Device-independent certification of the Hilbert-space dimension using a family of Bell expressions

A. K. Pan[*] and Shyam Sundar Mahato
National Institute of Technology Patna, Ashok Rajpath, Patna, Bihar 800005, India

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A dimension witness provides a device-independent certification of the minimal dimension required to reproduce the observed data without imposing assumptions on the functioning of the devices used to generate the experimental statistics. In this paper, we provide a family of Bell expressions where Alice and Bob perform 2^{n-1} and n number of dichotomic measurements, respectively, which serve as the device-independent dimension witnesses of Hilbert space of 2^m dimensions with $m = 1, 2, \ldots, 2^{\lfloor n/2 \rfloor}$. The family of Bell expressions considered here determines the success probability of a communication game known as the n-bit parity-oblivious random access code. The parity obliviousness constraint is equivalent to the preparation noncontextuality assumption in an ontological model of an operational theory. For any given $n \geqslant 3$, if such a constraint is imposed on the encoding scheme of the random access code, then the local bound of the Bell expression reduces to the preparation noncontextual bound. We provide explicit examples for the $n = 4$ and 5 case to demonstrate that the relevant Bell expressions certify the qubit and two-qubit system, and for the $n = 6$ case to demonstrate that the relevant Bell expression certifies the qubit, two-qubit, and three-qubit systems. We further demonstrate the sharing of quantum preparation contextuality by multiple Bobs sequentially to examine whether the number of Bobs sharing the preparation contextuality is dependent on the dimension of the system. We provide explicit examples of $n = 5$ and 6 to demonstrate that the number of Bobs sequentially sharing the contextuality remains the same for any of the 2^m-dimensional systems.

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I. INTRODUCTION

Dimensionality is a fundamental property of a quantum system. The Hilbert space in which the quantum state belongs is an abstract construction but the number of dimensions available to a system is a physical quantity and is considered to be a resource for quantum computation and quantum information theory [1]. A higher-dimensional system can make a given protocol more efficient and, alternatively, the security of many cryptographic protocols relies on the dimensional characteristics of the system. For example, instead of a qubit system, if four-dimensional states are used then the celebrated Bennett-Brassard (BB84) cryptographic protocol [2] can be shown to be entirely compromised [3,4]. From the fundamental perspective, there are quantum correlations the simulations of which by classical resources inevitably require dimensional superiority. The quantum dimension witness is a criterion that provides a lower bound on the dimension that is needed to reproduce a given measurement statistics. Of late, the device-independent dimension witness has become an important research area where the dimension of a quantum system is certified without requiring *a priori* knowledge about the devices used in the experiment.

The notion of the dimension witness was first introduced in a seminal paper by Brunner *et al.* [5] in the context of the bipartite Bell scenario, which involves two spatially separated observers Alice and Bob, who access uncharacterized devices (black boxes). Alice and Bob receive inputs $x \in \{1, 2, \ldots, n_A\}$ and $y \in \{1, 2, \ldots, n_B\}$, respectively, and the uncharacterized measurement device yielding respective outputs $a \in \{0, 1\}$ and $b \in \{0, 1\}$. The conditional probability $P(ab|xy)$ admits a d-dimensional representation if it can be written as $P(ab|xy) = \mathrm{tr}[\rho_{AB}(M_a^x \otimes M_b^y)]$ for the state $\rho_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^d$ shared between two parties and the local measurements M_a^x and M_b^y acting on $\mathbb{C}^d$. The reproduction of every joint probability $P(ab|xy)$ in quantum theory puts a lower bound on the dimension of the Hilbert space.

Since then a flurry of interesting works along this direction has been reported [6–24]. The work of Brunner *et al.* [5] was further generalized and extended to the prepare and measure scenario by Gallego *et al.* [7], who proposed a family of inequalities which serve as classical and quantum dimension witnesses given by

$$\mathbf{I}_N = \sum_{y=1}^{N-1} E_{1y} + \sum_{x=2}^{N} \sum_{y=1}^{N+1-x} \alpha_{xy} E_{xy} \tag{1}$$

where $\alpha_{xy} = 1$ if $x + y \leqslant N$, and $\alpha_{xy} = -1$ otherwise. Here, $x \in \{1, 2, \ldots, N\}$, $y \in \{1, 2, \ldots, N-1\}$, and E_{xy} is the correlation. The problem of the dimension witness is meaningful if the number of preparations (N) is greater than the Hilbert-space dimension of the system. For classical states of dimension $d \leqslant N$ it is found that algebraic bound $I_N \leqslant L_d$, where $L_d = \frac{N(N-3)}{2} + 2d - 1$. For example, for $N = 3$ and $d = 2$ one finds the classical value is 3 and the quantum value is $2\sqrt{2} + 1$. Further analysis found that the I_3 inequity achieves its optimal value for $d = 3$, which is 5. So this

[*]akp@nitp.ac.in

inequality has the ability to test the dimension as well as to distinguish between the classical and quantum system.

Later, by assuming independence of the prepare and measure devices a nonlinear dimension witness is also proposed [11] and a test of dimension in a communication network is also proposed [12]. A connection to the random access code (RAC) [6] and to state discrimination [9] are also pointed out. In a recent proposal, by employing binary-outcome measurements a certification of an arbitrary-dimensional quantum system is proposed [20]. Experimental verifications of dimension witnesses including higher-dimensional systems have also been performed [25–32].

In this paper we provide a family of quantum dimension witnesses based on the parity-oblivious random access code (PORAC) [33]. The parity-oblivious condition imposed on Alice's encoding scheme implies here that no parity information of the inputs of Alice is shared to Bob. It can be shown [34] that the success probability of a n-bit random access code can be solely determined by a family of Bell expressions ($\mathcal{B}_n$) where Alice and Bob use 2^{n-1} and n number of dichotomic measurements, respectively. Importantly, for a given n, the optimal quantum value of $\mathcal{B}_n$ can only be achieved for the quantum system having local Hilbert-space dimension $d = 2^{\lfloor n/2 \rfloor}$. The parity-oblivious constraint is shown [36] to be equivalent to the preparation noncontextuality assumption in an ontological model, and for a given n such a constraint on the encoding scheme reduces the local bound of the family of Bell expressions $\mathcal{B}_n$ to the preparation noncontextual bound [36]. This is due to the fact that such a condition puts further restriction on free choices of the values of Alice's observables.

For $n = 2$ and 3, the Bell expressions are well-known Clauser-Horne-Shimony-Holt (CHSH) [37] and elegant Bell expressions [38], respectively. Since both the Bell expressions for $n = 2$ and 3 can be optimized for the qubit system, they cannot serve as dimension witnesses of Hilbert space. However, each of the Bell expressions for $n \geqslant 4$ has the potential to distinguish the dimensions $d = 2^m$ of the Hilbert space with $m = 1, 2, \ldots, 2^{\lfloor n/2 \rfloor}$, thereby serving as dimension witnesses of the Hilbert space. We provide explicit examples for $n = 4, 5$, and 6 cases to demonstrate that the Bell expressions for both $n = 4$ and 5 certify qubit and two-qubit systems and for $n = 6$ the relevant Bell expression certifies the qubit, two-qubit, and three-qubit local systems.

Further, we examine the sharing of preparation contextuality by multiple sequential Bobs performing unsharp measurements. Using the family of Bell expressions mentioned above, it was shown [39] that the sharing of preparation contextuality can be demonstrated for an arbitrary number of Bobs by using the optimal quantum value of the family of Bell expressions, achieved for the $d = 2^{\lfloor n/2 \rfloor}$-dimensional Hilbert space. We argue that there is a possibility of sharing preparation contextuality by an arbitrary number of Bobs for the system in lower-dimensional Hilbert space. We provide an explicit example for $n = 5$ where the number of sequential Bobs sharing preparation contextuality remains the same for qubit and two-qubit systems. Similarly, for $n = 6$ the sharing is for a possible same number of sequential Bobs for qubit, two-qubit, and three qubit systems. However, the value of the unsharpness parameter required for demonstrating preparation contextuality by sequential Bobs is always higher in a lower-dimensional system, as expected.

This paper is organized as follows. In Sec. II, we briefly recapitulate the essence of the parity-oblivious random access code and derivation of the family of preparation noncontextual inequalities, i.e., the Bell inequalities that serve as the dimension witnesses. In Sec. III, we provide the sum-of-square (SOS) approach to optimize the dimension witnesses for various dimensional quantum systems. We provide specific examples for $n = 4, 5$, and 6 to demonstrate how the corresponding Bell inequalities certify the 2^m-dimensional systems in Sec. IV. For the dimension witnesses for $n = 4, 5$, and 6, we examine the sharing of preparation contextuality by multiple Bobs for qubit, two-qubit, and three-qubit systems in Sec. V. Finally, we summarize our work in Sec. VI.

