T$ denoting matrix transposition; $\mathbf{a} = (a_1, a_2, a_3)^T$, $\mathbf{b} = (b_1, b_2, b_3)^T$ denote Bloch vectors of the reduced density matrices $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ of qubits $A, B$, respectively, and $T$ corresponds to a $3 \times 3$ real correlation matrix, elements of which are given by $t_{ij} = \mathrm{Tr}(\rho_{AB}\sigma_i \otimes \sigma_j)$, $i, j = 1\text{–}3$. Thus, the $4 \times 4$ matrix $\Lambda$ is characterized by 15 real parameters (three each of the Bloch vectors $\mathbf{a}$, $\mathbf{b}$, and nine elements of the correlation matrix $T$) and provides a unique *real matrix parametrization* of the two-qubit density-matrix $\rho_{AB}$.

We give a brief outline on the Minkowski four-vectors and OPLG transformations in the following subsection before discussing the properties of the real parametrization $\Lambda$ of the two-qubit state.

### A. Minkowski space, four-vectors, and OPLG

The Minkowski space $\mathcal{M}$ is a four-dimensional real vector space consisting of four-vectors or Minkowski vectors [23,24], denoted by $\mathbf{x} = (x_0, x_1, x_2, x_3)^T$. The space is equipped with the metric,

$$G = \mathrm{diag}(1, -1, -1, -1), \tag{2.4}$$

and a scalar product,

$$\mathbf{x}^T G \mathbf{y} = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3. \tag{2.5}$$

As the Minkowski squared norm $\mathbf{x}^T G \mathbf{x}$ of an arbitrary four-vector $\mathbf{x}$ can assume positive, zero, or negative values, we employ the following nomenclature [21–25]:

(i) $\mathbf{x}^T G \mathbf{x} > 0$: positive four-vector,
(ii) $\mathbf{x}^T G \mathbf{x} = 0$: neutral (or null) four-vector,
(iii) $\mathbf{x}^T G \mathbf{x} < 0$: negative four-vector.

Consider the set of all real $4 \times 4$ matrices,

$$\{L \,|\, \det L = 1,\ L_{00} > 0\},$$

which constitutes the orthochronous proper Lorentz group SO(3,1). By definition, Lorentz matrix $L$ preserves the Minkowski norm, i.e., the four-vector $L\mathbf{x}$ is positive, neutral, or negative depending on whether $\mathbf{x} \in \mathcal{M}$ is positive, neutral, or negative, respectively. In particular, it is pertinent to highlight that the set $S_+$: $\{\mathbf{x} \in \mathcal{M} | \mathbf{x}^T G \mathbf{x} \geqslant 0,\ x_0 > 0\}$ of four-vectors gets mapped to itself under OPLG, i.e., $\tilde{S}_+$: $\{\tilde{\mathbf{x}} = L\mathbf{x} \in \mathcal{M} | \tilde{\mathbf{x}}^T G \tilde{\mathbf{x}} \geqslant 0,\ \tilde{x}_0 > 0\} \equiv S_+$. We would discuss, in Sec. II C, about encoding the set of all non-negative single-qubit operators in $\mathbb{C}_2$ in terms of four-vectors of the set $S_+$.

### B. SLOCC transformations and OPLG

Under the action of SLOCC, a two-qubit density matrix $\rho_{AB}$ transforms as [5,6,12]

$$\rho_{AB} \longrightarrow \widetilde{\rho}_{AB} = \frac{(A \otimes B)\rho_{AB}(A^\dagger \otimes B^\dagger)}{\mathrm{Tr}[\rho_{AB}(A^\dagger A \otimes B^\dagger B)]}, \tag{2.6}$$

where $A, B \in \mathrm{SL}(2, C)$ denote $2 \times 2$ complex matrices with a unit determinant. Owing to the homomorphism between the groups SL(2,C) and SO(3,1), one finds the correspondence $\pm A \mapsto L_A$, $\pm B \mapsto L_B$, where $A, B \in \mathrm{SL}(2, C)$ and $L_A, L_B \in \mathrm{SO}(3, 1)$. In particular, the basis matrices $\sigma_\mu \otimes \sigma_\nu$, $\mu$, $\nu = 0\text{–}3$ get transformed under $\mathrm{SL}(2, C) \otimes \mathrm{SL}(2, C)$ as

$$(A \otimes B)(\sigma_\mu \otimes \sigma_\nu)(A^\dagger \otimes B^\dagger) = A\sigma_\mu A^\dagger \otimes B\sigma_\nu B^\dagger$$
$$= \sum_{\alpha,\beta=0\text{–}3} (L_A)_{\alpha\mu} (L_B)_{\beta\nu} \sigma_\alpha \otimes \sigma_\beta. \tag{2.7}$$

Thus, SLOCC operation $\rho_{AB} \to \tilde{\rho}_{AB}$ on the two-qubit state is equivalent to the following transformation (up to normalization),

$$\Lambda \longrightarrow \tilde{\Lambda} = L_A \Lambda L_B^T, \tag{2.8}$$

on the real matrix $\Lambda$. So, it is evident that $\tilde{\Lambda}$—obtained after OPLG transformations $L_A, L_B$ on $\Lambda$ [see (2.8)]—parametrizes the two-qubit density matrix $\tilde{\rho}_{AB}$, which is physically realizable under SLOCC. Using suitable OPLG transformations $L_{A_c}, L_{B_c}$ one should be able to arrive at a canonical (normal) form $\Lambda^c$ associated with a given $\Lambda$, i.e.,

$$\Lambda^c = L_{A_c}\Lambda L_{B_c}^T. \tag{2.9}$$

It may be seen that (2.9) is the Minkowski space counterpart of the singular value decomposition in Euclidean space and is referred to as the Lorentz singular value decomposition [5,6].

### C. Real symmetric matrices $\Omega_A = \Lambda G \Lambda^T$ and $\Omega_B = \Lambda^T G \Lambda$

Let us denote the set of all non-negative operators acting on the Hilbert space $\mathbb{C}_2$ by $\mathcal{P}^+ := \{P|P \geqslant 0\}$. An element $P \in \mathcal{P}^+$ can be represented in the Pauli basis $\sigma_\mu = (\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3)$ as

$$P = \frac{1}{2}\sum_\mu p_\mu \sigma_\mu, \tag{2.10}$$

where $p_\mu = \mathrm{Tr}(P\sigma_\mu)$, $\mu = 0$–$3$ are the four real parameters characterizing $P$. With every $P \in \mathcal{P}^+$, we associate a four-vector $\mathbf{p} = (p_0, p_1, p_2, p_3)^T$. Non-negativity $P \geqslant 0$ of the operator $P$ is synonymous to the conditions $p_0 > 0$ and $p_0^2 - p_1^2 - p_2^2 - p_3^2 \geqslant 0$ on the four-vector $\mathbf{p}$. In the language of Minkowski space, non-negativity of the operator $P \geqslant 0$ reflects itself as the squared Minkowski norm condition $\mathbf{p}^T G \mathbf{p} \geqslant 0$ together with the restriction $p_0 > 0$ on the zeroth component of the four-vector $\mathbf{p}$.

Let us consider the map,

$$\begin{aligned} P_A \mapsto Q_B &= 2\,\mathrm{Tr}_A[(\sqrt{P_A}\otimes \mathbb{1}_2)\rho_{AB}(\sqrt{P_A}\otimes \mathbb{1}_2)] \\ &= 2\,\mathrm{Tr}_A[\rho_{AB}(P_A\otimes \mathbb{1}_2)], \end{aligned} \tag{2.11}$$

from the set of all non-negative operators $\mathcal{P}_A^+ := \{P_A|P_A \geqslant 0\}$ on the Hilbert space $\mathcal{H}_A$ to the set of non-negative operators $\mathcal{Q}_B^+ := \{Q_B = 2\,\mathrm{Tr}_A[\rho_{AB}(P_A\otimes \mathbb{1}_2)]\}$ acting on the Hilbert space $\mathcal{H}_B$. We have

$$\begin{aligned} Q_B &= 2\,\mathrm{Tr}_A[\rho_{AB}(P_A\otimes \mathbb{1}_2)] \\ &= \frac{1}{2}\sum_\nu (\Lambda^T \mathbf{p_A})_\nu \sigma_\nu, \end{aligned} \tag{2.12}$$

which results in the Minkowski four-vector transformation,

$$\mathbf{q}_B = \Lambda^T \mathbf{p}_A. \tag{2.13}$$

Thus, the map $P_A \mapsto Q_B$ is found to be identical to the four-vector map $\Lambda^T : \mathbf{p}_A \mapsto \mathbf{q}_B = \Lambda^T\,\mathbf{p}_A$. Non-negativity of the squared Minkowski norm of the four-vector $\mathbf{q}_B$ (which corresponds to $Q_B \geqslant 0$) leads to

$$\begin{aligned} \mathbf{q}_B^T G \mathbf{q}_B \geqslant 0 &\Longrightarrow \mathbf{p}_A^T \Lambda G \Lambda^T \mathbf{p}_A \geqslant 0 \\ &\Longrightarrow \mathbf{p}_A^T \Omega_A \mathbf{p}_A \geqslant 0, \end{aligned} \tag{2.14}$$

where

$$\Omega_A = \Lambda G \Lambda^T \tag{2.15}$$

denotes a real symmetric $4\times 4$ matrix, associated with the real parametrization $\Lambda$ of the two-qubit density matrix $\rho_{AB}$. Furthermore, positivity of the zeroth component of the four-vector $\mathbf{p}_A$ imposes that

$$p_{A_0} > 0 \quad \Longrightarrow \quad q_{B_0} = (\Lambda^T \mathbf{p}_A)_0 > 0. \tag{2.16}$$

Similarly, the map,

$$\begin{aligned} P_B \mapsto Q_A &= 2\,\mathrm{Tr}_B[(\mathbb{1}_2\otimes\sqrt{P_B})\rho_{AB}(\mathbb{1}_2\otimes\sqrt{P_B})] \\ &= 2\,\mathrm{Tr}_B[\rho_{AB}(\mathbb{1}_2\otimes P_B)], \end{aligned} \tag{2.17}$$

from the set of all non-negative operators $\mathcal{P}_B^+ := \{P_B\,|P_B \geqslant 0\}$ acting on the Hilbert space $\mathcal{H}_B$ to the set $\mathcal{Q}_A^+ := \{Q_A = 2\,\mathrm{Tr}_B[\rho_{AB}(\mathbb{1}_2\otimes P_B)] \subset \mathcal{H}_A$ on the Hilbert space $\mathcal{H}_A$ leads to the identification,

$$Q_A = \frac{1}{2}\sum_\mu (\Lambda \mathbf{p_B})_\mu \sigma_\mu. \tag{2.18}$$

In turn, we obtain the Minkowski four-vector transformation,

$$\mathbf{q}_A = \Lambda \mathbf{p}_B, \tag{2.19}$$

where the four-vector $\mathbf{q}_A$ characterizes a non-negative operator $Q_A \in \mathcal{Q}_A^+$ faithfully. The map $P_B \mapsto Q_A$ reflects itself in terms of the four-vector transformation $\Lambda\colon \mathbf{p}_B \mapsto \mathbf{q}_A = \Lambda \mathbf{p}_B$ in the Minkowski space such that

$$q_{A_0} > 0 \Longrightarrow (\mathbf{\Lambda p}_B)_0 > 0, \tag{2.20}$$

$$\begin{aligned} \mathbf{q}_A^T G \mathbf{q}_A \geqslant 0 &\Longrightarrow \mathbf{p}_B^T \Lambda^T G \Lambda \mathbf{p}_B \geqslant 0 \\ &\Longrightarrow \mathbf{p}_B^T \Omega_B \mathbf{p}_B \geqslant 0, \end{aligned} \tag{2.21}$$

where, we have denoted

$$\Omega_B = \Lambda^T G \Lambda. \tag{2.22}$$

The $4\times 4$ real symmetric matrices $\Omega_A = \Lambda G \Lambda^T$, $\Omega_B = \Lambda^T G \Lambda$ constructed from the real counterpart $\Lambda$ of the two-qubit density matrix $\rho_{AB}$ play a central role in our analysis.

