# Truncated moment sequences and a solution to the channel separability problem

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We consider the problem of separability of quantum channels via the Choi matrix representation given by the Choi-Jamiołkowski isomorphism. We explore three classes of separability across different cuts between systems and ancillae, and we provide a solution based on the mapping of the coordinates of the Choi state (in a fixed basis) to a truncated moment sequence (tms) $y$. This results in an algorithm which gives a separability certificate using semidefinite programming. The computational complexity and the performance of it depend on the number of variables $n$ in the tms and on the size of the moment matrix $M_t(y)$ of order $t$. We exploit the algorithm to numerically investigate separability of families of two-qubit and single-qutrit channels; in the latter case we can provide an answer for examples explored earlier through the criterion based on the negativity $N$, a criterion which remains inconclusive for Choi matrices with $N = 0$.

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## I. INTRODUCTION

Entanglement properties of quantum states have been at the center of many investigations in recent years. Meanwhile, first small-scale quantum processors have become available, and the problem of verifying that such devices work in a properly "quantum" way has become center stage. In that context, it is of high relevance to understand the way entanglement evolves under physical operations acting on quantum states [1–6]. Many contributions to tomography and benchmarking of quantum devices or, more generally, quantum channels can be found in the literature, e.g., recent approaches in the framework of resource and device-independent theories [7,8], schemes which aim at reducing the resources required for entanglement verification [9], methods based on quantum process tomography [10,11], approaches that detect inseparability based on witness operators [12] or separability based on theorems exploiting local operations and classical communication [13,14]. An uncontroversial requirement for a proper quantum operation is that the device is able to create entanglement, a resource on which quantum technology largely relies. In particular it is well known that a quantum computer that generates only limited amounts of entanglement can be simulated efficiently classically [15]. On the other hand the properties of devices which break entanglement turned out to be useful for proving relevant conjectures [16] for obtaining results for the problem of additivity of capacity [17,18] and for their connection with different types of quantum correlations [19]. The problem of deciding whether a quantum *state* is entangled or not has been solved in the sense of its reduction to matrix extensions and semidefinite programming [20], an approach that was later understood more generally within the theory of truncated moment sequences [21]. However, no corresponding algorithm that gives a definite outcome for quantum *channels* was known, i.e., an algorithm that takes as input an arbitrary quantum channel and outputs a definite answer whether the quantum channel can generate entanglement for some initial separable state. Although one might argue that with modern technology it is quite easy to entangle, e.g., two qubits and verify their entanglement, the entanglement is typically lost on relatively short timescales. The way entanglement is created and possibly destroyed again by the full channel, including storage and decoherence processes over longer times, depends on the input state. When trying to verify entanglement creation one would, thus, have to search for suitable input states. In such a situation it would be much more convenient to asses the possibility of entanglement creation directly on the level of the quantum channel. In the present paper we present such an algorithm for the channel separability problem. It generalizes to quantum channels the hierarchy of Refs. [20,21]. The resulting algorithm provides definiteness in the answer to the question whether a quantum channel is entangling or separable, even in cases where more straightforward separability criteria based on positive but not completely positive maps fail as we will demonstrate with explicit examples in Sec. IV C.

The mathematical object associated with a physical operation is a quantum channel, which acts on the joint state of a system $\mathcal{A}$ and its environment to produce an output state. The environment can be seen as an ancilla system $\mathcal{A}'$ with which the system $\mathcal{A}$ is possibly entangled. The system $\mathcal{A}$ itself may be bipartite and made of two subsystems $A$ and $B$ which may or may not be entangled with one another or with their respective ancillae $A'$ and $B'$. Since a channel acts on both the system and its ancilla, the output state may be entangled in different ways, which leads to different definitions of separability of quantum channels [22–26]. These definitions depend on whether the total state of the system and ancilla is separable for instance across the cut $\mathcal{A} - \mathcal{A}'$ or across the cut $A - B$. The algorithm that we present in Sec. III D allows one to investigate all different notions of separability with only small modifications needed in the input to go from one definition to the other, thus, giving a unifying framework for the separability problem in the case of quantum channels.

The Choi-Jamiołkowsi isomorphism relates completely positive trace-preserving maps with density matrices or equivalently completely positive maps with positive operators. Characterizing separability for channels can be investigated in the light of results obtained for quantum states. Many theoretical results have been obtained for states in terms of separability criteria [27]. One of the most well-known necessary conditions for separability is the positive partial transpose (PPT) criterion, which states that if a state $\rho$ is separable then $\rho^{\rm PT} \geqslant 0$ with $\rho^{\rm PT}$ the partial transpose with respect to one of the subsystems [28,29].

As was shown recently [30], the separability problem for states can be recast as a "truncated moment" problem, a problem well studied in recent years in the mathematical literature. The truncated moment problem consists of finding conditions under which a given sequence of numbers corresponds to moments of a probability distribution. The moment problem corresponds to the case where an infinite sequence is given, whereas in the truncated moment problem only the lowest moments are fixed and the aim is to find a measure matching these moments. Of relevance for the separability problem as we will see is the $K$-truncated moment problem where the measure is additionally required to have the set $K$ as support. In Ref. [30] we showed that asking whether a quantum state is separable along an arbitrary partition of Hilbert space can be cast in the form of a $K$-truncated moment problem, and we applied this approach to symmetric multiqubit states.

In the present paper our goal is to apply this formalism to the more general situation of the separability of quantum channels. Even though the problem of separability of channels can be related to the one of states through the Choi-Jamiołkowsi isomorphism, it is still relevant to explicitly formulate the mapping with the moment problem since it allows us to provide theorems that give necessary and sufficient conditions for a channel to be separable or entanglement breaking; moreover, the resulting necessary and sufficient criterion is also practically usable thanks to a quite simple algorithm that implements the theorems numerically. The paper is organized as follows. In Sec. II we recall some useful definitions about quantum channels and the various notions of separability. In Sec. III we explain in detail how the truncated moment problem maps to these separability problems, and we provide a theoretical solution in the form of a set of theorems (Sec. III C) and a numerical solution in terms of an algorithm (Sec. III D). In Sec. IV we consider various examples of application of this algorithm, which allow detection of separability in quantum channels. Finally we conclude in Sec. V.

## II. DEFINITIONS

We start by recalling some elementary definitions.

### A. Quantum channels

Let $\rho$ be a quantum state acting on a tensor product $H = H^{(1)} \otimes \cdots \otimes H^{(d)}$ of Hilbert spaces $H^{(i)}$ of finite dimension. Any physical transformation can be described by a completely positive map, that is, a map $\Phi$ such that $\Phi \otimes \mathbb{1}$ is positive on all states acting on an extended Hilbert space $H \otimes H'$ (where $H'$ is the Hilbert space of an ancillary system of arbitrary size). A quantum channel $\Phi$ is, therefore, defined as a completely positive trace-preserving linear map, which maps $\rho$ to a state $\rho' = \Phi(\rho)$ acting on some Hilbert space (that for simplicity we consider here equal to $H$ so that $\Phi\colon \mathcal{L}(H) \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ is the set of linear operators on $H$).

Let $N$ be the dimension of the Hilbert space $H$. A density matrix can be expanded as $\rho = \sum_{i,j} \rho_{ij} \left|i\right\rangle\left\langle j\right|$, with $\left|i\right\rangle$ as the vectors of the canonical basis of $H$. To any linear map $\Phi$ mapping $\rho$ to $\rho'$ one can associate a superoperator $M$ of size $N^2$ such that $\rho'_{ij} = M_{ij,kl}\rho_{kl}$ (with summation over repeated indices), and a dynamical matrix $D_\Phi$ defined [31] by a reshuffling of entries of $M$, namely, $(D_\Phi)_{ij,kl} = M_{ik,jl}$ [27]. Alternatively one can define the Choi matrix,

$$C_\Phi = \sum_{i,j} \Phi(\left|i\right\rangle\left\langle j\right|) \otimes \left|i\right\rangle\left\langle j\right|, \qquad (1)$$

[32], which coincides with $D_\Phi$ when written in the canonical basis. The Choi matrix $C_\Phi$ is Hermitian. The map $\Phi$ is positive if and only if the corresponding Choi matrix $C_\Phi$ is block positive (that is, positive on product states in $H \otimes H$) [33]. According to Choi's theorem [32], $\Phi$ is completely positive if and only if its Choi matrix is positive semidefinite. Finally, $\Phi$ is trace preserving if and only if the $N^2$ conditions $\sum_i (C_\Phi)_{ij,il} = \delta_{jl}$ are fulfilled. These conditions imply that $\operatorname{tr} C_\Phi = N$.

As a consequence, if $\Phi$ is a quantum channel, then $\frac{1}{N}C_\Phi$ can be seen as a density matrix acting on $H \otimes H$. Any completely positive trace-preserving map can be associated with a density matrix in that way. The Choi-Jamiołkowsi isomorphism is a bijection between a quantum channel $\Phi$ and its Choi matrix $C_\Phi$ [27,33]. We will also make use of the fact that a quantum channel can be written in Kraus form as

$$\Phi(\rho) = \sum_l E_l \rho E_l^\dagger, \qquad \sum_l E_l^\dagger E_l = \mathbb{1}. \qquad (2)$$

The Kraus operators $E_l$ are not unique, but a canonical form can be found by diagonalizing the Choi matrix and reshuffling its eigenvectors into square matrices in which case a set of at most $N^2$ Kraus operators suffices [27].