II. A FAMILY OF BELL EXPRESSIONS SERVING AS DIMENSION WITNESSES

Since the family of dimension witnesses is based on the PORAC and the parity-oblivious condition is equivalent to the preparation noncontextuality assumption in an ontological model of an operational theory, we first provide the essence of preparation noncontextuality and then provide the derivation of local and preparation noncontextual bounds of the aforementioned family of Bell expressions.

We start by encapsulating the notion of an ontological model reproducing the quantum statistics [36,40]. In quantum theory, a preparation procedure (P) produces a density matrix ρ and the measurement procedure (M), which is in general described by a suitable positive operator-valued measure (POVM) E_k, provides the probability of occurrence an outcome k is given by $p(k|P, M) = \mathrm{Tr}[\rho E_k]$, which is the Born rule. In an ontological model of quantum theory, the preparation of quantum state ρ by a specific preparation procedure P is equivalent to preparing a probability distribution $\mu_P(\lambda|\rho)$ in the ontic state space, satisfying $\int_\Lambda \mu_P(\lambda|\rho)d\lambda = 1$ where $\lambda \in \Lambda$ and Λ is the ontic state space. The probability of obtaining an outcome k is a response function $\xi_M(k|\lambda, E_k)$ satisfying $\sum_k \xi_M(k|\lambda, E_k) = 1$ where a measurement operator E_k is realized through a measurement procedure M. The primary requirement of such an ontological model is to reproduce the Born rule, i.e., $\forall\rho$, $\forall E_k$, and $\forall k$, $\int_\Lambda \mu_P(\lambda|\rho)\xi_M(k|\lambda, E_k)d\lambda = \mathrm{Tr}[\rho E_k]$.

The notion of noncontextuality was reformulated and generalized for any operational theory by Spekkens [36]. For our purpose, we focus on the quantum theory here. An ontological model of quantum theory can be considered to be preparation noncontextual if $\forall M : p(k|P, M) = p(k|P', M) \Rightarrow \mu_P(\lambda|\rho) = \mu_{P'}(\lambda|\rho)$ is satisfied where P and P' are two distinct preparation procedures but in the same equivalent class [41–44].

As mentioned, the family of dimension witnesses in the present paper is derived through a two-party communication game known as PORAC. It was shown by Spekkens *et al.* [33] that the parity-oblivious condition in an operational theory can equivalently be cast into the assumption of preparation noncontextuality in an ontological model. It is already shown in [34] that the success probability of a n-bit PORAC can be solely linked with a family of Bell's inequalities. For the

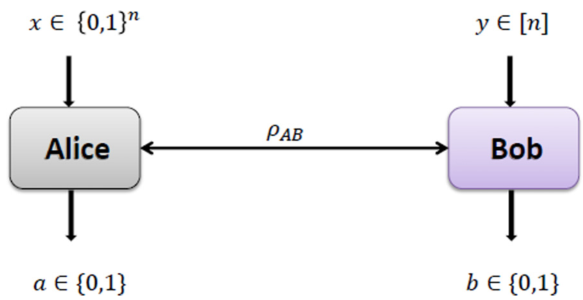

FIG. 1. Schematic diagram of bipartite Bell scenario.

sake of completeness, we briefly encapsulate the essence of the derivation.

In a n-bit PORAC, Alice chooses her bit x^δ randomly from $\{0,1\}^n$ with $\delta \in \{1, 2, \ldots, 2^n\}$. The relevant ordered set $\mathcal{D}_n$ can be written as $\mathcal{D}_n = (x^\delta | x^i \oplus x^j = 111, \ldots, 11 \text{ and } i + j = 2^n + 1)$ and $i \in \{1, 2, \ldots, 2^{n-1}\}$. Here, $x^1 = 00, \ldots, 00; x^2 = 00, \ldots, 01, \ldots$; and so on. Bob can choose any bit $y \in \{1, 2, \ldots, n\}$ and recover the bit x_y^δ with a probability. The condition of the task is the following: Bob's output must be the bit $b = x_y^\delta$. The parity-oblivious constraint here is that *no* information about any parity of x can be transmitted to Bob. Following [33], we define a parity set $\mathbb{P}_n = \{x | x \in \{0,1\}^n, \sum_r x_r \geqslant 2\}$ with $r \in \{1, 2, \ldots, n\}$. For any arbitrary $s \in \mathbb{P}_n$, no information about $s.x = \oplus_r s_r x_r$ (s parity) is to be transmitted to Bob, where $\oplus$ is sum modulo 2.

In an operational theory, Alice encodes her n-bit string of x^δ prepared by a procedure P_{x^δ}. Next, after receiving the particle, for every $y \in \{1, 2, \ldots, n\}$, Bob performs a two-outcome measurement $M_{n,y}$ and reports outcome b as his output. Then the probability of success is given by

$$p\big(b = x_y^\delta\big) = \frac{1}{2^n n} \sum_{x,y} p\big(b = x_y^\delta \big| P_{x^\delta}, M_{n,y}\big). \tag{2}$$

In quantum PORAC, Alice encodes her n-bit string of x^δ into quantum states ρ_{x^δ}, prepared by a procedure P_{x^δ}. On a suitable entangled state $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ with $|\psi_{AB}\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, Alice performs one of the 2^{n-1} projective measurements $\{P_{A_{n,i}}, \mathbb{I} - P_{A_{n,i}}\}$ where $i \in \{1, 2, \ldots, 2^{n-1}\}$ to encode her n bits into 2^n quantum states given by

$$\begin{aligned} \tfrac{1}{2}\rho_{x^i} &= \mathrm{Tr}_A[(P_{A_{n,i}} \otimes \mathbb{I})\rho_{AB}], \\ \tfrac{1}{2}\rho_{x^j} &= \mathrm{Tr}_A[(\mathbb{I} - P_{A_{n,i}} \otimes \mathbb{I})\rho_{AB}] \end{aligned} \tag{3}$$

with $i + j = 2^n + 1$.

In quantum theory, the parity-oblivious condition implies that

$$\forall s : \frac{1}{2^{n-1}} \sum_{x^\delta | x^\delta s = 0} \rho_{x^\delta} = \frac{1}{2^{n-1}} \sum_{x^\delta | x^\delta s = 1} \rho_{x^\delta}. \tag{4}$$

In the ontological model of quantum theory, the parity obliviousness in the Eq. (4) condition is equivalent to the preparation noncontextual assumption, i.e.,

$$\forall s : \frac{1}{2^{n-1}} \sum_{x^\delta | x^\delta . s = 0} \mu(\lambda | \rho_{x^\delta}) = \frac{1}{2^{n-1}} \sum_{x^\delta | x^\delta . s = 1} \mu(\lambda | \rho_{x^\delta}). \tag{5}$$

Note that the number of parity-oblivious conditions for the n-bit PORAC is the number of elements in $\mathbb{P}_n$ [34]. We noticed that there are two types of parity-oblivious conditions. One type arises from the natural construction, such as $\mathbb{I} = P^+_{A_{n,i}} + P^-_{A_{n,i}}$. In that case, $s \in \mathbb{P}_n$ follow the property $\sum_y s_y = 2m$ with $m \in \mathbb{N}$. For the rest of $s \in \mathbb{P}_n$ not satisfying the above property, nontrivial constraints on Alice's observables need to be satisfied and are given by

$$\sum_{i=1}^{2^{n-1}} (-1)^{s x^i} A_{n,i} = 0. \tag{6}$$

The total number of such nontrivial constraints on Alice's observables is $C_n = 2^{n-1} - n$.