## III. LORENTZ SINGULAR VALUE DECOMPOSITION OF $\Lambda$ AND CANONICAL FORMS OF TWO-QUBIT DENSITY MATRIX UNDER SLOCC

From the properties of the $4\times 4$ real matrix $\Lambda$, parametrizing the two-qubit density matrix $\rho_{AB}$, it is clear that: (i) under the map $\mathbf{p}_A \mapsto \mathbf{q}_B = \Lambda^T \mathbf{p}_A$ and $\mathbf{p}_B \mapsto \mathbf{q}_A = \Lambda \mathbf{p}_B$ [(see (2.13), (2.19)], four-vectors $\mathbf{p}_A$, $\mathbf{q}_A$ with Minkowski norms $\mathbf{p}_A^T G \mathbf{p}_A \geqslant 0$, $\mathbf{q}_A^T G \mathbf{q}_A \geqslant 0$ and positive zeroth components $p_{A_0} > 0$, $q_{A_0} > 0$ get transformed to four-vectors $\mathbf{q}_B$, $\mathbf{p}_B$ such that $\{\mathbf{q}_B^T G \mathbf{q}_B \geqslant 0,\ q_{B_0} > 0\}$, $\{\mathbf{p}_B^T G \mathbf{p}_B \geqslant 0,\ \ p_{B_0} > 0\}$, respectively. Furthermore, (ii) the sets $\{\Lambda \mathbf{p}|\mathbf{p}^T G \mathbf{p} \geqslant 0, p_0 > 0\}$ and $\{\tilde{\mathbf{p}} = L_A \Lambda L_B^T \mathbf{p}|\tilde{\mathbf{p}}^T G \tilde{\mathbf{p}} \geqslant 0,\ \tilde{p}_0 > 0\}$ are equivalent, as they are related to each other via SLOCC transformations on the two-qubit state $\rho_{AB}$.

Our interest is to look for a particularly simple canonical form as in (2.9) for $\Lambda$ by identifying suitable OPLG transformations $L_{A_c}, L_{B_c}$. In terms of the real symmetric matrices $\Omega_A = \Lambda G \Lambda^T$ and $\Omega_B = \Lambda^T G \Lambda$ introduced in (2.15), (2.22), we express

$$\begin{aligned} \Omega_A^c &= L_{A_c}\Lambda L_{B_c}^T G L_{B_c} \Lambda^T L_{A_c}^T \\ &= L_{A_c}\Omega_A L_{A_c}^T, \end{aligned} \tag{3.1}$$

$$\Omega_B^c = L_{B_c}\Lambda^T L_{A_c}^T G L_{A_c}\Lambda L_{B_c}^T = L_{B_c}\Omega_B L_{B_c}^T, \tag{3.2}$$

where we have used the defining property $L^T G L = G$ of Lorentz transformation matrix $L$ and denoted the canonical forms of the real symmetric matrices $\Omega_A, \Omega_B$ by $\Omega_A^c$, $\Omega_B^c$, respectively.

We would like to emphasize here that the canonical form $\Omega_A^c$ is determined completely by the real matrix $\Lambda$ and the OPLG tranformations $L_{A_c}$ [see (3.1)]. Similarly, $\Omega_B^c$ is entirely characterized by $\Lambda$ and $L_{B_c}$ [see (3.2)]. Therefore, it is possible to introduce the following canonical forms $\Lambda_A^c, \Lambda_B^c$ (up to normalization) for the real matrix $\Lambda$, associated with $\Omega_A^c$ and $\Omega_B^c$, respectively,

$$\Lambda_A^c = L_{A_c}\Lambda L_B^T, \quad \Lambda_B^c = L_{B_c}\Lambda^T L_A^T. \tag{3.3}$$

Note that in (3.3) the OPLG transformations $L_{A_c}, L_{B_c}$ correspond to physical SLOCC operations carried out by Alice, Bob on their parts of the two-qubit state; but the operations $L_A, L_B$ denote any arbitrary OPLG transformations, which, respectively, leave the structure of $\Omega_A^c, \Omega_B^c$ unaltered. Thus, we express

$$\Omega_A^c = \Lambda_A^c G\left(\Lambda_A^c\right)^T, \quad \Omega_B^c = \Lambda_B^c G\left(\Lambda_B^c\right)^T, \tag{3.4}$$

by substituting (3.3).

Continuing further, we note that the Lorentz congruent transformations,

$$\Omega_A \longrightarrow \widetilde{\Omega}_A = L_A\Omega_A L_A^T, \quad \Omega_B \longrightarrow \widetilde{\Omega}_B = L_B\Omega_B L_B^T \tag{3.5}$$

are not similarity transformations. But, the following pair of matrices

$$G\Omega_A = G\Lambda G\Lambda^T, \quad G\Omega_B = G\Lambda^T G\Lambda \tag{3.6}$$

do undergo similarity transformations,

$$G\Omega_A \longrightarrow GL_A\Omega_A L_A^T = \left(L_A^T\right)^{-1} G\Omega_A L_A^T, \tag{3.7}$$

$$G\Omega_B \longrightarrow GL_B\Omega_B L_B^T = \left(L_B^T\right)^{-1} G\Omega_B L_B^T, \tag{3.8}$$

when $\Omega_A, \Omega_B$ undergo OPLG transformations (3.5). In (3.7), (3.8), we have used $GL = (L^T)^{-1}G$ satisfied by every OPLG matrix $L$. It is evident that the eigenvalues of $G\Omega_A$ and $G\Omega_B$ remain invariant under OPLG transformations $L_A, L_B$, associated with the SLOCC operations on qubits $A$ and $B$, respectively. Furthermore, it is readily seen that the eigenvalues of $G\Omega_A$, $G\Omega_B$ are identical as

$$\mathrm{Tr}[(G\Omega_A)^n] = \mathrm{Tr}[(G\Omega_B)^n], \quad n = 1, 2, \ldots. \tag{3.9}$$

Based on a detailed algebraic analysis carried out by some of us [21,22,26] on $4 \times 4$ real matrices, satisfying the conditions (2.14), (2.16), (2.20), and (2.21), we state the following theorem on the nature of eigenvalues and eigenvectors of the matrices $G\Omega_A$ $(G\Omega_B)$:

*Theorem*. The $4 \times 4$ real matrix $G\Omega_A$ $(G\Omega_B)$ associated with the real form $\Lambda$ of a two-qubit density matrix $\rho_{AB}$ necessarily possesses

(i) non-negative eigenvalues;

(ii) either *positive* or *neutral* eigenvectors corresponding to its highest eigenvalue;

(iii) a set of eigenvectors consisting of either

(a) **one** positive four-vector belonging to the highest eigenvalue and **three** negative four-vectors

or

(b) **one** neutral four-vector belonging to the highest eigenvalue—at least, doubly degenerate—and **two** negative four-vectors.

From the above theorem (see Appendix A for a concise proof) it follows that two different cases arise, depending on whether the eigenvector belonging to the highest eigenvalue of $G\Omega_A$ $(G\Omega_B)$ is *positive* or *neutral*. Consequently, we have two types of canonical forms for $\Omega_A$ $(\Omega_B)$ and, consequently, for the real parametrization $\Lambda$, the corresponding two-qubit density matrix $\rho_{AB}$ under SLOCC transformations. Note that the eigenvalues, eigenvectors of $G\Omega_A$ $(G\Omega_B)$ are also referred to as $G$ eigenvalues and $G$ eigenvectors of the real symmetric matrix $\Omega_A$ $(\Omega_B)$.

Next, we proceed to find two different types of canonical forms of the real symmetric matrices $\Omega_A, \Omega_B$.

### A. Type-I canonical form

Let us arrange the eigenvalues of $G\Omega_A$, $G\Omega_B$ in the order $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ and denote the associated set of eigenvectors by $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, respectively. Suppose that $\mathbf{a}_0$ and $\mathbf{b}_0$ are positive four-vectors. From (iii a) of the theorem, it is clear that the set of eigenvectors $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ associated with the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ of the respective matrices $G\Omega_A$, $G\Omega_B$ form Minkowski $G$-orthoronal tetrads (see Appendix B for details) obeying

$$\mathbf{a}_\mu^T G\mathbf{a}_\nu = G_{\mu\nu}, \tag{3.10}$$

$$\mathbf{b}_\mu^T G\mathbf{b}_\nu = G_{\mu\nu}, \tag{3.11}$$

where $\mu, \nu = 0$–3, and $G_{\mu\nu}$ are elements of the Minkowski matrix $G$. We construct OPLG canonical transformation matrices $L_{A_{I_c}}^T$, $L_{B_{I_c}}^T$ explicitly (see Appendix B),

$$L_{A_{I_c}}^T = (\mathbf{a}_0 \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3), \tag{3.12}$$

$$L_{B_{I_c}}^T = (\mathbf{b}_0 \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3), \tag{3.13}$$

by arranging the eigenvectors $\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\{\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $G\Omega_A$, $G\Omega_B$ as columns of $L_{A_{I_c}}^T$, $L_{B_{I_c}}^T$, respectively.

