Quantum incompatibility of a physical context

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Pivotal within quantum physics, the concept of quantum incompatibility is generally related to algebraic aspects of the formalism, such as commutation relations and unbiasedness of bases. Recently, the concept was identified as a resource in tasks involving quantum state discrimination and quantum programmability. Here, we link quantum incompatibility with the amount of information that can be extracted from a system upon successive measurements of noncommuting observables, a scenario related to communication tasks. This approach leads us to characterize incompatibility as a resource encoded in a physical context, which involves both the quantum state and observables. Moreover, starting with a measure of context incompatibility we derive a measurement-incompatibility quantifier that is easily computable, admits a geometrical interpretation, and is maximum only if the eigenbases of the involved observables are mutually unbiased.

Introduction. One of the most intriguing phenomena involving microscopic systems, quantum incompatibility is commonly associated with the noncommutativity of self-adjoint operators. This means that, contrary to the state of affairs within the classical paradigm, when two observables do not commute, their eigenvalues cannot be simultaneously obtained through a single measurement. It is then natural to take violations of joint measurability—the hypothesis that a set of measurements can be decomposed in terms of a single "parent" measurement—as a faithful symptom of incompatibility [1–3].

Such an idea has shown to be very insightful, as it unveils interconnections between the so-called measurement incompatibility and nonlocal resources, as, for instance, Bell nonlocality [4–8] and Einstein-Podolsky-Rosen steering [9–11]. As for a quantitative assessment of the concept, incompatibility robustness measures have been introduced [12,13] with a basis on the amount of noise needed to render the measurements (or devices [14]) compatible. From that, further developments were accomplished within the contexts of device-independent characterizations [15–17], state-discrimination tasks [18–21], and quantum programmability [22], through which operational interpretations were conceived to measurement incompatibility. Recently, however, unexpected features have been noted for some widely used robustness-based measures of incompatibility [23].

Intuition requires that quantum incompatibility should vanish as the system approaches the classical domain—an instance that is usually accomplished through the quantum state. Accordingly, measurement incompatibility has been shown to disappear under noise [24]. In such an approach, however, one can use the duality relation $\text{Tr}[\Gamma(\rho)X] = \text{Tr}[\rho\,\Gamma_*(X)]$ to maintain a state-independent notion of measurement incompatibility. Indeed, one can always interpret any local noisy channel Γ, leading ρ to a classical state, as implying some degree of fuzziness in the X measurement. Nevertheless, this concept seems to be related more to experimental imperfections [23] than to fundamental classicalization processes involving the discard of correlated systems [25,26]. A subtler classical scenario can be conceived as follows. As far as heavy bodies are concerned, measurements are expected to be nearly nondisturbing, so that the resulting physical state should be independent of the ordering with which two noncommuting observables are measured. We then have a clear dependence of the notion of measurement incompatibility with an intrinsic property (the mass) of the probed body. In this case, it is less obvious how to effectively rephrase classicality in the formal structure of the measurement operators.

In this Rapid Communication, by considering a scenario designed to test the safety of a communication channel, we link quantum incompatibility with information—the most fundamental resource for quantum information and quantum thermodynamics tasks [27–29]. Our approach employs a key principle powering quantum cryptography [30], namely, that no information can be extracted from a system without disturbing it [31]. Here, the crux is that disturbances can only occur if the measured observables and the quantum state do not commute with each other. We then introduce the concept of context incompatibility and show that it is a quantum resource for communication tasks and can be linked with a formulation of measurement-incompatibility geometry.