The measurements for the decoding scheme are taken to be

$$M_{n,y} = \begin{cases} M^i_{n,y}, & \text{when } b = x^i_y \\ M^j_{n,y}, & \text{when } b = x^j_y \end{cases}, \tag{7}$$

$$M^{i(j)}_{n,y} = \begin{cases} P_{B_{n,y}}, & \text{when } x^{i(j)}_y = 0 \\ \mathbb{I} - P_{B_{n,y}}, & \text{when } x^{i(j)}_y = 1 \end{cases}. \tag{8}$$

The quantum success probability can then be written as

$$\begin{aligned} p_Q &= \frac{1}{2^n n} \sum_{y=1}^{n} \sum_{i=1}^{2^{n-1}} p\big(b = x^i_y \big| \rho_{x^i}, M^i_y\big) + p\big(b = x^j_y \big| \rho_{x^j}, M^j_y\big) \\ &= \frac{1}{2} + \frac{\mathcal{B}_n}{2^n n} \end{aligned} \tag{9}$$

where $\mathcal{B}_n$ is the family of Bell expressions given by

$$\mathcal{B}_n = \sum_{y=1}^{n} \sum_{i=1}^{2^{n-1}} (-1)^{x^i_y} A_{n,i} \otimes B_{n,y}, \tag{10}$$

which serve as the family of dimension witnesses in the present paper. Note that the Bell expressions in Eq. (10) provide the CHSH and Gisin elegant Bell inequality for $n = 2$ and 3, respectively, and the corresponding local bounds are 2 and 6. The local bound of the family of Bell expressions for any arbitrary n is given by [34]

$$(\mathcal{B}_n)_{\text{local}} \leqslant n \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}. \tag{11}$$

However, the parity-oblivious condition in the usual RAC imposes a functional relationship between Alice's observables as given by Eq. (6). This means that Alice's choices of values are restricted and the local bound of Eq. (10) gets reduced (which we call the preparation-noncontextual bound [33,34]) to

$$(\mathcal{B}_n)_{\text{PNC}} \leqslant 2^{n-1}. \tag{12}$$

Since for any n we have $(\mathcal{B}_n)_{\text{PNC}} \leqslant (\mathcal{B}_n)_{\text{local}}$ then for a given n even if optimal quantum value $(\mathcal{B}_n)_Q$ does not violate the local bound in Eq. (11) it may still reveal nonclassicality by violating the preparation noncontextuality given by Eq. (12). Note that it is already known [34] that the optimal quantum value of $\mathcal{B}_n$ can be obtained for the local system having dimension $d = 2^{\lfloor n/2 \rfloor}$. Our purpose here is to examine the maximum quantum values of $\mathcal{B}_n$ that can be achieved for the lower-dimensional systems having dimensions $d < 2^{\lfloor n/2 \rfloor}$.

III. SUM-OF-SQUARE APPROACH FOR MAXIMIZATION

In order to find the quantum upper bound of the Bell expression $(\mathcal{B}_n)_{d=2^m}$ for various dimensions, we use the SOS approach (see, for example, [35]), so that $(\mathcal{B}_n)_Q \leqslant \beta_n$ for all possible quantum states ρ_{AB} and measurement operators $A_{n,i}$ and $B_{n,y}$. Here β_n is the upper bound on the quantum value of $(\mathcal{B}_n)_{d=2^m}$ for the system having dimension $d = 2^m$. This is equivalent to showing that there is a positive semidefinite operator $\gamma_n \geqslant 0$, that can be expressed as $\langle\gamma_n\rangle_Q = \beta_n - (\mathcal{B}_n)_Q$ where β_n is a number. This can be proven by considering a set of suitable positive operators M_n^i which are polynomial functions of $A_{n,i}$ and $B_{n,y}$, so that

$$\gamma_n = \sum_{i=1}^{2^{n-1}} \frac{\omega_{n,i}}{2} \left(M_n^i\right)^\dagger M_n^i \tag{13}$$

where $\omega_{n,i}$ is positive semidefinite and to be specified shortly. The maximum value of $(\mathcal{B}_n)_Q$ for any given dimension is obtained if $\langle\gamma_n\rangle_Q = 0$, implying that

$$M_n^i|\psi\rangle = 0. \tag{14}$$

For the family of Bell expressions given by Eq. (10), the operators M_n^i can be written as

$$M_n^i = \frac{1}{\omega_{n,i}} \sum_{y=1}^{n} (-1)^{x_y^i} B_{n,y} - A_{n,i} \tag{15}$$

where $\omega_{n,i} = ||\sum_{y=1}^{n}(-1)^{x_y^i} B_{n,y}||$. Plugging Eq. (15) into Eq. (13) and by noting that $A_{n,i}^\dagger A_{n,i} = B_{n,y}^\dagger B_{n,y} = \mathbb{I}$, we get

$$\langle\gamma_n\rangle_Q = -(\mathcal{B}_n)_Q + \sum_{i=1}^{2^{n-1}} \left[\frac{1}{2\omega_{n,i}}\left(\sum_{y=1}^{n}(-1)^{x_y^i} B_{n,y}\right)^2 + \frac{\omega_{n,i}}{2}\right], \tag{16}$$

which can be rewritten in a simple form as

$$\langle\gamma_n\rangle_Q = -(\mathcal{B}_n)_Q + \sum_{i=1}^{2^{n-1}} \omega_{n,i}. \tag{17}$$

The maximum quantum value of $(\mathcal{B}_n)_Q$ can be obtained when $\langle\gamma_n\rangle_Q = 0$ which in turn provides

$$\begin{aligned}(\mathcal{B}_n)_Q^{\max} &= \max_{B_{n,y}}\left(\sum_{i=1}^{2^{n-1}} \omega_{n,i}\right)\\ &= \max_{B_{n,y}}\left(\sum_{i=1}^{2^{n-1}} \left|\left|\sum_{y=1}^{n}(-1)^{x_y^i} B_{n,y}\right|\right|\right),\end{aligned} \tag{18}$$

and the explicit condition obtained from Eq. (14) is given by

$$\forall i \quad \sum_{y=1}^{n}(-1)^{x_y^i} B_{n,y}|\psi\rangle = \omega_{n,i} A_{n,i}|\psi\rangle. \tag{19}$$

To obtain the maximum quantum value from Eq. (18) for a given dimensional system, we use the concavity inequality, i.e.,

$$\sum_{i=1}^{2^{n-1}} \omega_{n,i} \leqslant \sqrt{2^{n-1}\sum_{i=1}^{2^{n-1}}(\omega_{n,i})^2}. \tag{20}$$

In Eq. (20), the equality can be obtained only when $\omega_{n,i}$ are equal for each i, when $B_{n,y}$ are mutually anticommuting, and thus for $n > 3$ the optimal value cannot be obtained for the qubit system.

Also, for satisfying the parity obliviousness conditions Eq. (6) has to be satisfied by Alice's observables $A_{n,i}$. This implies that the optimal quantum value of the Bell expression $(\mathcal{B}_n)$ can only be achieved for bounded dimension of the Hilbert space. By using Eqs. (6) and (19) the condition that is required to hold is given by

$$\sum_{i=1}^{2^{n-1}}\sum_{y=1}^{n}(-1)^{s.x^i + x_y^i}\frac{B_{n,y}}{\omega_{n,i}} = 0. \tag{21}$$

Since $B_{n,y}$s are dichotomic, the quantity $\omega_{n,i}$ can explicitly be written as

$$\begin{aligned}\omega_{n,i} = \Bigg[n &+ \left\{(-1)^{x_1^i}B_{n,1}, \sum_{y=2}^{n}(-1)^{x_y^i}B_{n,y}\right\}\\ &+ \left\{(-1)^{x_2^i}B_{n,2}, \sum_{y=3}^{n}(-1)^{x_y^i}B_{n,y}\right\} + \cdots\\ &+ \{(-1)^{x_{n-1}^i}B_{n,n-1}, (-1)^{x_n^i}B_{n,n}\}\Bigg]^{-1/2}\end{aligned} \tag{22}$$

where {, } denotes the anticommutation.

As already mentioned, the optimal quantum value $(\mathcal{B}_n)_Q^{\text{opt}}$ of the family of Bell expressions can only be achieved when Bob's observables $B_{n,y}$ are mutually anticommuting, and this in turn fixes the dimension of the Hilbert space $d = 2^{\lfloor n/2\rfloor}$. In such a case $\omega_{n,i} = \sqrt{n}$ for every i and from Eq. (18) the optimal quantum value can be calculated as

$$(\mathcal{B}_n)_{d=2^{\lfloor n/2\rfloor}}^{\text{opt}} = 2^{n-1}\sqrt{n}. \tag{23}$$

For this, the required maximally entangled state having local dimension $2^{\lfloor n/2\rfloor}$ is given by

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2^{\lfloor n/2\rfloor}}}\sum_{k=1}^{2^{\lfloor n/2\rfloor}}|k\rangle_A|k\rangle_B.$$

Thus, for the cases $n \geqslant 4$, the qubit system will not satisfy the purpose and one requires a higher-dimensional system. For example, the optimal value of the Bell expression for $n = 4$ requires at least a two-qubit system and for the qubit system an upper bound $(\mathcal{B}_4)_{d=2}^{\max}$ can be found, which is smaller than the optimal quantum value. Hence, $\mathcal{B}_4$ serves as a dimension witness for certifying the qubit system. Similarly, for any arbitrary $n \geqslant 4$, the Bell expression $\mathcal{B}_n$ given by Eq. (10) certifies the $d = 2^m$-dimensional local system with $m = 1, 2, \ldots, \lfloor n/2\rfloor$.