Using (3.10), (3.11), (3.12), (3.13), and the property,

$$G\Omega_A\,\mathbf{a}_\mu = \lambda_\mu\mathbf{a}_\mu \quad \Rightarrow \quad \Omega_A\mathbf{a}_\mu = \lambda_\mu G\mathbf{a}_\mu, \tag{3.14}$$

$$G\Omega_B\mathbf{b}_\mu = \lambda_\mu\mathbf{b}_\mu \quad \Rightarrow \quad \Omega_B\mathbf{b}_\mu = \lambda_\mu G\mathbf{b}_\mu, \tag{3.15}$$

of the eigenvectors of $G\Omega_A,\ G\Omega_B$, we arrive at the diagonal canonical forms $\Omega_{A_c},\ \Omega_{B_c}$,

$$\Omega_A^{\mathrm{I}_c}=L_{A_{\mathrm{I}_c}}\Omega_A L_{A_{\mathrm{I}_c}}^T=\begin{pmatrix}\lambda_0 & 0 & 0 & 0\\ 0 & -\lambda_1 & 0 & 0\\ 0 & 0 & -\lambda_2 & 0\\ 0 & 0 & 0 & -\lambda_3\end{pmatrix},$$

$$\Omega_B^{\mathrm{I}_c}=L_{B_{\mathrm{I}_c}}\Omega_B L_{B_{\mathrm{I}_c}}^T=\begin{pmatrix}\lambda_0 & 0 & 0 & 0\\ 0 & -\lambda_1 & 0 & 0\\ 0 & 0 & -\lambda_2 & 0\\ 0 & 0 & 0 & -\lambda_3\end{pmatrix}. \quad (3.16)$$

*Corollary 1.* Under the canonical OPLG transformations $L_{A_{\mathrm{I}_c}},\ L_{B_{\mathrm{I}_c}}$ as in (3.12), (3.13), the real matrix $\Lambda$, with $\mathrm{sgn}[\det(\Lambda)]=\pm$, assumes the following diagonal canonical forms:

$$\begin{aligned}\Lambda_A^{\mathrm{I}_c}&=\frac{L_{A_{\mathrm{I}_c}}\Lambda L_B^T}{(L_{A_{\mathrm{I}_c}}\Lambda L_B^T)_{00}}\\ &=\mathrm{diag}\left(1,\sqrt{\frac{\lambda_1}{\lambda_0}},\sqrt{\frac{\lambda_2}{\lambda_0}},\pm\sqrt{\frac{\lambda_3}{\lambda_0}}\right),\\ \Lambda_B^{\mathrm{I}_c}&=\frac{L_A\Lambda L_{B_{\mathrm{I}_c}}^T}{(L_A\Lambda L_{B_{\mathrm{I}_c}}^T)_{00}}\\ &=\mathrm{diag}\left(1,\sqrt{\frac{\lambda_1}{\lambda_0}},\sqrt{\frac{\lambda_2}{\lambda_0}},\pm\sqrt{\frac{\lambda_3}{\lambda_0}}\right)\\ &\times\lambda_0\geqslant\lambda_1\geqslant\lambda_2\geqslant\lambda_3,\end{aligned} \quad (3.17)$$

if and only if the eigenvectors corresponding to the highest eigenvalue $\lambda_0$ of $G\Omega_A,\ G\Omega_B$ are positive four-vectors in $\mathcal{M}$.

*Proof.* It follows from explicit evaluation that

$$\Omega_A^{\mathrm{I}_c}=\lambda_0\Lambda_A^{\mathrm{I}_c}G\left(\Lambda_A^{\mathrm{I}_c}\right)^T,$$
$$\Omega_B^{\mathrm{I}_c}=\lambda_0(\Lambda_B^{\mathrm{I}_c})^TG\left(\Lambda_B^{\mathrm{I}_c}\right).$$

■

Expressed in terms of the three-term factorization,

$$\Lambda=\left(L_{A_{\mathrm{I}_c}}\right)^{-1}\Lambda_A^{\mathrm{I}_c}\left(L_B^T\right)^{-1}$$

[or equivalently $\Lambda=(L_{B_{\mathrm{I}_c}})^{-1}\ \Lambda_B^{\mathrm{I}_c}(L_A^T)^{-1}$], it is evident that $\Lambda$ is characterized by 15 real parameters where six real parameters each are from the Lorentz transformations $L_{A_{\mathrm{I}_c}},\ L_B$ (or $L_{B_{\mathrm{I}_c}},\ L_A$) and the rest of the three real parameters are given by $\sqrt{\lambda_i/\lambda_0},\ i=1\text{–}3$.

It is easy to see that the two-qubit density matrix $\rho_{AB}^{\mathrm{I}_c}$ associated with both canonical forms $\Lambda_A^{\mathrm{I}_c},\ \Lambda_B^{\mathrm{I}_c}$ is a Bell-diagonal state,

$$\begin{aligned}\rho_{AB}^{\mathrm{I}_c}&=\frac{1}{4}\left(\mathbb{1}_2\otimes\mathbb{1}_2+\sum_{i=1,2}\sqrt{\frac{\lambda_i}{\lambda_0}}\sigma_i\otimes\sigma_i\mp\sqrt{\frac{\lambda_3}{\lambda_0}}\sigma_3\otimes\sigma_3\right)\\ &=\rho_{BA}^{\mathrm{I}_c}.\end{aligned} \quad (3.18)$$

### B. Type-II canonical forms

Suppose the maximum eigenvalue $\lambda_0$ of $G\,\Omega_A$, associated with the neutral eigenvector $\mathbf{u}_0$ is, at least, doubly degenerate. Let us denote the set of eigenvalues of $G\Omega_A$ by $\{\lambda_0,\lambda_0,\lambda_1,\lambda_2\}$ arranged as $\lambda_0\geqslant\lambda_1\geqslant\lambda_2$. From (iii b) of the theorem, we have a maximal $G$-orthogonal triad $\{\mathbf{u}_0,\widetilde{\mathbf{a}}_1,\widetilde{\mathbf{a}}_2\}$ of eigenvectors of $G\Omega_A$ obeying

$$\mathbf{u}_0^TG\mathbf{u}_0=0,\quad \mathbf{u}_0^TG\widetilde{\mathbf{a}}_i=0,$$
$$\widetilde{\mathbf{a}}_i^TG\widetilde{\mathbf{a}}_j=-\delta_{ij},\quad i,j=1,2.$$

As outlined in the Appendix B we construct a $G$-orthogonal tetrad $\{\widetilde{\mathbf{a}}_0,\widetilde{\mathbf{a}}_1,\widetilde{\mathbf{a}}_2,\widetilde{\mathbf{a}}_3\}$ of four-vectors from the given set $\{\mathbf{u}_0,\widetilde{\mathbf{a}}_1,\widetilde{\mathbf{a}}_2\}$ of the eigenvectors of $G\Omega_A$ which consists of one neutral and two negative four-vectors.

Choosing a four-vector $\mathbf{u}_3$ such that $\mathbf{u}_3^TG\mathbf{u}_0\neq 0$ and $\mathbf{u}_3^TG\widetilde{\mathbf{a}}_i=0,\ i=1,2$, we construct

$$\widetilde{\mathbf{a}}_0=\mathbf{u}_3+\tau_u\mathbf{u}_0,\quad \widetilde{a}_{00}\geqslant 0,$$
$$\widetilde{\mathbf{a}}_3=\mathbf{u}_3-\kappa_u\mathbf{u}_0, \quad (3.19)$$

where

$$\tau_u=\frac{1-\mathbf{u}_3^TG\mathbf{u}_3}{2\mathbf{u}_3^TG\mathbf{u}_0},$$
$$\kappa_u=\frac{1+\mathbf{u}_3^TG\mathbf{u}_3}{2\mathbf{u}_3^TG\mathbf{u}_0}. \quad (3.20)$$

The tetrad $\{\widetilde{\mathbf{a}}_0,\widetilde{\mathbf{a}}_1,\widetilde{\mathbf{a}}_2,\widetilde{\mathbf{a}}_3\}$ of four-vectors satisfy the $G$-orthogonality conditions: $\widetilde{\mathbf{a}}_\mu^TG\widetilde{\mathbf{a}}_\nu=G_{\mu\nu}$.

On arranging the $G$-orthogonal tetrad $\{\widetilde{\mathbf{a}}_0,\widetilde{\mathbf{a}}_1,\widetilde{\mathbf{a}}_2,\widetilde{\mathbf{a}}_3\}$ as columns, we construct the OPLG matrix,

$$L_{A_{\mathrm{II}_c}}=(\widetilde{\mathbf{a}}_0\quad\widetilde{\mathbf{a}}_1\quad\widetilde{\mathbf{a}}_2\quad\widetilde{\mathbf{a}}_3), \quad (3.21)$$

in order to transform $\Omega_A$ to its canonical form.

Let us denote the "00" element of the matrix $\Omega_A^{\mathrm{II}_c}$ by

$$\phi_0=\left(L_{A_{\mathrm{II}_c}}\Omega_AL_{A_{\mathrm{II}_c}}^T\right)_{00}$$
$$=\widetilde{\mathbf{a}}_0^T\Omega_A\widetilde{\mathbf{a}}_0. \quad (3.22)$$

Substituting (3.19), (3.20), (3.22), and simplifying "30" and "33" matrix elements of $\Omega_A^{\mathrm{II}_c}$ we obtain

$$\begin{aligned}\left(L_{A_{\mathrm{II}_c}}\Omega_AL_{A_{\mathrm{II}_c}}^T\right)_{30}&=\widetilde{\mathbf{a}}_3^T\Omega_A\widetilde{\mathbf{a}}_0\\ &=\phi_0-\lambda_0,\\ \left(L_{A_{\mathrm{II}_c}}\Omega_AL_{A_{\mathrm{II}_c}}^T\right)_{33}&=\widetilde{\mathbf{a}}_3^T\Omega_A\widetilde{\mathbf{a}}_3\\ &=\phi_0-2\lambda_0.\end{aligned} \quad (3.23)$$

We, thus, arrive at the nondiagonal type-II canonical form of the real symmetric matrix $\Omega_A$ as

$$\Omega_A^{\mathrm{II}_c}=L_{A_{\mathrm{II}_c}}\Omega_AL_{A_{\mathrm{II}_c}}^T$$
$$=\begin{pmatrix}\phi_0 & 0 & 0 & \phi_0-\lambda_0\\ 0 & -\lambda_1 & 0 & 0\\ 0 & 0 & -\lambda_2 & 0\\ \phi_0-\lambda_0 & 0 & 0 & \phi_0-2\lambda_0\end{pmatrix}, \quad (3.24)$$

where $\lambda_0\geqslant\lambda_1\geqslant\lambda_2$.

