Emergent dynamics from entangled mixed states

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Entanglement is at the core of quantum physics, playing a central role in quantum phenomena involving composite systems. According to the timeless picture of quantum dynamics, entanglement may also be essential for understanding the very origins of dynamical evolution and the flow of time. Within this point of view, the Universe is regarded as a bipartite entity comprising a clock C and a system R (or "rest of the Universe") jointly described by a global stationary state, and the dynamical evolution of R is construed as an emergent phenomena arising from the entanglement between C and R. In spite of substantial recent efforts, many aspects of this approach remain unexplored, particularly those involving mixed states. In the present contribution we investigate the timeless picture of quantum dynamics for mixed states of the clock-system composite, focusing on quantitative relations linking the clock-system entanglement with the emerging dynamical evolution experienced by the system.

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I. INTRODUCTION

One of the goals of science is to formulate the most economical description possible of natural phenomena. Guided by this desire for conceptual economy, scientists try to develop theories having the least possible number of basic assumptions or primitive elements. In this regard, research into the phenomenon of quantum entanglement has led to remarkable insights. For instance, the study of entanglement clarified the origin of the states describing systems in thermal equilibrium with a heat bath, without the need to invoke the microcanonical distribution for the system-bath composite [1,2]. More radically, research work revolving around quantum entanglement also provided a plausible explanation of the origins of dynamical evolution and the flow of time. The concomitant arguments, according to which time and dynamics are emergent phenomena arising from quantum correlations, were first articulated by Page and Wootters (PW) [3,4], although related ideas had been previously advanced in the context of the quantum theory of gravity [5,6].

Within the PW *timeless* picture of quantum mechanics [3,4], the whole Universe U is assumed to be in a global stationary state, which is an eigenstate of the total Hamiltonian with zero energy eigenvalue. Dynamical evolution arises from this static state as a result of the quantum entanglement between the degree of freedom of an appropriate subsystem C, called the *clock*, and the *rest of the Universe* R. According to this idea, time and dynamics are emergent features of the Universe rooted in the entanglement between two subsystems, R and C. The Schrödinger time-independent equation describing the global stationary state of the $R+C$ composite is reminiscent of the celebrated Wheeler-DeWitt equation in quantum cosmology, describing a stationary state with zero eigenvalue for the wave function of the entire (closed) Universe [5,6].

The PW timeless approach to quantum mechanics has been elaborated and extended in various directions, from both the theoretical and the experimental points of view [7–24]. Healthy controversy [9,11] has invigorated research into the PW proposal, stimulating the exploration of its possibilities. The timeless picture was criticized by Albrecht and Iglesias [9], who pointed out apparent ambiguities concerning nonequivalent choices for the clock subsystem. Subsequent counter arguments by Marletto and Vedral [11] showed that these ambiguities do not arise, if one takes carefully into account the properties needed by a subsystem to be acceptable as a clock. Recent work reported in the literature attests to the deep and manifold implications of the timeless picture of quantum mechanics. Research into this subject has led to the reconsideration of well-known foundational issues, such as Pauli's famous argument for the impossibility of a time observable in quantum mechanics [15]. New facets of time in quantum mechanics have been discovered, such as its basic connection with quantum coherence [16]. Interesting forays into relativistic scenarios have also been made, with the implementation of the PW scheme for Dirac [17] and scalar [18] particles. A formalism akin to the one behind the timeless picture has led to the development of alternative computational techniques for problems in quantum dynamics, which are reformulated as ground-state eigenvalue problems [19]. Going beyond theoretical considerations, concrete experiments illustrating the timeless picture have been successfully conducted in recent years [20–22].

As already mentioned, the system-clock entanglement is central to the timeless approach to quantum dynamics. However, the quantitative relation between quantum correlations and specific, dynamic-related aspects of the evolving system R has received relatively little attention, with most efforts focusing on scenarios where the system-clock composite is in a pure state [13,14,24]. Our aim in this work is to explore the timeless picture of quantum dynamics for mixed global states of the bipartite system $R + C$.

Motivations to study mixed states within this context are manifold. First, the system R is, in general, itself composite. In realistic circumstances one may have access only to a subsystem R_a of R that, while weakly coupled to other parts of R, may nevertheless be entangled with them and, consequently, be in a mixed state. In this scenario the system $R = R_a + R_b$ has a total Hamiltonian $\hat{H}_R \approx \hat{H}_{R_a} + \hat{H}_{R_b}$, where $\hat{H}_{R_a}$ and $\hat{H}_{R_b}$ are the Hamiltonians of R_a and R_b, respectively. If the state of R is σ_R, the subsystem R_a is described by the reduced, marginal state $\sigma_{R_a} = \mathrm{Tr}_{R_b}[\sigma_R]$ obeying the von Neumann equation

$$\frac{d\sigma_{R_a}}{dt} = \frac{1}{i\hbar}[\hat{H}_{R_a}, \sigma_{R_a}]. \tag{1}$$

The subsystem R_a is then, for all intents and purposes, our rest of the Universe. The PW approach, in its standard formulation (pure-state version), studies how the dynamics of an isolated system R described by a pure state and obeying the Schrödinger equation can be embedded into a stationary pure state of the $R + C$ system. It is legitimate and pertinent to inquire if the dynamics of a system R_a evolving according to von Neumann's equation (1), which is the most general equation of motion for a closed quantum system, can similarly be embedded into a stationary *mixed* state of $R_a + C$. One can, of course, circumvent the need to consider mixed states by implementing the pure-state version of the PW picture for the complete composite $R_a + R_b + C$, assumed to be in a pure state. But that procedure entails carrying the excess baggage of describing all the degrees of freedom of subsystem R_b, which may be inaccessible and irrelevant. Avoidance of that extra load is the main motivation for considering the PW approach for mixed states, which, by the way, coincides with the very reason for using the von Neumann equation (1) to study the dynamics of entangled, but dynamically isolated, subsystems. In a cosmological context, the aforementioned picture is consistent with the one advanced by Bunyi and Hsu in Ref. [25]. According to these authors, the standard Big-Bang cosmological model implies that a given subsystem R_a of the Universe is likely to be entangled to other subsystems with which R_a is not currently interacting.

