Shareability of quantum steering and its relation with entanglement

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Steerability is a characteristic of quantum correlations lying between entanglement and Bell nonlocality. Understanding how these steering correlations can be shared between different parties has profound applications in ensuring the security of quantum communication protocols. Here we show that at most two bipartite reduced states of a three-qubit state can violate the three-setting CJWR linear steering inequality, contrary to the two-setting linear steering inequality. This result explains that quantum steering correlations have limited shareability properties apart from the conventional 'nonshareable' monogamy constraint. In contrast to the two-measurement-setting scenario, the three-setting scenario turns out to be more useful for developing a deeper understanding of the shareability of tripartite steering correlations. Apart from the distribution of steering correlations, several relations between reduced bipartite steering, different measures of bipartite entanglement of reduced states, and genuine tripartite entanglement are presented here. The results enable the detection of different kinds of tripartite entanglement.

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I. INTRODUCTION

The success of a secure quantum network depends on quantum correlations distributed and shared among different parties over many sites [1]. Different kinds of quantum correlations, for instance, multipartite entanglement [2,3] and multipartite nonlocality [4], have been used extensively as a resource to perform many tasks in such networks. A key property of these quantum correlations used to secure quantum networks is that they have limited shareability properties and sometimes can even be monogamous. For example, when a quantum system A is entangled with another system B, then this entanglement puts a constraint on the amount of entanglement that can exist between one of those parties (B, say) and a third party, C. This limited shareability phenomenon is termed monogamy. This is one of the fundamental differences between quantum entanglement and classical correlations, where all classical probability distributions can be shared [5]. Monogamy of entanglement was first quantified by Coffman, Kundu, and Wootters (CKW) in [6], where it was shown that the sum of the individual pairwise entanglement between A and B and C cannot exceed the entanglement between A and the remaining parties together. Since then much research work has been done on such monogamy relations of quantum entanglement [7–12]. This characteristic of quantum entanglement has found potential applications in various quantum information processing tasks such as quantum key distribution [13,14], classification of quantum states [15–17], and study of black-hole physics [18] and frustrated spin systems [19]. Similarly to monogamy of entanglement, if any two quantum systems A and B are correlated in such a way that they violate Bell-CHSH inequality [20], then neither A nor B can be Bell-CHSH nonlocal with the other system, C. For a given Bell inequality, this 'nonshareability' feature of quantum correlations is termed 'monogamy of nonlocality.' Otherwise, these correlations are shareable. In the last few years, several fundamental results on shareability of nonolocal correlations have been proven that constrain the distribution of nonlocal correlations in terms of violation of some Bell-type inequalities among the subsystems of a multipartite system [5,21–32] and they play a key role in the applications of quantum nonlocal correlations to cryptography [13,14]. Monogamy relations have also been studied for quantum discord [33], indistinguishability [34], coherence [35], measurement-induced nonlocality [36], and other nonclassical correlations [36–38].

Despite the importance of shareability in quantum information, the knowledge of shareability for quantum steering is so far rather limited [39–41]. The objective of this paper is to achieve a better understanding of the shareability associated with quantum steering. The notion of steering was introduced by Schrödinger in 1935 [42] and the effect was recently formalized from the foundational as well as the quantum information perspective [43,44]. Considering two distant observers, Alice and Bob, sharing an entangled state, steering captures the fact that Alice, by performing a local measurement on her subsystem, can remotely steer Bob's state. This is not possible if the shared state is only classically correlated. This kind of quantum correlation is known as steering [45]. It can be understood as a form of quantum nonlocality intermediate between entanglement and Bell nonlocality [46]. Quantum steering is certified by the violation of steering inequalities. A number of steering inequalities have been designed to observe steering [47–56]. Violation of such steering inequalities certifies the presence of entanglement in a one-sided device-independent way. Steerable states were shown

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to be beneficial for tasks involving randomness generation [57], subchannel discrimination [58], quantum information processing [59], and one-sided device-independent processing in quantum key distribution [60]. However, comparatively little is known about the shareability of this type of nonlocality. By deriving shareability relations, one can understand how this special type of nonlocal correlation (steering) can be distributed over different subsystems. In this paper, by using the three-setting linear steering inequality formulated by Cavalcanti, Jones, Wiseman, and Reid (CJWR) [48,56], we derive different kinds of trade-off relations that quantify the amount of bipartite steering that can be shared among three-qubit systems. In turn, these trade-off relations help us to prove that at most two of three reduced states of an arbitrary three-qubit state can violate the three-setting CJWR linear steering inequality, contrary to the two-setting CJWR linear steering inequality or Bell-CHSH inequality, where at most one of the reduced states can violate those inequalities. Consequently, in general, steering correlations turn out to be shareable.

Over the past few years it has become clear that correlation statistics of two-body subsystems can be very fruitful for inferring the multipartite properties of a composite quantum system [61–67]. In this context, we have also studied how the reduced bipartite steering of a three-qubit state depends on the bipartite and genuine tripartite entanglement of the three-qubit state. Interestingly, criteria for detecting different kinds of entanglement of pure three-qubit states are obtained based on these shareability relations. We illustrate the relevance of our results with different examples.

II. PRELIMINARIES

In this section, we briefly discuss the concept of steering and the three-setting CJWR linear steering inequality that we use in this work.

Steering

Steering is usually formulated by considering a quantum information task [43,44]. Suppose that two spatially separated observers, say Alice and Bob, share a bipartite state ρ_{AB} and they can perform measurements in the sets M_A and M_B, respectively. In a steering test, Bob, who trusts his own but not Alice's apparatus, wants to verify whether the shared state between them is entangled. He will be convinced that the shared state ρ_{AB} is entangled only if his system is genuinely influenced by Alice's measurement, instead of some preexisting local hidden states (LHSs) which Alice may have access to. To make sure that Bob must exclude the LHS model

$$P(a,b|A,B,\rho_{AB})=\sum_{\lambda}p_{\lambda}P(a|A,\lambda)P_Q(b|B,\rho_{\lambda}), \tag{1}$$

in which $P(a,b|A,B,\rho_{AB})=\mathrm{Tr}(A_a\otimes B_b\,\rho_{AB})$ is the probability of getting outcomes a and b when measurements A and B are performed on ρ_{AB} by Alice and Bob, respectively; A_a and B_b are their corresponding measurement operators; λ is the hidden variable; ρ_λ is the state that Alice sends with probability $p_\lambda(\sum_\lambda p_\lambda=1)$; $P(a|A,\lambda)$ is the conditioned probability of Alice obtaining outcome a under λ; and $P_Q(b|B,\rho_\lambda)$ denotes the quantum probability of outcome b, given by measuring B on the local hidden state ρ_λ. Now, if Bob determines that such a correlation, $P(a,b|A,B,\rho_{AB})$, cannot be explained by any LHS model, then he will be convinced that Alice can steer his state, and thus the corresponding bipartite state is entangled. In short, the bipartite state ρ_{AB} is unsteerable by Alice to Bob if and only if the joint probability distributions satisfy Eq. (1) for all measurements A and B. The assumption of this LHS model leads to certain steering inequalities, violation of which indicates the occurrence of steering.

The simplest way of constructing steering inequality is to find constraint for the correlations between Alice's and Bob's measurement statistics. In this work, we are interested in using the CJWR type of linear steering inequality [48]. They proposed the following series of steering inequalities to check whether a bipartite state is steerable from Alice to Bob when both parties are allowed to perform n dichotomic measurements on their respective subsystems:

$$F_n(\rho_{AB},\mu)=\frac{1}{\sqrt{n}}\left|\sum_{k=1}^{n}\langle A_k\otimes B_k\rangle\right|\leqslant 1, \tag{2}$$

where $A_k=\hat{a}_k\cdot\overrightarrow{\sigma}$, $B_k=\hat{b}_k\cdot\overrightarrow{\sigma}$, $\overrightarrow{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ is a vector composed of the Pauli matrices, $\hat{a}_k\in\mathbb{R}^3$ are unit vectors, $\hat{b}_k\in\mathbb{R}^3$ are orthonormal vectors, $\mu=\{\hat{a}_1,\hat{a}_2,\ldots,\hat{a}_n,\hat{b}_1,\hat{b}_2,\ldots,\hat{b}_n\}$ is the set of measurement directions, $\langle A_k\otimes B_k\rangle=\mathrm{Tr}(\rho_{AB}(A_k\otimes B_k))$, and $\rho_{AB}\in\mathbb{H}_{\mathbb{A}}\otimes\mathbb{H}_{\mathbb{B}}$ is any bipartite quantum state.

Here our attention is confined to the qubit case. In the Hilbert-Schmidt representation any two-qubit state can be expressed as

$$\rho_{AB}=\frac{1}{4}\left[\mathbf{I}\otimes\mathbf{I}+\vec{a}\cdot\vec{\sigma}\otimes\mathbf{I}+\mathbb{I}\otimes\vec{b}\cdot\vec{\sigma}+\sum_{i,j}t_{ij}^{AB}\sigma_i\otimes\sigma_j\right], \tag{3}$$

$\vec{a}$ and $\vec{b}$ being the local bloch vectors and $T_{AB}=[t_{ij}^{AB}]$ the correlation matrix. The components t_{ij}^{AB} are given by $t_{ij}^{AB}=[\rho_{AB}\mathrm{Tr}\sigma_i\otimes\sigma_j]$ and $\vec{a}^2+\vec{b}^2+\sum_{i,j}(t_{ij}^{AB})^2\leqslant 3$. In [68], Luo showed that any two-qubit state can be reduced, by local unitary equivalence, to

$$\rho'_{AB}=\frac{1}{4}\left[\mathbf{I}\otimes\mathbf{I}+\vec{a}'\cdot\vec{\sigma}\otimes\mathbf{I}+\mathbb{I}\otimes\vec{b}'\cdot\vec{\sigma}+\sum_{i}u'_i\sigma_i\otimes\sigma_i\right], \tag{4}$$

where the correlation matrix of ρ'_{AB} is $T'_{AB}=\mathrm{diag}(u'_1,u'_2,u'_3)$. In [56], for any two-qubit state ρ'_{AB}, the authors derived an analytical expression for the maximum value of the two-setting and three-setting CJWR linear steering inequalities in terms of diagonal elements of the correlation matrix $T'_{AB}=\mathrm{diag}(u'_1,u'_2,u'_3)$.