In an analogous manner, we consider the triad $\{\mathbf{v}_0,\widetilde{\mathbf{b}}_1,\widetilde{\mathbf{b}}_2\}$ of eigenvectors of $G\Omega_B$ corresponding, respectively, to the eigenvalues $\lambda_0$ (doubly degenerate) $\lambda_1$ and $\lambda_2$. The eigenvectors satisfy $G$-orthogonality conditions,

$$\mathbf{v}_0^TG\mathbf{v}_0=0,\quad \mathbf{v}_0^TG\widetilde{\mathbf{b}}_k=0,$$
$$\widetilde{\mathbf{b}}_k^TG\widetilde{\mathbf{b}}_l=-\delta_{kl},\quad k,l=2,3. \quad (3.25)$$

Starting from this eigenvector set containing one neutral and two negative four-vectors, we pick a four-vector $\mathbf{v}_3$ such that $\mathbf{v}_3^T G\mathbf{v}_0 \neq 0$ and $\mathbf{v}_3^T G\widetilde{b}_i = 0,\ i = 1, 2$, and construct (see Appendix B for details),

$$\widetilde{\mathbf{b}}_0 = \mathbf{v}_3 + \tau_v \mathbf{v}_0, \quad \widetilde{b}_{00} \geqslant 0,$$
$$\widetilde{\mathbf{b}}_3 = \mathbf{v}_3 - \kappa_v \mathbf{v}_0, \tag{3.26}$$

where

$$\tau_v = \frac{1 - \mathbf{v}_3^T G\mathbf{v}_3}{2\mathbf{v}_3^T G\mathbf{v}_0},$$
$$\kappa_v = \frac{1 + \mathbf{v}_3^T G\mathbf{v}_3}{2\mathbf{v}_3^T G\mathbf{v}_0}. \tag{3.27}$$

This helps us to identify the tetrad $\{\widetilde{\mathbf{b}}_0, \widetilde{\mathbf{b}}_1, \widetilde{\mathbf{b}}_2, \widetilde{\mathbf{b}}_3\}$ of four-vectors obeying $G$-orthogonality conditions $\widetilde{\mathbf{b}}_\mu^T G\widetilde{\mathbf{b}}_\nu = G_{\mu\nu}$. So, we can explicitly construct the canonical OPLG matrices,

$$L^T_{B_{\rm II_c}} = (\widetilde{\mathbf{b}}_0 \quad \widetilde{\mathbf{b}}_1 \quad \widetilde{\mathbf{b}}_2 \quad \widetilde{\mathbf{b}}_3), \tag{3.28}$$

and obtain the canonical form of the real symmetric matrix $\Omega_B = \Lambda^T G\Lambda$ as

$$\Omega_B^{\rm II_c} = L_{B_{\rm II_c}} \Omega_B L^T_{B_{\rm II_c}} = \begin{pmatrix} \chi_0 & 0 & 0 & \chi_0 - \lambda_0 \\ 0 & -\lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 \\ \chi_0 - \lambda_0 & 0 & 0 & \chi_0 - 2\,\lambda_0 \end{pmatrix}. \tag{3.29}$$

Here, we have denoted the "00" element of $\Omega_B^{\rm II_c}$ by

$$\chi_0 = \left(L_{B_{\rm II_c}} \Omega_B L^T_{B_{\rm II_c}}\right)_{00} = \widetilde{\mathbf{b}}_0^T \Omega_B \widetilde{\mathbf{b}}_0. \tag{3.30}$$

Then, we evaluate 30 and 33 elements of $\Omega_B^{\rm II_c}$ by substituting (3.26), (3.27), and (3.30) to obtain

$$\left(L_{B_{\rm II_c}} \Omega_B L^T_{B_{\rm II_c}}\right)_{30} = \widetilde{\mathbf{b}}_3^T \Omega_B \widetilde{\mathbf{b}}_0 = \chi_0 - \lambda_0,$$
$$\left(L_{B_{\rm II_c}} \Omega_B L^T_{B_{\rm II_c}}\right)_{33} = \widetilde{\mathbf{b}}_3^T \Omega_B \widetilde{\mathbf{b}}_3 = \chi_0 - 2\lambda_0. \tag{3.31}$$

*Corollary 2.* When the eigenvectors corresponding to—at least, doubly degenerate—largest eigenvalue $\lambda_0$ of the matrix $G\Omega_A(G\Omega_B)$ is a neutral four-vector in $\mathcal{M}$, there exist canonical OPLG transformations $L_{A_{\rm II_c}}, L_{B_{\rm II_c}}$ yielding the following *nondiagonal* canonical forms of the real matrix $\Lambda$,

$$\Lambda_A^{\rm II_c} = \frac{L_{A_{\rm II_c}} \Lambda L_B^T}{\left(L_{A_{\rm II_c}} \Lambda L_B^T\right)_{00}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{\lambda_1}{\phi_0}} & 0 & 0 \\ 0 & 0 & \pm\sqrt{\frac{\lambda_2}{\phi_0}} & 0 \\ 1 - \frac{\lambda_0}{\phi_0} & 0 & 0 & \frac{\lambda_0}{\phi_0} \end{pmatrix}, \tag{3.32}$$

where $L_B$ is an OPLG transformation and

$$\Lambda_B^{\rm II_c} = \frac{L_A \Lambda L^T_{B_{\rm II_c}}}{\left(L_A \Lambda L^T_{B_{\rm II_c}}\right)_{00}} = \begin{pmatrix} 1 & 0 & 0 & 1 - \frac{\lambda_0}{\chi_0} \\ 0 & \sqrt{\frac{\lambda_1}{\chi_0}} & 0 & 0 \\ 0 & 0 & \pm\sqrt{\frac{\lambda_2}{\chi_0}} & 0 \\ 0 & 0 & 0 & \frac{\lambda_0}{\chi_0} \end{pmatrix}, \tag{3.33}$$

with $L_A$ being a suitable OPLG transformation.

Depending on if $\text{sgn}(\det \Lambda) = \pm$, one obtains "$\pm$" sign in the diagonal element $(\Lambda^{\rm II_c}_{A,{\rm or}B})_{22}$ in (3.32), (3.33).

*Proof.* It readily follows from explicit evaluations that

$$\Omega_A^{\rm II_c} = \phi_0 \Lambda_A^{\rm II_c} G\left(\Lambda_A^{\rm II_c}\right)^T,$$

and

$$\Omega_B^{\rm II_c} = \chi_0 \left(\Lambda_B^{\rm II_c}\right)^T G \Lambda_B^{\rm II_c}.$$

■

*Remark.* From our discussions in Sec. II, which resulted in the identification of real symmetric matrices $\Omega_A, \Omega_B$ associated with the real parametrization $\Lambda$ of the two-qubit density-matrix $\rho_{AB}$ [see (2.1), (2.3), (2.12)–(2.15), (2.18)–(2.22)], we observe that

(i) although the real matrix $\Lambda$ parametrizes the two-qubit density-matrix $\rho_{AB}$, its transpose $\Lambda^T$ characterizes $\rho_{BA}$, which is obtained by swapping $A$ and $B$;

(ii) canonical SLOCC transformations $\rho_{AB} \longrightarrow \rho^c_{AB}$ and $\rho_{BA} \longrightarrow \rho^c_{BA}$ are governed by the eigenvalues and the eigenvectors of the real matrices $G\Lambda G\Lambda^T = G\Omega_A$, $G\Lambda^T G\Lambda = G\Omega_B$, respectively;

(iii) even though $G\Omega_A, G\Omega_B$ share same eigenspectrum, the associated set of eigenvectors is different, in general, and, hence, one may expect different canonical structures $\rho^c_{AB}, \rho^c_{BA}$ for the density-matrices $\rho_{AB}, \rho_{BA}$;

(iv) exactly identical canonical forms $\Lambda_A^{\rm I_c} = \Lambda_B^{\rm I_c}$ [see (3.17)] and correspondingly $\rho^c_{AB} = \rho^c_{BA}$ [see (3.18)] are obtained when the eigenvectors of $G\Omega_A, G\Omega_B$ corresponding to their highest eigenvalue are positive four-vectors in $\mathcal{M}$;

(v) when neutral four-vectors in $\mathcal{M}$ happen to be one of the eigenvectors of $G\Omega_A, G\Omega_B$ (corresponding to, at least, doubly repeated highest eigenvalue $\lambda_0$) there are two different OPLG canonical forms [see (3.32), (3.33)] $\Lambda_A^{\rm II_c}, \Lambda_B^{\rm II_c}$, and, hence, SLOCC canonical forms $\rho_{AB}^{\rm II_c}, \rho_{BA}^{\rm II_c}$ of the corresponding density-matrix $\rho_{AB}$ differ, in general;

(vi) when $\Omega_A^{\rm II_c} = \Omega_B^{\rm II_c}$ one obtains $\Lambda_A^{\rm II_c} = (\Lambda_B^{\rm II_c})^T$.

Corresponding to the type-II canonical form $\Lambda_A^{\rm II_c}$ given by (3.32) we obtain explicit matrix form of $\rho_{AB}^{\rm II_c}$ (in the standard two-qubit basis $\{|0_A, 0_B\rangle, |0_A, 1_B\rangle, |1_A, 0_B\rangle, |1_A, 1_B\rangle\}$),

$$\rho_{AB}^{\rm II_c} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \frac{r_1 - r_2}{2} \\ 0 & (1 - r_0) & \frac{r_1 + r_2}{2} & 0 \\ 0 & \frac{r_1 + r_2}{2} & 0 & 0 \\ \frac{r_1 - r_2}{2} & 0 & 0 & r_0 \end{pmatrix}, \tag{3.34}$$

where we have denoted

$$\frac{\lambda_0}{\phi_0} = r_0, \quad \sqrt{\frac{\lambda_i}{\phi_0}} = r_i, \quad i = 1, 2. \tag{3.35}$$

Non-negativity condition $\rho_{AB}^{\mathrm{II_c}} \geqslant 0$ of the density matrix demands that

$$r_1 = -r_2, \qquad r_0 \geqslant r_1^2. \tag{3.36}$$

Similarly, the explicit matrix structure of the two-qubit density-matrix $\rho_{BA}^{\mathrm{II}_c}$ associated with the type-II canonical form $\Lambda_B^{\mathrm{II}_c}$ [see (3.33)] is given by

$$\rho_{BA}^{\mathrm{II_c}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & \frac{s_1-s_2}{2} \\ 0 & 0 & \frac{s_1+s_2}{2} & 0 \\ 0 & \frac{s_1+s_2}{2} & (1-s_0) & 0 \\ \frac{s_1-s_2}{2} & 0 & 0 & s_0 \end{pmatrix}, \tag{3.37}$$

where we have denoted

$$\frac{\lambda_0}{\chi_0} = s_0, \quad \sqrt{\frac{\lambda_i}{\chi_0}} = s_i, \quad i = 1, 2. \tag{3.38}$$

It is readily seen that $\rho_{BA}^{\mathrm{II_c}} \geqslant 0$ if and only if

$$s_1 = -s_2, \qquad s_0 \geqslant s_1^2. \tag{3.39}$$

Substituting (3.36), (3.39), we get *bona fide* type-II Lorentz canonical forms $\Lambda_A^{\mathrm{II_c}}$, $\Lambda_B^{\mathrm{II_c}}$ and the associated density-matrices $\rho_{AB}^{\mathrm{II_c}}$, $\rho_{BA}^{\mathrm{II_c}}$ as

$$\Lambda_A^{\mathrm{II_c}} = \frac{L_{A_{\mathrm{II_c}}}\Lambda L_B^T}{(L_{A_{\mathrm{II_c}}}\Lambda L_B^T)_{00}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & -r_1 & 0 \\ 1-r_0 & 0 & 0 & r_0 \end{pmatrix},$$

$$\rho_{AB}^{\mathrm{II_c}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & r_1 \\ 0 & 1-r_0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r_1 & 0 & 0 & r_0 \end{pmatrix}, \quad 0 \leqslant r_1^2 \leqslant r_0 \leqslant 1,$$

$$\Lambda_B^{\mathrm{II_c}} = \frac{L_A\Lambda L_{B_{\mathrm{II_c}}}^T}{(L_A\Lambda L_{B_{\mathrm{II_c}}}^T)_{00}} = \begin{pmatrix} 1 & 0 & 0 & 1-s_0 \\ 0 & s_1 & 0 & 0 \\ 0 & 0 & -s_1 & 0 \\ 0 & 0 & 0 & s_0 \end{pmatrix},$$

$$\rho_{BA}^{\mathrm{II_c}} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & s_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-s_0 & 0 \\ s_1 & 0 & 0 & s_0 \end{pmatrix}, \quad 0 \leqslant s_1^2 \leqslant s_0 \leqslant 1 \tag{3.40}$$

It is pertinent to point out that type-II canonical forms are associated with SLOCC transformations on the two-qubit density matrices of rank less than or equal to 3. Based on the three-term factorization (up to normalization) $\Lambda = (L_{A_{\mathrm{II_c}}})^{-1}\Lambda_A^{\mathrm{II}_c}(L_B^T)^{-1}$, it is clear that the 14 real parameters characterizing $\Lambda$ are expressed in terms of 12 parameters of the OPLG transformations $L_{A_{\mathrm{II_c}}}$, $L_B$ and the two parameters of the canonical form, i.e., $r_0 = \lambda_0/\phi_0,\ r_1 = \sqrt{\lambda_1/\phi_0}$. [Similarly, $(L_A)^{-1}\Lambda_B^{\mathrm{II}_c}(L_{B_{\mathrm{II_c}}}^T)^{-1}$ is characterized by 12 real parameters of transformations $L_A, L_{B_{\mathrm{II_c}}}$ and the canonical parameters $s_0 = \lambda_0/\chi_0,\ s_1 = \sqrt{\lambda_1/\chi_0}$].