Second, relevant motivations for developing a PW approach for mixed states are not limited to scenarios, such as those discussed above, where mixed states describe subsystems of a composite quantum Universe that, as a whole, may be in a pure state. Indeed, the Universe itself (that is, the whole system $R + C$) may conceivably be in a mixed state [26,27]. This possibility was entertained by Page in Ref. [26], where a quantum description of the Universe was proposed which, in contrast with the celebrated Hartle-Hawking wave function, corresponds to a density matrix describing an impure quantum state. Physical effects depending on the degree of mixture of the density matrix of the Universe were considered by Gurzadyan and Kocharyan in Ref. [27]. In the present work, according to the above discussion, the expression "rest of the Universe" may refer either to the "total rest" R, or to an appropriate subsystem R_a. In the latter case we shall drop the subindex a.

Additionally, considering the more general framework given by mixed states may also help in adapting the timeless PW picture to extensions or modifications of quantum mechanics motivated by research into the interface between quantum and gravitational phenomena. In this regard, we can mention Deutch's proposal for a formulation of quantum mechanics in the presence of closed timelike curves (CTCs), which explicitly requires density matrices describing mixed states and cannot be formulated in terms of wave functions [28]. Last, the analysis of mixed states in connection with the timeless approach to quantum mechanics may shed new light on the problem of the ontological status of mixed states [29].

All the above motivations can be encompassed by a single aim: To formulate the PW picture in a fashion that incorporates the most general description of the dynamics of a closed quantum system, which is the one given by von Neumann's equation for the evolution of time-dependent mixed states. Therefore, in the present work we advance a PW-like static scenario involving mixed global states of composite $R + C$. Our proposal is compatible with general time-dependent mixed states of the evolving system R, but otherwise keeps the main assumptions of the PW scenario. This mixed-state version of the PW approach constitutes a proof of principle showing that a consistent mixed-state PW scenario can be developed, and provides a testing ground to explore possible physical features of such a scenario, particularly in connection with quantum entanglement. We shall analyze quantitatively the entanglement between R and C and its relation with the emerging time evolution experienced by R. We investigate a quantitative indicator of entanglement, based on comparing an entropic measure evaluated on the global system with the corresponding entropic measure evaluated on the reduced state associated with R. Using this indicator of entanglement we prove that (under the PW constraint of a definite total energy of $R + C$ equal to zero) the composite system $R + C$ necessarily has to be entangled for the system R to exhibit dynamical evolution, meaning that other forms of nonclassical correlations alone are not sufficient for time and evolution to arise. We establish an upper bound, as well as the asymptotic value, of the entanglement indicator, expressing these quantities in terms of an entropic measure of the spread of the energy probability distribution associated with the system R. We also investigate the connection between the entanglement present in the $R + C$ composite, and a measure of the energy uncertainty of the system R.

The paper is organized as follows. A brief summary of the timeless approach is given in Sec. II. The connection between time evolution and entanglement for mixed states of the whole system $R + C$ is analyzed in Sec. III, on the basis of an appropriate entropic entanglement indicator. An upper bound and the asymptotic limit of this quantity is discussed in

Sec. IV. Its connection with the energy dispersion of the system is investigated in Sec. V. Finally, some concluding remarks are given in Sec. VI.

II. TIMELESS APPROACH TO DYNAMICS FOR PURE STATES OF THE SYSTEM-CLOCK COMPOSITE

As a starting point, we consider a bipartite quantum system (the *Universe* U) comprising a *clock* (C) and the *rest* of the Universe (R). The Hilbert spaces corresponding to these two subsystems and to the total system are, respectively, $\mathcal{H}_C$, $\mathcal{H}_R$, and $\mathcal{H}_U = \mathcal{H}_R \otimes \mathcal{H}_C$. Global states of $R + C$ are spanned in a product orthonormal basis $\{|\boldsymbol{x}\rangle \otimes |t\rangle = |\boldsymbol{x}\rangle|t\rangle\}$ of $\mathcal{H}_U$, where $\{|t\rangle\}$ and $\{|\boldsymbol{x}\rangle\}$ are the orthonormal bases of $\mathcal{H}_C$ and $\mathcal{H}_R$, respectively. The continuous label $t \in \mathbb{R}$ characterizing the basis states of $\mathcal{H}_C$ corresponds to the eigenvalues of an observable $\hat{\mathcal{T}}$ associated with the position of the clock's hands, so that $\hat{\mathcal{T}}|t\rangle = t|t\rangle$. Likewise, the label $\boldsymbol{x}$ characterizing the basis states of $\mathcal{H}_R$ represents the position, or any other degrees of freedom, of the particle or particles constituting the system R. Throughout the paper we will assume that $\boldsymbol{x}$ is a continuous variable, yet it may also denote a discrete one, provided integrals are properly substituted by discrete sums.

To analyze the behavior of the complete system during a finite time interval $[0, T]$, we assume that U is in the pure state

$$|\Psi\rangle = \frac{1}{\sqrt{T}} \int \Psi(\boldsymbol{x}, t)|\boldsymbol{x}\rangle|t\rangle d\boldsymbol{x}dt, \tag{2}$$

described by a wave function $\Psi(\boldsymbol{x}, t) = (\langle \boldsymbol{x}|\langle t|)|\Psi\rangle$ that is spatially normalized, $\int |\Psi(\boldsymbol{x}, t)|^2 d\boldsymbol{x} = 1$. The state (2) is then properly normalized, both spatially and temporally,

$$\langle\Psi|\Psi\rangle = \frac{1}{T}\int_0^T \underbrace{\left(\int |\Psi(\boldsymbol{x}, t)|^2 d\boldsymbol{x}\right)}_{=1} dt = 1. \tag{3}$$

The state of R for a given configuration of the clock's hands (that is, for a particular value of t) is described by the *Everett relative* state [30]

$$|\Phi_t\rangle = \langle t|\Psi\rangle = \frac{1}{\sqrt{T}} \int \Psi(\boldsymbol{x}, t)|\boldsymbol{x}\rangle d\boldsymbol{x} = \frac{1}{\sqrt{T}}|\widetilde{\Phi}_t\rangle, \tag{4}$$

obtained by projecting $|\Psi\rangle$ onto $|t\rangle$. In Eq. (4), $|\widetilde{\Phi}_t\rangle$ stands for the *normalized* relative state, satisfying

$$\langle\widetilde{\Phi}_t|\widetilde{\Phi}_t\rangle = T\langle\Phi_t|\Phi_t\rangle = 1. \tag{5}$$

Restricting our analysis to the finite time interval $[0, T]$ corresponds to considering a part of the history of the Universe that, from the standard time-based viewpoint, is perceived as having a duration T. Quantum states normalized within the range $[0, T]$ result from projecting the state of the Universe onto the subspace spanned by the eigenstates of $\hat{\mathcal{T}}$ having eigenvalues $t \in [0, T]$. These states can be regarded as the result of postselecting the measurement value 1 when measuring the observable (projector) $\Pi = \int_0^T dt|t\rangle\langle t|$.