IV. DIMENSION WITNESSES FOR ONE-, TWO-, AND THREE-QUBIT SYSTEMS

In the following, we provide several examples starting from the case of $n = 3$ to 6. As already discussed, the optimal values of Bell expressions $\mathcal{B}_n$ for $n = 2$ and 3 require two and three mutually anticommuting observables, respectively, which can be obtained for the qubit system, and thus $\mathcal{B}_2$

and $\mathcal{B}_3$ cannot serve as dimension witness. But from $n \geqslant 4$ the Bell expression $\mathcal{B}_n$ serves as the dimension witness. We explicitly demonstrate that the Bell expressions $\mathcal{B}_4$ and $\mathcal{B}_5$ for $n = 4$ and 5, respectively, serve as dimension witnesses for qubit and two-qubit systems, and the Bell expression for $n = 6$ serves as the dimension witness for qubit, two-qubit, and three-qubit systems. We first provide the analysis for $n = 3$ to make the reader familiar with the optimization technique and how parity-oblivious conditions are satisfied by Alice's observables when the optimal quantum value is achieved for the qubit system.

A. Analysis for $n = 3$

For $n = 3$, from Eq. (10) we obtain Gisin's elegant Bell expression [38] given by

$$\begin{aligned}\mathcal{B}_3 = A_{3,1} &\otimes (B_{3,1} + B_{3,2} + B_{3,3}) \\ &+ A_{3,2} \otimes (B_{3,1} + B_{3,2} - B_{3,3}) \\ &+ A_{3,3} \otimes (B_{3,1} - B_{3,2} + B_{3,3}) \\ &+ A_{3,4} \otimes (-B_{3,1} + B_{3,2} + B_{3,3}).\end{aligned}$$

The local bound of $\mathcal{B}_3$ is 6. The parity-oblivious condition derived from Eq. (6) provides a functional relation between Alice's observables, i.e., $A_{3,1} - A_{3,2} - A_{3,3} - A_{3,4} = 0$. If this condition is imposed, the local bound reduces to the preparation noncontextual bound 4. The optimal quantum value of the relevant Bell expression is $(\mathcal{B}_3)_Q^{\text{opt}} = \max(\sum_{i=1}^{4} \omega_{3,i})$ where $\omega_{3,i}$ can be written as

$$\begin{aligned}\omega_{3,1} &= \sqrt{3 + \{B_{3,1}, (B_{3,2} + B_{3,3})\} + \{B_{3,2}, B_{3,3}\}}, \\ \omega_{3,2} &= \sqrt{3 + \{B_{3,1}, (B_{3,2} - B_{3,3}) - \{B_{3,2}, B_{3,3}\}}, \\ \omega_{3,3} &= \sqrt{3 + \{B_{3,1}, (B_{3,2} - B_{3,3})\} - \{B_{3,2}, B_{3,3}\}}, \\ \omega_{3,4} &= \sqrt{3 - \{B_{3,1}, (B_{3,2} + B_{3,3})\} + \{B_{3,2}, B_{3,3}\}}. \end{aligned} \tag{24}$$

For $n = 3$, by noting a symmetry, we have $\sum_{i=1}^{4} (\omega_{3,i})^2 = 12$ which can only be available if $B_{3,1}$, $B_{3,2}$, and $B_{3,3}$ are mutually commuting. This in turn provides $\omega_{3,i} = \sqrt{3}$ for each i and thereby provides $(\mathcal{B}_3)_{d=2}^{\text{opt}} = 4\sqrt{3}$.

From Eq. (19), one can find the observables $A_{3,i}$ required for Alice to obtain the optimal violation of the elegant Bell inequality. Such a choice can be available for the qubit system by taking mutually anticommuting observables of Bob, viz., $B_{3,1} = \sigma_x$, $B_{3,2} = \sigma_y$, and $B_{3,3} = \sigma_z$. Using Eq. (19), Alice's choices of observables are the following: $A_{3,1} = (\sigma_x + \sigma_y + \sigma_z)/\sqrt{3}$, $A_{3,2} = (\sigma_x + \sigma_y - \sigma_z)/\sqrt{3}$, $A_{3,3} = (\sigma_x - \sigma_y + \sigma_z)/\sqrt{3}$, and $A_{3,4} = (-\sigma_x + \sigma_y + \sigma_z)/\sqrt{3}$.

Such choices of observables by Alice need to satisfy the parity-oblivious condition given by Eq. (21). Using Eqs. (19) and (21) we find that the following conditions have to be satisfied by $\omega_{3,i} = 1/\alpha_{3,i}$ and are given by

$$\begin{aligned}\alpha_{3,1} - \alpha_{3,2} - \alpha_{3,3} + \alpha_{3,4} = 0, \\ \alpha_{3,1} - \alpha_{3,2} + \alpha_{3,3} - \alpha_{3,4} = 0, \\ \alpha_{3,1} + \alpha_{3,2} - \alpha_{3,3} - \alpha_{3,4} = 0. \end{aligned} \tag{25}$$

The solutions of the Eq. (25) are $\omega_{3,1} = \omega_{3,2} = \omega_{3,3} = \omega_{3,4} \equiv \omega'_3$. This is only possible if $B_{3,y}$ are mutually anticommuting, and in this case $\omega'_3 = \sqrt{3}$, as expected. As mentioned, for $n = 3$ the relevant Bell expression can be optimized for the qubit system and hence does not serve as a dimension witness. However, we demonstrate below that for $n \geqslant 4$ the family of Bell expressions serves as the dimension witnesses of the Hilbert space.

B. Dimension witness for $n = 4$

Next, we demonstrate that for $n \geqslant 4$ the Bell expressions Eq. (10) serve as witnesses of the Hilbert space having dimension $d = 2^m$ where $m = 1, 2, \ldots, \lfloor n/2 \rfloor$. We first demonstrate that for $n = 4$ the maximum quantum value of the Bell expression for the qubit system is smaller than the optimal value obtained for the two-qubit system, i.e., $(\mathcal{B}_4)_{d=2}^{\max} \leqslant (\mathcal{B}_4)_{d=2^2}^{\text{opt}}$.

The Bell expression for $n = 4$ can be written as

$$\begin{aligned}\mathcal{B}_4 = A_{4,1} &\otimes (B_{4,1} + B_{4,2} + B_{4,3} + B_{4,4}) \\ &+ A_{4,2} \otimes (B_{4,1} + B_{4,2} + B_{4,3} - B_{4,4}) \\ &+ A_{4,3} \otimes (B_{4,1} + B_{4,2} - B_{4,3} + B_{4,4}) \\ &+ A_{4,4} \otimes (B_{4,1} - B_{4,2} + B_{4,3} + B_{4,4}) \\ &+ A_{4,5} \otimes (-B_{4,1} + B_{4,2} + B_{4,3} + B_{4,4}) \\ &+ A_{4,6} \otimes (B_{4,1} + B_{4,2} - B_{4,3} - B_{4,4}) \\ &+ A_{4,7} \otimes (B_{4,1} - B_{4,2} + B_{4,3} - B_{4,4}) \\ &+ A_{4,8} \otimes (B_{4,1} - B_{4,2} - B_{4,3} + B_{4,4}), \end{aligned} \tag{26}$$

the local bound of which is 12 and the preparation noncontextual bound of which is 8. As already mentioned, by using the concavity inequality Eq. (20) one finds the optimal value $(\mathcal{B}_3)_{2^2}^{\text{opt}} = 16$ for the two-qubit system when all the $\omega_{4,i}$ are equal to 2. This happens when all four $B_{4,y}$ are mutually anticommuting in the two-qubit system. One such choice is $B_{4,1} = \sigma_x \otimes \sigma_x$, $B_{4,2} = \sigma_x \otimes \sigma_y$, $B_{4,3} = \sigma_x \otimes \sigma_z$, and $B_{4,4} = \sigma_y \otimes \mathbb{I}$, and three more such sets are possible. However, for the qubit system there are only three mutually commuting observables available and then $\mathcal{B}_3$ cannot reach the optimal value for the qubit system.

We derive the maximum quantum value of $(\mathcal{B}_4)_{d=2}^{\max}$ for the qubit system. It is straightforward to understand that all of the eight $\omega_{4,i}$ from Eq. (18) cannot be equal for the qubit system. The question is how many of them are equal to each other. Using the concavity inequality in Eq. (20) two times one finds that there are two optimal sets for which at most four of them are equal to each other. For example, $\omega_{4,1} = \omega_{4,4} = \omega_{4,5} = \omega_{4,6} \equiv \omega'_4$ and $\omega_{4,2} = \omega_{4,3} = \omega_{4,7} = \omega_{4,8} \equiv \omega''_4$. This provides the condition on Bob's observables given by

$$\begin{aligned}\{B_{4,1}, B_{4,2}\} = \{B_{4,1}, B_{4,3}\} = \{B_{4,1}, B_{4,4}\} = 0, \\ \{B_{4,2}, B_{4,3}\} = \{B_{4,2}, B_{4,4}\} = 0. \end{aligned} \tag{27}$$

From Eq. (18), we can then write

$$\begin{aligned}(\mathcal{B}_4)_Q &\leqslant 4(\omega'_4 + \omega''_4) \\ &= 4(\sqrt{4 + \{B_{4,3}, B_{4,4}\}} + \sqrt{4 - \{B_{4,3}, B_{4,4}\}}). \end{aligned} \tag{28}$$

It is easy to check from Eq. (28) that the optimal value $(\mathcal{B}_4)_Q^{\text{opt}}$ can be obtained only when $\{B_{4,3}, B_{4,4}\} = 0$ along with the relations in Eq. (27), i.e., all four $B_{4,y}$ are mutually anticommuting. Such a requirement cannot be fulfilled for a qubit system and one needs at least a two-qubit system.