### C. Nondiagonal SLOCC normal form of Verstraete *et al.*

Verstraete *et al.* [5] had obtained two different types of Lorentz canonical forms of the real matrix $\Lambda$ under the transformation $\Lambda \longrightarrow L_A\Lambda L_B^T,\ L_A, L_B \in SO(3,1)$, by making use of Theorem (5.3) of Ref. [20] on matrix decompositions in $n$-dimensional space with an indefinite metric. One of the canonical forms of real matrix $\Lambda$ of Ref. [5] is diagonal (type I) and the corresponding SLOCC structure of the two-qubit density matrix is Bell diagonal. Our type-I canonical form (3.17) for the real matrix $\Lambda$ agrees identically with this result given by Ref. [5]. The nondiagonal canonical form of the real matrix $\Lambda$, corresponding to two-qubit states of rank less than 4, has the following explicit structure [5]:

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & d & 0 & 0 \\ 0 & 0 & -d & 0 \\ c & 0 & 0 & 1+c-b \end{pmatrix}, \tag{3.41}$$

where $b,\ c$, and $d$ are real parameters. The two-qubit density-matrix $\rho_{AB}^{\Sigma}$ associated with the real matrix $\Sigma$ is given (in the standard two-qubit basis) by

$$\rho_{AB}^{\Sigma} = \frac{1}{2}\begin{pmatrix} 1+c & 0 & 0 & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b-c & 0 \\ d & 0 & 0 & 1-b \end{pmatrix}. \tag{3.42}$$

It is clearly seen that the eigenvalues of $\rho_{AB}^{\Sigma}$ are non-negative if

$$(1+c)(1-b) \geqslant d^2, \quad 0 \leqslant (b-c) \leqslant 2, \quad -1 \leqslant b, c, d \leqslant 1. \tag{3.43}$$

In order to establish a connection between the nondiagonal form (3.41) with the type-II canonical forms (3.40) we evaluate the symmetric $4 \times 4$ matrices $\Omega_A = \Sigma G \Sigma^T$ and $\Omega_B = \Sigma^T G \Sigma$ associated with the nondiagonal canonical form (3.41), which are given explicitly by

$$\Omega_A^{\Sigma} = \Sigma G \Sigma^T = \begin{pmatrix} 1-b^2 & 0 & 0 & -(1-b)(b-c) \\ 0 & -d^2 & 0 & 0 \\ 0 & 0 & -d^2 & 0 \\ -(1-b)(b-c) & 0 & 0 & (1-b)(b-2c-1) \end{pmatrix}$$

$$\Omega_B^{\Sigma} = \Sigma^T G \Sigma = \begin{pmatrix} 1-c^2 & 0 & 0 & (b-c)(1+c) \\ 0 & -d^2 & 0 & 0 \\ 0 & 0 & -d^2 & 0 \\ (b-c)(1+c) & 0 & 0 & (1+c)(2b-c-1) \end{pmatrix}.$$

Note that when $b = c$, the symmetric matrices $\Omega_A^\Sigma, \Omega_B^\Sigma$ are diagonal and, thus, one obtains type-I diagonal canonical form (see Sec. III A) for $\Sigma$. Moreover, for $b = \pm 1$ or $c = \pm 1$ the density-matrix (3.42) reduces to a product form $\rho_{AB}^\Sigma = \rho_A \otimes \rho_B$ where $\rho_A$ or $\rho_B$ are pure states. It is easy to see that the eigenvalues of $G\Omega_A^\Sigma$ and $G\Omega_B^\Sigma$ are zero in the cases $b = \pm 1$ or $c = \pm 1$. We, thus, confine our attention to $b \neq c,\ b, c \neq \pm 1$.

Eigenvalues of $G\Omega_A^\Sigma, G\Omega_B^\Sigma$ are readily obtained as

$$\lambda_0 = \lambda_3 = (1+c)(1-b), \quad \lambda_1 = \lambda_2 = d^2. \tag{3.44}$$

From the non-negativity constraint $(1+c)(1-b) \geqslant d^2$ on the density-matrix $\rho_{AB}^\Sigma$ [see (3.43)] it follows that $\lambda_0$ happens to be the highest eigenvalue and the corresponding eigenvectors of $G\Omega_A^\Sigma, G\Omega_B^\Sigma$ are neutral four-vectors. This confirms that under SLOCC operations on the two-qubit density-matrix $\rho_{AB}^\Sigma$ of (3.42) the real matrix $\Sigma$ can be transformed to Lorentz canonical forms of type II [see (3.40), (3.35), and (3.38)], which we denote by $\Lambda_{A_\Sigma}^{\mathrm{II}_c}$ or $\Lambda_{B_\Sigma}^{\mathrm{II}_c}$. Now we proceed further to obtain explicit matrices corresponding to these type-II canonical forms of $\Sigma$.

We identify that $\Omega_B^\Sigma$ already exhibits a canonical form as given in (3.29) if we substitute

$$\chi_0 = \left(\Omega_B^\Sigma\right)_{00} = 1 - c^2. \tag{3.45}$$

Thus, we recognize that $L_{B_{\mathrm{II}_c}} = \mathbb{1}_4$, i.e., a $4\times 4$ identity matrix. With the help of an OPLG transformation matrix,

$$L_A = \begin{pmatrix} \frac{1}{\sqrt{1-c^2}} & 0 & 0 & \frac{-c}{\sqrt{1-c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-c}{\sqrt{1-c^2}} & 0 & 0 & \frac{1}{\sqrt{1-c^2}} \end{pmatrix}, \tag{3.46}$$

we obtain one of the type-II canonical stuctures (3.40) for $\Sigma$,

$$\Lambda_{B_\Sigma}^{\mathrm{II}_c} = \frac{L_A\Sigma}{(L_A\Sigma)_{00}} = \begin{pmatrix} 1 & 0 & 0 & \frac{b-c}{1-c} \\ 0 & \frac{d}{\sqrt{1-c^2}} & 0 & 0 \\ 0 & 0 & \frac{-d}{\sqrt{1-c^2}} & 0 \\ 0 & 0 & 0 & \frac{1-b}{1-c} \end{pmatrix}. \tag{3.47}$$

In other words our type-II canonical form $\Lambda_{B_\Sigma}^{\mathrm{II}_c}$ is Lorentz equivalent to the real matrix $\Sigma$ [see (3.41)] of Ref. [5].

Following the method outlined in the Sec. III B and in Appendix B, for the construction of the explicit OPLG transformation matrix $L_{A_{\mathrm{II}_c}}$, we obtain

$$L_{A_{\mathrm{II}_c}} = \begin{pmatrix} \frac{1-b+c}{\sqrt{(1+c)(1+c-2b)}} & 0 & 0 & \frac{-b}{\sqrt{(1+c)(1+c-2b)}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-b}{\sqrt{(1+c)(1+c-2b)}} & 0 & 0 & \frac{1-b+c}{\sqrt{(1+c)(1+c-2b)}} \end{pmatrix}, \tag{3.48}$$

and verify that

$$\Lambda_{A_\Sigma}^{\mathrm{II}_c} = \frac{L_{A_{\mathrm{II}_c}}\Sigma}{\left(L_{A_{\mathrm{II}_c}}\Sigma\right)_{00}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{d^2(1+c-2b)}{\lambda_0(1-b)}} & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{d^2(1+c-2b)}{\lambda_0(1-b)}} & 0 \\ \frac{c-b}{1-b} & 0 & 0 & \frac{1-2b+c}{1-b} \end{pmatrix} \tag{3.49}$$

exhibits type-II canonical form $\Lambda_A^{\mathrm{II}_c}$ given in (3.40). This proves that the nondiagonal normal form $\Sigma$ [given by (3.41)] is SLOCC equivalent to the type-II canonical form $\Lambda_{A_\Sigma}^{\mathrm{II}_c}$ of (3.49) in conformity with our approach.

## IV. GEOMETRIC REPRESENTATION OF SLOCC CANONICAL FORMS OF TWO-QUBITS

It is shown in Sec. III that the real matrix $\Lambda$, parametrizing a two-qubit density-matrix $\rho_{AB}$, can be reduced to two algebraically distinct types of canonical forms (3.17), (3.40) under OPLG transformations. The algebraically distinct canonical forms are determined via the eigenvalues and eigenvectors of the matrices $G\Omega_A = G\Lambda G\Lambda^T$ and $G\Omega_B = G\Lambda^T G\Lambda$ constructed from $\Lambda$ and the Minkowski space metric tensor $G$. In this section we discuss the geometrical representation captured by the canonical forms of $\Lambda$, which, in turn, offer visualization of the SLOCC invariant families of two-qubit density matrices on and within the Bloch ball. To this end we recall (see Sec. II C) that a map $P_A \mapsto Q_B$ from the set $\mathcal{P}_A^+ := \{P_A = \frac{1}{2}\sum_\mu p_{A_\mu}\sigma_\mu | P_A \geqslant 0\}$ of all non-negative operators acting on the Hilbert space $\mathcal{H}_A$ to another set of non-negative operators $\mathcal{Q}_B^+ := \{Q_B = 2\,\mathrm{Tr}_A[\rho_{AB}(P_A\otimes \mathbb{1}_2)] | Q_B \geqslant 0\}$ on the Hilbert space $\mathcal{H}_B$ can be expressed alternately as a linear transformation on Minkowski four-vectors, i.e., $\Lambda^T\colon \mathbf{p}_A \mapsto \mathbf{q}_B = \Lambda^T\mathbf{p}_A$, where $\mathbf{p}_A, \mathbf{q}_B$ are non-negative ( positive/neutral ) four-vectors with their zeroth components positive $p_{A_0} > 0$, $q_{B_0} > 0$. Similarly, the real matrix $\Lambda$ induces a linear transformation $\Lambda\colon \mathbf{p}_B \mapsto \mathbf{q}_A = \Lambda\mathbf{p}_B$ from the set of all non-negative four-vectors $\{\mathbf{p}_B | \mathbf{p}_B^T G\mathbf{p}_B \geqslant 0,\ p_{B_0} > 0\}$ to the set $\{\mathbf{q}_A = \Lambda\mathbf{p}_B | \mathbf{q}_A^T G\mathbf{q}_A^T \geqslant 0,\ q_{A_0} > 0\}$.