Context incompatibility. Let $\mathbb{C} \equiv \{\rho, X, Y\} \subset \mathfrak{B}(\mathcal{H})$ be a context such that $X = \sum_j x_j X_j$ and $Y = \sum_k y_k Y_k$ are nondegenerate discrete observables, with respective eigenbases $\{|x_j\rangle\}_{j=1}^d$ and $\{|y_k\rangle\}_{k=1}^d$, $X_j = |x_j\rangle\langle x_j|$ and $Y_k = |y_k\rangle\langle y_k|$ are projectors, ρ is a generic quantum state, $\mathcal{H}$ is a d-dimensional Hilbert space, and $\mathfrak{B}(\mathcal{H})$ is the set of linear bounded operators acting on $\mathcal{H}$. Let us consider the generic protocol depicted in

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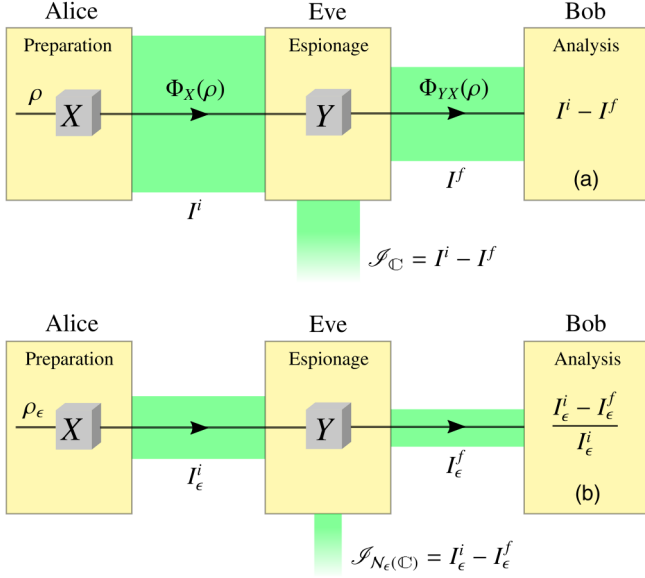†msavi@fisica.ufpr.br
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FIG. 1. (a) For a preparation ρ, Alice measures an observable X and thus sets an amount $I^i = I(\Phi_X(\rho))$ of information (depicted by the first green thick stripe). Aware of the calibration procedure performed by Alice, the trusted partner Bob makes state tomography and then checks the information received. Upon the action of an eavesdropper, Eve, who measures Y, the received information actually is just $I^f = I(\Phi_{YX}(\rho))$. The incompatibility of a context $\mathbb{C} \equiv \{\rho, X, Y\}$ is a resource, quantified as $\mathscr{I}_{\mathbb{C}} := I^i - I^f$, that allows for Alice and Bob to detect, via information leakage, Eve's espionage. (b) Aiming at discouraging potential eavesdroppers, Alice now prepares a highly noisy state $\rho_\epsilon = \mathcal{N}_\epsilon(\rho)$ [Eq. (6)] and injects a very limited amount of information I^i_ϵ in the channel. Bob receives an even more restrictive amount I^f_ϵ of information. However, because the information leakage is proportional to the injected information, by looking at the ratio $(I^i_\epsilon - I^f_\epsilon)/I^i_\epsilon$ he still succeeds to detect Eve's intervention.

Fig. 1. Alice prepares a state ρ, with informational content

$$I(\rho) := \ln d - S(\rho), \quad (1)$$

where $S(\rho) = -\text{Tr}(\rho \ln \rho)$ is the von Neumann entropy of ρ. After measuring X, without registering the outcome of any particular run of the experiment, Alice transforms the initial preparation into

$$\Phi_X(\rho) := \sum_{j=1}^{d} X_j \rho X_j = \sum_{j=1}^{d} p_j X_j, \quad (2)$$

where $p_j = \text{Tr}(\rho X_j)$. The completely positive trace-preserving (CPTP) unital map Φ_X removes quantum coherence from ρ in the basis $\{|x_j\rangle\}$. At this stage, the informational resource is reduced to the value $I^i \equiv I(\Phi_X(\rho)) = \ln d - H(p_j)$, where $H(p_j) = -\sum_j p_j \ln p_j$ is the Shannon entropy of the probability distribution p_j. The system is then delivered to Bob, who expects to receive an amount I^i of informational resource, as prearranged with Alice. Upon a successful verification, the trusted partners will have ascertained that the channel is safe from information leakage. Now, suppose that an eavesdropper, Eve, intercepts the system sent by Alice and probes it by measuring Y. The procedure is conducted by means of a