Now, for a qubit system, it can be checked that $\mathcal{B}_4$ reaches its maximum value

$$(\mathcal{B}_4)^{max}_{d=2} = 4(\sqrt{2}+\sqrt{6}) \leqslant (\mathcal{B}_4)^{opt}_{d=2^2} = 16 \tag{29}$$

when $\{B_{4,3}, B_{4,4}\} = \pm 1$. Explicitly, the choices of $B_{4,y}$ are $B_{4,1} = \sigma_x$, $B_{4,1} = \sigma_y$, $B_{4,1} = \sigma_z$, and $B_{4,1} = \pm\sigma_z$. Thus, the Bell expression $\mathcal{B}_4$ is a dimension witness distinguishing the dimension between qubit and two-qubit Hilbert space. In Appendix A, we demonstrate how Alice's choices of observables required to obtain the maximum quantum values of $\mathcal{B}_4$ for qubit and two-qubit systems satisfy the parity-oblivious condition.

C. Dimension witness for $n = 5$

We now demonstrate that the Bell expression in Eq. (10) for $n = 5$ also serves as a dimension witness for the qubit system. The explicit form of $\mathcal{B}_5$ is given in Eq. (B1) of Appendix B. Following the same technique adopted for $n = 3$ and 4 we can find that for optimizing $\mathcal{B}_5$ the following relations between $B_{5,y}$ have to be satisfied: $\{B_{5,1}, B_{5,2}\} = \{B_{5,2}, B_{5,3}\} = \{B_{5,1}, B_{5,3}\} = \{B_{5,3}, B_{5,4}\} = \{B_{5,3}, B_{5,5}\} = \{B_{5,4}, B_{5,5}\} = 0$, $\{B_{5,1}, B_{5,4}\} = \{B_{5,1}, B_{5,5}\}$, and $\{B_{5,2}, B_{5,4}\} = -\{B_{5,2}, B_{5,5}\}$. Using those relations, from Eq. (18), we find

$$(\mathcal{B}_5)_Q \leqslant 4\sqrt{5 + \{(B_{5,1}+B_{5,2}), (B_{5,4}+B_{5,5})\}} + 4\sqrt{5 + \{(B_{5,1}-B_{5,2}), (B_{5,4}-B_{5,5})\}}. \tag{30}$$

Note that the optimal quantum value can be reached if $\{B_{5,1}, B_{5,4}\} = \{B_{5,1}, B_{5,5}\} = \{B_{5,2}, B_{5,4}\} = \{B_{5,2}, B_{5,5}\} = 0$ which means all $B_{5,y}$ are mutually anticommuting observables providing $(\mathcal{B}_5)^{opt}_{d=2^2} = 16\sqrt{5}$. Again, such choices cannot be obtained for a qubit system and one requires at least a two-qubit system. A choice of such a set of observables is given by $B_{5,1} = \sigma_x \otimes \sigma_x$, $B_{5,2} = \sigma_x \otimes \sigma_y$, $B_{5,3} = \sigma_x \otimes \sigma_z$, $B_{5,4} = \sigma_y \otimes \mathbb{I}$, and $B_{5,5} = \sigma_z \otimes \mathbb{I}$.

Now, for the qubit system the maximum quantum value can be obtained for the following choices of the observables: $B_{5,1} = \sigma_x$, $B_{5,2} = \sigma_y$, $B_{5,3} = \sigma_y$, $B_{5,4} = (\sigma_x + \sigma_z)/\sqrt{2}$, and $B_{5,5} = (\sigma_x - \sigma_z)/\sqrt{2}$ and the maximum quantum value is

$$(\mathcal{B}_5)^{max}_{d=2} = 8\left(\sqrt{5+2\sqrt{2}} + \sqrt{5-2\sqrt{2}}\right) \leqslant (\mathcal{B}_5)^{opt}_{d=2^2} = 16\sqrt{5}. \tag{31}$$

Alice's observables can be found by using Eq. (19). In order to examine whether Alice's observables satisfy the parity-oblivious conditions at the maximum quantum values for qubit and two-qubit systems, we follow similar procedures as those adopted for the cases $n = 3$ and 4. The details of the argument are given in Appendix B.

D. Dimension witness (for $n = 6$)

We have just demonstrated that the Bell inequalities for $n = 4$ and 5 certify the qubit and two-qubit systems. Next, we demonstrate that the Bell expression $\mathcal{B}_6$ for $n = 6$ certifies qubit, two-qubit, and three-qubit systems. The explicit form of $\mathcal{B}_6$ is quite lengthy but can easily be obtained from Eq. (10). Once again, to obtain the optimal quantum value of $(\mathcal{B}_6)^{opt}_Q = 32\sqrt{6}$ one requires all the 32 values of $\omega_{6,i}$ in Eq. (22) to be equal. This can only be obtained if all six $B_{6,y}$ are mutually anticommuting and we thus require at least a three-qubit system. For qubit and two-qubit systems we can obtain two upper bounds. Using the concavity relation eight times, we find that the following relations between $B_{6,y}$ have to be satisfied: $\{B_{6,1}, B_{6,2}\} = \{B_{6,3}, B_{6,4}\} = \{B_{6,5}, B_{6,6}\} = 0$ and $\{B_{6,1}, B_{6,3}\} = \{B_{6,1}, B_{6,4}\} = -\{B_{6,2}, B_{6,3}\} - \{B_{6,2}, B_{6,4}\}$. This provides

$$\begin{aligned}(\mathcal{B}_6)_Q \leqslant\ & 4\left[\sqrt{6 + \{(B_{6,1}+B_{6,2}+B_{6,3}+B_{6,4}), (B_{6,5}+B_{6,6})\}} + \sqrt{6 - \{(B_{6,1}+B_{6,2}+B_{6,3}+B_{6,4}), (B_{6,5}+B_{6,6})\}}\right]\\ &+ 4\left[\sqrt{6 + \{(B_{6,1}-B_{6,2}+B_{6,3}+B_{6,4}), (B_{6,5}+B_{6,6})\}} + \sqrt{6 - \{(B_{6,1}-B_{6,2}+B_{6,3}+B_{6,4}), (B_{6,5}+B_{6,6})\}}\right]\\ &+ 4\left[\sqrt{6 + \{(B_{6,1}+B_{6,2}+B_{6,3}-B_{6,4}), (B_{6,5}-B_{6,6})\}} + \sqrt{6 - \{(B_{6,1}+B_{6,2}+B_{6,3}-B_{6,4}), (B_{6,5}-B_{6,6})\}}\right]\\ &+ 4\left[\sqrt{6 + \{(B_{6,1}+B_{6,2}-B_{6,3}+B_{6,4}), (B_{6,5}-B_{6,6})\}} + \sqrt{6 - \{(B_{6,1}+B_{6,2}-B_{6,3}+B_{6,4}), (B_{6,5}-B_{6,6})\}}\right].\end{aligned} \tag{32}$$

The optimal value for the qubit is $(\mathcal{B}_6)^{max}_{d=2} = 12\sqrt{2} + 8\sqrt{6} + 12\sqrt{10}$ and choices of observables required are given by

$$B_{6,1} = \frac{(\sigma_y+\sigma_x)}{\sqrt{2}}, \quad B_{6,2} = \frac{(\sigma_y-\sigma_x)}{\sqrt{2}}, \quad B_{6,3} = \frac{(\sigma_x+\sigma_z)}{\sqrt{2}},$$
$$B_{6,4} = \frac{(\sigma_x-\sigma_z)}{\sqrt{2}}, \quad B_{6,5} = \frac{(\sigma_z+\sigma_y)}{\sqrt{2}}, \quad B_{6,6} = \frac{(\sigma_z-\sigma_y)}{\sqrt{2}}. \tag{33}$$