Using the fact that every positive four-vector can always be expressed as a sum of neutral four-vectors [21,22,24], we conveniently restrict ourselves to the maps,

(i) $\mathbf{p}_n \mapsto \mathbf{q} = \Lambda\mathbf{p}_n$,
(ii) $\mathbf{p}_n \mapsto \mathbf{q} = \Lambda^T\mathbf{p}_n$

induced by the real matrix $\Lambda$, on the set of all neutral four-vectors $\{\mathbf{p}_n | \mathbf{p}_n^T G\mathbf{p}_n = 0,\ p_{n0} > 0\}$.

Let us consider the set of all neutral four-vectors $\{\mathbf{p}_n = (1, x_1, x_2, x_3)^T,\ x_1^2 + x_2^2 + x_3^2 = 1\}$ with $(x_1, x_2, x_3)$ representing the entire Bloch sphere (i.e., the unit sphere $\mathcal{S}^2 \in \mathbb{R}^3$). The type-I canonical form $\Lambda_A^{\mathrm{I}_c}$ given in (3.17) transforms $\mathbf{p}_n = (1, x_1, x_2, x_3)^T$ to a non-negative four-vector $\mathbf{q} = \Lambda^{\mathrm{I}_c}\mathbf{p}_n = (1, y_1, y_2, y_3)$ where

$$y_1 = \sqrt{\frac{\lambda_1}{\lambda_0}}x_1, \quad y_2 = \sqrt{\frac{\lambda_2}{\lambda_0}}x_2, \quad y_3 = \pm\sqrt{\frac{\lambda_3}{\lambda_0}}x_3. \tag{4.1}$$

Evidently, the transformed three-vector $(y_1, y_2, y_3)$ obeys the equation of a point on the surface of an ellipsoid,

$$\frac{y_1^2}{\xi_1^2} + \frac{y_2^2}{\xi_2^2} + \frac{y_3^2}{\xi_3^2} = 1, \tag{4.2}$$

where $\xi_i = \sqrt{\lambda_i/\lambda_0},\ i = 1$–3. Geometric intuition of the canonical form $\Lambda_A^{\mathrm{I}_c}$ is, thus, clear: The map $\Lambda^{\mathrm{I}_c}\colon (1, x_1, x_2, x_3)^T \mapsto (1, y_1, y_2, y_3)^T$ transforms the Bloch sphere to an ellipsoidal surface described by (4.2). It may be recognized that the ellipsoidal surface described by (4.2) geometrically represents the set of all steered Bloch vectors [6,10–12] of Alice's (Bob's) qubit after Bob (Alice) performs

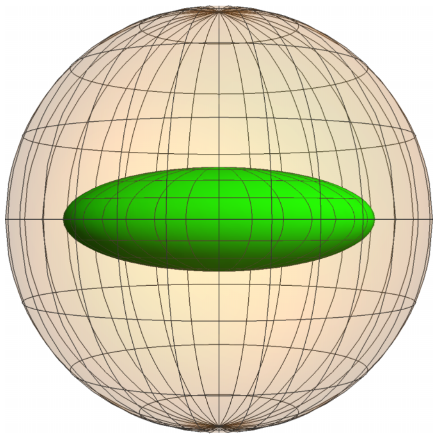

FIG. 1. Ellipsoid representing type-I canonical form $\Lambda^{\mathrm{I}_c}$ given by (3.17). Semiaxes lengths of this ellipsoid [see (4.2)] are given by $(\sqrt{\lambda_1/\lambda_0}, \sqrt{\lambda_2/\lambda_0}, \sqrt{\lambda_3/\lambda_0})$, where $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ denote eigenvalues of $G\Omega_A, G\Omega_B$ [see (3.6)]. The ellipsoid is centered at the origin (0,0,0), and it provides geometric insight for the set of all two-qubit states, which are on the SLOCC orbit of Bell-diagonal states (3.18).

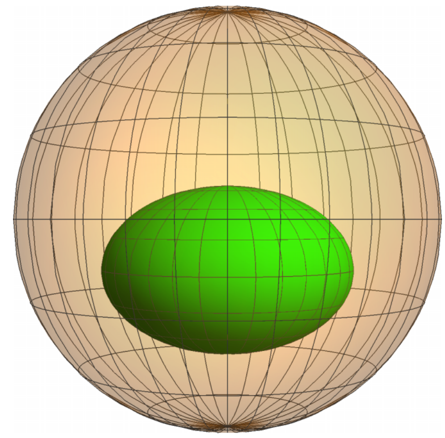

FIG. 2. Steering spheroid (4.6) offering pictorial representation of type-II canonical form $\Lambda_A^{\mathrm{II}_c}$, which characterizes the two-qubit state $\rho_{AB}^{\mathrm{II}_c}$ [see (3.40)]. The spheroid is centered at $(0, 0, 1 - r_0)$ and has semiaxes lengths $(r_1, r_1, r_0)$, $0 \leqslant r_1^2 \leqslant r_0 \leqslant 1$.

projective measurements on his (her) qubit [see (2.17), (2.20), and (2.21)], given that the two-qubit state shared between them is in the canonical Bell-diagonal form (3.18), which is achieved by SLOCC on $\rho_{AB}$.

In Fig. 1 we have depicted the ellipsoid with lengths of its semiaxes given by [see (4.2)] $(\sqrt{\lambda_1/\lambda_0}, \sqrt{\lambda_2/\lambda_0}, \sqrt{\lambda_3/\lambda_0})$. Here $\lambda_0 \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$. The ellipsoid is centered at the origin (0,0,0).

Associated with the type-II canonical forms $\Lambda_A^{\mathrm{II}_c}$, $\Lambda_B^{\mathrm{II}_c}$ [see (3.40)] one obtains

$$\Lambda_A^{\mathrm{II}_c}(1, x_1, x_2, x_3)^T = (1, y_{A_1}, y_{A_2}, y_{A_3})^T$$
$$\left(\Lambda_B^{\mathrm{II}_c}\right)^T(1, x_1, x_2, x_3)^T = (1, y_{B_1}, y_{B_2}, y_{B_3})^T. \qquad (4.3)$$

Here $x_1^2 + x_2^2 + x_3^2 = 1$ represents the Bloch sphere and

$$y_{A_1} = r_1 x_1, \quad y_{A_2} = -r_1 x_2,$$
$$y_{A_3} = (1 - r_0) + r_0 x_3, \quad 0 \leqslant r_1^2 \leqslant r_0 \leqslant 1, \qquad (4.4)$$

$$y_{B_1} = s_1 x_1, \quad y_{B_2} = -s_1 x_2$$
$$y_{B_3} = (1 - s_0) + s_0 x_1, \quad 0 \leqslant s_1^2 \leqslant s_0 \leqslant 1 \qquad (4.5)$$

represent the set of all qubit states (Bloch vectors) that can be steered to by projective measurements performed on another qubit of the two-qubit state $\rho_{AB}^{\mathrm{II}_c}$ of (3.40), where $r_0, r_1$, are specified by (3.35) and (3.36) and $s_0, s_1$ are defined via (3.38) and (3.39), together with (3.22) and (3.30). From (4.4) it is seen that $(y_{A_1}, y_{A_2}, y_{A_3})$ and $(y_{B_1}, y_{B_2}, y_{B_3})$ satisfy the equations,

$$\frac{y_{A_1}^2 + y_{A_2}^2}{r_1^2} + \frac{[y_{A_3} - (1 - r_0)]^2}{r_0^2} = 1,$$
$$\frac{y_{B_1}^2 + y_{B_2}^2}{s_1^2} + \frac{[y_{B_3} - (1 - s_0)]^2}{s_0^2} = 1, \qquad (4.6)$$

which represent surfaces of spheroids centered, respectively, at $(0, 0, 1 - r_0)$, $(0, 0, 1 - s_0)$. The spheroidal surfaces (4.6) provide geometric visualization of the collection of all Bloch vectors of one of the qubits after projective measurements are performed on the other qubit [6,10–12] when the two-qubit state $\rho_{AB}$ is SLOCC equivalent to the type-II canonical density-matrix $\rho_{AB}^{\mathrm{II}_c}$ of (3.40). In Fig. 2 steering spheroid representing type-II states $\rho_{AB}^{\mathrm{II}_c}$ of (3.40) is shown.

## V. SUMMARY

In this paper we have presented a complete analysis to obtain two different types of SLOCC canonical forms and the associated geometric visualization of two-qubit states—which happen to be the simplest composite systems. Using the established result that the action of SLOCC on a two-qubit state $\rho_{AB} = \frac{1}{4}\sum_{\mu,\nu=0}^{3} \Lambda_{\mu,\nu}\sigma_\mu \otimes \sigma_\nu$ manifests itself in terms of Lorentz transformation on its $4 \times 4$ real matrix parametrization $\Lambda$, two different types of canonical forms had been obtained previously by Verstraete *et al.* [5] and Verstraete [6]. However, the approach employed by Refs. [5,6] to arrive at the SLOCC canonical forms involved highly technical results on matrix decompositions in spaces with indefinite metric.

Based on a different approach, inspired by the techniques developed in classical polarization optics by some of us [21,22], we have given here a simple procedure to explicitly evaluate two different types of SLOCC canonical forms of the real matrix $\Lambda$ and the associated two-qubit density matrix. Equivalence between the canonical forms obtained via our approach with the ones obtained in Ref. [5] has also been established here. Finally, our approach leads to an elegant geometric representation aiding visualization of two different types of canonical forms associated with the entire family of two-qubit states on the respective SLOCC orbits. We believe that our comprehensive analysis offers new insights in the study of SLOCC canonical forms of higher-dimensional and multipartite composite systems too.

## ACKNOWLEDGMENTS

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## APPENDIX A

For the sake of completeness we give a brief outline covering essential elements of the proof of the theorem stated in Sec. III. For a detailed proof with all its nuances addressed, see Ref. [21].

To begin with, note that the real matrix $\Lambda$ parametrizing a two qubit density matrix induces a linear trasnformation $\Lambda:\mathbf{p}\mapsto\mathbf{q}=\Lambda\mathbf{p}$ from the set of all non-negative four-vectors,

$$\{\mathbf{p}|\mathbf{p}^TG\mathbf{p}\geqslant 0,\quad p_0>0\},\tag{A1}$$

to another identical set,

$$\{\mathbf{q}=\Lambda\mathbf{p}|\mathbf{q}^TG\mathbf{q}\geqslant 0,\quad q_0>0\}.\tag{A2}$$

Since every positive four-vector can always be expressed as a sum of neutral four-vectors [21], we restrict ourselves to the set,

$$\{\mathbf{p}_n=(1,x)^T,\quad x^Tx=x_1^2+x_2^2+x_3^2=1;\mathbf{p}_n^TG\mathbf{p}_n=0\},$$

without any loss of generality. We then express the non-negativity condition (A2) as

$$\{\mathbf{q}=\Lambda\mathbf{p}_n,\ \mathbf{p}_n^TG\mathbf{p}_n=0\Rightarrow\mathbf{q}^TG\mathbf{q}=\mathbf{p}_n^T\Omega\mathbf{p}_n\geqslant 0\},\tag{A3}$$

where

$$\Omega=\Lambda^TG\Lambda=\Omega^T\tag{A4}$$

is a $4\times 4$ real symmetric matrix.