Within the timeless formalism it is assumed that

$$\hat{H}_U|\Psi\rangle = 0, \tag{6}$$

where $\hat{H}_U$ is the total Hamiltonian $\hat{H}_U = \hat{H}_R \otimes \mathbb{I}_C + \mathbb{I}_R \otimes \hat{H}_C$, with $\hat{H}_R$ being an arbitrary Hamiltonian of R, and $\hat{H}_C$ the Hamiltonian of the clock. Notice that a good clock, in order to appropriately keep track of time, has to be dynamically isolated and should not be perturbed by interactions with other systems. The absence of interaction between R and C also plays a crucial role in guaranteeing the uniqueness of the $R|C$ bipartition of the complete system U [11], solving the ambiguity problem raised in Ref. [9]. As discussed in Ref. [11], we consider here an *ideal* clock that does not interact at all with R. Furthermore, it is considered that the clock's observable $\hat{\mathcal{T}}$ and the Hamiltonian $\hat{H}_C$ satisfy the commutation relation $[\hat{\mathcal{T}}, \hat{H}_C] = i\hbar$. Under these conditions, it follows from Eq. (6) [10] that the relative state $|\Phi_t\rangle$ (whether normalized or not) obeys the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\Phi_t\rangle = \hat{H}_R|\Phi_t\rangle. \tag{7}$$

We thus see that the usual dynamical scenario—embodied in the (time-dependent) Schrödinger equation—ensues from the static image of the nonevolving state $|\Psi\rangle$. The important point here to be noticed is that the evolution emerges if and only if C and R are entangled. Otherwise, $\Psi(\boldsymbol{x}, t)$ factorizes as $\Psi(\boldsymbol{x}, t) = \Psi_C(t)\Psi_R(\boldsymbol{x})$, $\Psi_R(\boldsymbol{x})$ is an eigenstate of $\hat{H}_R$, and therefore $\Psi(\boldsymbol{x}, t)$ is a stationary (nonevolving) state. Such an intimate relationship between entanglement and time evolution has been explored previously [13,14,24] in this pure-case scenario. In what follows we will analyze the more general case of mixed states and show that the relation still holds.

Before ending this section, let us add a few comments regarding the eigenvalue spectra of the observables $\hat{\mathcal{T}}$ and $\hat{H}_C$. Both of them have continuous spectra. The allowed eigenvalues of $\hat{H}_C$, however, are restricted to a discrete set, because the complete system is assumed to be in a zero-energy eigenstate of the total Hamiltonian $\hat{H}_R \otimes \mathbb{I}_C + \mathbb{I}_R \otimes \hat{H}_C$. Indeed, if $\hat{H}_R$ has a discrete spectrum consisting of the eigenvalues $\{E_n\}$, $n = 0, 1, 2, \ldots$ (as we assume here), then the set of allowed eigenvalues of $\hat{H}_C$ is discrete too: these eigenvalues, and their associated eigenstates, are $\{-E_n\}$ and $\{e^{-iE_nt/\hbar}\}$, respectively. We have, therefore, an *effectively* discrete Hilbert space for C, spanned by this discrete set of states. This resembles what happens with the operators $\hat{x}$ and $\hat{p}$ of a particle in a finite box. The commutation relation satisfied by these operators coincides with the one satisfied by $\hat{\mathcal{T}}$ and $\hat{H}_C$. Imposing appropriate boundary conditions on the walls of the box yields a discrete set $\{p_n\}$ of accessible eigenvalues for $\hat{p}$, with a corresponding discrete set of eigenvalues $\{e^{-ip_nx/\hbar}\}$, that span the Hilbert space of the system. Of course, the above similarity is only formal, since in the present situation the (effective) discrete spectra of $\hat{H}_C$ results from the constraint of total zero energy of the $C + R$ composite, instead of arising from boundary conditions.

III. EVOLUTION AND ENTANGLEMENT FOR MIXED STATES OF THE SYSTEM-CLOCK COMPOSITE

To extend the above ideas beyond scenarios corresponding to pure global states of the $R + C$ system, we shall assume a mixed global state ρ that is stationary under the dynamics determined by the total Hamiltonian $\hat{H}_U$, and has a definite total energy equal to zero. That is, we shall assume that $\langle\hat{H}_U\rangle = \mathrm{Tr}(\rho\hat{H}_U) = 0$ and $\langle\hat{H}_U^2\rangle - \langle\hat{H}_U\rangle^2 = \mathrm{Tr}[\rho(\hat{H}_U - \langle\hat{H}_U\rangle)^2] = 0$.

The state ρ is then of the form

$$\rho = \sum_j p_j |\Psi_j\rangle\langle\Psi_j|, \tag{8}$$

where $p_j \geqslant 0$ for all j, $\sum_j p_j = 1$, and $\{|\Psi_j\rangle\}$ is a set of stationary pure states of U with $\hat{H}_U|\Psi_j\rangle = 0$ for all j. The density matrix (8) describes thus a statistical mixture of the pure states $|\Psi_j\rangle$ with (probability) weights p_j.