unitary transformation $U \in \mathfrak{B}(\mathcal{H} \otimes \mathcal{H}_\mathcal{E})$ that entangles the system, which left Alice's laboratory in the state $\Phi_X(\rho)$, with Eve's apparatus $\mathcal{E}$ initially prepared in a state $\rho_0^\mathcal{E} \in \mathfrak{B}(\mathcal{H}_\mathcal{E})$. The composite state $\Omega_0 = \Phi_X(\rho) \otimes \rho_0^\mathcal{E}$ thus evolves into $\Omega_t = U\Omega_0 U^\dagger$. Using Eq. (1) we can rewrite the mutual information, defined by $\mathcal{I}(\Omega_t) = S(\rho_t) + S(\rho_t^\mathcal{E}) - S(\Omega_t)$, in the form $I(\Omega_t) = I(\rho_t) + I(\rho_t^\mathcal{E}) + \mathcal{I}(\Omega_t)$, where $\rho_t^\mathcal{E}$ is the reduced state of the apparatus and $\rho_t = \text{Tr}_\mathcal{E}\Omega_t = \Phi_{YX}(\rho)$ (via the Stinespring theorem). Unitary invariance of the von Neumann entropy guarantees that $I(\Omega_0) = I(\Omega_t)$, from which we obtain $I(\Phi_X(\rho)) + I(\rho_0^\mathcal{E}) = I(\Phi_{YX}(\rho)) + I(\rho_t^\mathcal{E}) + \mathcal{I}(\Omega_t)$. With the notation $\Delta I_\mathcal{E} \equiv I(\rho_t^\mathcal{E}) - I(\rho_0^\mathcal{E})$ and $I^f \equiv I(\Phi_{YX}(\rho))$, we arrive at

$$I^i - I^f = I(\Phi_X(\rho)) - I(\Phi_{YX}(\rho)) = \Delta I_\mathcal{E} + \mathcal{I}(\Omega_t), \quad (3)$$

where $\Phi_{YX}(\rho) = \sum_{j,k} \wp_{k|j} p_j Y_k$, $\wp_{k|j} = |\langle x_j | y_k \rangle|^2 = \text{Tr}(X_j Y_k)$, and $I^f = \ln d - H(\sum_j \wp_{k|j} p_j)$. Clearly, the resource consumed from Alice's system, $\mathscr{I}_{\mathbb{C}} \equiv I^i - I^f$, was used to change the local information of Eve's apparatus and to increase the correlations between the system and the apparatus. If $\mathscr{I}_{\mathbb{C}} > 0$, Alice and Bob then discover that the channel is being spied upon [Fig. 1(a)]. Now, using $\sum_k Y_k^2 = \mathbb{1}$ and $Y_k \Phi_Y(\sigma) = \Phi_Y(\sigma) Y_k$, one shows that $\text{Tr}[\sigma g(\Phi_Y(\sigma))] = \text{Tr}[\Phi_Y(\sigma) g(\Phi_Y(\sigma))]$ for any state σ and function g. This allows us to write $S(\sigma || \Phi_Y(\sigma)) = S(\Phi_Y(\sigma)) - S(\sigma)$, with $S(\sigma || \varrho) = \text{Tr}[\sigma(\ln \sigma - \ln \varrho)] \geqslant 0$ being the relative entropy (equality holding if and only if $\sigma = \varrho$). We then arrive at the form

$$\mathscr{I}_{\mathbb{C}} = I(\Phi_X(\rho)) - I(\Phi_{YX}(\rho)) = S(\Phi_X(\rho) || \Phi_{YX}(\rho)), \quad (4)$$