For the two-qubit system, we additionally have $\{B_{6,1}, B_{6,5}\} = \{B_{6,2}, B_{6,5}\} = \{B_{6,3}, B_{6,5}\} = \{B_{6,4}, B_{6,5}\} = \{B_{6,2}, B_{6,6}\} = \{B_{6,3}, B_{6,6}\} = \{B_{6,4}, B_{6,6}\} = \{B_{6,5}, B_{6,6}\} = 0$. We can then write Eq. (33) as

$$(\mathcal{B}_6)_{d=2^2} \leqslant 16\left(\sqrt{6 + \{B_{6,1}, B_{6,6}\}} + \sqrt{6 - \{B_{6,1}, B_{6,6}\}}\right). \tag{34}$$

In order to get the maximum quantum value of $\mathcal{B}_6$ for the two-qubit system we need to choose $B_{6,y} = B_{5,y}$ for $y = 1, 2, \ldots, 5$ and $B_{6,6} = \pm\sigma_z \otimes \mathbb{I}$ and the maximum quantum value will be $(\mathcal{B}_6)^{max}_{d=2^2} = 32(1+\sqrt{2})$. For a three-qubit system, we additionally have $\{B_{6,1}, B_{6,6}\} = 0$, i.e., six mutually anticommuting observables available for the three-qubit system are given by

$$B_{6,y} = \sigma_x \otimes B_{5,y} \text{ for } y = 1, 2, \ldots, 5,$$
$$B_{6,6} = \sigma_y \otimes \mathbb{I} \otimes \mathbb{I}. \tag{35}$$

TABLE I. The maximum quantum values of the Bell expressions for $n = 2$ to 6 are provided for qubit, two-qubit, and three-qubit systems. Here PNC in the second column denotes the preparation noncontextuality. It is shown that the Bell expressions for $n =$ 4 and 5 certify the qubit and two-qubit systems, and the Bell expression for $n = 6$ certifies the qubit, two-qubit, and three-qubit systems.

n value	PNC bound	Qubit	Two-qubit	Three-qubit
2	2	$2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$
3	4	$4\sqrt{3}$	$4\sqrt{3}$	$4\sqrt{3}$
4	8	15.45	16	16
5	16	34.17	35.77	35.77
6	32	71.79	77.25	78.11

We can then summarize the quantum values of $(\mathcal{B}_6)_Q$ for qubit, two-qubit, and three-qubit systems as

$$\begin{aligned}(\mathcal{B}_6)^{\max}_{d=2} &= 12\sqrt{2} + 8\sqrt{6} + 12\sqrt{10}\\ &\leqslant (\mathcal{B}_6)^{\max}_{d=2^2} = 32(1+\sqrt{2})\\ &\leqslant (\mathcal{B}_6)^{\text{opt}}_{d=2^3} = 32\sqrt{6}. \qquad (36)\end{aligned}$$

Hence, the Bell expression $\mathcal{B}_6$ can certify the qubit, two-qubit, and three-qubit system.

The relevant results obtained for $n = 3$ to 6 are given in Table I.

V. SHARING OF PREPARATION CONTEXTUALITY BY MULTIPLE BOBS

Let us now examine how many Bobs can sequentially share preparation contextuality if the dimension of the system is lower than the dimension required in obtaining the optimal quantum value $(\mathcal{B}_n)_Q$. The notion of sharing of nonlocal quantum correlation by multiple Bobs was recently put forward [45], where an entangled pair of particles is shared between a single Alice and multiple Bobs performing unsharp measurements on the same particle sequentially. Sharing the nonlocal correlation by a larger number of sequential Bobs requires their sequential measurements to be as unsharp as possible but just enough for violating the preparation noncontextual bound of the family of Bell expressions. In [45], it is demonstrated that nonlocality through the violation the CHSH inequality can be shared by at most two Bobs for the unbiased choices of measurement settings and experimental verification is also reported [46,47]. This initiated the studies of sharing of entanglement [48] and steering [49]. For a suitable choice of entangled state in higher dimensions, steering can be shared by an unbounded number of Bobs [50]. One of the authors has earlier demonstrated [39] the sharing of preparation contextuality by an arbitrary number of Bobs by using the family of Bell expressions given by Eq. (10). However, the $2^{\lfloor n/2\rfloor}$-dimensional system was taken for optimizing the Bell expression in Eq. (10).

A relevant question could be to examine the sharing of preparation contextuality by considering the dimension of the system lower than the dimension $d = 2^{\lfloor n/2\rfloor}$. Here, we demonstrate the sharing of preparation contextuality using the Bell expressions in Eq. (10) for $n = 4$, 5, and 6 for the system having dimensions $d = 2^m$ where $m = 1, 2, \dots, \lfloor n/2\rfloor$.

In order to find the number of independent sequential Bobs (k) who can share the preparation contextuality, let us consider that there is one Alice who performs sharp measurement and k number of Bobs who perform unsharp measurement sequentially. However, the kth Bob may perform a projective measurement. For the Bell expression $\mathcal{B}_n$ Alice and each Bob perform the measurements for 2^{n-1} and n number of dichotomic observables, respectively. Given a n value, each Bob is required to perform the same set of n number of observables. We also consider that Bob's choices of measurement settings are completely random. Consider that Alice and first Bob (say, Bob$_1$) share a maximally entangled state and $k-1$ number of Bobs perform the unsharp measurements of the observables $B_{n,y}$ given by

$$E^{\pm}_{B_n,y,j} = \frac{1\pm\lambda_{n,j}}{2} M^0_{B_n,y} + \frac{1\mp\lambda_{n,j}}{2} M^1_{B_n,y} \qquad (37)$$

where $E^{\pm}_{B_n,y,j}$ are unbiased POVMs and $M^0_{B_n,y}$ and $M^1_{B_n,y}$ are the projectors of Bob's observable. Here, $\lambda_{n,j} \in [0,1]$ is the unsharpness parameter for jth Bob where $j = 1, 2, \dots, k-1$ [51,52]. We consider that for a given n the unsharpness parameters are the same for each of Bob's observables $B_{n,y}$ and independent of y.

The shared state between Alice and kth Bob is obtained after the unsharp measurements of $k-1$ Bobs given by

$$\begin{aligned}\rho_{n,k} &= \frac{1}{n}\sum_{b\in\{+,-\}}\sum_{y=1}^{n}\Big(\mathbb{I}\otimes\sqrt{E^b_{B_{n,y,k-1}}}\Big)\rho_{n,k-1}\Big(\mathbb{I}\otimes\sqrt{E^b_{B_{n,y,k-1}}}\Big)\\ &= \sqrt{1-\lambda^2_{n,k-1}}\,\rho_{n,k-1} + \frac{\Big(1-\sqrt{1-\lambda^2_{n,k-1}}\Big)}{n}\\ &\quad\times\sum_{b\in\{+,-\}}\sum_{y=1}^{n}(\mathbb{I}\otimes\Pi^b_{B_{n,y,k-1}})\rho_{n,k-1}\big(\mathbb{I}\otimes\Pi^b_{B_{n,y,k-1}}\big) \qquad (38)\end{aligned}$$

where $\rho_{n,k-1}$ is the state shared between Alice and $(k-1)$th Bob before $(k-1)$th Bob's unsharp measurement. For kth sequential Bob the maximum quantum value of the Bell expression given by Eq. (10) for the $d = 2^m$-dimensional system can be written as

$$\big(\mathcal{B}^k_n\big)^{\lambda}_Q = (\mathcal{B}_n)^{\max}_{d=2^m}\left(\prod_{j=1}^{k-1}\Big[1+(n-1)\sqrt{1-\lambda^2_{n,j}}\Big]\right)\lambda_{n,k} \qquad (39)$$

where $m = 1, 2, \dots, 2^{\lfloor n/2\rfloor}$. Now, by considering the preparation noncontextual bound $(\mathcal{B}_n)_{\text{PNC}} = 2^{n-1}$, the condition on the unsharpness parameter for sharing the preparation contextuality by kth Bob is given by

$$\lambda_{n,k} > \frac{2^{n-1}}{(\mathcal{B}_n)^{\max}_{d=2^m}\Big\{\prod_{j=1}^{k-1}\Big[1+(n-1)\sqrt{1-\lambda^2_{n,j}}\Big]\Big\}}. \qquad (40)$$

In order to find how many Bobs can sequentially share preparation contextuality for a given dimension $m = 1, 2, \dots, 2^{\lfloor n/2\rfloor}$, we just need to find the values of $\lambda_{n,j}$ within its valid range [0,1]. For this, one needs to use the critical value of the $\lambda_{n,j}$ for jth Bob so that it is just enough to violate the preparation contextuality.