Using the same notation as in the previous section, we have

$$|\Psi_j\rangle = \frac{1}{\sqrt{T}} \int \Psi_j(\boldsymbol{x}, t)|\boldsymbol{x}\rangle|t\rangle d\boldsymbol{x}dt, \tag{9}$$

and the corresponding relative state

$$|\Phi_{j,t}\rangle = \langle t|\Psi_j\rangle = \frac{1}{\sqrt{T}} \int \Psi_j(\boldsymbol{x}, t)|\boldsymbol{x}\rangle d\boldsymbol{x} = \frac{1}{\sqrt{T}}|\widetilde{\Phi}_{j,t}\rangle, \tag{10}$$

where $|\widetilde{\Phi}_{j,t}\rangle = \sqrt{T}|\Phi_{j,t}\rangle$ stands for the normalized relative states ($\langle\widetilde{\Phi}_{j,t}|\widetilde{\Phi}_{j,t}\rangle = 1$).

Now, in this case the Everett relative state, describing the state of R given that the clock's hands state is $|t\rangle$, is obtained according to

$$\sigma_{R,t} = \frac{\mathrm{Tr}_C(|t\rangle\langle t|\rho)}{\mathrm{Tr}(|t\rangle\langle t|\rho)} = T\sum_j p_j|\Phi_{j,t}\rangle\langle\Phi_{j,t}| = \sum_j p_j|\widetilde{\Phi}_{j,t}\rangle\langle\widetilde{\Phi}_{j,t}|. \tag{11}$$

This is a mixture of the states $|\Phi_{j,t}\rangle$, each of which satisfies the Schrödinger equation (7). Therefore, the relative state of R satisfies the von Neumann equation,

$$\frac{d}{dt}\sigma_{R,t} = \frac{1}{i\hbar}[\hat{H}_R, \sigma_{R,t}]. \tag{12}$$

We thus verify that the quantum dynamical equations of R are recovered also in the mixed state case.

To investigate the relation between the evolution and the entanglement in this more general scenario, we shall use an entanglement criteria based on the reduced, marginal, density matrix ρ_R of the system R, obtained by taking the partial trace over C of the global density matrix ρ:

$$\rho_R = \mathrm{Tr}_C\rho = \int_0^T \langle t|\rho|t\rangle dt = \frac{1}{T}\int_0^T \sigma_{R,t}dt = \overline{\sigma_{R,t}}, \tag{13}$$

where $\overline{(\cdot)}$ denotes the time average: $\overline{(\cdot)} = \frac{1}{T}\int_0^T(\cdot)dt$. It is worth emphasizing that the density matrices $\sigma_{R,t}$ and ρ_R, although both referring to system R, represent different states. The former represents the state of R *conditioned* to the state $|t\rangle$ of the clock and is a mixed state that evolves unitarily as a function of the parameter t. On the other hand, the (in general) mixed state ρ_R is obtained through taking, on the global state of $R + C$, the partial trace over the degrees of freedom of C. It represents a time-averaged state (over the interval $[0, T]$) and does not depend on t.

Now, the entropies $S[\rho]$ and $S[\rho_R]$ of the global (ρ) and the marginal (ρ_R) density matrices, respectively, provide an entanglement criterion for the global state as follows (see Refs. [31–35]):

$$S[\rho_R] > S[\rho] \Rightarrow \rho \text{ is entangled}. \tag{14}$$

That is, if we have less information about the subsystem R than information about the composite system $R + C$, then R and C are entangled. This entropic entanglement criterion can be implemented irrespective of the particular entropic measure used. Possible choices are von Neumann entropy, or the *linear entropy* defined, for a generic density matrix ϱ, as

$$S_L[\varrho] \equiv 1 - \mathrm{Tr}\varrho^2. \tag{15}$$

Since this latter has some computational advantages, we choose it for our calculations, and thus compare $S_L[\rho]$ with $S_L[\rho_R]$. Our entanglement indicator is thus

$$\Delta S \equiv S_L[\rho_R] - S_L[\rho], \tag{16}$$

in terms of which the entanglement criterion reads

$$\Delta S > 0 \Rightarrow \rho \text{ is entangled}. \tag{17}$$

The linear entropy of the global state ρ is given by

$$S_L[\rho] = 1 - \mathrm{Tr}\rho^2 = 1 - \sum_{jk} p_j p_k |\langle\Psi_j|\Psi_k\rangle|^2 = 1 - \sum_{jk} p_j p_k \left|\frac{1}{T}\int_0^T \langle\widetilde{\Phi}_{j,t}|\widetilde{\Phi}_{k,t}\rangle dt\right|^2. \tag{18}$$

Since the inner product $\langle\widetilde{\Phi}_{j,t}|\widetilde{\Phi}_{k,t}\rangle$ is invariant under the unitary evolution determined by the Schrödinger equation, we can substitute $\langle\widetilde{\Phi}_{j,t}|\widetilde{\Phi}_{k,t}\rangle = \langle\widetilde{\Phi}_{j,0}|\widetilde{\Phi}_{k,0}\rangle$ in the above equation and get

$$\begin{aligned} S_L[\rho] &= 1 - \sum_{jk} p_j p_k |\langle\widetilde{\Phi}_{j,0}|\widetilde{\Phi}_{k,0}\rangle|^2 \\ &= 1 - \sum_{jk} p_j p_k |\langle\widetilde{\Phi}_{j,t}|\widetilde{\Phi}_{k,t}\rangle|^2 \\ &= 1 - \mathrm{Tr}\sigma_{R,t}^2 = S_L[\sigma_{R,t}] \\ &= \overline{S_L[\sigma_{R,t}]}, \end{aligned} \tag{19}$$

where the last equality is due to the fact that (as follows from the first two lines) $S_L[\sigma_{R,t}]$ is a time-independent quantity.

As for the linear entropy of the marginal state ρ_R, Eq. (13) gives

$$S_L[\rho_R] = S_L[\overline{\sigma_{R,t}}]. \tag{20}$$

It follows from the above expressions that comparing the entropies of ρ and ρ_R amounts to compare the (time) average entropy of $\sigma_{R,t}$ with the entropy of the (time) average of $\sigma_{R,t}$.