### B. Separability of channels

A bipartite quantum state $\rho$ acting on a Hilbert space $H_A \otimes H_B$ is separable if it admits a decomposition,

$$\rho = \sum_i w_i \rho_i^{(A)} \otimes \rho_i^{(B)}, \qquad (3)$$

with $w_i \geqslant 0$ and $\rho_i^{(A)}, \rho_i^{(B)}$ acting on $H_A, H_B$ respectively. More generally, a positive semidefinite matrix $M$ is said to be separable if it can be written as

$$M = \sum_k P_k \otimes Q_k, \qquad (4)$$

with $P_k$ and $Q_k$ positive semidefinite matrices.

Various kinds of channel separability have been introduced in the literature. Consider the Hilbert space $H = H_A \otimes H_B$ describing a system partitioned into two subsystems $A$ and $B$ and let $\Phi\colon \mathcal{L}(H_A \otimes H_B) \to \mathcal{L}(H_A \otimes H_B)$ be a completely positive map. As a criterion for complete positivity one must consider the extended Hilbert state $H \otimes H'$ with $H' = H$ where here and in the following the prime is used to denote the ancilla system. The corresponding Choi matrix $C_\Phi$ can be seen as a density matrix acting on Hilbert space $\mathcal{H} = H_A \otimes H_B \otimes H_{A'} \otimes H_{B'}$. Following Eq. (1) it can be expressed as $C_\Phi = \sum_{ijrs} \Phi(|ir\rangle\langle js|) \otimes |ir\rangle\langle js|$.

#### 1. Separable channels

$\Phi$ is called separable (SEP) if it takes the form $\Phi(\rho) = \sum_l (A_l \otimes B_l)\rho(A_l \otimes B_l)^\dagger$ [22]. In other words, the Kraus operators for the channel $\Phi$ in (2) can be factored as $E_l = A_l \otimes B_l$. Such channels map separable states to separable states. In terms of these Kraus operators, the Choi matrix of a separable map $\Phi$ is given by

$$C_\Phi = \sum_{i,j,r,s} \sum_l A_l|i\rangle\langle j|A_l^\dagger \otimes B_l|r\rangle\langle s|B_l^\dagger \otimes |i\rangle\langle j| \otimes |r\rangle\langle s|. \quad (5)$$

Swapping $H_{A'}$ and $H_B$ we can interpret $C_\Phi$ as an operator in $H = H_A \otimes H_{A'} \otimes H_B \otimes H_{B'}$ and reexpress it as

$$C_\Phi = \sum_l \sum_{i,j} A_l|i\rangle\langle j|A_l^\dagger \otimes |i\rangle\langle j| \otimes \sum_{r,s} B_l|r\rangle\langle s|B_l^\dagger \otimes |r\rangle\langle s|. \quad (6)$$

It is clear that $\sum_{i,j} A_l|i\rangle\langle j|A_l^\dagger \otimes |i\rangle\langle j|$ is positive semidefinite for all $l$'s because it is the Choi matrix of the completely positive map $\rho \mapsto A_l\rho A_l^\dagger$; and the same holds for $B$. Therefore, $C_\Phi$ can be written as a sum $\sum_l M_{\mathcal{A}}^{(l)} \otimes M_{\mathcal{B}}^{(l)}$ with $M_{\mathcal{A}}^{(l)}$ and $M_{\mathcal{B}}^{(l)}$ positive semidefinite: It is, thus, a separable matrix across the $(A - A') - (B - B')$ cut. It was shown in Ref. [6] that the converse is true, namely, $C_\Phi$ is separable across the $(A - A') - (B - B')$ cut if and only if $\Phi$ is a separable map. We will use this characterization of separable channels in Sec. III C.

We will call $\Phi$ fully separable (FS) if the corresponding $C_\Phi$ is separable across all possible cuts.

#### 2. Entanglement-breaking channels

$\Phi$ is called entanglement breaking (EB) [23] if $(\Phi \otimes \mathbb{1})(\rho)$ is a separable state across the $H - H'$ cut whatever the initial state $\rho \in \mathcal{L}(\mathcal{H})$. It does not address the separability of the bipartite system $H$ into $A$ and $B$ but rather the separability between the system and its environment (it can, therefore, be defined for one-qubit channels). Various necessary and sufficient conditions for entanglement breaking have been obtained in Ref. [23]. One necessary and sufficient criterion is that there exist a Kraus form where all Kraus operators have rank 1. In terms of the Choi matrix, a necessary and sufficient condition for EB is that $C_\Phi$ be separable across the $(A - B) - (A' - B')$ cut. Physically these channels correspond to the case in which the output state is prepared according to the measurement outcomes made by the sender and sent via a classical channel to the receiver. We point out the difference between separable and entanglement-breaking channels in Fig. 1.

Channels which become entanglement breaking after a sufficient number of compositions with themselves are called eventually entanglement-breaking channels [25,26].

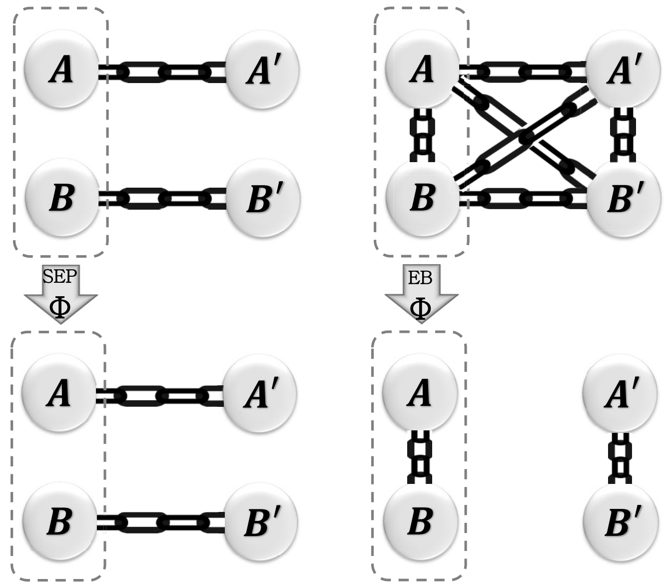


FIG. 1. Difference between separable (left) and entanglement-breaking (right) channels for a bipartite system $AB$ with ancillae $A'B'$. The chains represent entanglement. A separable channel preserves separability between $(A - A')$ and $(B - B')$, whereas an entanglement-breaking channel destroys entanglement between $A$ and all the ancillae and $B$ and all the ancillae, giving separability between $(A - B)$ and $(A' - B')$.

#### 3. Entanglement annihilating channels

$\Phi$ is called entanglement annihilating [34] if it destroys any entanglement within the system $H$ (but it does not necessarily destroy entanglement between $H$ and $H'$). A necessary and sufficient condition for entanglement annihilating channels in terms of the Choi matrix is that $C_\Phi \geqslant 0$ and that its partial trace over $A$ and $B$ is proportional to the identity matrix (see Corollary 1 of Ref. [24]). Such a condition on partial trace is not implementable in truncated moment sequence (tms) form, so we will not address this type of separability.

## III. TRUNCATED MOMENT SEQUENCES

In the present section, we introduce the mathematical framework of truncated moment sequences (Sec. III A) and then apply it to quantum states (Sec. III B) and channels (Sec. III C). In general, to some nonnegative measure $\mu$ on $\mathbb{R}^n$ one can associate its moments, which are the average values of the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The moment $y_\alpha$ of order $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ is defined as $y_\alpha = \int x^\alpha d\mu(x)$, where $x^\alpha$ denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. If we are given a finite set $y$ of real numbers, i.e., a *truncated sequence*, a natural question is to ask whether these numbers are the moments of a certain probability distribution. If the measure $\mu$ is constrained to be supported by a semialgebraic set $K$, the moment $y_\alpha$ is given as

$$y_\alpha = \int_K x^\alpha d\mu(x). \quad (7)$$

The tms problem deals with the characterization of the truncated sequences $y = (y_\alpha)_{\alpha \in \mathbb{Z}_+^n}$ that are sequences of moments of a measure $\mu$. Solutions to this problem have been put forward in the mathematical literature. As we will see, the separability problem can be expressed exactly in the form of Eq. (7). The reader interested in the mathematical results for the solution of the moment problem should continue with the next section; otherwise, jumping to Secs. III B and III C will directly give its connection with the physical problem of separability of quantum states and quantum channels, respectively.

### A. The tms problem

In order to be as self-contained and pedagogical as possible for a physics-oriented audience, we start by reviewing and explaining some results from the mathematical literature [35–41]. We follow the nice presentation from Ref. [42]. We then recall the theorems obtained in Ref. [30] for quantum states and formulate them in the case of quantum channels.

