# Quantum particles that behave as free classical particles

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The existence of nonvanishing Bohm potentials, in the Madelung-Bohm version of the Schrödinger equation, allows for the construction of particular solutions for states of quantum particles interacting with nontrivial external potentials that propagate as free classical particles. Such solutions are constructed with phases which satisfy the classical Hamilton-Jacobi for free particles and whose probability densities propagate with constant velocity, as free classical particles do.

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## I. INTRODUCTION

In 1979, Berry and Balazs [1] showed that a quantum-free wave packet can show unexpected accelerating characteristics. In this article we address the opposite question, that is, whether the wave function of the interacting particle, that satisfies the Schrödinger equation for a potential $V$, may propagate as free particles in the sense that their probability densities propagate with constant velocity as if they were free classical particles. Here, we establish the conditions which make this behavior possible and present numerous examples. The theoretical results presented in this article are in the same spirit as several others that have had experimental confirmation in optics and quantum mechanics [2–7].

We prove that such a possibility indeed exists in the framework of nonrelativistic quantum mechanics and its relation to the existence of the so-called Bohm potential. In other words, there are quantum solutions, for families of external potentials $V$, in which the wave function for the particle propagates as a free classical particle. This is only possible for a nonvanishing Bohm potential, which in turn implies that the amplitude of the wave function is not constant. We focus on one-dimensional systems, although our results can be generalized to higher dimensions [8–10], or to relativistic regimes [11] following the ideas presented here.

By a free classical particle, we understand it to be any particle of mass $m$ satisfying the free Hamilton-Jacobi (HJ) equation

$$\frac{1}{2m}(S')^2+\dot{S}=0, \tag{1}$$

for an action $S=S(x,t)$, where $'\equiv\partial_x$, and $\dot{}\equiv\partial_t$. The propagating solutions of this equation have a constant velocity. Thus, this action may be considered as the phase of a solution to the Schrödinger equation. Therefore, we are looking for wave functions with a phase satisfying Eq. (1), and with an amplitude that allows us to solve the Schrödinger equation for a given potential $V$.

Let us consider the wave function $\psi=\psi(x,t)$ of a one-dimensional Schrödinger equation (and its complex conjugate) for a real potential $V(x,t)$,

$$-\frac{\hbar^2}{2m}\psi''+V\psi-i\hbar\dot{\psi}=0. \tag{2}$$

The wave function may be written in terms of a polar decomposition as $\psi=A\exp{(iS/\hbar)}$, where the amplitude $A(x,t)$ and the phase $S(x,t)$ are real functions. Thereby, the Schrödinger equations become [12–17]

$$\frac{1}{2m}(S')^2+V_B+V+\dot{S}=0, \tag{3}$$

$$\frac{1}{m}(A^2S')'+(A^2)^{\cdot}=0, \tag{4}$$

where the Bohm potential is given by

$$V_B\equiv-\frac{\hbar^2}{2m}\frac{A''}{A}. \tag{5}$$

Equation (3) is the quantum Hamilton-Jacobi (QHJ) equation for the (external) potential $V$. The quantum modification consists in the addition of the Bohm potential to the classical HJ equation. Equation (4) is the continuity (probability conservation) equation. To enforce that the probability density of a quantum interacting particle propagates as a free classical particle, we need to require that the Bohm potential cancels out any contribution of the external potential,

$$V_B+V=0, \tag{6}$$

allowing the phase, from Eq. (3), to fulfill the HJ equation (1). The above condition implies that the external potential determines completely the dynamics of the amplitude $A$, through the Bohm potential. This also must be consistent with the continuity equation (4).

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The continuity equation (4) is identically solved by defining the arbitrary potential function $f = f(x,t)$, such that $A^2 = f'$, and $A^2 S' = -m\dot{f}$. For a one-dimensional system, once the free-particle action $S$ is found by solving HJ equation (1), $f = f(x,t)$ can be determined by the relation

$$f'S' + m\dot{f} = 0. \tag{7}$$

This equation states that $f$ depends on $x$ and $t$ through one variable only. On the other hand, the amplitude of the wave function is found to be given by the relation (with $f' > 0$)

$$A^2 = f'. \tag{8}$$

This allows, in principle, to have negative amplitudes. Examples of this are shown below, as well as others where the amplitude is always positive (without nodes in $x$) and they are also normalizable.