Specifically, $\max_\mu F_2(\rho'_{AB},\mu)$ and $\max_\mu F_3(\rho'_{AB},\mu)$ have been evaluated to be the following:

$$\max_{\mu}F_2(\rho'_{AB},\mu)=\sqrt{u_1'^2+u_2'^2} \tag{5}$$

and

$$\max_{\mu}F_3(\rho'_{AB},\mu)=\sqrt{\mathrm{Tr}\left[T_{AB}'^{T}T'_{AB}\right]}, \tag{6}$$

where $u_1'^2$ and $u_2'^2$ are the two largest diagonal elements of $T_{AB}'^2$. Here we consider only the three-setting linear steering inequality, as under two measurement settings the notions of steering and Bell-CHSH nonlocality are indistinguishable [56]. Since the states given in Eqs. (3) and (4) are local unitary equivalent, we must have

$$\max_{\mu} F_3(\rho'_{AB},\mu) = \sqrt{\mathrm{Tr}\big[T'^{T}_{AB}T'_{AB}\big]} = \sqrt{\mathrm{Tr}\big[T^{T}_{AB}T_{AB}\big]} = \max_{\mu} F_3(\rho_{AB},\mu).$$

Consequently, the linear inequality, (2) (for three measurement settings), implies that any state ρ_{AB} is F_3 steerable if and only if

$$S_{AB} = \mathrm{Tr}\big[T^{T}_{AB}T_{AB}\big] > 1. \tag{7}$$

Note that this condition is just a sufficient criterion to check steerability. There exist steerable states which satisfy $S_{AB} \leqslant 1$.

III. SHAREABILITY AND MONOGAMY OF STEERING CORRELATIONS

Consider a scenario in which Alice, Bob, and Charlie share a three-qubit state ρ_{ABC}. Let ρ_{AB}, ρ_{AC}, and ρ_{BC} denote the three bipartite reduced states of ρ_{ABC}. In general, for tripartite states, monogamy relations have the form

$$Q(\rho_{AB}) + Q(\rho_{AC}) \leqslant Q(\rho_{A|BC}) \tag{8}$$

or

$$Q(\rho_{AB}) + Q(\rho_{AC}) \leqslant K \tag{9}$$

for some bipartite quantum measure Q and positive real number K. Here $Q(\rho_{A|BC})$ represents the correlation between subsystem A and subsystem BC. Entanglement, Bell-CHSH nonlocality, and steering (via the two-setting linear steering F_2 inequality) are examples of correlation measures satisfying this monogamy relation [Eq. (9)]. Particularly, for Bell-CHSH inequality and F_2 inequality, the monogamy relation, (9), takes the form [5,23,26,30]

$$Q(\rho_{AB}) + Q(\rho_{AC}) \leqslant 2. \tag{10}$$

From Eqs. (2) and (5), it is easy to see that violation of the F_2 inequality by any bipartite state ρ implies $\max_\mu F_2(\rho,\mu) > 1$ [i.e., $Q(\rho) > 1$]. Thus, the above trade-off relation [Eq. (10)] implies that at most one bipartite reduced state with respect to a certain observer (say A) can violate the linear steering F_2 inequality. This shows that quantum steering correlations (obtained by the violation of the F_2 inequality) between party A and party B cannot be shared with parties A and C. This 'nonshareability' feature is known as "monogamy of steering correlations." For a given bipartite steering inequality and a three-party quantum state ρ_{ABC} we consider the state to be monogamous for the steering inequality if the violation of the steering inequality among any two of its subparts (say ρ_{AB}) is not shareable with any other party. Monogamy of the linear steering inequality violation thus implies that only one among three reduced states of ρ_{ABC} can violate the steering inequality. Otherwise, the bipartite steering correlations obtained from the state ρ_{ABC} are shareable for this steering inequality.

It is a fact that entanglement is a property of a quantum state; now correlations generated due to measurements performed on any entangled quantum state are not solely determined by the state of the system under consideration. It is also dependent on the specific setup used to determine the correlations. Consequently, in general, steerability of a quantum state varies from one measurement scenario to another. In this context, an obvious question arises: Can the addition of one more observable per party change the monogamous nature of steering? An affirmative answer to this query is given by the following theorem.

Theorem 1. For any three-qubit state $\rho_{ABC} \in \mathbb{H}^A \otimes \mathbb{H}^B \otimes \mathbb{H}^C$, at most two reduced states can violate the three-setting CJWR linear steering inequality, i.e., steering correlations can be shareable when each party measures three dichotomic observables.

Proof. Any three-qubit state ρ_{ABC} can be represented as

$$\rho_{ABC} = \frac{1}{8}\Bigg[\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I} + \vec{a}\cdot\vec{\sigma}\otimes\mathbb{I}\otimes\mathbb{I} + \mathbb{I}\otimes\vec{b}\cdot\vec{\sigma}\otimes\mathbb{I} + \mathbb{I}\otimes\mathbb{I}\otimes\vec{c}\cdot\vec{\sigma} + \sum_{ij} t^{AB}_{ij}\sigma_i\otimes\sigma_j\otimes\mathbb{I} + \sum_{ik} t^{AC}_{ik}\sigma_i\otimes\mathbb{I}\otimes\sigma_k + \sum_{jk} t^{BC}_{jk}\mathbb{I}\otimes\sigma_j\otimes\sigma_k + \sum_{ijk} t^{ABC}_{ijk}\sigma_i\otimes\sigma_j\otimes\sigma_k\Bigg]. \tag{11}$$

In the following ρ_i denotes the reduced density matrices for subsystem $i = A, B, C$. One computes from Eq. (11) that

$$\mathrm{tr}\big(\rho_A^2\big) = \frac{1+\vec{a}^2}{2}, \quad \mathrm{Tr}\big(\rho_{BC}^2\big) = \frac{1}{4}(1+\vec{b}^2+\vec{c}^2+S_{BC}). \tag{12}$$

Similarly,

$$\mathrm{tr}\big(\rho_B^2\big) = \frac{1+\vec{b}^2}{2}, \quad \mathrm{Tr}\big(\rho_{AC}^2\big) = \frac{1}{4}(1+\vec{a}^2+\vec{c}^2+S_{AC}),$$

$$\mathrm{tr}\big(\rho_C^2\big) = \frac{1+\vec{c}^2}{2}, \quad \mathrm{Tr}\big(\rho_{AB}^2\big) = \frac{1}{4}(1+\vec{a}^2+\vec{b}^2+S_{AB}). \tag{13}$$

First, consider ρ_{ABC} a pure state. Then from Schimdt decomposition, we have $\mathrm{Tr}(\rho_i^2) = \mathrm{Tr}(\rho_{jk}^2)$ for $i \neq j \neq k$, $i, j, k = A, B, C$. Using these relations and Eqs. (12) and (13), it is straightforward to calculate S_{ij} for each pair of qubits, yielding

$$S_{AB} = 1 + 2\vec{c}^2 - \vec{a}^2 - \vec{b}^2, \tag{14}$$

$$S_{AC} = 1 + 2\vec{b}^2 - \vec{a}^2 - \vec{c}^2, \tag{15}$$

and

$$S_{BC} = 1 + 2\vec{a}^2 - \vec{b}^2 - \vec{c}^2. \tag{16}$$

Adding these three relations and simplifying, one obtains the following relation:

$$S_{AB} + S_{AC} + S_{BC} = 3. \tag{17}$$

This relation is derived by a method similar to that used in [69] for developing Bell monogamy relations.

Now, taking the mixed state ρ_{ABC} as $\sum_n p_n|\psi_n\rangle\langle\psi_n|$, one has $S_{AB} \leqslant \sum_n p_n S^n_{AB}$, and similar relations for S_{AC}, S_{BC}. Adding these relations and using Eq. (17), we obtain

$$S_{AB} + S_{AC} + S_{BC} \leqslant 3. \tag{18}$$

This is a trade-off relation among two qubits of any three-qubit state ρ_{ABC}. Now $S_{AB} > 1$ is sufficient for Alice and Bob to witness violation of the F_3 inequality. Hence, inequality Eq. (18) imposes a constraint on the quantum steering: It is impossible for all pairs of qubits to violate the F_3 inequality.

But the trade-off relation, (18), is unable to assure us about the number of two-qubit reduced states that can violate the F_3 inequality. To complete the proof, we still have to find two reduced states of ρ_{ABC} which violate the F_3 inequality.

Using Eqs. (14)–(16), one can easily find that the reduced states ρ_{AB} and ρ_{AC} of the pure three-qubit state ρ_{ABC} will violate the F_3 inequality iff the following inequality is satisfied:

$$\vec{c}^2 > \frac{\vec{a}^2 + \vec{b}^2}{2}, \quad \vec{b}^2 > \frac{\vec{a}^2 + \vec{c}^2}{2}. \tag{19}$$

One can similarly obtain the condition of violation for other pairs of reduced states. Now consider the fully entangled three-qubit state,

$$|\psi_{ABC}\rangle = \frac{1}{2}(|100\rangle + |010\rangle + \sqrt{2}|001\rangle). \tag{20}$$

By using the above conditions, one can find that bipartite correlations between party A and party C of subsystem AC and between party B and party C of subsystem BC violate the F_3 inequality: $S_{BC} = S_{AC} = 1 + \frac{1}{4} > 1$. This shows that some of the steering correlations between party A and party C can thus be shared with parties B and C. Thus, under some conditions [for example, Eq. (19) and its permutations], steering correlations are shareable with respect to the F_3 inequality. ■

The above result for symmetric states leads to the following corollary.