A tms $y = (y_\alpha)_{|\alpha|\leqslant 2d}$ of degree $2d$ is a finite set of real numbers indexed by $n$-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ of integers $\alpha_i \geqslant 0$ such that $|\alpha| = \sum_i \alpha_i \leqslant 2d$ (here we only consider tms of even degree: indeed, although the definition would extend trivially to odd-degree tms, even-degree tms are the only ones involved in the theorems below, so this slightly simplifies notations). We denote by $S_{2d}$ the set of $n$-tuples $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \leqslant 2d$ so that $y$ is a vector in $\mathbb{R}^{S_{2d}}$. The number of such $n$ tuples is

$$\sum_{k=0}^{2d} \binom{k+n-1}{n-1} = \binom{n+2d}{2d}. \tag{8}$$

A moment sequence corresponds to a situation where all $y_\alpha$ are known to arbitrary order, which we denote by $y \in \mathbb{R}^{S_\infty}$.

The truncated moment problem (tms problem) is the problem of finding whether there exists a representing measure for a given sequence $y$, that is, a positive measure $d\mu$ such that $y_\alpha = \int x^\alpha d\mu(x)$ for all $\alpha$ with $|\alpha| \leqslant 2d$. Here the notation $x^\alpha$ stands for $\prod_{i=1}^n x_i^{\alpha_i}$.

The $K$-tms problem addresses the case where the measure $d\mu$ is additionally required to be supported by a semialgebraic set $K$, that is, a set defined by polynomial inequalities. We will use the notation $K = \{x \in \mathbb{R}^n | g_1(x) \geqslant 0, \dots, g_m(x) \geqslant 0\}$ with $g_j(x)$ multivariate polynomials. The sequence $y$ has a representing measure for the $K$-tms problem if for all $\alpha$'s with $|\alpha| \leqslant 2d$, Eq. (7) holds.

Necessary and sufficient conditions for the solution of the tms problem can be obtained in terms of moment matrices. Given a tms $(y_\alpha)_{|\alpha|\leqslant 2d}$, its moment matrix of order $t$ is the matrix $M_t(y)$ indexed by $\alpha, \beta$ with $|\alpha|, |\beta| \leqslant t$ and defined as $M_t(y)_{\alpha\beta} = y_{\alpha+\beta}$. The entries of the matrix involve indices of $y$ up to order $2t$ and since the highest index of $y$ is $2d$ (by definition of the tms) such a matrix is defined only if $t \leqslant d$. The size of $M_t(y)$ is given by the number of moments up to order $t$, that is, $\binom{n+t}{t}$. In the case of an infinite moment sequence, the matrix $M(y)$ is infinite.

Necessary and sufficient conditions for the solution of the $K$-tms problem additionally involve the localizing matrices associated with polynomials $g_j$ specifying $K$, which are defined as follows. Any polynomial $g$ of $n$ variables $x_1, \dots, x_n$ can be decomposed over monomials as $g = \sum_{|\alpha|\leqslant \deg(g)} g_\alpha x^\alpha$, where $\deg(g)$ is the degree of the multivariate polynomial $g$. It can, thus, be seen as a vector in $\mathbb{R}^{S_{\deg(g)}}$. For a tms $(y_\alpha)_{|\alpha|\leqslant 2d}$ and a polynomial $g$, we define a shifted sequence $g \star y$ by setting $(g \star y)_\alpha = \sum_\gamma g_\gamma y_{\alpha+\gamma}$. The localizing matrix of order $t$ associated with $g$ is defined as the moment matrix of order $t$ of the shifted sequence, that is, $M_t(g \star y)$. Explicitly, its components read $M_t(g \star y)_{\alpha\beta} = \sum_\gamma g_\gamma y_{\alpha+\beta+\gamma}$. The highest index of $y$ involved here is $2t + \deg(g)$ so that the matrix is defined only for $2t + \deg(g) \leqslant 2d$, that is, $t \leqslant d - \deg(g)/2$. The $m$ polynomials defining $K$ give rise to $m$-localizing matrices $M_t(g_j \star y)$. In order that all of them be defined, the order $t$ has to be such that $t \leqslant d - d_0$ with

$$d_0 = \max_{1\leqslant j\leqslant m} \{1, \lceil \deg(g_j)/2 \rceil\}, \tag{9}$$

that is, the degree of $y$ has to be greater than or equal to $2(t + d_0)$.

The three theorems below give necessary and sufficient conditions for a tms (or a full moment sequence) to have a representing measure, supported on $K$ or not. In all cases, the representing measure is $r$ atomic, meaning that it is a sum of $r$ $\delta$ functions with positive weights, $d\mu(x) = \sum_j \omega_j \delta(x - x_j)$. The central criterion is the existence of extensions. An extension of a tms $y$ of degree $2d$ is a tms of degree $2d'$ with $d' > d$ whose restriction to indices of order $2d$ or less coincides with $y$. We denote it again by $y$. One can define the moment matrix of order $t$ of such an extension for all $t \leqslant d'$, and we then say that for $t' > t$, $M_{t'}(y)$ is an extension of $M_t(y)$. An extension $M_{t'}(y)$ is said to be a flat extension of $M_t(y)$ if it satisfies the condition that its rank is equal to the rank of $M_t(y)$, that is,

$$\mathrm{rk}\, M_{t'}(y) = \mathrm{rk}\, M_t(y). \tag{10}$$

In particular, if (10) holds then $M_{t'}(y) \geqslant 0 \Leftrightarrow M_t(y) \geqslant 0$ (see Appendix B). Theorem 1 below deals with the moment problem, Theorem 2 with the tms problem, and Theorem 3 with the $K$-tms problem.

*Theorem 1.* ( Ref. [35]; see Theorem 1.2 of Ref. [42]) Let $y \in \mathbb{R}^{S_\infty}$. If $M(y) \geqslant 0$ and rk $M(y) = r$ is finite, then $y$ has a unique representing measure, which is $r$ atomic.

*Theorem 2.* (Ref. [35]; see Theorem 1.3 and Corollary 1.4 of Ref. [42]) Let $y \in \mathbb{R}^{S_{2t}}$. If $M_t(y) \geqslant 0$ and $M_t(y)$ is a flat extension of $M_{t-1}(y)$, then $y$ can be extended to $y \in \mathbb{R}^{S_{2t+2}}$ in such a way that $M_{t+1}(y)$ is a flat extension of $M_t(y)$.

From induction and using Theorem 1, one concludes that the tms in $\mathbb{R}^{S_{2t}}$ can be, in fact, extended to $y \in \mathbb{R}^{S_\infty}$ and has a unique representing measure, which is $r$ atomic with $r = \mathrm{rk}\, M_t(y)$. Moreover one can show (see Ref. [42] for detail) that the $r$ atoms $x_i$ which support the measure can be obtained from the kernel of $M_t(y)$, that is, the set of polynomials $p = \sum_\alpha p_\alpha x^\alpha$ such that $\sum_\beta M_t(y)_{\alpha\beta} p_\beta = 0$. More specifically, the set of $x_i$ is the variety $\mathcal{V}[\ker M_t(y)] = \{x \in \mathbb{C}^n;\ f(x) = 0\ \forall f \in \ker M_t(y)\}$, that is, the set of common roots of polynomials in the kernel of $M_t(y)$. In words, what the above results say is that in order to find a representing measure for $y \in \mathbb{R}^{S_{2d}}$ one has to start from the moment matrix $M_{t=d}(y)$ (which is the smallest moment matrix containing all the data) and look for extensions of higher and higher order, until for some order $t$ one has $\mathrm{rk}\, M_t(y) = \mathrm{rk}\, M_{t-1}(y)$. If such an extension exists then the representing measure exists and is supported by the common roots of polynomials of ker $M_t(y)$.

*Theorem 3.* (Ref. [35]; see Theorem 1.6 of Ref. [42]) Let $y \in \mathbb{R}^{S_{2t}}$ and $r = \mathrm{rk}\, M_t(y)$. Then $y$ has a $r$ atomic representing measure supported on $K$ if and only if $M_t(y) \geqslant 0$ and there exists a flat extension $M_{t+d_0}(y)$ with $M_t(g_j \star y) \geqslant 0$ for $1 \leqslant j \leqslant m$ and $d_0$ defined in (9).

This theorem can be decrypted as follows. Starting from the moment matrix of order $d$ and looking for higher-order extensions of order $t$, if there exists an extension $M_{t+d_0}(y)$ with $\mathrm{rk}\, M_{t+d_0}(y) = \mathrm{rk}\, M_t(y) = r$ then all its submatrices $M_{t+1}(y), M_{t+2}(y), \dots$ are also flat extensions of $M_t(y)$. From Theorems 1 and 2 one readily concludes that there exists a unique $r$-atomic representing measure; the atoms are given by the variety associated with the kernel of the first extension where the flatness condition is achieved. However these atoms may not be located on $K$. The conditions $M_t(g_j \star y) \geqslant 0$ on the localizing matrices precisely enforce that additional condition (see Appendix A for an insight into the proof). As mentioned above, these matrices are only defined if the degree of $y$ is greater than $2(t + d_0)$, which is why, in order to fulfill these conditions, one has to find extensions in $y \in \mathbb{R}^{S_{2(t+d_0)}}$. Therefore, although an extension to $M_{t+1}(y)$ is enough to guarantee the existence of a $r$-atomic representing measure, an extension to $M_{t+d_0}(y)$ is required so that it is supported by $K$. As a consequence, achieving the flatness condition requires to go quickly to matrices of high order, which has an impact in terms of computational complexity.