When the exact form of the amplitude $A$ (or function $f$) is found by solving Eq. (6), a quantum particle in the presence of a potential $V$ propagates as a free particle, in the sense that its phase is equal to the action for a free classical particle, while its amplitudes have a $x$ and $t$ dependence completely determined by the phase. Therefore, this solution is a quantum state with a phase that coincides with the action for a free classical particle, while its probability density propagates as a free classical particle with constant velocity. These solutions are nondiffracting wave packets and they do not correspond to a solution in a classical limit [17], and neither can they be found by a Galilean transformation [18].

## II. SEPARABLE ACTION FOR FREE CLASSICAL PARTICLE

Let us study the simplest case for a classical free particle, in which the spatial and temporal dependence are separated. The phase (action) is given by

$$S(x,t) = k\,x - \frac{k^2}{2m}t. \tag{9}$$

This action is a solution of (1) for any constant $k$. Equation (7) allows us to find that $f$ depends on $x$ and $t$ through one variable $z$ only. In this case, we obtain that it has the form $f(x,t) = f(z) = f(x - kt/m)$, and thus, by Eq. (8), we obtain that the amplitude depends on the same variable $z$ as

$$A(x,t)^2 = A(z)^2 = \frac{df}{dz}, \tag{10}$$

where $z$ is defined as

$$z \equiv x - \frac{k}{m}t. \tag{11}$$

A quantum particle interacting with an external potential $V(x,t) = V(z)$, propagates with phase (9), if the amplitude fulfills Eq. (6), in the form

$$V(z) = \frac{\hbar^2}{2mA(z)}\frac{d^2A(z)}{dz^2}. \tag{12}$$

For this case, all considered external potentials $V$ must depend on the $z$ variable, and therefore they are not static. In this form, any solution of Eq. (12) corresponds to a quantum particle which propagates with the action of a free classical particle, and its probability density moves with the velocity $k/m$. Several different solutions are described below.

*Constant force.* Consider a constant force $F = -V'$, with potential $V(z) = -Fz$. Thus, Eq. (12) produces an amplitude given in terms of Airy functions

$$A(x,t) = \mathrm{Ai}\left(-\left(\frac{2mF}{\hbar^2}\right)^{1/3} z\right). \tag{13}$$

This Airy wave packet propagates as a free classical particle (without acceleration) under a constant force.

*Moving potential trap.* An attractive potential with the form $V(z) = -\gamma\,\delta(z)$ is used to manipulate particles [19,20]. Here, $\gamma$ is a constant, and $\delta$ is the Dirac delta function. The amplitude solution of Eq. (12) becomes

$$A(x,t) = \frac{m\gamma\beta}{\hbar^2} z\,\mathrm{sgn}(z) - \beta, \tag{14}$$

for an arbitrary constant $\beta$, and where sgn is the sign function.

*Coulomb potential for a moving charge.* Let us assume a potential with the form $V(z) = \alpha/z$, for a moving charge with constant nonrelativistic velocity ($\alpha$ is a constant). This corresponds to the nonrelativistic expression for the Liénard-Wiechert four-potential [21]. In this case, Eq. (12) gives an amplitude in terms of Bessel functions $K_1$,

$$A(x,t) = \frac{\sqrt{2m\alpha\, z}}{\hbar} K_1\left(\frac{2\sqrt{2m\alpha\, z}}{\hbar}\right). \tag{15}$$

Solutions in terms of Bessel functions $I_1$ are also possible. Thus, this Coulomb potential produces Bessel wave packets that allow the particles to propagate freely.

*Electromagnetic wave.* A particle interacting with an electromagnetic wave (with wave number $\kappa$ and frequency $\kappa k/m$) experience a potential of the form $V(z) = \gamma\cos(\kappa z)$ (with constant $\gamma$). In this case, Eq. (12) becomes a Mathieu equation

$$\frac{d^2A}{dz^2} - \frac{2m\gamma}{\hbar^2}\cos(\kappa z)A = 0. \tag{16}$$

Explicit solutions are written in terms of the recurrence relations [22,23]. In this form, Mathieu beam wave packets support quantum solutions that propagate in a free classical fashion.