Corollary 3.1. None of the three reduced states of any three-qubit symmetric state ρ_{ABC} violates the F_3 inequality, i.e., steering is monogamous for such states with respect to the F_3 inequality. Theorem 1 guarantees the existence of three-qubit states for which all two-party reduced states with respect to a certain observer violate the F_3 inequality (Fig. 1). This shareable nature of steering allows one to reveal the shareable nature of the entanglement of bipartite mixed states. As far as the shareability of quantum correlations is concerned, quantum entanglement is, strictly speaking, only monogamous in the case of pure entangled states. If the state of two systems, say ρ_{AB}, is a mixed entangled state, then it is possible that both of the systems, A and B, are entangled with a third system, say C. For example, the so-called W state [15] $|W\rangle = \frac{(|001\rangle+|010\rangle+|100\rangle)}{\sqrt{3}}$ has bipartite reduced states that are all identical and entangled. Thus, entanglement of these reduced bipartite mixed states is shareable, however, the steering correlations obtainable from these states follow the monogamy inequality Eq. (10). So, by considering the F_2 inequality, one cannot reveal shareability of entanglement of bipartite mixed states. To reveal this, steering correlations obtainable from these states must be shareable. As shown in Theorem 1, the state $|\psi_{ABC}\rangle$ [Eq. (20)] provides steerable bipartite reduced

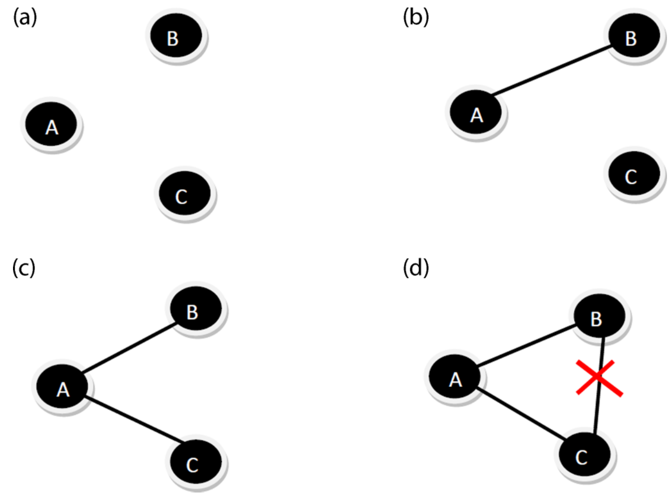

FIG. 1. Steering graphs: Here each circle represents a physical system and a solid line connecting two systems describes the bipartite steering correlation between them. Different possibilities of sharing bipartite steering among three distant physical systems are depicted. (a) No bipartite steering is detected between individual parties. For example, the tripartite GHZ state [70] $|\phi_{\rm GHZ}\rangle = \frac{|000\rangle+|111\rangle}{\sqrt{2}}$ has no bipartite steering. (b) Bipartite steering of one reduced state is detected. One such state of this kind is the pure biseparable state. (c) Two bipartite reduced states are steerable. As we show in Sec. III, state $|\psi_{\rm ABC}\rangle$ belongs to this group. (d) The trade-off relation, Eq. (18), prevents bipartite steering between all pairs of systems.

states between subsystem AC and subsystem BC. Therefore the corresponding reduced mixed states ρ_{AC} and ρ_{BC} are also entangled and entanglement of the two-qubit mixed entangled state ρ_{AC} is shareable with at least one other qubit. This in turn indicates that the F_3 inequality is an appropriate ingredient to reveal shareability of entanglement of mixed states.

Unlike the standard $|W\rangle$ state, the state $|\psi_{ABC}\rangle$ can be used as a resource for deterministic teleportation and dense coding [71]. As another application of the shareable nature of steering correlations, consider that a pure three-qubit state is provided to experimentalists, which they have to use as a resource in deterministic teleportation or dense coding. They are also provided with the information that the state is either $|\psi_{ABC}\rangle$ or $|W\rangle$. We show that the shareability phenomenon as described in Theorem 1 can be used to determine the desired state. For $|W\rangle$ state, $S_{ij} = 1$ for all reduced states, so monogamy is preserved. On the other hand, state $|\psi_{ABC}\rangle$ does not obey monogamy as shown in Theorem 1. Thus, the above result distinguishes the two types of states, though they belong to the same class (W-like states [15]).

Keeping in mind the usefulness of shareability relations, one naturally is interested to know which of the three-qubit states obey monogamy (or shareability) of steering. An explicit evaluation of the number of reduced steerable states along with the monogamy (or shareability) in each class of three-qubit pure states as classified by Sabín and García-Alcaine [72] is reported in the Appendix, where we see that only the steering correlation obtained from star-shaped states and W-like states can be shared. Next we ask whether the shareability behavior of those two classes of pure states is robust against white-noise admixture. The results are presented in the Appendix, where it is shown that less entangled states are more robust against white-noise admixture in comparison to more highly entangled states.

We note that the mere existence of the shareable nature of steering already follows from the work in [73]. This is due to the fact that shareability of nonlocality implies shareability of steering. Nevertheless, the present results are stronger in the sense that the F_3 inequality can detect much larger classes of shareable states compared to all facet inequalities in the three-setting Bell scenario [73,74]. For instance, the F_3 inequality detects the shareable nature of $|\psi_{ABC}\rangle$, which is not the case for any facet inequality in the three-setting Bell scenario, as each of the reduced states of $|\psi_{ABC}\rangle$ satisfies all facet inequalities under projective measurements.

Other than the constraint given by Eq. (19) and its permutations, few other conditions under which F_3 steering is shareable are derived in the following sections.

IV. REDUCED STEERING VERSUS ENTANGLEMENT

In two-qubit systems, the more entangled a pure state is, the more it can violate the Bell-CHSH inequality. In this context, a relevant study is to find the relation between violation of the F_3 inequality by the reduced bipartite states of a pure state and their corresponding entanglement (with respect to some measure). The relation between S_{AB} and concurrence $\mathcal{C}_{AB}$ (a measure of entanglement) [75,76] can be derived with methods similar to those used in [77]. For pure bipartite states the relation is $S_{AB} = 1 + 2\,\mathcal{C}_{AB}^2$. Hence, for pure states, more entanglement generates a larger violation of the F_3 inequality. However, from this relation we cannot infer anything about mixed bipartite reduced states of a three-qubit pure state. In the theorem below we derive a relation between them.

Theorem 2. The triples (S_{AB}, S_{AC}, S_{BC}) of three reduced states obtained from a pure three-qubit state and $(\mathcal{C}_{AB}, \mathcal{C}_{AC}, \mathcal{C}_{BC})$ maintain the same ordering, i.e.,

$$S_{AB} > S_{AC} > S_{BC} \quad \text{iff} \quad \mathcal{C}_{AB} > \mathcal{C}_{AC} > \mathcal{C}_{BC}. \tag{21}$$

Proof. By eliminating $\vec{a}$ from Eqs. (14) and (15), we have

$$S_{AB} - S_{AC} = 3(\vec{c}^2 - \vec{b}^2). \tag{22}$$

Now, the three tangle τ [6], for a three-qubit pure state, is given by [30]

$$\begin{aligned} \tau &= 1 - \vec{a}^2 - \mathcal{C}_{AB}^2 - \mathcal{C}_{AC}^2 \\ &= 1 - \vec{b}^2 - \mathcal{C}_{AB}^2 - \mathcal{C}_{BC}^2 \\ &= 1 - \vec{c}^2 - \mathcal{C}_{AC}^2 - \mathcal{C}_{BC}^2. \end{aligned} \tag{23}$$

Comparing these equalities, we obtain

$$\mathcal{C}_{AB}^2 - \mathcal{C}_{AC}^2 = \vec{c}^2 - \vec{b}^2 \tag{24}$$

and its permutations, which immediately lead to

$$S_{AB} - S_{AC} = 3\big(\mathcal{C}_{AB}^2 - \mathcal{C}_{AC}^2\big) \tag{25}$$

and its permutations. Thus, we have developed the ordering relation as per Eq. (21). ∎

It is interesting to note that (S_{AB}, S_{AC}, S_{BC}) and $(\vec{c}^2, \vec{b}^2, \vec{a}^2)$ follow the same ordering for all pure three-qubit states.

Distribution of bipartite quantum entanglement (i.e., entanglement of reduced bipartite states) of any pure three-qubit state is subjected to certain shareability laws. In particular, the addition of squared concurrence of all bipartite reduced states cannot be greater than $\frac{4}{3}$ [15]:

$$\mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 + \mathcal{C}_{BC}^2 \leqslant \tfrac{4}{3}. \tag{26}$$

This shareability constraint indicates that the shareability of the reduced bipartite steerability as well as the individual bipartite steerability of any pure three-qubit state might depend on concurrence of the reduced bipartite states. This is in fact the case. We next discuss a few results in this direction.

Theorem 3. If the squared concurrence of any bipartite reduced state for a pure three-qubit state is greater than $\frac{4}{9}$, the corresponding reduced state is F_3 steerable, i.e., if $\mathcal{C}_{ij}^2 > \frac{4}{9}$ $(i \neq j,\, i, j = A, B, C)$, the corresponding reduced state ρ_{ij} is F_3 steerable.

Proof. By using Eqs. (24) and (14)–(16), each of S_{ij} can be expressed in terms of $\mathcal{C}_{ij}$,

$$S_{AB} = 1 + 2\,\mathcal{C}_{AB}^2 - \mathcal{C}_{AC}^2 - \mathcal{C}_{BC}^2, \tag{27}$$

and its permutations. Let $\mathcal{C}_{AB}^2 = \frac{4}{9} + \epsilon$, where ϵ is a sufficiently small positive number. This immediately restricts the sum of squared concurrence of the other two bipartite reduced states,

$$\mathcal{C}_{AC}^2 + \mathcal{C}_{BC}^2 \leqslant \tfrac{8}{9} - \epsilon. \tag{28}$$

Applying these to the expression of S_{AB}, this leads to the sharp inequality $S_{AB} \geqslant 1 + 3\,\epsilon$. So, if $\mathcal{C}_{AB}^2 > \frac{4}{9}$, the F_3 inequality is violated. Similarly, it can be proved for other bipartite reduced states. ∎

This result holds for all pure three-qubit states. As an example, consider the pure state $|\psi_{ABC}\rangle$ which has two F_3 steerable reduced states, ρ_{AC} and ρ_{BC}, with $\mathcal{C}_{AC}^2 = \mathcal{C}_{BC}^2 = \frac{1}{2} > \frac{4}{9}$. However, one should note that the above inequality $\mathcal{C}_{ij}^2 > \frac{4}{9}$ is only a sufficient condition for F_3 steerability of the reduced bipartite state ρ_{ij}, because there are reduced states which violate the inequality $\mathcal{C}_{ij}^2 > \frac{4}{9}$ but still give rise to F_3 steerability. One such example is $|\phi_{\rm con}\rangle = \frac{\sqrt{3}}{2}|000\rangle + \frac{1}{2\sqrt{2}}|101\rangle + \frac{1}{2\sqrt{2}}|110\rangle$. For this state, from the above formulas one can obtain $\mathcal{C}_{AB}^2 = \frac{3}{8} < \frac{4}{9}$ and $S_{AB} = 1 + \frac{5}{16}$. Clearly, the reduced state ρ_{AB} violates the F_3 inequality, while it violates the inequality $\mathcal{C}_{AB}^2 > \frac{4}{9}$. Although an obvious necessary and sufficient condition can be derived from Eq. (27) and its permutations.