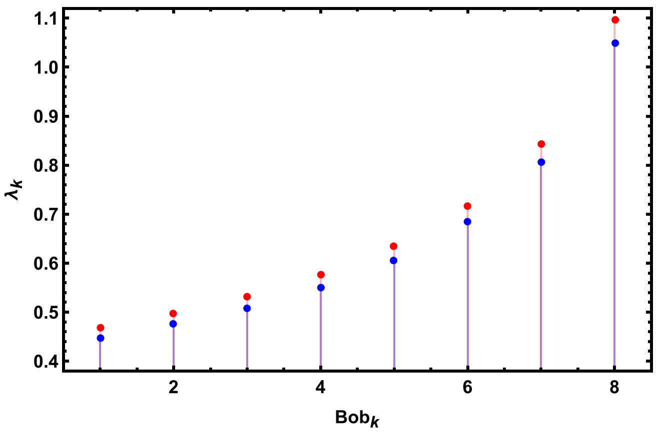

FIG. 2. Critical values of unsharpness parameter λ_k required for violating the preparation noncontextual bound are plotted for kth Bob in the case of $n = 5$. Here, blue and red dots denote the critical values corresponding to the two-qubit and qubit system, respectively.

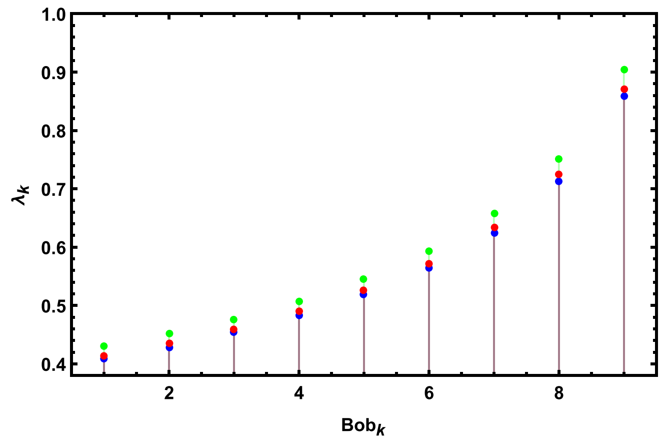

FIG. 3. Critical values of unsharpness parameter λ_k required for violating the preparation noncontextual bound are plotted for kth Bob in the case of $n = 6$. Here, blue, red, and green dots denote the critical values corresponding to the three-qubit, two-qubit, and qubit system, respectively.

For $n = 5$, if the dimension of Hilbert space for the local system is $d = 2$, i.e., the qubit system, we find that the sharing of preparation contextuality is possible for at most seven Bobs (Fig. 1). Importantly, instead of the qubit system, if the two-qubit system is taken, the number of Bobs sharing preparation contextuality remains the same. Note that the Bell expression for $n = 5$ reaches its optimal value for the two-qubit system. However, for every j, the value of the unsharpness parameter required for sequential violation of the preparation noncontextual bound is larger for the qubit system, as shown in Fig. 1. A similar feature is also obtained from the Bell expression for $n = 6$. It can be seen from Fig. 2 that an equal number of Bobs can sequentially share the preparation contextuality for qubit, two-qubit, and three-qubit systems. We may conjecture that this feature remains the same for any arbitrary n. For this, one needs to find the maximum quantum value of $(\mathcal{B}_n)_Q$ for the $d = 2^m$-dimensional systems where $m = 1, 2, \ldots, \lfloor n/2 \rfloor$. Thus, sharing of preparation contextuality using the qubit system is advantageous in the sense that one is required to deal with a lower-dimensional system.

VI. SUMMARY AND DISCUSSIONS

In summary, we have provided a family of Bell expressions that can certify various dimensions of the quantum system. Such Bell expressions were derived based on a two-party communication game known as the n-bit parity-oblivious random access code [33]. It can be shown that the success probability of that game can be solely determined by the aforementioned family of Bell expressions $(\mathcal{B}_n)$ where Alice and Bob use 2^{n-1} and n number of dichotomic measurements, respectively. In a RAC, the parity-oblivious condition implies that Alice may communicate any number ($<n$) of bits but such a communication does not allow Bob to retrieve the information about the parity of Alice's inputs. It is shown [33] that the parity-oblivious constraint is [36] equivalent to the preparation noncontextuality assumption in an ontological model. We have shown that for a given n such a constraint on the encoding scheme reduces the local bound of the family of Bell expressions $\mathcal{B}_n$ to the preparation noncontextual bound [36]. For a given n, the optimal quantum value of the Bell expression $\mathcal{B}_n$ can only be achieved for the quantum system having local Hilbert-space dimension $d = 2^{\lfloor n/2 \rfloor}$.

Note that the Bell expressions for $n = 2$ and 3 reduce to CHSH [37] and elegant Bell expressions [38], respectively, which cannot serve as dimension witnesses as they can be optimized for the qubit system. However, each of the Bell inequalities for $n \geqslant 4$ distinguishes the dimensions $d = 2^m$ of the Hilbert space with $m = 1, 2, \ldots, 2^{\lfloor n/2 \rfloor}$. We provided explicit examples by considering the Bell expressions for $n =$ 4, 5, and 6. It is shown that for both $n = 4$ and 5 the respective Bell expressions $\mathcal{B}_4$ and $\mathcal{B}_5$ certify qubit and two-qubit systems. But, for $n = 6$, we have found that the relevant Bell expression $\mathcal{B}_6$ certifies the qubit, two-qubit, and three-qubit local systems, as can be seen in Fig. 3.

Further, we have examined the sharing of preparation contextuality by multiple sequential Bobs through the violation of $(\mathcal{B}_n)_{\text{PNC}}$ in Eq. (12) when the dimension of the system is lower than what is required for achieving the optimal quantum value $(\mathcal{B}_n)_Q^{\text{opt}}$. One of us has shown [39] that the sharing of preparation contextuality can be demonstrated by an arbitrary number of Bobs by using the optimal quantum value $(\mathcal{B}_n)_Q^{\text{opt}}$ for $d = 2^{\lfloor n/2 \rfloor}$-dimensional Hilbert space. Here, by providing the examples of $n = 5$ and 6, we have demonstrated that even for a lower-dimensional system the number of Bobs who can share the preparation contextuality remains the same but the value of the unsharpness parameter required is always higher in a lower-dimensional system.

Finally, we remark that from our paper it is straightforward to understand that for any arbitrary n the family of Bell expressions $\mathcal{B}_n$ can certify the Hilbert space having dimension 2^m with $m = 1, 2, \ldots, 2^{\lfloor n/2 \rfloor}$. For this, following the scheme presented here the maximum quantum value $(\mathcal{B}_n)_{d=2^m}$ for different m values has to be derived. The analytical derivation can be lengthy with increasing value of n but it is doable to some extent. The numerical technique can also be an obvious option. It would then be interesting to examine if the sharing of preparation contextuality can be demonstrated through the Bell expression $\mathcal{B}_n$ by an unbounded number of Bobs even for a qubit system. This calls for an experimental test of the sharing of preparation contextuality for an unbounded number of sequential observers. Studies along this line could be an interesting avenue for further research.

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APPENDIX A: EXPLICIT PARITY-OBLIVIOUS CONDITIONS FOR $n = 4$

Here we provide the details of the calculation of how the choices of Alice's observables that provide the maximum quantum value of $\mathcal{B}_{4,y}$ for qubit and two-qubit systems satisfy the parity-oblivious condition given by Eq. (6). Alice's choices of observables can be found by using Eq. (19) given the choices of $B_{4,y}$s. In order to examine this issue, we consider the nontrivial elements of the parity set $s_4 \in \mathbb{P}_4$ where $s_4 = 1110$, 1101, 1011, and 0111. This dictates that corresponding to each element of s_4 we have four different functional relations between Alice's observables satisfying the parity-oblivious condition in Eq. (6), which are the following:

$$\begin{aligned}
&A_{4,1}+A_{4,2}-A_{4,3}-A_{4,4}-A_{4,5}-A_{4,6}-A_{4,7}+A_{4,8}=0,\\
&A_{4,1}-A_{4,2}+A_{4,3}-A_{4,4}-A_{4,5}-A_{4,6}+A_{4,7}-A_{4,8}=0,\\
&A_{4,1}-A_{4,2}-A_{4,3}+A_{4,4}-A_{4,5}+A_{4,6}-A_{4,7}-A_{4,8}=0,\\
&A_{4,1}-A_{4,2}-A_{4,3}-A_{4,4}+A_{4,5}+A_{4,6}+A_{4,7}+A_{4,8}=0.
\end{aligned} \tag{A1}$$