Now, given a time-dependent density matrix ϱ_t and a concave function $f(x)$, the following inequality holds:

$$\mathrm{Tr}[f(\overline{\varrho_t})] \geqslant \overline{\mathrm{Tr}[f(\varrho_t)]}, \tag{21}$$

with the equality satisfied only if ϱ_t is constant in time [36]. In particular, for $f(x) = x - x^2$, we get $S_L(\varrho_t) = \mathrm{Tr}f(\varrho_t)$, the inequality (21) leads to

$$S_L[\overline{\varrho_t}] \geqslant \overline{S_L[\varrho_t]}, \tag{22}$$

and, therefore, putting $\varrho_t = \sigma_{R,t}$, it follows from Eqs. (19) and (20) that

$$S_L[\rho_R] \geqslant S_L[\rho], \tag{23}$$

with the two entropies appearing in the above equation being equal only if there is no time evolution. In other words, ΔS vanishes only if $\sigma_{R,t}$ does *not* evolve in time; that is, $\Delta S = 0 \Rightarrow$ no time evolution.

Since Eq. (23) holds for *all* states belonging to the subspace spanned by the eigenstates of zero energy of the total Hamiltonian $\hat{H}_U$, it follows from the criterion (17) that *all* these states are entangled, provided $\sigma_{R,t}$ evolves in time. In other words, if the (relative) state of R changes with the ticking of the clock's hands, then R is necessarily entangled with C. Put another way, in the absence of entanglement between the clock and the system R, the state of R remains independent of t, and no evolution occurs. This means that, under the conditions of our mixed-state PW scenario, quantum correlations other than entanglement that may be present in mixed states, such as quantum discord, are not enough for dynamics and the flow of time to arise. Therefore, the study of mixed states within the timeless approach to quantum dynamics provides further evidence for the intimate link existing between entanglement and evolution.

Finally, it is interesting to ask whether, for the type of density matrices arising in the present PW context, the condition $\Delta S = 0$ implies that the entanglement between R and C vanishes [this would amount to stating that an entangled state implies $\Delta S > 0$, and consequently the criterion (17) would be not only sufficient but also necessary]. For the special case of pure states of the $R + C$ system, the answer is yes. For mixed states, the situation is more subtle. In such cases, as we have just seen, $\Delta S = 0$ implies that the relative state of R conditional to a given value of t, does not depend on this t value (that is, R does not evolve). However, this condition does not seem to imply that the joint density matrix of $R + C$ is nonentangled. It might happen that there are entangled joint states of $R + C$ for which R does not evolve. The existence or not of such states remains an open question, certainly worthy of further investigation.

IV. UPPER BOUND AND ASYMPTOTIC LIMIT OF THE ENTANGLEMENT INDICATOR

Now we determine an upper bound for the indicator ΔS of entanglement between the system and the clock, and also its asymptotic limit for large lengths of the interval $[0, T]$ within which the joint state of the system-clock composite is defined. We consider a d-level system with a Hamiltonian $\hat{H}_R$ having eigenstates $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ with corresponding eigenvalues $\{E_0, E_1, \ldots, E_{d-1}\}$. The relative state $\sigma_{R,t}$ evolves according to Eq. (12), and its matrix elements in the basis $\{|n\rangle\}$ (with $n = 0, \ldots, d-1$) can thus be written as

$$\sigma_{nm}(t) \equiv \langle n|\sigma_{R,t}|m\rangle = e^{-i(E_n - E_m)t/\hbar}\sigma_{nm}(0). \tag{24}$$

The matrix elements of the reduced state ρ_R are given, according to Eqs. (13) and (24), by

$$\begin{aligned}\langle n|\rho_R|m\rangle &= \langle n|\overline{\sigma_{R,t}}|m\rangle \\ &= \sigma_{nm}(0)e^{i(E_n - E_m)T/2\hbar}\,\mathrm{sinc}[(E_n - E_m)T/2\hbar],\end{aligned} \tag{25}$$

with $\mathrm{sinc}x = x^{-1}\sin x$.

From these expressions the linear entropies $S_L[\rho]$ and $S_L[\rho_R]$ can be computed directly as follows:

$$\begin{aligned}S_L[\rho] = S_L[\sigma_{R,t}] = 1 - \mathrm{Tr}\sigma_{R,t}^2 &= 1 - \sum_{nm}\sigma_{nm}\sigma_{mn} \\ &= 1 - \sum_{nm}|\sigma_{nm}(0)|^2,\end{aligned} \tag{26}$$

and

$$\begin{aligned}S_L[\rho_R] = 1 - \mathrm{Tr}\rho_R^2 &= 1 - \sum_{nm}\langle n|\rho_R|m\rangle\langle m|\rho_R|n\rangle \\ &= 1 - \sum_{nm}|\sigma_{nm}(0)|^2\,\mathrm{sinc}^2(\omega_{nm}T/2),\end{aligned} \tag{27}$$

where $\omega_{nm} = |E_n - E_m|/\hbar$. Decomposing the sum in Eq. (27) into those terms for which $\omega_{nm} = 0$ and those for which $\omega_{nm} \neq 0$, we get

$$\begin{aligned}S_L[\rho_R] &= \left(1 - \sum_{\substack{nm \\ (\omega_{nm}=0)}}|\sigma_{nm}(0)|^2\right) \\ &\quad - \sum_{\substack{nm \\ (\omega_{nm}\neq 0)}}|\sigma_{nm}(0)|^2\mathrm{sinc}^2(\omega_{nm}T/2).\end{aligned} \tag{28}$$

The entanglement indicator is thus

$$\begin{aligned}\Delta S &= S_L[\rho_R] - S_L[\rho] \\ &= \sum_{nm}|\sigma_{nm}(0)|^2\big[1 - \mathrm{sinc}^2(\omega_{nm}T/2)\big] \\ &= \sum_{\substack{nm \\ (\omega_{nm}\neq 0)}}|\sigma_{nm}(0)|^2\big[1 - \mathrm{sinc}^2(\omega_{nm}T/2)\big] \qquad (29) \\ &\leqslant \sum_{\substack{nm \\ (\omega_{nm}\neq 0)}}|\sigma_{nm}(0)|^2, \qquad (30)\end{aligned}$$

and its maximum value—which coincides with its asymptotic value when $T \to \infty$—is

$$\Delta S_{\max} = \sum_{nm\ (\omega_{nm}\neq 0)}|\sigma_{nm}(0)|^2. \tag{31}$$

Notice that the condition $\omega_{nm} \neq 0$ introduced above is not necessarily equivalent to $n \neq m$, due to possible degeneracies of the energy eigenvalues. The label n should therefore be understood as representing a (possibly compound) index containing all the quantum numbers required to completely characterize the eigenstates of $\hat{H}_R$. The set of possible values of this index (even if it is compound) is at most denumerable and can thus be regarded as ordered in the sequence $n = 0, 1, \ldots$.