Corollary 3.1. Any reduced state ρ_{ij} of a pure three-qubit state will violate the F_3 inequality if and only if the squared concurrence of the corresponding reduced state is greater than the average of the squared concurrence of the remaining two reduced states, i.e., $S_{ij} > 1$ if and only if $\mathcal{C}_{ij}^2 > \frac{\mathcal{C}_{ik}^2 + \mathcal{C}_{jk}^2}{2}$, where $i \neq j \neq k$ and $i, j, k = A, B, C$.

Due to the shareability constraint, Eq. (26), violation of one of the reduced states (say ρ_{AB}) puts a strong restriction on the average of squared concurrences of the remaining reduced states.

Corollary 3.2. For any F_3 steerable reduced state ρ_{ij}, the inequality $\frac{\mathcal{C}_{ik}^2 + \mathcal{C}_{jk}^2}{2} < \frac{4}{9}$ holds, where $i \neq j \neq k$ and $i, j, k = A, B, C$.

As shown in [6], the sum of the squared concurrence between i and k and the squared concurrence between j and k cannot be greater than 1, i.e., $\mathcal{C}_{ik}^2 + \mathcal{C}_{jk}^2 \leqslant 1$. Hence, from the above corollary it is observed that this restriction is further strengthened if one considers the F_3 steerability of ρ_{ij}.

Since Corollary 3.2 imposes a more stringent restriction, using it we get the following sufficient condition for monogamy of F_3 steering:

Corollary 3.4. For any pure three-qubit state ρ_{ABC}, steering correlations will obey monogamy if $\mathcal{C}_{ik}^2 + \mathcal{C}_{jk}^2 \geqslant \frac{8}{9}$, where $i \neq j \neq k$ and $i, j, k = A, B, C$, holds for at least two of three possible cases.

It may be noted that Theorem 3 gives rise to a sufficient condition for shareability of F_3 steerability.

Corollary 3.5. F_3 steering is shareable if $\mathcal{C}_{ij}^2 > \frac{4}{9}$ ($i \neq j$ and $i, j = A, B, C$) for any two pairs of i, j.

Now we discuss how the F_3 inequality violation by the reduced bipartite states depends on the genuine entanglement of the three-qubit state. As shown in Sec. III, a maximum of two bipartite reduced states of ρ_{ABC} can violate the F_3 inequality, so the bipartite steering of ρ_{ABC} implies that it comes from one component of either the triple (S_{AB}, S_{AC}, S_{BC}) or $((S_{AB}, S_{AC}), (S_{AB}, S_{BC}), (S_{AC}, S_{BC}))$. Considering both possibilities, we adopt two measures, $S^{\max}(\rho_{ABC})$ and $S_{\text{total}}^{\max}(\rho_{ABC})$, where $S^{\max}(\rho_{ABC}) = \max\{S_{AB}, S_{AC}, S_{BC}\}$ and $S_{\text{total}}^{\max}(\rho_{ABC}) = \max\{S_{AB} + S_{AC}, S_{AB} + S_{BC}, S_{AC} + S_{BC}\}$.

In each case, we now derive a relation with tripartite entanglement of ρ_{ABC}.

Theorem 4. For an arbitrary three-qubit state ρ_{ABC}, the three tangle $\tau(\rho_{ABC})$ and maximum bipartite steering $[S^{\max}(\rho_{ABC})]$ with respect to the F_3 inequality obey the following complementary relation:

$$S^{\max}(\rho_{ABC}) + 2\tau(\rho_{ABC}) \leqslant 3. \tag{29}$$

Proof. Note that for a pure three-qubit state Eq. (14) provides $S_{AB} = 1 + 2\vec{c}^2 - \vec{a}^2 - \vec{b}^2$. Incorporating this with the third equality of the three tangle in Eq. (23), we obtain

$$\begin{aligned} S_{AB} + 2\tau(\rho_{ABC}) &= 3 - \vec{a}^2 - \vec{b}^2 - 2\mathcal{C}_{AC}^2 - 2\mathcal{C}_{BC}^2 \\ &\leqslant 3. \end{aligned} \tag{30}$$

Similarly, one has $S_{AC} + 2\tau(\rho_{ABC}) \leqslant 3$ and $S_{BC} + 2\tau(\rho_{ABC}) \leqslant 3$. Hence for the pure state $S^{\max}(\rho_{ABC}) + 2\tau(\rho_{ABC}) \leqslant 3$. As the three tangle τ and $S^{\max}(\rho_{ABC})$ both are convex under mixing, it implies that the relation in Eq. (29) holds for all three-qubit states. ■

This complementary relation suggests that the F_3 inequality violation by the reduced bipartite states depends on the tripartite entanglement present in the tripartite system. We determine a class of three-qubit genuinely entangled states which saturates the above-mentioned relation. This single-parameter class of states is given by $|\phi_m\rangle = \frac{|000\rangle + m(|101\rangle + |010\rangle) + |111\rangle}{\sqrt{2+2m^2}}$, where $m \in [0, 1]$. The above class of states has been identified in [78] as the maximum dense-coding-capable class of states. For this class of states, $S^{\max}(|\phi_m\rangle) = 1 + \frac{8m^2}{(1+m^2)^2}$ and $\tau(|\phi_m\rangle) = 1 - \frac{4m^2}{(1+m^2)^2}$. Hence, for this class of states, one can show the relation $S^{\max}(|\phi_m\rangle) + 2\tau(|\phi_m\rangle) = 3$.

Theorem 5. For an arbitrary three-qubit state ρ_{ABC}, the three tangle $\tau(\rho_{ABC})$ and maximum bipartite steering $[S_{\text{total}}^{\max}(\rho_{ABC})]$ satisfy the following complementary relation:

$$S_{\text{total}}^{\max}(\rho_{ABC}) + \tau(\rho_{ABC}) \leqslant 3. \tag{31}$$

Proof. Combining Eqs. (14) and (15) and the last two equalities of Eq. (23), we get

$$\begin{aligned} S_{AB} + S_{AC} + 2\tau(\rho_{ABC}) &= 4 - 2\vec{a}^2 - \mathcal{C}_{AC}^2 - \mathcal{C}_{AC}^2 - 2\,\mathcal{C}_{BC}^2 \\ &= 3 + \tau - \vec{a}^2 - 2\,\mathcal{C}_{BC}^2. \end{aligned}$$

Thus,

$$S_{AB} + S_{AC} + \tau(\rho ABC) \leqslant 3. \tag{32}$$

Considering all permutation of parties, we get $S_{AB} + S_{BC} + \tau(\rho ABC) \leqslant 3$ and $S_{AC} + S_{BC} + \tau(\rho_{ABC}) \leqslant 3$. ■

Now, by using the convexity property of the left-hand sides of these inequalities, we claim that relation (29) holds for all three-qubit states.

We have identified a class of genuinely entangled states which saturates the afore-mentioned relation. This class of states is given by $|\phi_q\rangle = \frac{1}{\sqrt{2}}|000\rangle + \sqrt{\frac{1}{2} - q^2}|101\rangle + q|111\rangle$, where $q \in (0, \frac{1}{\sqrt{2}})$. For $|\phi_q\rangle$, $S_{\text{total}}^{\max} = 3 - 2q^2$ and $\tau = 2q^2$. Hence, $S_{\text{total}}^{\max}(\rho_{ABC}) + \tau(\rho_{ABC}) = 3$. However, $|\phi_q\rangle$ has only one reduced state which violates the F_3 inequality. Since among all pure three-qubit GHZ states, only star-shaped states can have two reduced steerable states (see the Appendix) and, for this class of states, $S_{\text{total}}^{\max}(\rho_{ABC}) + \tau(\rho_{ABC}) < 3$, there is no pure three-qubit state with $\tau \neq 0$ having two reduced bipartite steerable states which saturates the above inequality.

All the above-mentioned relations are obtained with respect to the three tangle. However, the three tangle is not a good measure of genuine tripartite entanglement even for pure states, as there exist a large number of pure states (W-like states [15]) for which it becomes 0. Hence, none of the relations are meaningful for these W-like states.

To obtain such relations for W-like states, we consider the measure for W entanglement introduced by Dur *et al.* [15], defined as $E_W = \min\{\mathcal{C}_{AB}^2, \mathcal{C}_{AC}^2, \mathcal{C}_{BC}^2\}$. Any pure state ρ_{ABC} contains W entanglement if $E_W > 0$. The W entanglement E_W achieves its maximum value $\frac{4}{9}$ in the $|W\rangle$ state.

Theorem 6. For an arbitrary pure three-qubit state $|\phi_{ABC}\rangle$, the W entanglement (E_W) and maximum bipartite steering $[S_{\text{total}}^{\max}(\rho_{ABC})]$ satisfy the following complementary relation:

$$S_{\text{total}}^{\max}(|\phi_{ABC}\rangle) + 3E_W(|\phi_{ABC}\rangle) \leqslant \frac{10}{3}. \tag{33}$$

Proof. Using Eq. (27) and its permutations, we have

$$S_{AB} + S_{AC} + 3\,\mathcal{C}_{BC}^2 = 2 + \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2 + \mathcal{C}_{BC}^2. \tag{34}$$

If one uses Eq. (26), the above equality immediately leads to

$$S_{AB} + S_{AC} + 3\,\mathcal{C}_{BC}^2 \leqslant \tfrac{10}{3}. \tag{35}$$

Similarly, permutation of parties gives $S_{AB} + S_{BC} + 3\,\mathcal{C}_{AC}^2 \leqslant \frac{10}{3}$ and $S_{AC} + S_{BC} + 3\,\mathcal{C}_{AB}^2 \leqslant \frac{10}{3}$. The above equations confirm the validity of the claim made in Eq. (33). ■

This relation imposes a restriction on the bipartite steering for a given amount of W entanglement and it is saturated by the $|W\rangle$ state.