*Harmonic oscillator.* For a shifted harmonic oscillator $V(z) = m\omega^2 z^2/2$ [24], with frequency $\omega$, Eq. (12) has a solution of parabolic cylinder functions [23],

$$A(x,t) = D_{-\frac{1}{2}}\left(\sqrt{\frac{2m\omega}{\hbar}}z\right). \tag{17}$$

*Pöschl-Teller potential.* Consider the moving potential $V(z) = -\gamma\,\mathrm{sech}^2 z$, with a constant $\gamma$. The amplitude solution of (12) is written in terms of a Legendre polynomial $P$ and a Legendre function of the second kind $Q$ as

$$A(x,t) = a_1 P_n(\tanh z) + a_2 Q_n(\tanh z) \tag{18}$$

with arbitrary $a_1$ and $a_2$, and $n = (\sqrt{1 + 8m\gamma/\hbar^2} - 1)/2$.

*Constant modified harmonic oscillator.* The above examples have the feature that the wave functions have nodes in $x$. However, normalizable states without nodes can also be found satisfying Eq. (12). An example of this is the system subject to a (shifted) harmonic oscillator, with $V(z) = m\omega^2 z^2/2 - \hbar\omega/2$, and arbitrary frequency $\omega$. This potential solves Eq. (12) for the amplitude

$$A(x,t) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar} z^2\right), \tag{19}$$

which gives rise to a square-integrable wave function.

*Constant modified Pöschl-Teller potential.* Examples of physical states can also be obtained for the following Pöschl-Teller potential $V(z) = -(\hbar^2/m)\,\mathrm{sech}^2 z + \hbar^2/(2m)$. In this case, the solution of Eq. (12) gives a propagating solitonic amplitude

$$A(x,t) = \frac{1}{\sqrt{2}}\,\mathrm{sech}\, z, \tag{20}$$

which produces a square-integrable wave function with a probability amplitude that propagates as a free classical particle with action (9).

## III. NONSEPARABLE ACTION FOR FREE CLASSICAL PARTICLE

Another very well-known solution for the classical HJ (1) for classical free particles is

$$S(x,t) = \frac{m\,(x-x_0)^2}{2(t-t_0)}, \tag{21}$$

for an arbitrary initial position $x_0$ and initial time $t_0 < t$. This action is a nonseparable function of space and time.

In this case, Eq. (7) allows us to find that any function with the functionality $f(x,t) = f(y)$ solves the continuity equation, where we have introduced the variable

$$y \equiv \frac{x-x_0}{t-t_0}. \tag{22}$$

Therefore, the amplitude is given by

$$A(x,t) = \frac{1}{\sqrt{t-t_0}}\mathcal{A}(y), \tag{23}$$

with $\mathcal{A}^2 = df/dy$. In this case, any external potential with the form

$$V(x,t) = \frac{1}{(t-t_0)^2}\mathcal{V}(y), \tag{24}$$

allows us to rewrite Eq. (6) as

$$\mathcal{V}(y) = \frac{\hbar^2}{2m\mathcal{A}(y)}\frac{d^2\mathcal{A}(y)}{dy^2}. \tag{25}$$

Potentials with the exact space and time dependence of the form (24) have been shown to produce exact Feynman propagators [25]. In this form, any solution of Eq. (25) produces a quantum particle that propagates classically with action (21) and a probability density that propagates with constant velocity $(x-x_0)/(t-t_0)$. Below we study some of them in our context.

*Time-decreasing force.* For a force decreasing in time with the form $F(t) = F_0/(t-t_0)^3$, a potential $\mathcal{V}(y) = -F_0\,y$ can be used. In this case, Eq. (25) produces Airy solutions, and amplitude (23) is

$$A(x,t) = \frac{1}{\sqrt{t-t_0}}\mathrm{Ai}\left(-\left(\frac{2mF_0}{\hbar^2}\right)^{1/3} y\right). \tag{26}$$

Thus, for such forces, the quantum system is solved exactly, and the particle propagates as if it were free.