through which we can check that there are only two instances in which $\mathscr{I}_{\mathbb{C}} = 0$: (i) $[X, Y] = 0$ ($\forall \rho$) and (ii) $\Phi_X(\rho) = \mathbb{1}/d$ ($\forall Y$). In case (i), the operators share the same set of eigenstates and $\Phi_{YX}(\rho) = \Phi_X(\rho)$. Case (ii) implies that $I^i = I^f = 0$. On the other hand, the consumed resource $\mathscr{I}_{\mathbb{C}}$ reaches its maximum value, $\ln d$, when $\rho = X_j$ (an eigenstate of X) and, in addition, the X and Y eigenbases form mutually unbiased bases (MUB) [32], that is, $|\langle x_j | y_k \rangle|^2 = 1/d$. Therefore, from Bob's (Eve's) viewpoint, noncommutativity and $I^i > 0$ are necessary ingredients—resources—for a successful leakage detection (information acquisition). Thus, with respect to the protocol depicted by Fig. 1(a), the following concept is introduced.

Definition. Context incompatibility is the resource encoded in a context $\mathbb{C} \equiv \{\rho, X, Y\}$ that allows one to test the safety of a communication channel against information leakage. Quantified via $\mathscr{I}_{\mathbb{C}} = I^i - I^f$ [Eq. (4)], it is operationally related to the amount of information subtracted from the system upon an external measurement.

Before proceeding with the proof that context incompatibility can be framed in the formal structure of a resource theory, it is interesting to note that a connection can be made with quantum coherence—a well-established quantum resource quantified by the $\{|y_k\rangle\}$-basis relative entropy of coherence, $C_Y(\rho) = S(\rho || \Phi_Y(\rho))$ [33,34]. As one can readily check, $\mathscr{I}_{\mathbb{C}} = C_Y(\Phi_X(\rho))$, meaning that context incompatibility can be viewed as the amount of Y coherence that is encoded in an X-incoherent state $\Phi_X(\rho)$. This is how the incompatibility of the set $\{X, Y\}$ is captured by $\mathscr{I}_{\mathbb{C}}$.

Resource theory of context incompatibility. We now formally characterize context incompatibility as a resource. Unlike the usual account of resource theories, where the resource is encoded in the quantum state, here the resource is encoded in the whole physical context $\mathbb{C}$. Following standard approaches [28,29], we devise a formal structure composed of (i) resourceless contexts, (ii) resourceful contexts, (iii) free operations, and (iv) a resource monotone. The last object is naturally identified with the measure (4), but any other contractive distance function involving the states $\Phi_{YX}(\rho)$ and $\Phi_X(\rho)$ would work as a proper monotone. The resourceless (free) contexts, defined as $\mathbb{C}^{\text{free}}$ such that $\mathscr{I}_{\mathbb{C}^{\text{free}}} = 0$, are

$$\mathbb{C}_1^{\text{free}} = \{\rho, X, Y\} \text{ s.t. } [X, Y] = 0 \ (\forall \rho), \tag{5a}$$

$$\mathbb{C}_2^{\text{free}} = \{\rho, X, Y\} \text{ s.t. } \Phi_X(\rho) = \mathbb{1}/d \ (\forall Y). \tag{5b}$$

Notice that for $\mathbb{C}_2^{\text{free}}$ one has $I^i = 0$. The proof that the above are the only existing free contexts is given in the Supplemental Material [35]. Apart from them, any other context is termed resourceful. With regard to free operations, since the relative entropy is nonincreasing under generic CPTP maps Γ, it follows that $\mathscr{I}_{\mathbb{C}} \geqslant \mathscr{I}_{\Gamma(\mathbb{C})}$, where $\Gamma(\mathbb{C}) \equiv \{\Gamma(\rho), X, Y\}$, provided that Γ commutes with the maps Φ_X and Φ_{YX}. In this case, it is clear that resource is never created upon the action of Γ. Also, to ensure that $\mathscr{I}_{\Gamma(\mathbb{C}^{\text{free}})} = 0$, we need to require Γ to be unital, so as not to make $\mathbb{C}_1^{\text{free}}$ resourceful upon Γ. Altogether, these aspects characterize the free operations Γ with respect to context incompatibility. In our approach we do not admit any operations on $\{X, Y\}$, as this would imply aspects of measurement fuzziness that have been disregarded from the outset.