We have also investigated such complementary relations for bipartite nonlocality (with respect to Bell-CHSH violation), bipartite steering, and the three tangle. Following the same procedure as before, a similar trade-off relation can be obtained for them,

$$S^{\max}(\rho_{ABC}) + \mathcal{M}^{\max}(\rho_{ABC}) + 3\,\tau(\rho_{ABC}) \leqslant 5, \tag{36}$$

where $\mathcal{M}^{\max}(\rho_{ABC}) = \max\{\mathcal{M}_{AB}, \mathcal{M}_{AC}, \mathcal{M}_{BC}\}$ and $\mathcal{M} = u_1^2 + u_2^2$ is the Horodecki parameter [79] used for measuring the degree of Bell-CHSH violation. u_1^2 and u_2^2 are the largest two eigenvalues of $T_{AB}^T T_{AB}$.

V. COMPLEMENTARY RELATIONS FOR LOCAL AND NONLOCAL INFORMATION CONTENTS

The total information content of a three-qubit state can be divided into two forms: Local and nonlocal information contents. Local information can be defined as [80]

$$I_{\text{local}} = \vec{a}^2 + \vec{b}^2 + \vec{c}^2. \tag{37}$$

To derive the complementary relation between local and nonlocal information contents, we consider only bipartite nonlocal information present in the three-qubit state. Bipartite nonlocal information content can be defined as

$$I_{\text{nonlocal}} = \max\{N_{AB} + N_{AC}, N_{AB} + N_{BC}, N_{AC} + N_{BC}\}, \tag{38}$$

where $N_{ij} = \max\{0, S_{ij} - 1\}$, $i \neq j$ and $i, j = A, B, C$, quantifies the amount of F_3 inequality violation and hence the steering nonlocal correlations of the two-qubit state ρ_{ij}.

Theorem 7. For an arbitrary three-qubit state ρ_{ABC},

$$I_{\text{local}} + I_{\text{nonlocal}} \leqslant 3. \tag{39}$$

Proof. For pure three-qubit states, it is straightforward to check that

$$\begin{aligned} I_{\text{local}} + (S_{AB} - 1) + (S_{AC} - 1) &= 2(\vec{b}^2 + \vec{c}^2) - \vec{a}^2 \\ &\leqslant 2(1 + \vec{a}^2) - \vec{a}^2 \\ &\leqslant 3, \end{aligned} \tag{40}$$

where in the first inequality we have used the fact that relation $\vec{b}^2 + \vec{c}^2 \leqslant 1 + \vec{a}^2$ holds for all pure three-qubit states [81]. Since $I_{\text{local}} \leqslant 3$, the above inequality [Eq. (40)] also holds when both N_{AB} and N_{AC} are equal to 0. Hence $I_{\text{local}} + N_{AB} + N_{AC} \leqslant 3$. Similarly, one gets $I_{\text{local}} + N_{AB} + N_{BC} \leqslant 3$ and $I_{\text{local}} + N_{AC} + N_{BC} \leqslant 3$. Note that the left-hand sides of these inequalities are convex under mixing. This confirms the relation presented in Eq. (39). ■

The above trade-off relation links local information and bipartite steering. One can easily show that $I_{\text{local}} = 3$ and $I_{\text{nonlocal}} = 0$ for the product state. On the other hand, in order for bipartite steering to exist, I_{local} must be less than 3. For $|\psi_{ABC}\rangle$ [Eq. (20)], $I_{\text{local}} = \frac{1}{2}$, $I_{\text{nonlocal}} = 2 + \frac{1}{2}$, and it is the state which saturates this trade-off. In this context, it may be noted that to obtain a larger violation of the F_3 inequality (characterizing a larger amount of steering), the local information content must be reduced. This fact is confirmed in the next section, where we show that the amount of local information content must be less than 1 for any three-qubit pure state to have two F_3-steerable bipartite reduced states.

VI. ENTANGLEMENT DETECTION

We now illustrate the relevance of the above results with some applications. By using the shareability relations, we derive criteria for detecting different types of tripartite entanglement.

Theorem 8. For any three-qubit pure state $|\phi_{ABC}\rangle \in \mathbb{H}^A \otimes \mathbb{H}^B \otimes \mathbb{H}^C$, if at least one of the following conditions holds:

$$\text{(i) } \vec{a}^2 \neq \frac{\vec{b}^2 + \vec{c}^2}{2}, \quad \text{(ii) } \vec{b}^2 \neq \frac{\vec{a}^2 + \vec{c}^2}{2}, \quad \text{(iii) } \vec{c}^2 \neq \frac{\vec{a}^2 + \vec{b}^2}{2}, \tag{41}$$

the state is entangled.

Proof. Let $|\phi_{ABC}\rangle$ be a separable state; then all bipartite reduced states are also separable and $S_{AB}, S_{AC}, S_{BC} \leqslant 1$. Hence violation of the F_3 inequality by any bipartite reduced state entails entanglement of $|\phi_{ABC}\rangle$. It is clear from Eqs. (14)–(16) that if S_{AB}, S_{AC}, $S_{BC} > 1$, then $\vec{c}^2 > \frac{\vec{a}^2+\vec{b}^2}{2}$, $\vec{b}^2 > \frac{\vec{a}^2+\vec{c}^2}{2}$, and $\vec{a}^2 > \frac{\vec{b}^2+\vec{c}^2}{2}$ hold, respectively. Again, by adding Eq. (14) and Eq. (15), we have $S_{AB} + S_{AC} = 2 + \vec{b}^2 + \vec{c}^2 - 2\vec{a}^2$. By noting that $S_{AB} + S_{AC} > 2$ implies steerability of at least one of ρ_{AB} or ρ_{AC}, $|\phi_{ABC}\rangle$ is entangled if $\vec{a}^2 < \frac{\vec{b}^2+\vec{c}^2}{2}$. Similarly, permutation of the parties gives $\vec{b}^2 < \frac{\vec{a}^2+\vec{c}^2}{2}$ and $\vec{c}^2 < \frac{\vec{a}^2+\vec{b}^2}{2}$. Combining all these expressions, we arrive at Eq. (41). ■

Now one may enquire whether condition (41) is also necessary for entanglement. Unfortunately, this is not the case. For example, consider the $|W\rangle$ state, which does not satisfy (41) but is entangled.

Theorem 9. For any three-qubit pure state $|\phi_{ABC}\rangle \in \mathbb{H}^A \otimes \mathbb{H}^B \otimes \mathbb{H}^C$, if at least one of the following conditions holds:

$$\begin{aligned} &\text{(i)} \quad \vec{a}^2 > \frac{\vec{b}^2 + \vec{c}^2}{2}, \quad \vec{b}^2 > \frac{\vec{a}^2 + \vec{c}^2}{2}, \\ &\text{(ii)} \quad \vec{a}^2 > \frac{\vec{b}^2 + \vec{c}^2}{2}, \quad \vec{c}^2 > \frac{\vec{a}^2 + \vec{b}^2}{2}, \\ &\text{(iii)} \quad \vec{b}^2 > \frac{\vec{a}^2 + \vec{c}^2}{2}, \quad \vec{c}^2 > \frac{\vec{a}^2 + \vec{b}^2}{2}, \end{aligned} \tag{42}$$

the state is genuinely entangled.

Proof. Let $|\phi_{ABC}\rangle$ be any biseparable state in which AB is independent of C; then it can be expressed as $(\cos\theta|00\rangle + \sin\theta|11\rangle)_{AB} \otimes |0\rangle_C$, where $0 \leqslant \theta \leqslant \frac{\pi}{4}$. For this state, $\vec{c}^2 = 1$ and $\vec{a}^2 = \vec{b}^2$. Using Eqs. (14)–(16), one can find that $S_{AB} = 3 - 2\vec{a}^2$, $S_{AC} = \vec{a}^2$, $S_{BC} = \vec{a}^2$. So only S_{AB} can be greater than 1. Similarly, one can show that only one reduced state will violate the F_3 inequality in which a system other than the C system factorizes. This immediately leads to a simple sufficient condition for genuinely entangled pure states: Violation of the F_3 inequality by two reduced states indicates genuine entanglement of $|\phi_{ABC}\rangle$. Then, from Eqs. (14)–(16), we obtain conditions (42). ■

It is important to note that for a pure biseparable state $\vec{a}^2 + \vec{b}^2 + \vec{c}^2 \geqslant 1$ and exactly one of the reduced bipartite states is F_3 steerable. Therefore, for the existence of two F_3-steerable bipartite reduced states of a three-qubit pure state, $\vec{a}^2 + \vec{b}^2 + \vec{c}^2 < 1$ must hold. This condition can be treated as necessary for a three-qubit pure state to have two F_3 steerable bipartite reduced states. However, this is not sufficient, for example, $\vec{a}^2 + \vec{b}^2 + \vec{c}^2 = \frac{1}{3} < 1$ for the $|W\rangle$ state, but no reduced bipartite state of this state is F_3 steerable.

At this stage a pertinent question would be whether there exists any biseparable mixed state which has more than one reduced steerable state. Let us consider the example

$$\begin{aligned}|\phi_b\rangle = {} & \tfrac{4}{9}(1+\epsilon)|\phi^+\rangle\langle\phi^+|_{AB} \otimes |0\rangle\langle 0|_C + \tfrac{4}{9}(1+\epsilon) \\ & \times |\phi^+\rangle\langle\phi^+|_{AC} \otimes |0\rangle\langle 0|_B + \tfrac{1}{9}(1-8\epsilon)|\phi^+\rangle\langle\phi^+|_{BC} \\ & \otimes |0\rangle\langle 0|_A, \end{aligned} \tag{43}$$

where $0 \leqslant \epsilon \leqslant 1$ and $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$. For this biseparable mixed state, the bipartite reduced states ρ_{AB} and ρ_{AC} are F_3 steerable if $\epsilon > \frac{9}{4\sqrt{3}} - 1$. Thus, genuine entanglement is not necessary to reveal the shareable nature of steering correlations.