### B. Tms for quantum states

Let us now apply these theorems to quantum states, following Ref. [30]. Consider a quantum state $\rho$ acting on the tensor product $H = H^{(1)} \otimes \dots \otimes H^{(p)}$ of Hilbert spaces $H^{(i)}$ with $\dim \mathcal{L}(H^{(i)}) = \kappa_i + 1$. Let $S^{(i)}_{\mu_i}$ $(0 \leqslant \mu_i \leqslant \kappa_i)$ be a set of Hermitian matrices forming an orthogonal basis for $\mathcal{L}(H^{(i)})$, and $S_{\mu_1\mu_2\cdots\mu_p} = S^{(1)}_{\mu_1} \otimes \dots \otimes S^{(p)}_{\mu_p}$ an orthogonal basis of $\mathcal{L}(H)$. We expand $\rho$ as

$$\rho = X_{\mu_1\mu_2\cdots\mu_p} S_{\mu_1\mu_2\cdots\mu_p} \tag{11}$$

(with implicit summation over repeated indices), where $X_{\mu_1\mu_2\cdots\mu_p} = \mathrm{tr}(\rho S_{\mu_1\mu_2\cdots\mu_p})$ are the (real) coordinates of the state. Here each index $\mu_i$ runs from 0 to $\kappa_i$, and we will use latin letters $a_i$ for indices running from 1 to $\kappa_i$. It will prove convenient to take $S^{(i)}_0$ as the identity matrix of size the dimension of $H^{(i)}$. Actually, as detailed in Ref. [30], the matrices $S_{\mu_1\mu_2\cdots\mu_p}$ need not be an orthogonal basis: It suffices that they be a tight frame (a mathematical structure bearing some analogy with orthogonal bases), which proves useful, for example, in the case of symmetric states where some redundancy of the matrices in the expansion (11) is handy.

One can associate with $\rho$ a tms $y = (y_\alpha)_{|\alpha| \leqslant p}$ of degree $p$ in the following way. A density matrix acting on Hilbert space $H^{(i)}$ can be expanded as $\sum_{\mu_i=0}^{\kappa_i} x^{(i)}_{\mu_i} S^{(i)}_{\mu_i}$. We associate to $H^{(i)}$ a set of $\kappa_i$ variables $x^{(i)}_{a_i}$, $1 \leqslant a_i \leqslant \kappa_i$. Let $x = (x_1, x_2, \dots, x_n)$ be the vector of all these variables. In the general case $(x_1, x_2, \dots, x_n) := (x^{(1)}_1, x^{(1)}_2, \dots, x^{(p)}_{\kappa_p})$ and $n = \sum_i \kappa_i$, and each $x_k$ corresponds to a certain $x^{(i)}_{a_i}$, whereas if we consider symmetric states (i.e., mixtures of pure states invariant under permutation of the $H^{(i)}$) only one set of variables, say $x^{(1)}_{a_1}$, should be considered, and then $n$ is the common value $\kappa_1 = \kappa_2 = \cdots$.

An arbitrary monomial of these variables $x_k$ can be written as $x^\alpha \equiv \prod_{k=1}^n x_k^{\alpha_k}$, where $\alpha_k$ counts the number of variables $x_k$ in the monomial. We then define a tms by $y_\alpha = X_{\mu_1\mu_2\cdots\mu_p}$, where $\alpha$ is the index such that $x^\alpha = \prod_{i=1}^p x^{(i)}_{\mu_i}$. Since $X$ has $p$ indices we have $|\alpha| \leqslant p$ so that $y_\alpha$ is a tms of degree $p$. In fact, in order to define a moment matrix, an even-degree tms is required. Thus, we set $p = 2d$ if $p$ is even or $p = 2d - 1$ if $p$ is odd. Thus, $X_{\mu_1\mu_2\cdots\mu_p}$ is mapped to a tms $(y_\alpha)_{\alpha \leqslant 2d}$ (and in the case where $p$ is odd the moments of order exactly $2d$ remain unspecified).

As an example, let us consider the case of a state of two spins 1. We expand it as $\rho = X_{\mu_1\mu_2} S_{\mu_1\mu_2}$, where indices $\mu_i$ run from 0 to 8 (since a spin-1 density matrix is a $3 \times 3$ Hermitian matrix and can be described by nine real numbers). We then introduce the vector of variables $x = (x_1, x_2, \dots, x_{16})$, where $x_1, \dots, x_8$ are associated with the first spin and $x_9, \dots, x_{16}$ with the second. Entries $X_{\mu_1\mu_2}$ define a tms $y_\alpha$ of degree 2 where each $\alpha$ is a vector of integers of length 16 with all entries equal to 0 if $\mu_1 = \mu_2 = 0$, a single nonzero entry $\alpha_{\mu_1} = 1$ if $\mu_1 \neq 0$ and $\mu_2 = 0$, a single entry $\alpha_{\mu_2+8} = 1$ if $\mu_2 \neq 0$ and $\mu_1 = 0$, and two entries equal to 1 if both $\mu_1$ and $\mu_2$ are nonzero. Each of these $\alpha$'s is associated with a monomial, for instance, $X_{3;8}$ corresponds to $\alpha = (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$ or to $x_3 x_{16}$.

As shown in Ref. [30], the problem of finding whether $\rho$ is separable across the multipartition $H^{(1)} \otimes \dots \otimes H^{(p)}$ is equivalent to a $K$-tms problem. Indeed, projecting the separability condition on the basis $S_{\mu_1\mu_2\cdots\mu_p}$, coordinates of a separable state can be written as

$$X_{\mu_1\mu_2\cdots\mu_p} = \int_K x^{(1)}_{\mu_1} x^{(2)}_{\mu_2} \cdots x^{(p)}_{\mu_p} d\mu(x), \tag{12}$$

with $x^{(i)}_0 = 1,$ $;x = (x^{(1)}, x^{(2)}, \dots, x^{(p)}) \in \mathbb{R}^n$ $(n = \sum_i \kappa_i)$, $x^{(i)} = (x^{(i)}_a)_{1 \leqslant a \leqslant t_i} \in \mathbb{R}^{\kappa_i}$, and $d\mu(x) = \sum_j \omega_j \delta(x - z_j)$ a measure supported on a semialgebraic set $K \subset \mathbb{R}^n$ defined by the positivity of the density matrices on each local Hilbert space (that is, the measure is an atomic measure with atoms $z_j \in K$). This tms problem is equivalent to asking whether there exists a positive measure $d\mu$ with support $K$ for a tms whose moments are the $y_\alpha$ given as explained above by the coordinates $X_{\mu_1\mu_2\cdots\mu_p}$ of the state $\rho$. In this language, Eq. (12) precisely takes the form (7). As a consequence, separability of $\rho$ can be addressed in the following way: Given a state $\rho$, we can map its coordinates $X_{\mu_1\mu_2\dots\mu_p}$ to a tms $(y_\alpha)_{\alpha \leqslant 2d}$ and look for extensions $(y_\alpha)_{\alpha \leqslant 2t}$, starting from $t = d$. State $\rho$ is separable if and only if there exists a flat extension $(y_\alpha)_{\alpha \leqslant 2(t+d_0)}$ of $(y_\alpha)_{\alpha \leqslant 2t}$ with $M_t(y) \geqslant 0$ and $M_t(g_j \star y) \geqslant 0$ for $j = 1, \dots, m$.

### C. Tms for quantum channels

We will now reformulate the theorem above to give a necessary and sufficient criterion for the separability of quantum channels. Let $\Phi: \mathcal{L}(H_A \otimes H_B) \rightarrow \mathcal{L}(H_A \otimes H_B)$ be a completely positive map and $C_\Phi$ its corresponding Choi matrix acting on $\mathcal{H} = H_A \otimes H_B \otimes H_{A'} \otimes H_{B'}$; an orthogonal basis of $\mathcal{H}$ is then given by matrices $S_{\mu_A\mu_B\mu_{A'}\mu_{B'}} = S^{(A)}_{\mu_A} \otimes S^{(B)}_{\mu_B} \otimes S^{(A')}_{\mu_{A'}} \otimes S^{(B')}_{\mu_{B'}}$, where $S^{(\bullet)}_\mu$ are Hermitian matrices forming an orthogonal basis of the set of bounded linear operators on $H_\bullet$. Let us translate the above tms theorems as necessary and sufficient conditions on the Choi matrix to be separable.

The compact $K$ is defined according to the decomposition we are interested in. In the EB case, one wants to decompose the Choi matrix as $\sum_k P_k \otimes Q_k$, where $P_k$ and $Q_k$ are positive operators acting on $H_A \otimes H_B$ and $H_{A'} \otimes H_{B'}$, respectively. Expanding the $P_k$ over a basis of operators $S_\lambda^{AB}$ (these $S_\lambda^{AB}$ could be taken as the $S_{\mu_A}^{(A)} \otimes S_{\mu_B}^{(B)}$) and $Q_k$ over a basis $S_{\lambda'}^{A'B'}$ and expressing the condition that they must be positive, we obtain a definition of the compact $K$ as the set of real expansion coefficients $c_\lambda, d_{\lambda'}$ such that

$$\sum_\lambda c_\lambda S_\lambda^{AB} \geqslant 0, \tag{13}$$

$$\sum_{\lambda'} d_{\lambda'} S_{\lambda'}^{A'B'} \geqslant 0. \tag{14}$$

These positivity conditions can be rewritten as inequalities on the coefficients of the corresponding characteristic polynomials using the Descartes sign rule (see Sec. III D below). In the SEP case, the Choi matrix now has to be decomposed as $\sum_k P_k \otimes Q_k$ with $P_k$ and $Q_k$ acting on $H_A \otimes H_{A'}$ and $H_B \otimes H_{B'}$, respectively. The same reasoning applies for the positivity conditions as in the EB case.