*Harmonic oscillator.* Consider the harmonic oscillator potential $V = m\omega^2 x^2/2$. For a time-decreasing frequency in the form $\omega = \omega_0/(t-t_0)^2$ [25] (with constant $\omega_0$), then the harmonic oscillator with potential $\mathcal{V}(y) = m\omega_0^2 y^2/2$ can be solved exactly. Using Eq. (25), amplitudes are given in terms of the parabolic cylinder functions [23]

$$A(x,t) = \frac{1}{\sqrt{t-t_0}} D_{-\frac{1}{2}}\left(\sqrt{\frac{2m\omega_0}{\hbar}}\,y\right). \tag{27}$$

*Coulomb-like potentials.* Consider a potential with the form $V(x,t) = Z(t)/x$. When $Z$ decreases in time as $Z(t) = Z_0/(t-t_0)$ [25], then $\mathcal{V} = Z_0/y$, and there exist solutions using our approach. The amplitude of the wave function is again given in terms of the Bessel functions $K_1$,

$$A(x,t) = \frac{\sqrt{2mZ_0\,y}}{\hbar\sqrt{t-t_0}} K_1\left(\frac{2\sqrt{2mZ_0\,y}}{\hbar}\right). \tag{28}$$

*Constant modified harmonic oscillator.* It is possible to construct square-integrable wave functions satisfying condition (25) for all of the examples presented here and in Sec. II. A simple example of this is for the harmonic oscillator with potential $V(x,t) = m\omega^2 x^2/2 - \hbar\omega/2$, with time-decreasing frequency $\omega = \omega_0/(t-t_0)^2$. This allows us to define the potential $\mathcal{V}(y) = m\omega_0^2 y^2/2 - \hbar\omega_0/2$, which according to Eq. (25) produces an amplitude of the form

$$A(x,t) = \left(\frac{m\omega_0}{\hbar\,\pi\,(t-t_0)^2}\right)^{1/4} \exp\left(-\frac{m\omega_0}{2\hbar} y^2\right). \tag{29}$$

This amplitude defines a physical normalizable wave function that propagates with action (21) under this harmonic oscillator potential.

## IV. DISCUSSION

With the above several examples and calculations we have shown that is possible for interacting quantum particles to have a probability density that propagates as a free classical particle for a wide range of known potentials. These quantum solutions have a phase that coincides with the action for a free classical particle, and therefore they are nontrivial solutions of quantum mechanics. This is only achieved because the Bohm potential of the wave function cancels out the external potential. By doing this, the external potential completely determines the amplitude of the wave packets, as it can be seen in Eqs. (12) and (25).

The condition (6) allows us to describe our solutions as quantum particles that propagate as classical ones, as they satisfy the HJ equation (1). However, they also satisfy the free-space Liouville equation $\partial_t\mathcal{F} + (p/m)\partial_x\mathcal{F} = 0$, for the phase-space density [26]

$$\mathcal{F}(x, p, t) = A(x, t)^2 \delta\left(p - \frac{\partial S}{\partial x}\right), \quad (30)$$

where $\delta$ is the Dirac delta function, and $A^2 = \int \mathcal{F} dp$ is the probability density of the studied solutions in each section. Here, $p = \partial_x S$ is the constant momentum, which for solutions of Sec. II is $p = k$, while for solutions of Sec. III is $p = m(x - x_0)/(t - t_0)$. The solutions presented above satisfy the free-space Liouville equation which implies that they behave as free classical particles.

It is remarkable the solutions explored in this work occur for the large family of potentials treated here. We think they can bring different insights in the propagation of quantum particles, as the quantum characteristics remain confined to the amplitude, while the phase is associated with the action of a free classical particle. Furthermore, we have shown that square-integrable wave functions can be obtained for known potentials, thus representing physical states.

Any solution fulfilling condition (6) can now be interpreted as a nondiffracting wave packet that modified its own probability density in order to propagate as if it were free. The implications of this behavior are not difficult to be envisaged as very interesting, as other quantum wave packets, with similar features, such as accelerating and curved properties, have been constructed and measured in laboratories [2–7].

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