VII. DISCUSSION

Analysis of the shareability of correlations between parties sharing a quantum system is an effective way of interpreting quantum theory. In this paper, we have investigated the shareability properties of quantum steering correlations. For our purpose, we have considered the three-setting linear steering (F_3) inequality. Interestingly it is observed that at most two reduced states of any arbitrary three-qubit state can violate the F_3 inequality. This in turn reveals the shareable nature of steering correlations. This observation is, however, contrary to the monogamous nature of steering obtained when using the two-setting linear steering inequality or Bell-CHSH inequality. This indicates that steering correlations can be shareable depending on the measurement scenario. Now steering correlations in a setup with two settings per party cannot be shared, whereas this is possible when a setup with three settings per party is considered. So it might be tempting to think that an increase in the number of settings per party could provide more steerable reduced states. Consequently, it would be interesting to investigate this shareability phenomenon in a scenario with more than three settings

We have also addressed the question how different measures of genuine entanglement and also entanglement of reduced states relate to reduced bipartite steering of three-qubit states. We have established several relations between reduced bipartite steering and different measures of entanglement. The relation between bipartite steering, Bell-CHSH nonlocality, and genuine entanglement for three-qubit states has also been analyzed.

Next we have determined the complementarity relation between the local information content and bipartite steering. We believe that this will be helpful in designing some appropriate information-theoretic measures of steering. Moreover, we have shown that the shareability constraints allow us to detect different types of tripartite entanglement. Now, monogamy is the essential part in ensuring the security of quantum cryptographic protocols [13]. For this reason, it is beneficial to capture precisely under what conditions the steering correlations are monogamous. So, our observations may be used in framing some more secure quantum cryptographic protocols.

We hope that our results will be useful for further understanding the formalism underlying steering correlations and their distribution in multipartite states. Apart from investigating our work in a scenario with more than three settings, it will be interesting to generalize the shareability concept of steering correlations and relations between different quantum correlations for reduced states of more than two parties. Also, investigation of the same beyond qubit systems is a potential topic for future research.

APPENDIX: REDUCED BIPARTITE STEERING OF THREE-QUBIT STATES

To check the number of reduced steerable states of any pure three-qubit state, we consider the general Schmidt decomposition (GSD) of three-qubit pure states as [82]

$$|\phi\rangle = \lambda_0|000\rangle + \lambda_1 e^{\iota\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle, \tag{A1}$$

where $\lambda_i \geqslant 0$, $\sum_i \lambda_i = 1$, and ϕ is a phase between 0 and π. It is direct to derive that [30] $\vec{a} = (2\lambda_0\lambda_1\cos\phi, 2\lambda_0\lambda_1\sin\phi, 2\lambda_0^2 - 1)$, $\vec{b} = (2\lambda_1\lambda_3\cos\phi + 2\lambda_2\lambda_4, -2\lambda_1\lambda_3\sin\phi, 1 - 2\lambda_3^2 - 2\lambda_4^2)$, and $\vec{c} = (2\lambda_1\lambda_2\cos\phi + 2\lambda_2\lambda_4, -2\lambda_1\lambda_2\sin\phi, 1 - 2\lambda_2^2 - 2\lambda_4^2)$. From the formulas for calculating S_{ij} presented in Eqs. (14)–(16), one can provide the following expressions of S_{ij} for any three-qubit state in $|\phi\rangle$:

$$\begin{aligned} S_{AB} = {} & 1 + 8\lambda_0^2\lambda_3^2 - 4\lambda_0^2\lambda_2^2 - 4\lambda_1^2\lambda_4^2 - 4\lambda_2^2\lambda_3^2 \\ & + 8\lambda_1\lambda_2\lambda_3\lambda_4\cos\phi, \end{aligned} \tag{A2}$$

$$\begin{aligned} S_{AC} = {} & 1 + 8\lambda_0^2\lambda_2^2 - 4\lambda_0^2\lambda_3^2 - 4\lambda_1^2\lambda_4^2 - 4\lambda_2^2\lambda_3^2 \\ & + 8\lambda_1\lambda_2\lambda_3\lambda_4\cos\phi, \end{aligned} \tag{A3}$$

and

$$\begin{aligned} S_{BC} = {} & 1 - 4\lambda_0\lambda_2^2 - 4\lambda_0^2\lambda_3^2 + 8\lambda_1^2\lambda_4^2 + 8\lambda_2^2\lambda_3^2 \\ & - 16\lambda_1\lambda_2\lambda_3\lambda_4\cos\phi. \end{aligned} \tag{A4}$$

By somewhat tedious but straightforward calculations, we obtain the concurrence of each bipartite reduced state [76]: $\mathcal{C}_{AB}^2 = 4\lambda_0^2\lambda_3^2$, $\mathcal{C}_{AC}^2 = 4\lambda_0^2\lambda_2^2$, and $\mathcal{C}_{BC}^2 = 4\lambda_2^2\lambda_3^2 + 4\lambda_1^2\lambda_4^2 - 8\lambda_1\lambda_2\lambda_3\lambda_4\cos\phi$.

In [72], Sabín and García-Alcaine proposed a classification of three-qubit states based on the existence of bipartite and tripartite entanglements. Here we investigate the number of reduced steerable states in each of these classes of states. Different types of reduced steering are summarized in Fig. 1.

(i) *Type 0-0 (fully separable states)*: A pure state $|\phi\rangle$ is fully separable if it can be written as $|\phi_1\rangle \otimes |\phi_2\rangle \otimes |\phi_3\rangle$. Clearly all reduced states are separable, thereby implying the absence of F_3-steerable reduced states. The corresponding steering graph [Fig. 1(a)] has three vertices without any edge.

(ii) *Subtype 1^1-1 (biseparable states)*: Any state in this class has one of the following GSD forms:

(a) $|\phi_{\rm BS}\rangle = \lambda_1 e^{\iota\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$, where $\lambda_1\lambda_4 \neq \lambda_2\lambda_3$ and $\lambda_1\lambda_4$ or $\lambda_2\lambda_3$ can be 0 if $\lambda_1\lambda_4 = \lambda_2\lambda_3$, the state is of type 0-0;

(b) $|\phi'_{\rm BS}\rangle = \lambda_0|000\rangle + \lambda_1 e^{\iota\phi}|100\rangle + \lambda_2|101\rangle$; or

(c) $|\phi''_{\rm BS}\rangle = \lambda_0|000\rangle + \lambda_1 e^{\iota\phi}|100\rangle + \lambda_3|110\rangle$, where λ_1 can be 0 in the latter two cases.

In each case, exactly one of the reduced states is F_3 steerable. For example, ρ_{BC} is the only reduced F_3-steerable state of $|\phi_{\rm BS}\rangle$. So any biseparable pure state will obey monogamy of steering. The corresponding steering graph has only one edge connecting two circles [see Fig. 1(b)].

(iii) *Subtype 2-0 (GHZ-like states)*: This class of states has the form $|\phi_{\rm GGHZ}\rangle = \alpha|000\rangle + \beta|111\rangle$, where $\alpha^2 + \beta^2 = 1$. It includes the $|\phi_{\rm GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ state. Entanglement of this class of states cannot be maintained if one of the qubits is traced out. Hence, none of the reduced states can be F_3 steerable. Three circles without any edge [see Fig. 1(c)] correspond to this class of states. Thus, we see that two types of states (GHZ-like states and separable states) have the same graph.

(iv) *Subtype 2-1 (extended GHZ states)*: Any state in this class has one of the following GSD forms:

(a) $|\phi_{\rm EGHZ}\rangle = \lambda_0|000\rangle + \lambda_1 e^{\iota\phi}|100\rangle + \lambda_4|111\rangle$,
(b) $|\phi'_{\rm EGHZ}\rangle = \lambda_0|000\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle$, or
(c) $|\phi''_{\rm EGHZ}\rangle = \lambda_0|000\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$,

with three nonzero coefficients in each case.

Any state in this class has only one entangled reduced state. For example, the entangled reduced state ρ_{AC} of $|\phi_{\rm EGHZ}\rangle$ is given by $\rho_{AC} = |\alpha\rangle\langle\alpha| + \lambda_4^2|11\rangle\langle 11|$, where $|\alpha\rangle = \lambda_0|00\rangle + \lambda_2|11\rangle$, with concurrence $\mathcal{C}_{AC}^2 = 4\lambda_0^2\lambda_2^2$. Since $\mathcal{C}_{AB}^2$ and $\mathcal{C}_{BC}^2$ are both equal to 0, S_{AC} can be obtained straightforwardly from Eq. (27) and its permutations as $S_{AC} = 1 + 2\,\mathcal{C}_{AC}^2$. Thus, any extended GHZ state has only one reduced F_3-steerable state and thereby maintains a monogamous nature. Hence, biseparable states and extended GHZ states have the same graph.

(v) *Subtype 2-2 (star-shaped states)*: This class of states takes one of the following GSD forms:

(a) $|\phi_{\rm STAR}\rangle = \lambda_0|000\rangle + \lambda_1 e^{\iota\phi}|100\rangle + \lambda_2|101\rangle + \lambda_4|111\rangle$

or

(b) $|\phi'_{\rm STAR}\rangle = \lambda_0|000\rangle + \lambda_1 e^{\iota\phi}|100\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle$,

with all coefficients nonzero.

These states belong to class GHZ [83], since it contains genuine entanglement with $\tau = 4\lambda_0^2\lambda_4^2$. It is the only class of states among all GHZ classes that can have two entangled reduced states. We find that for $|\phi'_{\rm STAR}\rangle$, $\mathcal{C}_{AC}$ is always 0, while $\mathcal{C}_{AB}^2 (= 4\lambda_0^2\lambda_3^2)$ and $\mathcal{C}_{BC}^2 (= 4\lambda_1^2\lambda_4^2)$ are nonzero. Combining these with Eq. (27) and its permutations, one finds that a state belonging to this class will obey shareability if and only if $4\lambda_1^2\lambda_4^2 > 2\lambda_0^2\lambda_2^2 > \lambda_1^2\lambda_4^2$ holds.

One simple example of such a state is $\sqrt{\frac{11}{64}}|000\rangle + \sqrt{\frac{5}{64}}|100\rangle + \frac{1}{2}|110\rangle + \frac{1}{\sqrt{2}}|111\rangle$. Similarly, one can also find a state in this class which violates the above-mentioned inequality: $\sqrt{\frac{3}{32}}|000\rangle + \sqrt{\frac{5}{32}}|100\rangle + \frac{1}{2}|110\rangle + \frac{1}{\sqrt{2}}|111\rangle$. Thus, this class of states can be both monogamous and shareable. Also it is clear that any state in this class has at least one steerable reduced state, since $S_{AB} + S_{BC} = 2 + 4\lambda_0^2\lambda_3^2 + 4\lambda_1^2\lambda_4^2 > 2$ for every nonzero value of state parameters. This class of states is represented in Figs. 1(b) and 1(c).