Given a channel $\Phi$, we expand the corresponding Choi matrix as

(1) for EB, $C_\Phi = \sum_{\lambda,\lambda'} X_{\lambda\lambda'} S_\lambda^{AB} \otimes S_{\lambda'}^{A'B'}$ (with $S_\lambda^{AB}$ a basis of operators for the system and $S_{\lambda'}^{A'B'}$ for the ancilla)

(2) for SEP, $C_\Phi = \sum_{\lambda,\lambda'} \tilde{X}_{\lambda\lambda'} S_\lambda^{AA'} \otimes S_{\lambda'}^{BB'}$ (with $S_\lambda^{AA'}$ a basis of operators for the Hilbert space $H_A \otimes H_{A'}$, and $S_{\lambda'}^{BB'}$ for the Hilbert space $H_B \otimes H_{B'}$).

We can then map either the coordinates $X_{\lambda\lambda'}$ or the coordinates $\tilde{X}_{\lambda\lambda'}$ to a tms $(y_\alpha)_{\alpha\leqslant 2}$ (indeed, since we look for separability across a bipartition, the degree of the tms is 2). The necessary and sufficient conditions for channels are then given as follows:

*Theorem 4.*

(i) The channel $\Phi$ is EB if and only if, considering extensions $(y_\beta)_{\beta\leqslant 2t}$ of $(y_\beta)_{\beta\leqslant 2}$, there exists a flat extension $(y_\beta)_{\beta\leqslant 2(t+d_0)}$ of $(y_\beta)_{\beta\leqslant 2t}$ (possibly with $t=1$) with $M_t(y) \geqslant 0$ and $M_t(g_j \star y) \geqslant 0$ for $j = 1, \ldots, m$ where the $g_j$'s are polynomials of variables $c_\lambda$ and $d_{\lambda'}$ defined by the conditions $\sum_\lambda c_\lambda S_\lambda^{AB} \geqslant 0$, $\sum_{\lambda'} d_{\lambda'} S_{\lambda'}^{A'B'} \geqslant 0$, and $d_0 = \max_{1\leqslant j\leqslant m}\{1, \lceil \deg(g_j)/2 \rceil\}$.

(ii) The channel $\Phi$ is SEP if and only if, considering extensions $(y_\beta)_{\beta\leqslant 2t}$ of $(y_\beta)_{\beta\leqslant 2}$, there exists a flat extension $(y_\beta)_{\beta\leqslant 2(t+d_0)}$ of $(y_\beta)_{\beta\leqslant 2t}$ (possibly with $t=1$), with $M_t(y) \geqslant 0$ and $M_t(g_j \star y) \geqslant 0$ for $j = 1, \ldots, m$ where the $g_j$'s are polynomials of variables $c_\lambda$ and $d_{\lambda'}$ defined by the conditions $\sum_\lambda c_\lambda S_\lambda^{AA'} \geqslant 0$, $\sum_{\lambda'} d_{\lambda'} S_{\lambda'}^{BB'} \geqslant 0$, and $d_0 = \max_{1\leqslant j\leqslant m}\{1, \lceil \deg(g_j)/2 \rceil\}$.

In the case of fully separable channels, the Choi matrix must be separable across any cut. We expand the matrix $C_\Phi$ as $C_\Phi = X_{\mu_A\mu_B\mu_{A'}\mu_{B'}} S_{\mu_A}^{(A)} \otimes S_{\mu_B}^{(B)} \otimes S_{\mu_{A'}}^{(A')} \otimes S_{\mu_{B'}}^{(B')}$. The coefficients $X_{\mu_A\mu_B\mu_{A'}\mu_{B'}}$ are now mapped to a tms of order 4, and the set $K$ is given by positivity conditions on each Hilbert space. The channel $\Phi$ is fully separable if and only if, looking for extensions of that tms, we find a flat extension (with positivity conditions on the moment and localizing matrices).

### D. The algorithm

Theorem 4 can be translated into an algorithm that characterizes separable or entangling channels with respect to a chosen partition. The algorithm is based on semidefinite programming (SDP). The inputs to the algorithm are the following. The first input is the Choi matrix of the specific channel that one wants to test; it acts on the system-ancilla Hilbert space $H = H_A \otimes H_B \otimes H_{A'} \otimes H_{B'}$, and its coordinates (in a basis depending on the partition chosen) provide a tms $y_\alpha$. The second input is the set of polynomials $g_j$ defining the compact $K$ via polynomial inequalities [as in Eqs. (13) and (14)], which allows one to define the localizing matrices. Keeping the second input fixed, we can change the Choi matrix by swapping Hilbert spaces so as to explore different separability problems (SEP, EB, or FS) as defined in Sec. II B. The SDP algorithm minimizes a linear function of the moments $y_\alpha$ under the constraints that the moment matrix and the localizing matrices are positive semidefinite.

Let $W$ be a matrix as in (13) and (14). It depends on the set of variables associated with each Hilbert space, for instance, the variables $c_\lambda$ in Eq. (13). To derive an explicit expression for the $g_j$, we express the coefficients of the characteristic polynomial $p(z) = \sum_{k=0}^n (-1)^{n-k} a_k z^k$ of $W$ through the recursive Faddeev-LeVerrier algorithm, i.e., for $1 \leqslant m \leqslant n$,

$$a_{n-m} = -\frac{1}{m}\sum_{k=1}^{m} (-1)^k a_{n-m+k} \mathrm{tr}(W^k), \tag{15}$$

with $a_n = 1$ and $a_0 = \det(W)$. From Descartes sign rule, positivity of $W$ is equivalent to having $a_k \geqslant 0$ for all $k$'s. Let us consider, for example, the case of two-qubit channels for which $i, j$ go from 0 to 1 in Eq. (6) and $C_\Phi$ is a $16 \times 16$ matrix and look for its separability as a tensor product of two $4 \times 4$ matrices. The characteristic polynomial for each factor is then of degree 4 [$n = 4$ in Eq. (15)], and the inequalities for positivity are given by Newton's identities (also known as Girard-Newton formulas). Besides $a_4 = 1$ and $a_3 = \mathrm{tr}\ W = 1$ (since $W$ is a density matrix), we get the conditions,

$$\begin{aligned} a_2 &= \tfrac{1}{2}(1 - \mathrm{tr} W^2) \geqslant 0, \\ a_1 &= \tfrac{1}{6}(2\,\mathrm{tr}\,W^3 - 3\,\mathrm{tr}\,W^2 + 1) \geqslant 0, \\ a_0 &= \tfrac{1}{24}(-6\,\mathrm{tr}\,W^4 + 8\,\mathrm{tr}\,W^3 + 3(\mathrm{tr}\,W^2)^2 - 6\,\mathrm{tr}\,W^2 + 1) \geqslant 0, \end{aligned} \tag{16}$$

which yield polynomial inequalities on the $c_\lambda$.

The tms $y_\alpha$ associated with $C_\Phi$ is obtained from its coordinates in a certain basis. In the case of states (see Sec. III B), specifying the coordinates of the density matrix was equivalent to fixing some moments of the measure $d\mu(x)$ as being the expectation values of some physical observables, given by $\mathrm{tr}(\rho S_{\mu_1}^{(1)} \otimes \cdots \otimes S_{\mu_p}^{(p)})$. In the case of channels instead, the observables are relative to the enlarged space system ancilla, so in order to perform physical measurements on the system only one needs to express the values $\mathrm{tr}(C_\Phi S_{\mu_A}^{(A)} \otimes S_{\mu_B}^{(B)} \otimes S_{\mu_{A'}}^{(A')} \otimes S_{\mu_{B'}}^{(B')})$ in terms of the entries of the superoperator $M$ specifying the channel as $\rho'_{ij} = M_{ij,kl}\rho_{kl}$. This gives a direct relation with the input-output representation, i.e., the quantum channel $\Phi$ is seen as a dynamical process: If $\rho$ is the initial (input) state before the process, then $\Phi(\rho)$ is the final (output) state after the process occurs. We can go from one representation to the other considering that $M$ and $C_\Phi$ are related by the reshuffling operation in the computational basis; for a generic basis this will, in general, result in a linear combination of physical measurements on the system. The number of physical measurements needed to fix one entry of the moment matrix relative to $C_\Phi$ can be used, for instance, as a cost function to decide between efficiency of entanglement detection and experimental convenience. The system-ancilla approach is what is used in the so-called ancilla-assisted process tomography (see, e.g., Ref. [43]), whereas the input-output one is the standard quantum process tomography (see, e.g., Ref. [44]).