Measurement incompatibility. Let us come back to the protocol, now considering a noisy scenario [Fig. 1(b)]. To discourage any potential eavesdroppers, Alice introduces, in a controllable way, an amount $1 - \epsilon$ of noise in the input state, which then reads

$$\rho_\epsilon = \mathcal{N}_\epsilon(\rho) \coloneqq (1 - \epsilon)\frac{\mathbb{1}}{d} + \epsilon \rho, \tag{6}$$

where $\epsilon \in [0, 1]$ and $\mathcal{N}_\epsilon$ is a CPTP unital noise map. From the concavity of the entropy and the joint convexity of the relative entropy, one can check that $I_\epsilon^i \equiv I(\Phi_X(\rho_\epsilon)) \leqslant \epsilon I^i$ and $\mathscr{I}_{\mathcal{N}_\epsilon(\mathbb{C})} \leqslant \epsilon \mathscr{I}_{\mathbb{C}}$, where $\mathcal{N}_\epsilon(\mathbb{C}) = \{\rho_\epsilon, X, Y\}$ and $\mathcal{N}_{\epsilon=0}(\mathbb{C}) = \mathbb{C}$. Hence, the preparation ρ_ϵ implies, for $\epsilon \ll 1$, a very limited amount of information in the channel and an equally restrictive amount of consumable information. Aware of the amount of noise introduced, Bob can still check the security of the channel by looking at the amount of information that leaks per unit of injected information. Bob computes the ratio $\mathscr{R}_{\mathcal{N}_\epsilon(\mathbb{C})} \coloneqq (I_\epsilon^i - I_\epsilon^f)/I_\epsilon^i = \mathscr{I}_{\mathcal{N}_\epsilon(\mathbb{C})}/I_\epsilon^i$, with $I_\epsilon^f \equiv I(\Phi_{YX}(\rho_\epsilon))$, since, in the large-noise limit, it reads

$$\lim_{\epsilon \to 0} \mathscr{R}_{\mathcal{N}_\epsilon(\mathbb{C})} = \frac{||\Phi_{YX}(\rho) - \Phi_X(\rho)||^2}{||\Phi_X(\rho) - \mathbb{1}/d||^2} =: \mathscr{R}_{\mathbb{C}}, \tag{7}$$

where $||A|| \coloneqq \sqrt{\text{Tr}(A^\dagger A)}$ is the Hilbert-Schmidt norm of A. The limit is calculated as follows. Since $|x_j\rangle$ are eigenstates of $\Phi_X(\rho_\epsilon)$ with eigenvalues $\frac{1-\epsilon}{d} + \epsilon p_j$, we can explicitly compute the von Neumann entropy and the associated information, I_ϵ^i. For $d \ll 1/\epsilon$, we expand the formulas and retain only terms up to order ϵ^2. With this procedure, we find $I_\epsilon^i \cong$

$\frac{\epsilon^2}{2}[d||\Phi_X(\rho)||^2 - 1]$ and $I_\epsilon^f \cong \frac{\epsilon^2}{2}[d||\Phi_{YX}(\rho)||^2 - 1]$. From the result $\text{Tr}[\Phi_X(\rho)\Phi_{YX}(\rho)] = ||\Phi_{YX}(\rho)||^2$, one is able to show that $||\Phi_{YX}(\rho) - \Phi_X(\rho)||^2 = ||\Phi_X(\rho)||^2 - ||\Phi_{YX}(\rho)||^2$. The emerging expression for $\mathscr{I}_{\mathcal{N}_\epsilon(\mathbb{C})}/I_\epsilon^i$ results to be ϵ independent and the limit trivially follows. Therefore, by use of this ratio, Bob can still check information leakage for arbitrarily large noise. An interesting feature of formula (7) is that it is invariant upon noise maps of the form (6), that is, $\mathscr{R}_{\mathcal{N}_\epsilon(\mathbb{C})} = \mathscr{R}_{\mathbb{C}}$, for all ρ and $\epsilon \in [0, 1]$. This allows us to write, up to order ϵ^2,

$$\mathscr{I}_{\mathcal{N}_\epsilon(\mathbb{C})} \cong \mathscr{R}_{\mathbb{C}} I_\epsilon^i, \tag{8}$$

which shows that the amount of information that is extracted by Eve is directly proportional to the injected information [as suggested by the green thick stripes in Fig. 1(b)]. Being ϵ independent, the proportionality ratio $\mathscr{R}_{\mathbb{C}}$ might be expected to be more closely associated with the algebraic relations between X and Y solely (this will be shown to be true for any qubit context), but in general it provides an estimate for the context incompatibility, though in a norm-based way.