(vi) *Subtype 2-3 (W-like states)*: We now take W-like states into account. This class of states is given by $|\phi_W\rangle = \lambda_0|000\rangle + \lambda_1 e^{\iota\phi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle$, where $\lambda_0, \lambda_2, \lambda_3 > 0$ and $\lambda_1^2 \geqslant 0$. For W-like states, all bipartite entanglements are nonzero, with $\mathcal{C}_{AB}^2 = 4\lambda_0^2\lambda_3^2$, $\mathcal{C}_{AC}^2 = 4\lambda_0^2\lambda_2^2$, and $\mathcal{C}_{BC}^2 = 4\lambda_2^2\lambda_3^2$. At this point one might wonder whether the W class contains states with no reduced steering. Let us consider the $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ state. As shown in Sec. III, it has no reduced steering. This is in contrast to GHZ states $|\phi_{\rm GHZ}\rangle$, which are less bipartite entangled but have the same steering graph. Let us now address the question of monogamy (or shareability) for states in the W class. From the criterion presented in Corollary 3.1, monogamy holds for this class of states if and only if $H(\lambda_i^2, \lambda_j^2) < \lambda_k^2$ $(i \neq j \neq k, \quad i, j, k = 0, 2, 3)$ for any two sets of values of (i, j, k), where $H(\lambda_i^2, \lambda_j^2)$ denotes the harmonic mean of λ_i^2 and λ_j^2. This is the the only class of states where one can get all types of steering graphs, i.e., no reduced steering states, one reduced steering state, and also two reduced steering states. Examples of one reduced steering state and two reduced steering states are given below: One reduced steering state, $\frac{1}{\sqrt{6}}(|000\rangle + |100\rangle + |101\rangle) + \frac{1}{\sqrt{2}}|110\rangle$; and two reduced steering states, $|\psi_{ABC}\rangle = \frac{1}{2}(|100\rangle + |010\rangle + \sqrt{2}|001\rangle)$. From the above analysis, it is clear that this class of states can correspond to any steering graph [Figs. 1(a)–1(c)].

Regarding the above classification, we want to remark that only star-shaped states (subtype 2-2) and W-like states (subtype 2-3) can violate monogamy of steering correlations. We believe that our classification of three-qubit pure states in terms of reduced steering and monogamy (or shareabilty) can be useful in many areas of quantum information.

Now we investigate the effect of admixing white noise to these two classes of pure states (star-shape states and W-like states), which can exhibit the shareable nature of steering correlations. In order to analyze it, we define a critical value v $(0 \leqslant v \leqslant 1)$ for which the mixed states defined by

$$\rho_{\rm star} = v(|\phi'_{\rm star}\rangle\langle\phi'_{\rm star}|) + (1-v)\frac{\mathbf{I}}{8} \tag{A5}$$

and

$$\rho_W = v(|\phi_W\rangle\langle\phi_W|) + (1-v)\frac{\mathbf{I}}{8} \tag{A6}$$

lose the shareable nature of the original pure states. For a given noisy state, we intend to find the critical value $v_{\rm crit}$ such that, if $v > v_{\rm crit}$, the shareable nature is preserved for steering correlations, i.e., there exist two steerable reduced states of the given noisy state.

For the $\rho_{\rm star}$ state, one has $S_{AB} = v(1 + 8\lambda_0^2\lambda_2^2 - 4\lambda_1^2\lambda_4^2)$, $S_{BC} = v(1 + 8\lambda_1^2\lambda_4^2 - 4\lambda_0^2\lambda_2^2)$, and $S_{AC} = 1 - 4\lambda_1^2\lambda_4^2 - 4\lambda_0^2\lambda_2^2$. Consequently, the state $\rho_{\rm star}$ leads to the critical visibility

$$v_{\rm crit} = \max\left[\frac{1}{1 + 8\lambda_0^2\lambda_2^2 - 4\lambda_1^2\lambda_4^2}, \frac{1}{1 + 8\lambda_1^2\lambda_4^2 - 4\lambda_0^2\lambda_2^2}\right]. \tag{A7}$$

Note that $v_{\rm crit}$ is minimized for $\lambda_i^2 = \frac{1}{4}$ $(i = 0, 1, 2, 3)$, which corresponds to the state $|\phi_v^{\rm star}\rangle = \frac{1}{2}|000\rangle + \frac{1}{2}|100\rangle + \frac{1}{2}|110\rangle + \frac{1}{2}|111\rangle$ and leads to $v_{\rm crit} = 0.8$. Thus, the state $|\phi_v^{\rm star}\rangle$ is more robust against white noise than any other shareable $|\phi'_{\rm star}\rangle$ state.

Similarly, one can find

$$v_{\rm crit} = \min[\max\{w_1, w_2\}, \max\{w_1, w_3\}, \max\{w_2, w_3\}] \tag{A8}$$

for the ρ_W state, where $w_1 = \frac{1}{1+8\lambda_0^2\lambda_3^2-4\lambda_0^2\lambda_2^2-4\lambda_2^2\lambda_3^2}$, $w_2 = \frac{1}{1+8\lambda_0^2\lambda_2^2-4\lambda_0^2\lambda_3^2-4\lambda_2^2\lambda_3^2}$, and $w_3 = \frac{1}{1+8\lambda_2^2\lambda_3^2-4\lambda_0^2\lambda_2^2-4\lambda_0^2\lambda_3^2}$. The most robust shareability property is observed for the state $|\phi_v^W\rangle = \sqrt{\frac{2}{3}}|000\rangle + \sqrt{\frac{1}{6}}|101\rangle + \sqrt{\frac{1}{6}}|110\rangle$ and the corresponding $v_{\text{crit}} = 0.75$. Intuitively it can be expected that highly entangled states might have greater robustness of nonmonogamy compared to less entangled states. Let us take the example of the $|\psi_{ABC}\rangle$ state, which has $v_{\text{crit}} = 0.8$. Now if we compare the efficiency of $|\phi_v^W\rangle$ with $|\psi_{ABC}\rangle$, we find that the less entangled state $|\phi_v^W\rangle$ with $E_W = \frac{1}{9}$ is more robust in comparison to the more highly entangled state $|\psi_{ABC}\rangle$ having $E_W = \frac{1}{4}$.

[1] H. J. Kimble, The quantum Internet, Nature (London) **453**, 1023 (2008).

[2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. **81**, 865 (2009).

[3] O. Gühne and G. Tóth, Entanglement detection, Phys. Rep. **474**, 1 (2009).

[4] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 839(E) (2014).

[5] B. Toner, Monogamy of non-local quantum correlations, Proc. R. Soc. A **465**, 59 (2008).

[6] V. Coffman,, J. Kundu, and W. K. Wootters, Distributed entanglement, Phys. Rev. A **61**, 052306 (2000).

[7] T. J. Osborne and F. Verstraete, General Monogamy Inequality for Bipartite Qubit Entanglement, Phys. Rev. Lett. **96**, 220503 (2006).

[8] Y. C. Ou, H. Fan, and S. M. Fei, Proper monogamy inequality for arbitrary pure quantum states, Phys. Rev. A **78**, 012311 (2008).

[9] Y. K. Bai, Y. F. Xu, and Z. D. Wang, General Monogamy Relation for the Entanglement of Formation in Multiqubit Systems, Phys. Rev. Lett. **113**, 100503 (2014).

[10] X. N. Zhu and S. M. Fei, Entanglement monogamy relations of qubit systems, Phys. Rev. A **90**, 024304 (2014).

[11] B. Regula, S. Di Martino, S. Lee, and G. Adesso, Strong Monogamy Conjecture for Multiqubit Entanglement: The Four-Qubit Case, Phys. Rev. Lett. **113**, 110501 (2014).

[12] B. M. Terhal, Is entanglement monogamous? IBM J. Res. Dev. **48**, 71 (2004).

[13] M. Pawlowski, Security proof for cryptographic protocols based only on the monogamy of Bell's inequality violations, Phys. Rev. A **82**, 032313 (2010).

[14] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, Rev. Mod. Phys. **74**, 145 (2002).

[15] W. Dür, G. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **62**, 062314 (2000).

[16] G. L. Giorgi, Monogamy properties of quantum and classical correlations, Phys. Rev. A **84**, 054301 (2011).

[17] R. Prabhu, A. K. Pati, A. Sen(De), and U. Sen, Conditions for monogamy of quantum correlations: Greenberger-Horne-Zeilinger versus *W* states, Phys. Rev. A **85**, 040102(R) (2012).

[18] S. Lloyd and J. Preskill, Unitarity of black hole evaporation in final-state projection models, J. High Energy Phys. 08 (2014) 126.

[19] K. R. K. Rao, H. Katiyar, T. S. Mahesh, A. Sen(De), U. Sen, and A. Kumar, Multipartite quantum correlations reveal frustration in a quantum Ising spin system, Phys. Rev. A **88**, 022312 (2013).

[20] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).

[21] V. Scarani and N. Gisin, Quantum Communication Between N Partners and Bell's Inequalities, Phys. Rev. Lett. **87**, 117901 (2001).

[22] V. Scarani and N. Gisin, Quantum key distribution between N partners: Optimal eavesdropping and Bell's inequalities, Phys. Rev. A **65**, 012311 (2001).

[23] B. Toner and F. Verstraete, Monogamy of Bell correlations and Tsirelson's bound, arXiv:quant-ph/0611001.

[24] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts, Nonlocal correlations as an information-theoretic resource, Phys. Rev. A **71**, 022101 (2005).

[25] Ll Masanes, A. Acin, and N. Gisin, General properties of nonsignaling theories, Phys. Rev. A **73**, 012112 (2006).

[26] M. P. Seevinck, Monogamy of correlations versus monogamy of entanglement, Quant. Info. Proc. **9**, 273 (2010).

[27] M. Pawlowski and C. Brukner, Monogamy of Bell's Inequality Violations in Nonsignaling Theories, Phys. Rev. Lett. **102**, 030403 (2009).