For a particular case, say, $s = 1110$, using Eq. (21) we find that the relations between $\omega_{4,i}$ required to be satisfied are given by

$$\begin{aligned}
&\alpha_{4,1}+\alpha_{4,2}-\alpha_{4,3}-\alpha_{4,4}+\alpha_{4,5}-\alpha_{4,6}-\alpha_{4,7}+\alpha_{4,8}=0,\\
&\alpha_{4,1}+\alpha_{4,2}-\alpha_{4,3}+\alpha_{4,4}-\alpha_{4,5}-\alpha_{4,6}+\alpha_{4,7}-\alpha_{4,8}=0,\\
&\alpha_{4,1}+\alpha_{4,2}+\alpha_{4,3}-\alpha_{4,4}-\alpha_{4,5}+\alpha_{4,6}-\alpha_{4,7}-\alpha_{4,8}=0,\\
&\alpha_{4,1}-\alpha_{4,2}-\alpha_{4,3}-\alpha_{4,4}-\alpha_{4,5}+\alpha_{4,6}+\alpha_{4,7}+\alpha_{4,8}=0.
\end{aligned} \tag{A2}$$

Here $\alpha_{4,i} = 1/\omega_{4,i}$. A similar set of four relations between $\alpha_{4,i}$ can be found for each of the other elements of s_4. There are two conditions on $\omega_{4,i}$s available for which four equations in Eq. (A2) will be simultaneously satisfied: first, if $\omega_{4,1} = \omega_{4,2} = \omega_{4,5} = \omega_{4,6} = \omega'_4$ and $\omega_{4,2} = \omega_{4,3} = \omega_{4,7} = \omega_{4,8} = \omega''_4$; and second, if for every i the $\omega_{4,i}$ is the same. Note that the second condition requires four mutually anticommuting observables and thus cannot be satisfied for the qubit system. Similar relations can be obtained for the other three elements of s_4. Using the first restriction on $\omega_{4,i}$, we obtain Eq. (28) and this in turn proves that the parity-oblivious condition is satisfied by Alice's choices of observables.

APPENDIX B: DETAILED CALCULATION FOR $n = 5$

For $n = 5$ the Bell expression involves the measurements of eight and five dichotomic observables by Alice and Bob, respectively. From Eq. (10), the Bell expression can be written as

$$\begin{aligned}
\mathcal{B}_5 = {}& A_1\otimes(B_1+B_2+B_3+B_4+B_5)+A_2\otimes(B_1+B_2+B_3+B_4-B_5)+A_3\otimes(B_1+B_2+B_3-B_4+B_5)\\
&+A_4\otimes(B_1+B_2+B_3-B_4-B_5)+A_5\otimes(B_1+B_2-B_3+B_4+B_5)+A_6\otimes(B_1+B_2-B_3+B_4-B_5)\\
&+A_7\otimes(B_1+B_2+B_3-B_4-B_5)+A_8\otimes(B_1+B_2-B_3-B_4+B_5)+A_9\otimes(B_1-B_2-B_3+B_4+B_5)\\
&+A_{10}\otimes(-B_1-B_2+B_3+B_4+B_5)+A_{11}\otimes(B_1+B_2-B_3+B_4-B_5)+A_{12}\otimes(B_1-B_2+B_3-B_4+B_5)\\
&+A_{13}\otimes(B_1-B_2+B_3+B_4-B_5)+A_{14}\otimes(-B_1+B_2-B_3+B_4+B_5)+A_{15}\otimes(-B_1+B_2+B_3+B_4-B_5)\\
&+A_{16}\otimes(-B_1+B_2+B_3-B_4+B_5),
\end{aligned} \tag{B1}$$

the preparation noncontextual bound of which is $(\mathcal{B}_5)_{\text{PNC}} \leqslant 16$. As mentioned in the main text, Alice's observables need to satisfy the parity-oblivious conditions given by Eq. (21). The parity set $\mathbb{P}_5$ contains 11 nontrivial elements and each of them provides a functional relationship between Alice's choice of observables $A_{5,i}$. For example, if we take one of the elements, say, 11111, a functional relation between $A_{n,i}$s has to be satisfied by Alice's observables, given by

$$A_{4,1}-A_{4,2}-A_{4,3}-A_{4,4}-A_{4,5}-A_{4,6}+\sum_{i=7}^{16}A_{n,i}=0. \tag{B2}$$

A similar ten more such constraints can be found for other elements $s_5 \in \mathbb{P}_n$. Note that the condition for optimization required for the SOS approach is given by

$$\forall i \quad \sum_{y=1}^{n}(-1)^{x_y^i}\alpha_{n,i}B_{n,y}|\psi\rangle = A_{n,i}|\psi\rangle \tag{B3}$$

where $\alpha_{5,i} = 1/\omega_{5,i}$. In the present case of $n = 5$, by using Eq. (B3), from Eq. (B2) we have the following conditions on $\alpha_{5,i}$s that need to be satisfied, given by

$$\alpha_{5,1} - \alpha_{5,2} - \alpha_{5,3} - \alpha_{5,4} - \alpha_{5,5} + \alpha_{5,6} + \alpha_{5,7} + \alpha_{5,8} + \alpha_{5,9} - \alpha_{5,10} + \alpha_{5,11} + \alpha_{5,12} + \alpha_{5,13} - \alpha_{5,14} - \alpha_{5,15} - \alpha_{5,16} = 0, \tag{B4}$$

$$\alpha_{5,1} - \alpha_{5,2} - \alpha_{5,3} - \alpha_{5,4} + \alpha_{5,5} - \alpha_{5,6} + \alpha_{5,7} + \alpha_{5,8} - \alpha_{5,9} - \alpha_{5,10} + \alpha_{5,11} - \alpha_{5,12} - \alpha_{5,13} + \alpha_{5,14} + \alpha_{5,15} + \alpha_{5,16} = 0, \tag{B5}$$

$$\alpha_{5,1} - \alpha_{5,2} - \alpha_{5,3} + \alpha_{5,4} - \alpha_{5,5} - \alpha_{5,6} + \alpha_{5,7} - \alpha_{5,8} - \alpha_{5,9} + \alpha_{5,10} - \alpha_{5,11} + \alpha_{5,12} + \alpha_{5,13} - \alpha_{5,14} + \alpha_{5,15} + \alpha_{5,16} = 0, \tag{B6}$$

$$\alpha_{5,1} - \alpha_{5,2} + \alpha_{5,3} - \alpha_{5,4} - \alpha_{5,5} - \alpha_{5,6} - \alpha_{5,7} - \alpha_{5,8} + \alpha_{5,9} + \alpha_{5,10} + \alpha_{5,11} - \alpha_{5,12} + \alpha_{5,13} + \alpha_{5,14} + \alpha_{5,15} - \alpha_{5,16} = 0, \tag{B7}$$

$$\alpha_{5,1} + \alpha_{5,2} - \alpha_{5,3} - \alpha_{5,4} - \alpha_{5,5} - \alpha_{5,6} - \alpha_{5,7} + \alpha_{5,8} + \alpha_{5,9} + \alpha_{5,10} - \alpha_{5,11} + \alpha_{5,12} - \alpha_{5,13} + \alpha_{5,14} - \alpha_{5,15} + \alpha_{5,16} = 0. \tag{B8}$$

The functional relations between $\omega_{5,i}$ given by Eqs. (B4)–(B8) provide two solutions: first, $\omega_{5,1} = \omega_{5,2} = \omega_{5,4} = \omega_{5,5} = \omega_{5,9} = \omega_{5,11} = \omega_{5,12} = \omega_{5,15} = \omega'_5$ and $\omega_{5,3} = \omega_{5,6} = \omega_{5,7} = \omega_{5,8} = \omega_{5,10} = \omega_{5,13} = \omega_{5,14} = \omega_{5,16} = \omega''_5$; and second, $\omega_{5,i}$ values are equal to each other for every i. Note that the second condition cannot be satisfied by the observables in a qubit system. Using the first solution, we find that the following relations between $B_{5,y}$ need to be satisfied: $\{B_{5,1}, B_{5,2}\} = \{B_{5,2}, B_{5,3}\} = \{B_{5,1}, B_{5,3}\} = \{B_{5,3}, B_{5,4}\} = \{B_{5,3}, B_{5,5}\} = \{B_{5,4}, B_{5,5}\} = 0$, $\{B_{5,1}, B_{5,4}\} = \{B_{5,1}, B_{5,5}\}$, and $\{B_{5,2}, B_{5,4}\} = -\{B_{5,2}, B_{5,5}\}$. Using those relations and Eq. (18), we can write

$$(\mathcal{B}_5)_Q \leqslant 4(\omega'_5 + \omega''_5) = [\sqrt{5 + \{(B_{5,1} + B_{5,2}), (B_{5,4} + B_{5,5})\}} + \sqrt{5 + \{(B_{5,1} - B_{5,2}), (B_{5,4} - B_{5,5})\}}] \tag{B9}$$

which is Eq. (30) in the main text. Thus, Alice's observables maximizing $(\mathcal{B}_5)_Q$ satisfy the parity-oblivious condition.

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