The SDP algorithm then consists of minimizing a function $\sum_\alpha R_\alpha y_\alpha$ with $R_\alpha$ an arbitrary polynomial under the constraint that $M_t(y)$ and the localizing matrices $M_t(g_j \star y)$ are positive semidefinite and look for an extension such that the flatness condition is fulfilled. The algorithm is implemented using GLOPTIPOLY [45] and the MOSEK optimization toolbox [46]. Note that if the rank condition is not met the SDP can still yield a solution to the minimization problem [47], but it does not tell us anything *a priori* on the representing measure problem. To describe all the ingredients in the algorithm, to study its complexity and its efficiency, we will apply it in the next section to different examples: the spin-1 channels mentioned already above, and specific two-qubit channels, which are relevant in many experimental settings.

## IV. EXAMPLES

In the general case, the number of moments involved, and, thus, the size of the moment matrices, scales very fast with the extension order $t$ so that numerically the SDP soon becomes intractable. More specifically, whereas full separability of two-qubit channels is a problem that is still tractable numerically, already the SEP and EB cases turn out to be too complex if we consider arbitrary qubit channels. Indeed, in that case the variables involved are $(x_\mu)_{1\leqslant\mu\leqslant 15}$ for the system and $(x'_\mu)_{1\leqslant\mu\leqslant 15}$ for the ancilla. The number of decision variables in the SDP is the number of free entries of the extension of the moment matrix we are looking for; in the order-$t$ extension $M_t(y)$, it is the number of monomials from 30 variables up to degree $2t$, given by $\binom{30+2t}{2t}$ [see Eq. (8)]. Moreover, the polynomials defining the compact $K$ for a two-qubit Hilbert space (of dimension 4) are the ones given in Eq. (17), that is, their degree is 4, and, thus, $d_0 = 2$. Since the smallest moment matrix containing all given moments is $M_1(y)$, the smallest extension we have to consider in Theorem 4 is $M_3(y)$. The size of this matrix is $\binom{33}{3} = 5456$, and the number of decision variables is $\binom{36}{6} \geqslant 10^6$. Therefore, the size of the SDP grows very quickly, and, thus, the number of semidefinite constraints requires too much time and memory.

Nevertheless, the algorithm can still be applied to families of channels for which the number of variables involved is smaller than in the general case. In the following we present different examples of such families. We highlight their complexities and computational cost, and explain in more detail the role of the different factors mentioned above. We finally outline some numerical results on their entangling or separable properties.

### A. Fully symmetric Choi matrix

We start with a simple example which allows us to highlight the connection between the TMS algorithm for channels and for states. We consider quantum channels $\Phi$ such that the Choi matrix $C_\Phi$ has components only on the symmetric subspace. In other words, we impose that the four-qubit state associated with the two-qubit channel $\Phi$ via the Choi-Jamiołkowski isomorphism be fully symmetric under permutation of the qubits (in the sense that it is a mixture of fully symmetric pure states). In that case, the Choi matrix only has components on the subspace spanned by Dicke states $|D_j^{(m)}\rangle$, which are the symmetrized tensor products of $2j$ qubits with $j = 2$ (four qubits) and $-j \leqslant m \leqslant j$. This means that

$$(\mathbb{1} - P)C_\Phi(\mathbb{1} - P) = (\mathbb{1} - P)C_\Phi P = PC_\Phi(\mathbb{1} - P) = 0, \tag{17}$$

where $P = \sum_{m=-2}^{2} |D_4^{(m)}\rangle\langle D_4^{(m)}|$ is the projection operator onto the symmetric subspace. The constraints in Eq. (17) fix conditions on the superoperator $M$ of which $C_\Phi$ is a reshuffling. For $j = 2$, only $(2j + 1)^2$ real independent parameters remain.

Such a restriction has a clear physical interpretation in the case of one-qubit channels. Indeed, the Choi matrix of a nonunital one-qubit channel can be put in the form

$$\frac{1}{2}\begin{pmatrix} 1+\lambda_3+t_3 & 0 & t_1+it_2 & \lambda_1+\lambda_2 \\ 0 & 1-\lambda_3+t_3 & \lambda_1-\lambda_2 & t_1+it_2 \\ t_1-it_2 & \lambda_1-\lambda_2 & 1-\lambda_3-t_3 & 0 \\ \lambda_1+\lambda_2 & t_1-it_2 & 0 & 1+\lambda_3-t_3 \end{pmatrix}, \tag{18}$$

in the canonical basis [27]. Imposing that the matrix is associated with a symmetric state is equivalent to imposing that it has no component over the singlet state; this leads to the conditions $t_1 = t_2 = t_3 = 0$ (i.e., the channel is unital) and $\lambda_1 - \lambda_2 + \lambda_3 = 1$, which correspond to a face of the tetrahedron of admissible values of the $\lambda_i$ corresponding to unital channels, given by the Fujiwara-Algoet conditions $1 \pm \lambda_3 \geqslant |\lambda_1 \pm \lambda_2|$ [48]. Such points on a face of the tetrahedron correspond to channels whose Kraus rank is 3, which are characterized by the fact that they are the only indivisible channels (that is, they cannot be written as the composition of two nonunitary channels) [49,50].

In the two-qubit channel case there is no such clear geometrical picture of the fully symmetric Choi matrix. However, since the Choi state is a fully symmetric state of $N = 4$ qubits, if it is separable with respect to an arbitrary partition, then it is fully separable, and it can be written as a convex sum of $N$ projectors on pure symmetric states (see, e.g., Ref. [51]). This means that in this case we only need to consider the fully separable case, which coincides with exploring the case of spin-2 states (since those states can be seen as symmetric states of four qubits). The tms algorithm for states was exploited in Ref. [21] to investigate multipartite entanglement of such states. The problem can be formulated as in Eq. (7) with a tms of degree 4 [thus, the smallest moment matrix to consider in Theorem 4 is $M_2(y)$] and a vector of variables $(x_1, x_2, x_3)$ (as explained in Sec. III B since the state is fully symmetric we only need the three variables associated with a single qubit). The semialgebraic set $K$ is the Bloch sphere so that $d_0 = 1$. Thus, the first flatness condition in Theorem 4 reads $\mathrm{rk}\, M_3(y) = \mathrm{rk}\, M_2(y)$ with $M_2(y)$ and $M_3(y)$ of sizes $10 \times 10$ and $20 \times 20$, respectively. The algorithm usually stops at the first extension, and it takes at about 1 s to give a certificate of separability or entanglement of the channel (the time here reported refers to running the algorithm on a standard computer with a 64-bit Windows operating system, 4-GB RAM and Intel Core i7 CPU 2.00–2.60 GHz). We refer to the results obtained for states in Refs. [21,30] for more detail on the implementation in that case.

### B. Two-qubit planar channels

We now consider the case where the two-qubit channel is a linear combination of tensor products of single-qubit planar channels. Such one-qubit channels $\phi_{\mathrm{pl}}$ send the (three-dimensional) Bloch ball into a (two-dimensional) ellipse. Note that, according to the so-called "no-pancake theorem" a planar channel cannot map the Bloch ball to a disk touching the sphere unless it reduces to a point or a line (see Refs.[49,52] ).

Any one-qubit channel can be described by a $4 \times 4$ matrix of the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_2 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}, \tag{19}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i \geqslant 0$ is the distortion vector and $\boldsymbol{t} = (t_1, t_2, t_3)$ is the translation vector. Geometrically, the channel maps the Bloch vector $\boldsymbol{r}$ to $M\boldsymbol{r} + \boldsymbol{t}$, that is, the sphere becomes an ellipsoid whose half-axes are given by the $\lambda_i$ and centered at $\boldsymbol{t}$.

Planar channels are those where one of the $\lambda_i$ is zero. Geometrically, this means that they map the Bloch ball to a disk. In Ref. [53] this type of channel was investigated, but with focus on their entanglement-annihilating properties. In what follows, we consider planar channels $\phi_{\mathrm{pl}}$ with $\lambda_2 = 0$. We investigate whether linear combinations, such as

$$\Phi = a\phi_{\mathrm{pl}}^{(1)} \otimes \phi_{\mathrm{pl}}^{(1)} + b\phi_{\mathrm{pl}}^{(2)} \otimes \phi_{\mathrm{pl}}^{(2)}, \tag{20}$$

with $a, b \in \mathbb{R}$ result in separable channels. We consider the case in which both $\phi_{\mathrm{pl}}^{(1)}$ and $\phi_{\mathrm{pl}}^{(2)}$ are unital, one unital, the other nonunital, and both nonunital. Note that states (20) are not symmetric states, in general, as they are symmetrizations of mixed states but not mixtures of symmetric pure states. The condition of complete positivity in the case of a unital planar channel ($\boldsymbol{t} = 0$) is given by $|\lambda_1| \leqslant 1 - |\lambda_3|$ with $|\lambda_1|$, $|\lambda_3|$ the half-axes of the ellipse. In the case of nonunital channels the conditions for complete positivity can be found in Theorem IV.1 of Ref. [49]. Here for simplicity we consider the case where $\lambda_2 = 0$ and $\boldsymbol{t} = (0, 0, t_3)$. In such a case these conditions simplify to

$$\begin{aligned} 1 + \lambda_1 + \lambda_3 \geqslant 0, \quad 1 + \lambda_1 - \lambda_3 \geqslant 0 \\ 1 - \lambda_1 - \lambda_3 \geqslant 0, \quad 1 - \lambda_1 + \lambda_3 \geqslant 0, \\ t_3^2 \leqslant 1 - \lambda_1^2 + \lambda_3^2 - 2|\lambda_3|. \end{aligned} \tag{21}$$

The Choi matrix $C_\Phi$ is then properly normalized ($b = \frac{1}{16} - a$) in order to obtain a valid quantum state with trace 1, giving the Choi state on which we apply our algorithm. The basis over which $C_\Phi$ is expanded is chosen as the tensor product $\sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \sigma_{\mu_3} \otimes \sigma_{\mu_4}$ with $0 \leqslant \mu_i \leqslant 2$ and $\{\sigma_{\mu_i}\} = \{\mathbb{1}, \sigma_x, \sigma_z\}$, $\sigma_x, \sigma_z$ being the usual Pauli matrices (this is also reasonable from the experimental point of view since Pauli physical measurements are often used for multiqubit channels). The Choi states associated with states (20) turn out to be equal to their partial transpose with respect to any qubit. Invariance under partial transposition with respect to the first qubit in $2 \times N$ systems was shown in Ref. [54] to entail separability. Therefore, the four-qubit Choi state is separable across any bipartition into sets of one and three qubits.