Let us express $\Omega$ as a $1\oplus 3$ block matrix,

$$\Omega=\begin{pmatrix}n_0 & \tilde{n}^T\\ \tilde{n} & A\end{pmatrix},\tag{A5}$$

with $n_0>0$, $\tilde{n}=(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3)^T$ a three-componental column and $A^T=A$ is a $3\times 3$ real symmetric matrix.

With the help of a Lorentz transformation $L=1\oplus R$, where $R\in SO(3)$ denotes a three-dimensional rotation matrix, one can diagonalize the $3\times 3$ real symmetric matrix $A$ [see (A5)], i.e., $R^TAR=A_0=\text{diag}(\alpha_1,\alpha_2,\alpha_3)$. We, thus, obtain

$$\Omega_0=L^T\Omega L=\begin{pmatrix}n_0 & n_1 & n_2 & n_3\\ n_1 & \alpha_1 & 0 & 0\\ n_2 & 0 & \alpha_1 & 0\\ n_3 & 0 & 0 & \alpha_3\end{pmatrix}$$

where $(n_1,n_2,n_3)^T=n=R\tilde{n}$.

Let us denote

$$\mathbf{p}_n^T\Omega_0\,\mathbf{p}_n=D(\Omega_0;x).$$

The non-negativity condition $\mathbf{p}_n^T\Omega_0\mathbf{p}_n=D(\Omega_0;x)\geqslant 0$ assumes the form

$$D(\Omega_0;x)=n_0+2x^Tn+x^TA_0\,n\geqslant 0\quad\forall\,x^Tx=1.\tag{A6}$$

Note that the condition (A6) is ensured for all $x^Tx=1$, if the absolute minimum $D_{\min}$ of the function $D(\Omega_0;x)$, or equivalently, all the *critical values* $D_a$ of $D(\Omega_0;x)$ are non-negative. The method of Lagrange multipliers to evaluate the critical values of the function $D(\Omega_0;x)$, subject to the constraint $x^Tx=1$, leads to an auxiliary function,

$$K(\Omega_0;x)=D(\Omega_0;x)+\lambda(x^Tx-1),\tag{A7}$$

where $\lambda$ denotes the Lagrange multiplier. Critical values $D_a$ of the function $D(\Omega_0;x)$ can then be obtained by solving

$$\left.\frac{\partial K(\Omega_0;x)}{\partial\lambda}\right|_{\lambda_a,x_{a,i}}=0,$$

$$\left.\frac{\partial K(\Omega_0;x)}{\partial x_i}\right|_{\lambda_a,x_{a,i}}=0,\quad i=1\text{–}3.\tag{A8}$$

The equations determining $\lambda_a$, $x_{a,i}=(x_{a,1},x_{a,2},x_{a,3})$ can then be expressed as

$$(A_0+\lambda_a\mathbb{1}_3)x_a=-n,\quad x_a^Tx_a=1,\tag{A9}$$

where $\mathbb{1}_3$ denotes $3\times 3$ identity matrix. Solutions of (A9), in turn, determine the critical values $D(\Omega_0;\lambda_a,x_a)=D_a$ of the function $D(\Omega_0;x)$.

Substituting $A_0=\text{diag}(\alpha_1,\ \alpha_2,\alpha_3)$ in (A9) and simplifying, we obtain

$$x_{a,i}=\frac{-n_i}{(\alpha_i+\lambda_a)},\qquad i=1\text{–}3.\tag{A10}$$

Furthermore, the normalization condition $x_a^Tx_a=1$ leads to

$$\sum_{i=1}^{3}\frac{n_i^2}{(\alpha_i+\lambda_a)^2}=1.\tag{A11}$$

The critical values of $D(\Omega_0;x)$ are then given by

$$D_a=n_0-\lambda_a-\sum_{i=1}^{3}\frac{n_i^2}{\lambda_a+\alpha_i}.\tag{A12}$$

We focus on identifying the implications of the conditions $D_a\geqslant 0$, $a=1,2,\ldots$ on the eigenvalues and eigenvectors of the $4\times 4$ matrix $G\Omega$, which are termed as $G$ eigenvalues and $G$ eigenvectors of the real symmetric matrix $\Omega$. To this end, we study the behavior of the function,

$$h(\lambda)=n_0-\lambda-\sum_{i=1}^{3}\frac{n_i^2}{\lambda+\alpha_i},\tag{A13}$$

obtained by replacing $\lambda_a$ by a continuous real variable $\lambda$ in (A12).

We list some of the important properties of the function $h(\lambda)$, which follow from its definition (A13):

(a) The function $h(\lambda)$ is differentiable everywhere on the $\lambda$ axis except for a finite number of discontinuities at $\lambda=-\alpha_i$, $i=1\text{–}3$, whenever the corresponding $n_i\neq 0$.

(b) As $h(\lambda)$ changes sign across a discontinuity, it is positive to the immediate left and is negative to the immediate right of a discontinuity. This implies that there must be an odd number of real zeros of the function $h(\lambda)$ in between any two consecutive discontinuities.

(c) In the limit $\lambda \to \infty$ it is seen that $h(\lambda) \to -\infty$ and as $\lambda \to -\infty$ one finds $h(\lambda) \to \infty$. This observation along with the behavior of $h(\lambda)$ near a discontinuity leads to the conclusion that there must be an even number of zeros in the interval $(\alpha_{\rm max}, \infty)$.

(d) Since the largest zero $\lambda_{\rm max}$ occurs in the interval $(\alpha_{\rm max}, \infty)$, the slope of $h(\lambda)$ at $\lambda_{\rm max}$ is either negative or zero. In fact, when $h'(\lambda_{\rm max}) = 0$, both the zero and the critical value occur simultaneously at $\lambda_{\rm max}$. [Here $h'(\lambda)$ denotes differentiation of $h(\lambda)$ with respect to the variable $\lambda$].

(e) The function $h(\lambda)$ must have, at least, $k+1$ real zeros where $k \leqslant 3$ denotes the number of discontinuities.

(f) Depending on the number of nonzero values of $n_1, n_2, n_3$ and based on the degeneracies $\alpha_1, \alpha_2, \alpha_3$, there are 20 possible situations, each with a different number of discontinuities, zeros, and the critical values of $h(\lambda)$: (i) none of $n_1, n_2, n_3$ are zero; (ii) one of $n_1, n_2, n_3$ is zero; (iii) two of $n_1, n_2, n_3$ are zero; (iv) $n_1 = n_2 = n_3 = 0$. Each of these four cases fall under five different subclasses corresponding to the degeneracies of $\alpha_1, \alpha_2, \alpha_3$: nondegenerate, i.e., (A) $\alpha_1 \neq \alpha_2 \neq \alpha_3$, twofold degenerate, i.e, (B1) $\alpha_1 = \alpha_2 \equiv \alpha \neq \alpha_3$, (B2) $\alpha_1 \neq \alpha_2 = \alpha_3 \equiv \alpha$, (B3) $\alpha_1 = \alpha_3 \equiv \alpha \neq \alpha_2$, and fully degenerate, i.e., (C) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.

Associated with these $4 \times 5 = 20$ distinct possibilities one may list the number of discontinuities, zeros, local maxima, and local minima of $h(\lambda)$. As mentioned already there are $k+1$ real zeros associated with $k$ discontinuities of $h(\lambda)$. For instance, if there are $k = 3$ distinct discontinuities (realized when $\alpha_1 \neq \alpha_2 \neq \alpha_3$ and $n_1, n_2, n_3 \neq 0$), it can be seen that, at least, *two* zeros exist. Furthermore, in the region $(\alpha_{\rm max}, \infty)$ one should find, at least, *two* zeros. In other words, at least, *four* real zeros exist for the function $h(\lambda)$ when there are three distinct discontinuities. When there are *two* distinct discontinuities ($k = 2$), at least, *one* zero of the function $h(\lambda)$ occurs between them; in the region $(\alpha_{\rm max}, \infty)$ *two* zeros (either distinct or doubly repeated) exist. Thus, $1 + 2 = 3$ real zeros exist for $h(\lambda)$ when $k = 2$.

Interestingly, the function $h(\lambda)$ can be expressed in terms of the characteristic polynomial $\phi(\lambda) = \det(\Omega - \lambda G)$ of the real symmetric matrix $\Omega$ and $\psi(\lambda) = \prod_{i=1}^{3}(\alpha_i + \lambda) = \det(A_0 + \lambda \mathbb{1}_3)$ as

$$h(\lambda) = \frac{\phi(\lambda)}{\psi(\lambda)} = \frac{\det(\Omega_0 - \lambda G)}{\det(A_0 + \lambda \mathbb{1}_3)} = \frac{\det(\Omega - \lambda G)}{\det(A + \lambda \mathbb{1}_3)}. \tag{A14}$$

Furthermore, it is found convenient to express the characteristic polynomial $\phi(\lambda)$ as

$$\phi(\lambda) = \psi(\lambda) h(\lambda) = \phi_1(\lambda) g(\lambda) h(\lambda),$$

in terms of some simple polynomials $\phi_1(\lambda), g(\lambda)$ with real roots, chosen such that the roots of $\phi_1(\lambda)$ may be readily identified and $\phi_1(\lambda), g(\lambda)$ are finite at every real zero of the function $h(\lambda)$.

Examining the characteristic equation $\phi(\lambda) = 0$ and based on explicit evaluations of $\phi_1(\lambda)$, $g(\lambda)$, and $h(\lambda)$ in each of the 20 cases one arrives at the following conclusions [21]:

(1) Every real zero of $h(\lambda)$ is a $G$ eigenvalue of $\Omega$.

(2) If $r$ denotes the number of (real) roots of $\phi_1(\lambda)$ and $k$ denotes the number of discontinuities of $h(\lambda)$, then it is identified that $r + k + 1 = 4$ in all the 20 cases, thus, proving that $\phi(\lambda)$ has four real roots $\lambda_\mu$, $\mu = 0$–3. This proves that the $G$ eigenvalues of $\Omega$ are real.

(3) If $\mathbf{x}$ denotes the $G$ eigenvector of $\Omega$ belonging to $G$-eigenvalue $\lambda$, it can be seen that

$$\mathbf{x}^T G X = -h'(\lambda). \tag{A15}$$

Let us denote the largest $G$ eigenvalue of $\Omega$ by $\lambda_0$. As stated already [see property (d) of the function $h(\lambda)$] $h'(\lambda_0)$ must be either negative or zero. Thus, from (A15) it is clear that the $G$-eigenvector $\mathbf{x}_0$ belonging to the largest $G$-eigenvalue $\lambda_0$ obeys $\mathbf{x}^T G \mathbf{x} \geqslant 0$ implying that it is either positive or neutral.

It also follows that the largest $G$-eigenvalue $\lambda_0$ is doubly degenerate when $h'(\lambda_0) = 0$. In other words, $\mathbf{x}_0$ corresponding to a largest doubly degenerate eigenvalue $\lambda_0$ is a neutral four-vector.