[28] P. Kurzynski, T. Paterek, R. Ramanathan, W. Laskowski, and D. Kaszlikowski, Correlation Complementarity Yields Bell Monogamy Relations, Phys. Rev. Lett. **106**, 180402 (2011).

[29] H. H. Qin, S. M. Fei, and X. Li-Jost, Trade-off relations of Bell violations among pairwise qubit systems, Phys. Rev. A **92**, 062339 (2015).

[30] S. Cheng and M. J. W. Hall, Anisotropic Invariance and the Distribution of Quantum Correlations, Phys. Rev. Lett. **118**, 010401 (2017).

[31] M. C. Tran, R. Ramanathan, M. McKague, D. Kaszlikowski, and T. Paterek, Bell monogamy relations in arbitrary qubit networks, Phys. Rev. A **98**, 052325 (2018).

[32] R. Ramanathan1 and P. Mironowicz, Trade-offs in multiparty Bell-inequality violations in qubit networks, Phys. Rev. A **98**, 022133 (2018).

[33] A. Streltsov, G. Adesso, M. Piani, and D. Bruss, Are General Quantum Correlations Monogamous? Phys. Rev. Lett. **109**, 050503 (2012).

[34] M. Karczewski, D. Kaszlikowski, and P. Kurzyński, Monogamy of Particle Statistics in Tripartite Systems Simulating Bosons and Fermions, Phys. Rev. Lett. **121**, 090403 (2018).

[35] C. Radhakrishnan, M. Parthasarathy, S. Jambulingam, and T. Byrnes, Distribution of Quantum Coherence in Multipartite Systems, Phys. Rev. Lett. **116**, 150504 (2016).

[36] S. Cheng and L. Liu, Monogamy relations of nonclassical correlations for multi-qubit states, Phys. Lett. A **382**, 1716 (2018).

[37] S. Luo and W. Sun, Separability and entanglement in tripartite states, Theor. Math. Phys. **160**, 1316 (2009).

[38] S. Luo and N. Li, Quantum correlations reduce classical correlations with ancillary systems, Chin. Phys. Lett. **27**, 120304 (2010).

[39] M. D. Reid, Monogamy inequalities for the Einstein-Podolsky-Rosen paradox and quantum steering, Phys. Rev. A **88**, 062108 (2013).

[40] Y. Xiang, I. Kogias, G. Adesso, and Q. Y. He, Multipartite Gaussian steering: Monogamy constraints and quantum cryptographyapplications, Phys. Rev. A **95**, 010101(R) (2017).

[41] S. Cheng, A. Milne, M. J. W. Hall, and H. M. Wiseman, Volume monogamy of quantum steering ellipsoids for multiqubit systems, Phys. Rev. A **94**, 042105 (2016).

[42] E. Schrödinger, Discussion of probability relations between separated systems, Proc. Cambridge Philos. Soc. **31**, 555 (1935).

[43] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, Phys. Rev. Lett. **98**, 140402 (2007).

[44] S. J. Jones, H. M. Wiseman, and A. C. Doherty, Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering, Phys. Rev. A **76**, 052116 (2007).

[45] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, Rev. Mod. Phys. **92**, 015001 (2020).

[46] M. T. Quintino, T. Vértesi, D. Cavalcanti, R. Augusiak, M. Demianowicz, A. Acín, and N. Brunner, Inequivalence of entanglement, steering, and Bell nonlocality for general measurements, Phys. Rev. A **92**, 032107 (2015).

[47] M. D. Reid, A. Acín, and N. Brunner, Demonstration of the Einstein-Podolsky-Rosen paradox using nondegenerate parametric amplification, Phys. Rev. A **40**, 913 (1989).

[48] E. G. Cavalcanti, S. J. Jones, H. M. Wiseman, and M. D. Reid, Experimental criteria for steering and the Einstein-Podolsky-Rosen paradox, Phys. Rev. A **80**, 032112 (2009).

[49] S. P. Walborn, A. Salles, R. M. Gomes, F. Toscano, and P. H. Souto Ribeiro, Revealing Hidden Einstein-Podolsky-Rosen Nonlocality, Phys. Rev. Lett. **106**, 130402 (2011).

[50] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti, and J. C. Howell, Einstein-Podolsky-Rosen steering inequalities from entropic uncertainty relations, Phys. Rev. A **87**, 062103 (2013).

[51] J.-L. Chen, X.-J. Ye, C. Wu, H.-Y. Su, A. Cabello, L. C. Kwek, and C. H. Oh, All-versus-nothing proof of Einstein- Podolsky-Rosen steering, Sci. Rep. **3**, 2143 (2013).

[52] I. Kogias, P. Skrzypczyk, D. Cavalcanti, A. Acín, and G. Adesso, Hierarchy of Steering Criteria Based on Moments for All Bipartite Quantum Systems, Phys. Rev. Lett. **115**, 210401 (2015).

[53] IE. G. Cavalcanti, C. J. Foster, M. Fuwa, and H. M. Wiseman, Analog of the Clauser-Horne-Shimony-Holt inequality for steering, J. Opt. Soc. Am. B **32**, A74 (2015).

[54] M. Zukowski, A. Dutta, and Z. Yin, Geometric Bell-like inequalities for steering, Phys. Rev. A **91**, 032107 (2015).

[55] S. Jevtic, M. J. W. Hall, M. R. Anderson, M. Zwierz, and H. M. Wiseman, Einstein-Podolsky-Rosen steering and the steering ellipsoid, J. Opt. Soc. Am. B, **32**, A40 (2015).

[56] A. C. S. Costa and R. M. Angelo, Quantification of Einstein-Podolski-Rosen steering for two-qubit states, Phys. Rev. A **93**, 020103(R) (2016).

[57] Y. Z. Law, L. P. Thinh, J.-D. Bancal, and V. Scarani, Quantum randomness extraction for various levels of characterization of the devices, J. Phys. A: Math. Theor. **47**, 424028 (2016).

[58] M. Piani and J. Watrous, Necessary and Sufficient Quantum Information Characterization of Einstein-Podolsky-Rosen Steering, Phys. Rev. Lett. **114**, 060404 (2015).

[59] C. Branciard and N. Gisin, Quantifying the Nonlocality of Greenberger-Horne-Zeilinger Quantum Correlations by a Bounded Communication Simulation Protocol, Phys. Rev. Lett. **107**, 020401 (2011).

[60] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, Phys. Rev. A **85**, 010301(R) (2012).

[61] N. Brunner, J. Sharam, and T. Vertesi, Testing the Structure of Multipartite Entanglement with Bell Inequalities, Phys. Rev. Lett. **108**, 110501 (2012).

[62] M. Wiesniak, M. Nawareg, and M. Zukowski, N-particle nonclassicality without N-particle correlations, Phys. Rev. A **86**, 042339 (2012).

[63] J. Tura, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, and A. Acín, Detecting nonlocality in many-body quantum states, Science **344**, 1256 (2012).

[64] G. Tóth, Entanglement witnesses in spin models, Phys. Rev. A **71**, 010301(R) (2005).

[65] J. K. Korbicz, J. I. Cirac, J. Wehr, and M. Lewenstein, Hilbert's 17th Problem and the Quantumness of States, Phys. Rev. Lett. **94**, 153601 (2005).

[66] G. Tóth, C. Knapp, O. Günhe, and H. J. Briegel, Optimal Spin Squeezing Inequalities Detect Bound Entanglement in Spin Models, Phys. Rev. Lett. **99**, 250405 (2007).

[67] M. Markiewicz, W. Laskowski, T. Paterek, and M. Żukowski, Detecting Genuine Multipartite Entanglement of Pure States with Bipartite Correlations, Phys. Rev. A **87**, 034301 (2013).

[68] S. Luo, Quantum discord for two-qubit systems, Phys. Rev. A **77**, 042303 (2008).

[69] Ng K. Feng, Bell Monogamy, B.Sc. Final Year Project, Nanyang Technological University, 2015, https://hdl.handle.net/10356/63200.

[70] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Bell's theorem without inequalities, Am. J. Phys. **58**, 1131 (1990).

[71] P. Agrawal and A. Pati, Perfect teleportation and superdense coding with *W* states, Phys. Rev. A **74**, 062320 (2006).

[72] C. Sabín and G. García-Alcaine, A classification of entanglement in three-qubit systems, Eur. Phys. J. D **48**, 435 (2008).

[73] D. Collins and N. Gisin, A relevant two qubit Bell inequality inequivalent to the CHSH inequality, J. Phys. A: Math. Gen. **37**, 1775 (2004).

[74] C. Śliwa, Symmetries of the Bell correlation inequalities, Phys. Lett. A, **317**, 165 (2003).

[75] S. Hill and W. K. Wootters, Entanglement of a Pair of Quantum Bits, Phys. Rev. Lett. **78**, 5022 (1997).

[76] W. K. Wootters, Entanglement of Formation of an Arbitrary State of Two Qubits, Phys. Rev. Lett. **80**, 2245 (1998).

[77] F. Verstraete and M. M. Wolf, Entanglement Versus Bell Violations and Their Behavior Under Local Filtering Operations, Phys. Rev. Lett. **89**, 170401 (2002).

[78] R. Nepal, R. Prabhu, A. Sen(De), and U. Sen, Maximally-dense-coding-capable quantum states, Phys. Rev. A **87**, 032336 (2013).

[79] R. Horodecki, P. Horodecki, and M. Horodecki, Violating Bell inequality by mixed states: Necessary and sufficient condition, Phys. Lett. A **200**, 340 (1995).

[80] X. Zha, Z. Da, I. Ahmed, D. Zhang, and Y. Zhang, General monogamy equalities of complementarity relation and distributive entanglement for multi-qubit pure states, Laser Phys. Lett. **15**, 025202 (2018).

[81] A. Higuchi, A. Sudbery, and J. Szulc, One-Qubit Reduced States of a Pure Many-Qubit State: Polygon Inequalities, Phys. Rev. Lett. **90**, 107902 (2003).

[82] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Generalized Schmidt Decomposition and Classification of Three-Quantum-Bit States, Phys. Rev. Lett. **85**, 1560 (2000).

[83] M. Plesch and V. Bužek, Entangled graphs: Bipartite entanglement in multiqubit systems, Phys. Rev. A **67**, 012322 (2003).