Separability for the bipartitions into two sets of two qubits, required from the definition of EB and SEP channels, corresponds to the situation of Theorem 4 and can be explored with our algorithm as follows. In contrast to the symmetric case addressed in Subsec. IV A, there are now different variables $x_i$ in Eq. (12) for the system $A$ and the ancilla $A'$ (and equivalently for $B$ and $B'$)

Let us first consider the question of full separability. In that case, since each system qubit and ancilla qubit, respectively, is described by two variables $(x_\mu^A)_{1\leqslant\mu\leqslant2}$, $(x_\mu^B)_{1\leqslant\mu\leqslant2}$ and $(x_\mu^{A'})_{1\leqslant\mu\leqslant2}$, $(x_\mu^{B'})_{1\leqslant\mu\leqslant2}$, the vector of variables has length 8. The moments $y_\alpha$ are given by entries of the Choi matrix, the tms has degree 4, so that formula (8) applies with $n = 8$ and $2d = 4$. The semialgebraic set is given by the choice of basis matrices for the Choi matrix. Since we expanded it over Pauli matrices, the constraint for each set of variable is the one for qubits, i.e., the vector of variables is restricted to the Bloch ball. The compact $K$ is, therefore, the product of four unit disks.

Since all polynomials defining $K$ are of degree 2, we have $d_0 = 1$, and, thus, the first rank condition reads $\mathrm{rk}\, M_3(y) = \mathrm{rk}\, M_2(y)$ where the moment matrices have size $\binom{n+t}{t}$, i.e., respectively 165 and 45. A first hint on the computational complexity of the SDPs we need to solve is given by the number of decision variables of the optimization, which in our case corresponds to the number of monomials from eight variables up to degree 6, the latter being the degree of the extension of the tms needed to construct $M_3(y)$. Moreover, SDP are usually solved with the interior point method; each iteration in the primal-dual interior point algorithm requires the solution of a linear system, which is the most expensive operation with $O(N^3)$ complexity, solvable using Gaussian elimination. Here $N$ is the number of linear constraints in the SDP, and efficiency drops with the growing number of semidefinite terms involved in these linear constraints, which in the case here considered are $\sim 10^3$. This, in general, has a big impact on the time and memory requested for a single run of the algorithm [46]. Nevertheless, we could run our algorithm in that case, which allowed us to test for separability of channels of the form (20). The algorithm still performs very well (on a machine with same characteristics as described above in Sec. IV A); for all the examples tested a certificate of separability was found either at the first relaxation order $\mathrm{rk}\, M_3 = \mathrm{rk}\, M_2$ (with a time of ~10$s$ for a single run) or at the second relaxation order $\mathrm{rk}\, M_4 = \mathrm{rk}\, M_3$ (with a running time of ~6 min).

We tested $\sim 10^3$ cases, which were chosen uniformly at random in the range of parameters ($\lambda_1^{(1)}$, $\lambda_3^{(1)}$, $\lambda_1^{(2)}$, $\lambda_3^{(2)}$, $t_3^{(1)}$, $t_3^{(2)}$, and $a$) allowed by the complete-positivity conditions of the quantum channels considered [see Eq. (22) and above it]. All the Choi states tested result fully separable for all the three cases listed above (where channels $\phi_{\mathrm{pl}_i}$ can be unital or not); as a consequence, all these states are both EB and SEP. Based on the available numerical evidence we conjecture that all states of the form (20) are fully separable.

### C. Qutrit channels

We now study the case of qutrit channels. More specifically, we apply our algorithm to a family of channels presented in Ref. [55] where EB properties of qutrit gates were studied through the negativity $N(\rho) = \frac{1}{2}(\|\rho^{T_H}\|_1 - 1)$ with $\|\rho^{T_H}\|_1$ the trace norm of the partial transpose with respect to the system qutrit. The negativity $N(\rho)$ cannot detect PPT-entangled states; in other words there exist entangled states with $N(\rho) = 0$. For such states, our algorithm is able to give a certificate of separability as we illustrate below. Note that, even though in this case the system is not bipartite, the definition of entanglement breaking still applies since it involves the presence of an ancilla, as explored for one-qubit channels in Ref. [52]; on the other hand, the definition of SEP separability cannot be applied to this example.

As a basis for qutrit density operators, we use Gell-Mann matrices $\{\lambda_i\}_{i=1}^8$ together with $\lambda_0 = \sqrt{\frac{2}{3}}\mathbb{1}$. In this basis, an arbitrary qutrit density matrix can be written as

$$\rho = \frac{1}{3}\left(\mathbb{1} + \sum_{i=1}^{8} \zeta_i \lambda_i\right), \tag{22}$$

with $\zeta_i = \frac{3}{2}\mathrm{tr}(\rho\lambda_i)$.

The channel we consider is a damping qutrit channel, i.e., a channel that can be written as an affine transformation on the generalized (qutrit) Bloch vector as $\Phi_D\colon \boldsymbol{\zeta} \to \boldsymbol{\zeta}' = \Lambda\boldsymbol{\zeta}$, where $\Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_8)$ is the damping matrix. The $\Lambda_i$ cannot take any arbitrary value because $\Phi_D$ has to be completely positive, thus, leading to the constraints $|\Lambda_i| \leqslant 1$. More specifically, we consider the family of damping channels given in Ref. [55] and parametrized by $\Lambda_{i\neq 3,8} = x$, $\Lambda_{i=3} = y$, $\Lambda_{i=8} = y^2$. The Choi state corresponding to $\Phi_D$ can be written by transforming the propagator to the canonical basis, then reshuffling and normalizing (it corresponds to a maximally mixed state for $x = y = 0$ and to a maximally entangled state of two qutrits for $x = y = 1$). The region of parameters for which $C_{\Phi_D}$ is positive semidefinite together with the values of the corresponding negativity is shown in Fig. 2.

Any two-qutrit state can be expanded over the basis formed by tensor products of Gell-Mann matrices [56]. This setting is analogous to the one described in Sec. III B for two spin-1 states. The vector of variables is $x = (x_1, x_2, \dots, x_{16})$, where $x_1, \dots, x_8$ are the coordinates $\alpha_i$ associated with the system qutrit, and $x_9, \dots, x_{16}$ are associated with the ancilla qutrit. Since there are two subsystems, and the tms has degree 2. The characteristic polynomial for a qutrit density matrix has

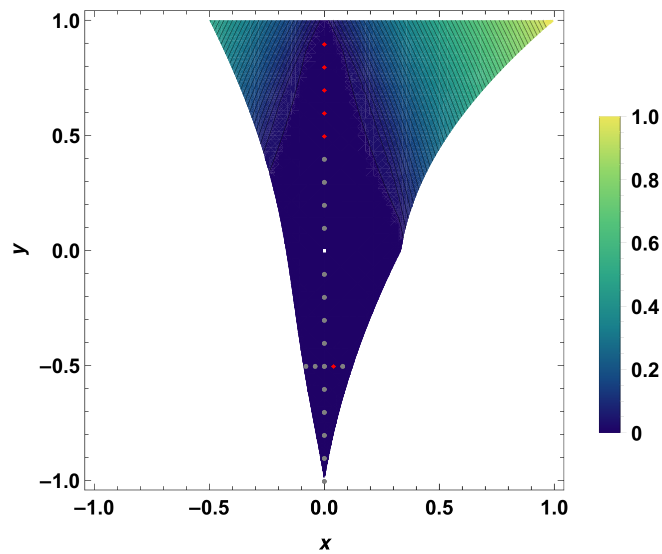


FIG. 2. Region of $x$ and $y$ parameters for which $C_{\Phi_D}$ of the damping qutrit channel is positive semidefinite; the color function corresponds to the negativity values in the range of [0,1] with steps for the contour lines of 0.02. The central plateau corresponds to the region of zero negativity where the PPT criterion remains inconclusive. Gray points correspond to states found separable by our algorithm, signifying entanglement breaking channels; red points correspond to states where the algorithm needs to go to a higher extension order and remains inconclusive with our numerical resources. The white point marks the maximally mixed state.

degree 3, therefore, the semialgebraic set is given by the conditions $\mathrm{tr}\,\rho^2 \leqslant 1$ and $\det\,\rho \geqslant 0$ with $\rho$ as the density operator in Eq. (22). It follows that the corresponding polynomials of the variable $x_i$ have maximal degree 3, and, thus, $d_0 = 2$. This gives the rank shift in Theorem 4: At the first iteration of the algorithm the flatness condition reads $\mathrm{rk}\,M_3(y) = \mathrm{rk}\,M_1(y)$. These moment matrices have size 969 and 17, respectively. The number of decision variables in the SDP corresponds to the number of monomials from 16 variables up to degree 6 ($\sim 7 \times 10^4$) and the number of semidefinite constraints is given by $\binom{n+t}{t} + m\binom{n+t-1}{t-1} + m\binom{n+t-2}{t-2}$, that is, the size of the moment matrix of the first extension ($t = 3$) and the size of the localizing matrices multiplied by the number $m$ of inequalities in the semialgebraic set for each set of variables.