Now, let us denote with $\sigma_{R|M}$ the state of R obtained when a *nonselective* energy measurement is performed on R, that is

$$\sigma_{R|M} = \sum_E p_E\sigma_{R|E} = \sum_E \Pi_E\sigma_{R,t}\Pi_E, \tag{32}$$

where $p_E = \mathrm{Tr}(\Pi_E\sigma_{R,t})$ is the probability of obtaining the result E when measuring the energy of R when it is in the state $\sigma_{R,t}$, $\sigma_{R|E} = \Pi_E \sigma_{R,t} \Pi_E / p_E$ is the (collapsed) state of R obtained when the energy measurement yields the result E, and $\Pi_E = \sum_{\substack{n \\ (E_n=E)}} |n\rangle\langle n|$ is the projector onto the subspace spanned by the degenerate eigenstates $|n\rangle$ that correspond to the same energy eigenvalue E. The projector satisfies

$$\Pi_E \Pi_{E'} = \sum_{\substack{nm \\ (E_n=E),(E_m=E')}} |n\rangle\langle n|m\rangle\langle m| = \delta_{EE'} \sum_{\substack{n \\ (E_n=E)}} |n\rangle\langle n| = \delta_{EE'}\Pi_E, \tag{33}$$

so that

$$\sigma_{R|E}\sigma_{R|E'} = \sigma^2_{R|E}\delta_{EE'}. \tag{34}$$

Taking into account the second equality in Eqs. (32) and (33), we get for the linear entropy of the state $\sigma_{R|M}$

$$\begin{aligned} S_L[\sigma_{R|M}] &= 1 - \mathrm{Tr}\sigma^2_{R|M} \\ &= 1 - \sum_E \mathrm{Tr}(\sigma_{R,t}\Pi_E\sigma_{R,t}\Pi_E) \\ &= 1 - \sum_E \sum_{\substack{nm \\ (E_n=E_m=E)}} |\sigma_{nm}(0)|^2 \\ &= 1 - \sum_{\substack{nm \\ (\omega_{nm}=0)}} |\sigma_{nm}(0)|^2, \end{aligned} \tag{35}$$

which, combined with Eqs. (26) and (31), leads to

$$\Delta S_{\max} = S_L[\sigma_{R|M}] - S_L[\sigma_{R,t}]. \tag{36}$$

This relation shows that the asymptotic value, when $T \to \infty$, of the entanglement indicator is given by the difference between the entropy of the state of R after and before a nonselective energy measurement is performed.

The entropy $S_L[\sigma_{R|M}]$ bears information regarding the possible states $\sigma_{R|E}$ that can be obtained after an energy measurement, and also regarding the energy probability distribution $\{p_E\}$. Such information can be extracted by recourse to the first equality in Eq. (32) and to Eq. (34), obtaining

$$\begin{aligned} S_L[\sigma_{R|M}] &= 1 - \mathrm{Tr}\sigma^2_{R|M} \\ &= 1 - \mathrm{Tr}\sum_{EE'} p_E p_{E'}\sigma_{R|E}\sigma_{R|E'} \\ &= 1 - \sum_E p_E^2 \mathrm{Tr}\sigma^2_{R|E} \\ &= S_L[\{p_E\}] + \sum_E p_E^2 S_L[\sigma_{R|E}], \end{aligned} \tag{37}$$

where $S_L[\sigma_{R|E}]$ stands for the linear entropy associated with the state $\sigma_{R|E}$, and $S_L[\{p_E\}] = 1 - \sum_E p_E^2$ is the linear entropy corresponding to the energy probability distribution $\{p_E\}$.

It is instructive to consider particular cases of the bound (36). When the spectrum of $\hat{H}_R$ has no degeneracy, one has $S_L[\sigma_{R|E}] = 0$, hence $S_L[\sigma_{R|M}]$ becomes $S_L[\{p_E\}]$, and the upper bound reduces to

$$(\Delta S_{\max})|_{\text{nondegenerate}} = S_L[\{p_E\}] - S_L[\sigma_{R,t}]. \tag{38}$$

It is also particularly interesting to see what happens if the global state ρ is pure, so that $S_L[\rho] = S_L[\sigma_{R,t}] = 0$. In this case also $\sigma_{R|E}$ is a pure state, whence $S_L[\sigma_{R|E}] = 0$, and again $S_L[\sigma_{R|M}] = S_L[\{p_E\}]$. Consequently, for pure states one recovers the expression [24]

$$(\Delta S_{\max})|_{\text{pure}} = S_L[\{p_E\}], \tag{39}$$

meaning that the upper bound of the S_L-based indicator of entanglement is given by the spread of the energy probability distribution p_E, as measured by its linear entropy. This is no longer the case for mixed states. In an extreme case, for example, in which ρ is diagonal in the energy eigenbasis, one has $\sigma_{nm} \sim \delta_{nm}$, and Eq. (31) leads straightforwardly to

$$(\Delta S_{\max})|_{\text{diagonal}} = 0, \tag{40}$$

meaning that diagonal states do not evolve in time [see below Eq. (23)]. Now, when ρ is diagonal in the energy eigenbasis, all the spread in the energy probability distribution is purely classical, whereas for pure states (that are not energy eigenstates) all the spread in the energy probability distribution is of a quantum nature. These observations, together with Eqs. (39) and (40), indicate that only the quantum component of the spread in the energy probability distribution contributes to the upper bound of the system-clock entanglement.