(4) The $G$-eigenvectors $\mathbf{x}_r$ corresponding to the $G$-eigenvalues $\lambda_r < \lambda_0$ of $\Omega$ are negative, i.e., $\mathbf{x}_r^T G \mathbf{x}_r < 0$. This follows essentially from the observation that $h'(\lambda_r) = -\mathbf{x}_r^T G \mathbf{x}_r$ [see (A15)] is positive when $\lambda_r < \lambda_0$.

(5) An explicit analysis of the $G$ eigenspace of $\lambda_0$ in each of the 20 different cases proves that the real symmetric $4 \times 4$ matrix $\Omega$, obeying the condition $\mathbf{p}_n^T \Omega \mathbf{p}_n \geqslant 0$, possesses either: (i) a positive $G$ eigenvector belonging to the largest $G$-eigenvalue $\lambda_0$ and three negative $G$ eigenvectors or (ii) a neutral $G$ eigenvector belonging to, *at least*, doubly degenerate $G$-eigenvalue $\lambda_0$ and two negative $G$ eigenvectors.

(6) A tetrad consisting of one positive and three negative $G$ eigenvectors constitute the columns of a Lorentz matrix which ensures the transformation $\Omega \to \Omega^c = L \Omega^{\rm I_c} L^T$ to a diagonal canonical form $\Omega^{\rm I_c}$. Based on a triad consisting of one neutral and two negative $G$ eigenvectors it is possible to construct a Lorentz matrix (see Appendix B where explicit construction of Lorentz matrix in this case is given) such that transformation $\Omega \to \Omega^c = L \Omega^{\rm II_c} L^T$ resulting in a nondiagonal canonical form $\Omega^{\rm II_c}$ can be obtained (when the largest eigenvalue $\lambda_0$ of $\Omega$ is doubly degenerate and the corresponding $G$ eigenvector is neutral).

(7) Using the explicit forms of the diagonal and nondiagonal canonical forms $\Omega^{\rm I_c}$ and $\Omega^{\rm II_c}$ of $\Omega$, it can be explicitly verified that the $G$ eigenvalues of $\Omega$ are *non-negative*.

## APPENDIX B

In this Appendix we discuss explicit construction of a Lorentz matrix $L$ belonging to OPLG in terms of a set of $G$-orthogonal four-vectors [21,22,24].

(i) Consider a positive four-vector $\mathbf{x}_0$ with its zeroth component $x_{00} > 0$ and three other negative four-vectors $\mathbf{x}_i$, $i = 1$–3, obeying Minkowski $G$-orthogonality conditions, i.e.,

$$\mathbf{x}_\mu^T G \mathbf{x}_\nu = G_{\mu\nu}, \quad \mu, \nu = 0\text{–}3, \tag{B1}$$

where $G_{\mu\,\nu}$ denotes elements of the Minkowski matrix $G$. The set $\{\mathbf{x}_\mu,\ \mu = 0\text{–}3\}$ of four-vectors obeying (B1) forms a *$G$-orthogonal tetrad* in $\mathcal{M}$.

It is readily seen that a real $4 \times 4$ matrix $L = (\mathbf{x}_0 \quad \mathbf{x}_1 \quad \mathbf{x}_2 \quad \mathbf{x}_3)$, with its columns forming a $G$-orthogonal set satisfies $L^T GL = G$ with $(L)_{00} = x_{00} \geqslant 0$, and hence, $L$ is a Lorentz matrix belonging to OPLG.

(ii) A set $\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2\}$ of four-vectors, consisting of a neutral vector $\mathbf{y}_0$ and two negative vectors $\mathbf{y}_1, \mathbf{y}_2$ obeying the property,

$$\mathbf{y}_0^T G\mathbf{y}_0 = 0, \quad \mathbf{y}_0^T G\mathbf{y}_i = 0,$$
$$\mathbf{y}_i^T G\mathbf{y}_j = -\delta_{ij}, \qquad i, j = 1, 2$$

forms a *G-orthogonal triad*. The neutral vector $\mathbf{y}_0$ is a self-orthogonal vector as its Minkowski norm $\mathbf{y}_0^T G\mathbf{y}_0$ is zero.

Given the $G$-orthogonal triad $\{\mathbf{y}_0, \widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_1\}$, consisting of a neutral vector $\mathbf{y}_0$, it is possible to construct a tetrad $\{\widetilde{\mathbf{y}}_0, \widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2, \widetilde{\mathbf{y}}_3\}$ of four-vectors obeying the $G$-orthonormality conditions $\widetilde{\mathbf{y}}_\mu^T G\widetilde{\mathbf{y}}_\nu = G_{\mu\nu},\ \mu, \nu = 0$–3. To this end, we construct a four-vector $\mathbf{y}_3$ such that

$$\mathbf{y}_3^T G\mathbf{y}_0 \neq 0, \quad \mathbf{y}_3^T G\widetilde{\mathbf{y}}_i = 0, \quad i = 2, 3, \tag{B2}$$

and define two four-vectors $\widetilde{\mathbf{y}}_0$ and $\widetilde{\mathbf{y}}_3$ as follows [21,22]:

$$\widetilde{\mathbf{y}}_0 = \mathbf{y}_3 + \tau_y\, \mathbf{y}_0, \quad y_{00} \geqslant 0,$$
$$\widetilde{\mathbf{y}}_3 = \mathbf{y}_3 - \kappa_y \mathbf{y}_0, \tag{B3}$$

where the real parameters $\tau_y, \kappa_y$ are given by

$$\tau_y = \frac{1 - \mathbf{y}_3^T G\mathbf{y}_3}{2\mathbf{y}_3^T G\mathbf{y}_0}, \quad \kappa_y = \frac{1 + \mathbf{y}_3^T G\mathbf{y}_3}{2\mathbf{y}_3^T G\widetilde{\mathbf{y}}_0}. \tag{B4}$$

By construction, the set $\{\widetilde{\mathbf{y}}_0, \widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2, \widetilde{\mathbf{y}}_3\}$ of four-vectors forms a $G$-orthonormal tetrad consisting of *one* positive and *three* negative four-vectors. By following the explicit procedure outlined above one can construct a Lorentz matrix $L_2 = (\widetilde{\mathbf{y}}_0 \quad \widetilde{\mathbf{y}}_1 \quad \widetilde{\mathbf{y}}_2 \quad \widetilde{\mathbf{y}}_3)$, starting from a $G$-orthogonal triad $\{\mathbf{y}_0, \widetilde{\mathbf{y}}_2, \widetilde{\mathbf{y}}_3\}$, consisting of a neutral four-vector $\widetilde{\mathbf{y}}_0$.


[1] G. Kimura, The Bloch vector for n-level systems, Phys. Lett. A **314**, 339 (2003).

[2] G. Kimura and A. Kossakowski, The bloch-vector space for n-level systems: The spherical-coordinate point of view, Open Syst. Inf. Dyn. **12**, 207 (2005).

[3] S. K. Goyal, B. N. Simon, R. Singh, and S. Simon, Geometry of the generalized Bloch sphere for qutrits, J. Phys. A: Math. Theor. **49**, 165203 (2016).

[4] R. Horodecki and M. Horodecki, Information-theoretic aspects of inseparability of mixed states, Phys. Rev. A **54**, 1838 (1996).

[5] F. Verstraete, J. Dehaene, and B. DeMoor, Local filtering operations on two qubits, Phys. Rev. A **64**, 010101(R) (2001).

[6] F. Verstraete, Quantum entanglement and quantum information, Ph.D. thesis, Katholieke Universiteit Leuven, 2002.

[7] I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, 2006).

[8] J. E. Avron, G. Bisker, and O. Kenneth, Visualizing two qubits, J. Math. Phys. **48**, 102107 (2007).

[9] J. E. Avron and O. Kenneth, Entanglement and the geometry of two qubits, Ann. Phys. (NY) **324**, 470 (2009).

[10] M. Shi, F. Jiang, C. Sun, and J. Du, Geometric picture of quantum discord for two-qubit quantum states, New J. Phys. **13**, 073016 (2011).

[11] M. Shi, C. Sun, F. Jiang, X. Yan, and J. Du, Optimal measurement for quantum discord of two-qubit states, Phys. Rev. A **85**, 064104 (2012).

[12] S. Jevtic, M. F. Pusey, D. Jennings, and T. Rudolph, Quantum Steering Ellipsoids, Phys. Rev. Lett. **113**, 020402 (2014).

[13] O. Gamel, Entangled Bloch spheres: Bloch matrix and two-qubit state space, Phys. Rev. A **93**, 062320 (2016).

[14] A. Milne, S. Jevtic, D. Jennings, H. Wiseman, and T. Rudolph, Quantum steering ellipsoids, extremal physical states and monogamy, New J. Phys. **16**, 083017 (2014).

[15] A. Milne, D. Jennings, S. Jevtic, and T. Rudolph, Quantum correlations of two-qubit states with one maximally mixed marginal, Phys. Rev. A **90**, 024302 (2014).

[16] S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, Einstein-Podolsky-Rosen steering and the steering ellipsoid, J. Opt. Soc. Am. B **32**, A40 (2015).

[17] H. C. Nguyen and T. Vu, Nonseparability and steerability of two-qubit states from the geometry of steering outcomes, Phys. Rev. A **94**, 012114 (2016).

[18] H. C. Nguyen and T. Vu, Necessary and sufficient condition for steerability of two-qubit states by the geometry of steering outcomes, Europhys. Lett. **115**, 10003 (2016).

[19] Q. Quan, H. Zhu, S.-Y. Liu, S.-M. Fei, H. Fan, and W.-L. Yang, Steering Bell-diagonal States, Sci. Rep. **6**, 22025 (2016).

[20] I. Gohberg, P. Lancaster, and L. Rodman, *Matrices and Indefinite Scalar Products* (Birkhauser-Verlag, Basel, 1983).

[21] A. V. Gopala Rao, K. S. Mallesh, and Sudha, On the algebraic characterization of a Mueller matrix in polarization optics: I. Identifying a Mueller matrix from its N matrix, J. Mod. Opt. **45**, 955 (1998).

[22] A. V. Gopala Rao, K. S. Mallesh, and Sudha, On the algebraic characterization of a Mueller matrix in polarization optics: II. Necessary and sufficient conditions for Jones-derived Mueller matrices, J. Mod. Opt. **45**, 989 (1998).

[23] J. L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1972), pp. 291–293.

[24] K. N. Srinivasa Rao, *The Rotation and Lorentz Groups and Their Representations for Physicists* (Wiley Eastern, New Delhi, 1988).


[25] It may be noted that a positive, neutral, or negative four-vector is referred to as *timelike, null, and spacelike*, respectively, in special relativity theory [24].

[26] A real $4 \times 4$ matrix $M$, transforming the set of all positive and neutral four-vectors $\mathbf{s} \in \mathcal{M}$ with positive zeroth component $\{\mathbf{s}|s_0 > 0\}$ to an identical set $\{M\mathbf{s}|(M\mathbf{s})_0 > 0\}$ is called a Mueller matrix in classical polarization optics [21,22].