The tms algorithm can be exploited to investigate, in particular, the Choi states with zero negativity for which the PPT criterion alone is inconclusive. The results for some pairs of parameters with ($x = 0$, $y \in [-1, 1]$) and ($x \in [-\frac{2}{25}, \frac{2}{25}]$, $y = -\frac{1}{2}$) are explored and they are shown in Fig. 2. The points highlighted in gray are the points tested with the algorithm which give a certificate of separability, including the white point which corresponds to a Choi state equal to the maximally mixed state of two qutrits. In the latter cases the SDP is feasible and the flatness condition $\mathrm{rk}\,M_3(y) = \mathrm{rk}\,M_1(y)$ is satisfied, meaning that the corresponding $\Phi_D$'s are EB; on the other hand, the algorithm remains inconclusive for the red points at the first iteration, leading to the necessity for higher-order extensions, which are beyond our computational resources. We did not detect PPT entangled states among the tests performed; the algorithm confirms entanglement for negativity greater than zero for all the states tested. A single run of the algorithm in this case takes about $5h$ and between 150 and 300 GB of RAM.

## V. CONCLUSIONS

In this paper we have discussed an algorithm that deterministically detects whether a quantum channel is separable or not, or whether it is entanglement breaking or not. Such an algorithm finds its motivation in important questions relative to modern quantum technology as the verification of devices which should work in a properly quantum way, often leading to the necessity of detecting whether a quantum channel is able to generate entanglement or not also over a certain time (as explained in the Introduction). We were able to explore in a unifying framework three classes of separability across different cuts between systems and ancillae (SEP, EB, or FS); indeed, with only a small modification in the input we can switch between these different classes. This algorithm is the numerical counterpart of a theorem that provides a necessary and sufficient separability criterion based on a mapping between coordinates of the Choi matrix of the channel, expressed in a given basis and a truncated moment sequence. Low-order moments are fixed by measurements performed on the channel, and the separability problem is equivalent to finding whether these moments are those of a measure supported on a certain compact set.

In the case of fully symmetric Choi matrices for qubit channels where the aim is to find a decomposition over the Bloch sphere, the number of variables in the tms is $n = 3$ so that the size of a moment matrix of order $t$ is $\binom{n+t}{t} \sim t^3/6$. On the other hand, in the simplest case of detection of EB or SEP in a generic two-qubit channel, there are $n = 30$ variables involved, and, thus, the size of the moment matrix is $\binom{n+t}{t} = 5456$ for $t = 3$. Moreover, the number of independent entries in $M_t(y)$ is given by $\binom{n+2t}{2t} \sim 2 \times 10^6$ for $t = 3$. Nevertheless, we can consider families of channels for which the number of free parameters in each subsystem is smaller than in the general case. Then, the number of variables involved in the mapping to tms is reduced and the matrices in the SDP become amenable to numerical investigation. As we showed here, this is the case for planar channels (where one dimension is suppressed) or qutrit channels (which live in the symmetric space of two qubits). Our algorithm is then able to decide whether the channel is EB or SEP. For instance, in the case of qutrit channels we were able to provide a certificate of separability in cases where the negativity of the Choi matrix vanishes and, thus, is unable to yield a conclusion. Since calculations are costly, this approach could be used as a numerical tool to explore possible conjectures or produce counterexamples.

## APPENDIX A: SKETCH OF THE PROOF OF THEOREM 3

Suppose $\operatorname{rk} M_t(y) = r$ with $M_t(y) \geqslant 0$ and there exists a flat extension $M_{t+d_0}(y)$ with $M_t(g_j \star y) \geqslant 0$ for $1 \leqslant j \leqslant m$. Then $M_{t+1}(y)$ is also a flat extension of $M_t(y)$, and we then know from Theorem 2 that $y$ admits a (unique) $r$-atomic representing measure supported by $x_k \in \mathcal{V}[\ker M_t(y)]$. All what remains to show is that positivity of the localizing matrices enforces that the $x_j$ belong to $K$, that is, $g_j(x_k) \geqslant 0$ for $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant r$.

This can be performed as follows. First, observe that since $M_t(y)$ is of rank $r$, one can find a nonsingular $r \times r$ principal submatrix of $M_t(y)$. If $\mathcal{B}$ is the set of labels $\alpha$ of the rows of that matrix, then the image of $M_t(y)$ is spanned by the $x^\alpha,\ \alpha \in \mathcal{B}$, and by definition these $x^\alpha$ are on the order less than or equal to $t$. Since the whole vector space of polynomials can be decomposed as a direct sum of the image and the kernel of $M_t(y)$, an arbitrary polynomial $p$ can be decomposed as $p = q + \tilde{p}$ with $q = \sum_{\alpha\in\mathcal{B}} q_\alpha x^\alpha \in \operatorname{Im} M_t(y)$ and $\tilde{p} \in \ker M_t(y)$.

Now let $p_k$ be interpolating polynomials of the $x_{k'}$, which are the atoms supporting the representing measure of $y$. That is, $p_k(x_{k'}) = \delta_{kk'}$ for $1 \leqslant k,\ k' \leqslant r$. One can decompose them as above as $p_k = q_k + \tilde{p}_k$ with $\tilde{p}_k \in \ker M_t(y)$ and $q_k$ of degree less than $t$. By definition, the $x_{k'}$ are roots of all polynomials in $\ker M_t(y)$, and, thus, one has $\tilde{p}_k(x_{k'}) = 0$, which implies $q_k(x_{k'}) = \delta_{kk'}$ for $1 \leqslant k,\ k' \leqslant r$.

Now, for $y = \int x^\alpha d\mu(x)$ and for arbitrary polynomials represented by vectors $p, q \in \mathbb{R}^{S_t}$,

$$\begin{aligned} q^T M_t(y)p &= q_\alpha M_{\alpha\beta} p_\beta \\ &= q_\alpha y_{\alpha+\beta} p_\beta \\ &= \int q_\alpha x^{\alpha+\beta} p_\beta d\mu(x) \\ &= \int p(x)q(x)d\mu(x) \end{aligned} \tag{A1}$$

(with Einstein summation convention) and

$$\begin{aligned} q^T M_t(g * y)p &= q_\alpha g_\gamma y_{\alpha+\beta+\gamma} p_\beta \\ &= \int q_\alpha g_\gamma p_\beta x^{\alpha+\beta+\gamma} d\mu(x) \\ &= \int p(x)q(x)g(x)d\mu(x). \end{aligned} \tag{A2}$$

Thus, $M_t(g_j \star y) \geqslant 0$ and $d\mu(x) = \sum_i \omega_i \delta(x - x_i)dx$ entail $\forall\, k, j$,

$$\begin{aligned} 0 &\leqslant q_k^T M_t(g_j \star y) q_k \\ &= \int q_k(x)^2 g_j(x) d\mu(x) \\ &= \sum_{i=1}^r \omega_i \int dx\, q_k(x)^2 g_j(x)\delta(x - x_i) \\ &= \sum_{i=1}^r \omega_i q_k(x_i)^2 g_j(x_i) \\ &= \omega_k g_j(x_k), \end{aligned} \tag{A3}$$

since $q_k(x_i) = \delta_{ki}$. As all $\omega_k > 0$ this implies that $g_j(x_k) \geqslant 0$ and, thus, $x_k \in K$, which completes the proof.

## APPENDIX B: RANK PROPERTY OF EXTENSIONS

Let us show that the rank condition $\operatorname{rk} M_{t'}(y) = \operatorname{rk} M_t(y)$ implies the fact that positivity of $M_t(y)$ and $M_{t'}(y)$ are equivalent.

Since $M_t(y)$ is a principal submatrix of $M_{t'}(y)$ one direction is obvious. To show the converse, suppose

$M_t(y) \geqslant 0$ and $\mathrm{rk}\, M_t(y) = r = \mathrm{rk}\, M_{t'}(y)$. Then, as in Appendix A, there exists a nonsingular $r \times r$ principal submatrix of $M_t(y)$ indexed by labels $\alpha \in \mathcal{B}$ with $|\alpha| \leqslant t$. This $r \times r$ submatrix is also a nonsingular principal submatrix of $M_{t'}(y)$. Since $M_{t'}(y)$ has rank $r$, the corresponding $r$ monomials $x^\alpha$ are, therefore, a basis of $\mathrm{Im}\, M_{t'}(y)$. Since the submatrix is positive because $M_t(y)$ is, then so is $M_{t'}(y)$.


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