An example: The qubit case

As an illustration of our previous results we consider now a qubit (two-level) system with a Hamiltonian $\hat{H}_R$ having eigenstates $|0\rangle$ and $|1\rangle$, with corresponding eigenvalues E_0 and E_1. Following Eq. (24), the relative state $\sigma_{R,t}$ in the basis $\{|0\rangle, |1\rangle\}$ reads

$$\sigma_{R,t} = \begin{pmatrix} \sigma_{00}(0) & e^{i\epsilon t/\hbar}\sigma_{01}(0) \\ e^{-i\epsilon t/\hbar}\sigma^*_{01}(0) & 1-\sigma_{00}(0) \end{pmatrix}, \tag{41}$$

where we wrote $\epsilon = E_1 - E_0$. The reduced density matrix ρ_R is given, according to Eq. (25), by

$$\rho_R = \overline{\sigma_{R,t}} = \begin{pmatrix} \sigma_{00}(0) & \sigma_{01}(0)e^{ix}\,\mathrm{sinc}x \\ \sigma^*_{01}(0)e^{-ix}\,\mathrm{sinc}x & 1-\sigma_{00}(0) \end{pmatrix}, \tag{42}$$

with $x = \epsilon T/2\hbar$. The corresponding linear entropies are [see Eqs. (26) and (27)]

$$S_L[\rho_R] = 2\{\sigma_{00}(0)[1-\sigma_{00}(0)] - |\sigma_{01}(0)|^2\,\mathrm{sinc}^2 x\}, \tag{43}$$

and

$$\begin{aligned} S_L[\rho] &= S_L[\sigma_{R,t}] \\ &= 2\{\sigma_{00}(0)[1-\sigma_{00}(0)] - |\sigma_{01}(0)|^2\}. \end{aligned} \tag{44}$$

The entanglement indicator is thus

$$\Delta S = 2|\sigma_{01}(0)|^2(1-\mathrm{sinc}^2 x), \tag{45}$$

which is greater than zero for $x > 0$, provided $\sigma_{01}(0) \neq 0$. That is, for any nonzero T, the evolution of R reflects its entanglement with the clock.

Basic features of the connection between the evolution of the qubit and its entanglement with the clock can be appreciated in Fig. 1. The dependence of the entropies $S_L[\rho_R]$ and

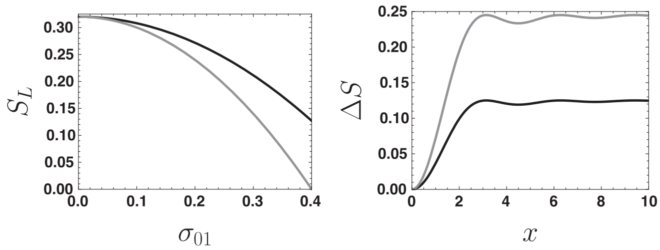

FIG. 1. (left panel) $S_L[\rho_R]$ (black) and $S_L[\rho]$ (gray) as a function of $\sigma_{01}(0)$ for a qubit state, setting $\sigma_{00} = 0.2$ and $x = 1.2$. (right panel) ΔS as a function of x, setting $\sigma_{00} = 0.2$ and $\sigma_{01}(0) = 0.25$ (black), 0.35 (gray). All depicted quantities are dimensionless.

$S_L[\rho]$ on the parameter $\sigma_{01}(0)$ is depicted in the left panel of the figure. The entanglement indicator ΔS as a function of $x = \epsilon T/2\hbar$ for two values of the parameter $\sigma_{01}(0)$ is shown in the right panel. Notice that the entanglement indicator ΔS approaches its asymptotic limit rather quickly, becoming close to its upper bound already at values of T corresponding to $\epsilon T/2\hbar \approx 2$.

V. RELATION BETWEEN THE ENTANGLEMENT INDICATOR AND ENERGY DISPERSION

As we have seen in the previous sections, the entanglement between the system R and the clock C is linked to the time evolution of R. On the other hand, the evolution of a quantum system is closely related to the system's energy uncertainty. Consequently, there has to be a connection between the energy uncertainty of R and the entanglement between R and C. In this section we shall investigate such a connection for mixed joint states of $R + C$.

By recourse to Eq. (29) and to the Taylor series of the sinc function,

$$\operatorname{sinc} z = \sum_{l=0}^{\infty} \frac{(-1)^l z^{2l}}{(2l+1)!}, \tag{46}$$

it can be verified that, to lowest order in T, the entanglement indicator ΔS is

$$\begin{aligned} \Delta S &= \frac{T^2}{12\hbar^2} \sum_{nm} |\sigma_{nm}(0)|^2 (E_n - E_m)^2 \\ &= -\frac{T^2}{12\hbar^2} \mathrm{Tr}([\hat{H}_R, \sigma_{R,t}]^2). \end{aligned} \tag{47}$$

Here we face a situation similar to the one analyzed in the previous section, but now referred to the energy dispersion:

$$\begin{aligned} \sigma_E^2 &\equiv \langle \hat{H}_R^2 \rangle - \langle \hat{H}_R \rangle^2 \\ &= \mathrm{Tr}(\hat{H}_R^2 \sigma_{R,t}) - \mathrm{Tr}^2(\hat{H}_R \sigma_{R,t}), \end{aligned} \tag{48}$$

instead of the spread in the energy probability distribution, as measured by $S_L[\{p_E\}]$. This can be seen as follows: For pure states $\sigma_{R,t} = |\tilde{\Phi}_t\rangle\langle\tilde{\Phi}_t|$ we have

$$\begin{aligned} \mathrm{Tr}([\hat{H}_R, \sigma_{R,t}]^2) &= \mathrm{Tr}([\hat{H}_R, |\tilde{\Phi}_t\rangle\langle\tilde{\Phi}_t|]^2) \\ &= 2(\langle\tilde{\Phi}_t|\hat{H}_R|\tilde{\Phi}_t\rangle)^2 - 2\langle\tilde{\Phi}_t|\hat{H}_R^2|\tilde{\Phi}_t\rangle \\ &= -2\sigma_E^2, \end{aligned} \tag{49}$$

and we obtain the expression

$$(\Delta S)|_{\text{pure}} = \frac{T^2}{6\hbar^2} \sigma_E^2, \tag{50}$$

relating, for pure states of $R + C$, the lowest-order expansion of the S_L-based entanglement indicator (describing its behavior for short-time intervals), with the energy dispersion. In the other extreme situation, for mixed states that are diagonal in the basis of eigenvectors of $\hat{H}_R$, one has $[\hat{H}_R, \sigma_{R,t}] = 0$, and Eq. (47) gives

$$(\Delta S)|_{\text{diagonal}} = 0. \tag{51}$$

Equations (50) and (51) are analogous to Eqs. (39) and (40). As happens with the spread of the energy probability distribution, the energy dispersion has both classical and quantum components. For pure states, all the energy dispersion is of quantum nature, whereas for mixed states that are diagonal in an energy eigenbasis, it is purely classical. Thus, the quantity

$$\mathcal{D} \equiv -\mathrm{Tr}([\hat{H}_R, \sigma_{R,t}]^2) \tag{52}$$

can be interpreted as a measure of the quantum contribution to the energy dispersion of the state $\sigma_{R,t}$.

We shall now illustrate the above results considering states of the form

$$\sigma_{R,t} = \alpha|\psi(t)\rangle\langle\psi(t)| + \frac{(1-\alpha)}{d}\mathbb{I}_d, \tag{53}$$

where $0 \leqslant \alpha \leqslant 1$, and $\mathbb{I}_d$ is the $d \times d$ identity matrix (recall that d is the dimension of $\mathcal{H}_R$). These states can be regarded as pure states perturbed by white noise. We decompose $|\psi(t)\rangle$ as

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |\phi_n\rangle, \tag{54}$$

where $\{|\phi_n\rangle\}$ is the set of (orthonormal) eigenstates of $\hat{H}_R$ with corresponding eigenvalues E_n, and the normalization condition $\sum_n |c_n|^2 = 1$ is satisfied.

Direct calculation gives

$$\mathrm{Tr}\sigma_{R,t}^2 = \alpha^2 + \frac{1}{d}(1-\alpha^2), \tag{55}$$

and

$$\begin{aligned} \mathrm{Tr}\sigma_{R|M}^2 &= \sum_E \left[\alpha^2 \sum_{\substack{n \\ (E_n = E)}} |c_n|^2 \sum_{\substack{m \\ (E_m = E)}} |c_m|^2 \right] \\ &\quad + 2\frac{\alpha(1-\alpha)}{d} \underbrace{\sum_E p_E}_{=1} + \frac{(1-\alpha)^2}{d} \\ &= \sum_E [\alpha^2 p_E^2] + \frac{1}{d}(1-\alpha)^2, \end{aligned} \tag{56}$$

where

$$p_E = \sum_{n\ (E_n = E)} |c_n|^2.$$

Using Eq. (34) we thus get

$$\Delta S_{\max} = \mathrm{Tr}\sigma_{R,t}^2 - \mathrm{Tr}\sigma_{R|M}^2 = S_L[\alpha p_E] + (\alpha^2 - 1) = \alpha^2 S_L[\{p_E\}], \quad (57)$$

and therefore recover the result (39) for $\alpha = 1$.

On the other hand, one also has

$$\mathrm{Tr}([\hat{H}_R, \sigma_{R,t}]^2) = \alpha^2 \mathrm{Tr}([\hat{H}_R, |\psi\rangle\langle\psi|]^2) = -2\alpha^2\sigma_E^2, \quad (58)$$

which reduces for $\alpha = 1$ to the expression (49) corresponding to pure states.

In summary, the entropic indicator ΔS that detects entanglement between the system R and the clock C is given, to lowest order in the length T of the interval within which the state of $R + C$ is defined, by a quantity representing the quantum contribution to the energy uncertainty of R.

VI. CONCLUDING REMARKS

Time evolution requires a composite consisting of at least two parts: a system R that evolves, and a system C, the clock, that keeps track of time. All the properties of the dynamical evolution of R can be encoded in the correlations (entanglement) exhibited by a stationary quantum state jointly describing the complete system $R + C$. In this sense, the origins of dynamics and of the flow of time are, perhaps, the most radical instances of the central role played by entanglement in the physics of composite quantum systems. These considerations constitute the gist of the timeless picture of quantum dynamics. According to this viewpoint, there have to be quantitative relations connecting the amount of entanglement between the clock and the evolving system, on the one hand, with, on the other hand, specific features of the dynamical evolution of the system.

In the present contribution we explore these relations for an extension of the PW proposal which, while allowing mixed quantum states of the $R + C$ composite, keeps the other PW main assumption, particularly that concerning a definite energy of the $R + C$ system equal to zero. By recourse to an entanglement indicator for the global state of $R + C$, it is possible to elucidate how entanglement relates to the time evolution of the system R. It turns out that, in our extension of the PW scenario, entanglement is indeed necessary for R to exhibit evolution. That is, mild forms of quantum correlations, such as quantum discord without entanglement, are not enough to give rise to time and dynamics. This conclusion follows from an entropic sufficient criterium for entanglement satisfied by the state (pure or mixed) of $R + C$ whenever the system R exhibits dynamical evolution.

It is a fact of the quantum world that dynamical evolution is always accompanied by energy uncertainty. Consistently, the system-clock entanglement is related to energy uncertainty as well. Indeed, the aforementioned entanglement indicator for global states of $R + C$ admits an upper bound and an asymptotic limit, both expressible in terms of the spread of the energy probability distribution associated with the system R, as measured by an entropic measure evaluated on that distribution. The entanglement indicator is also related to the energy dispersion of system R, in a way reminiscent of a time-energy uncertainty relation.

Our present developments suggest various possible lines for further inquiry. In would be interesting to explore mixed-state formulations of the PW timeless approach for systems with a clock having a discrete, finite Hilbert space. For this kind of system, mixed-state timeless scenarios may be amenable to experimental implementation, leading to extensions of the works reported in Refs. [21,22]. On the theoretical side, including mixed states within the timeless picture may contribute to elucidating a way in which the PW approach is related to quantum thermodynamics and to quantum coherence [16]. In this regard, it would be interesting to explore more general extensions of the PW formulation, allowing for mixed states of the system-clock compound having a nonvanishing dispersion of the total energy. These lines of inquiry, in turn, may be enriched by including relativistic effects, along the lines pioneered in Refs. [17,18].

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