In search of a link between context incompatibility and measurement incompatibility, the natural move is to restrict ourselves to the context $\mathbb{C}_j \equiv \{X_j, X, Y\}$. Then, setting $\rho = X_j$ in Eq. (6), we find $\mathscr{R}_{\mathbb{C}_j} = \frac{d}{d-1}[1 - ||\Phi_Y(X_j)||^2]$, which is just the linear entropy of $\Phi_Y(X_j) = \sum_k \wp_{k|j} Y_k = \sum_k \text{Tr}(Y_k X_j) Y_k$, and $I_\epsilon^i \cong \epsilon^2(d-1)/2$. It follows that $\mathscr{I}_{\mathcal{N}_\epsilon(\mathbb{C}_j)} \cong \mathscr{R}_{\mathbb{C}_j} I_\epsilon^i$, with I_ϵ^i keeping no dependence on the input state X_j. This result is relevant because it shows that the ratio $\mathscr{R}_{\mathbb{C}_j}$, which is an easily computable measure, suffices to capture the level of incompatibility in the context $\mathbb{C}_j$. However, it cannot be our definitive figure of merit for quantifying the measurement incompatibility of the set $\{X, Y\}$, since it considers only a single element of the X eigenbasis. We then examine the averaging

$$\mathscr{M}_{\{X,Y\}} \coloneqq \frac{1}{d}\sum_{j=1}^d \mathscr{R}_{\mathbb{C}_j}. \tag{9}$$

By construction, $\mathscr{M}_{\{X,Y\}}$ tends to be a more appropriate quantifier of measurement incompatibility, for it (i) encompasses the contribution of all X eigenstates and (ii) is symmetrical upon the ordering permutation $X \leftrightarrow Y$, that is, $\mathscr{M}_{\{X,Y\}} = \mathscr{M}_{\{Y,X\}}$ (a desirable property for a measure meant to describe an algebraic relation between two observables). This point can be checked from the manipulated form

$$\mathscr{M}_{\{X,Y\}} = \sum_{j,k=1}^d \frac{||[X_j, Y_k]||^2}{2(d-1)} = \frac{1}{d-1}\left(d - \sum_{j,k=1}^d |\langle x_j|y_k\rangle|^4\right), \tag{10}$$

The first equality makes it explicit a relation with the commutator $[X, Y] = \sum_{j,k} x_j y_k [X_j, Y_k]$. This is an important reference to the well-known fact that, when projective measurements are concerned, joint measurability and commutativity turn out to be equivalent notions, although this is not true in general [3]. Moreover, we have $0 \leqslant \mathscr{M}_{\{X,Y\}} \leqslant 1$, with the upper (lower) bound being reached for, and only for, MUB (commuting operators). As demonstrated in the Supplemental Material [35], $\mathscr{M}_{\{X,Y\}}$ can be derived via an independent

purely algebraic construction, which further reinforces the claim that this can be taken as a reasonable quantifier of measurement incompatibility.

Geometrical interpretation. We now build a geometrical picture for the incompatibility measures introduced above. To this end, we employ the generalized Bloch representation, which is based on the observation that the set of matrices $\{\mathbb{1}, \Lambda_1, \ldots, \Lambda_{d^2-1}\}$ form a basis for linear operators acting on the state space, where the $d \times d$ complex, traceless, orthogonal, self-adjoint matrices Λ_i are the generators of SU(d), the special group of degree d (see the Supplemental Material [35] for a very brief review of the complete formalism [36,37]). With the normalization $\text{Tr}(\Lambda_i \Lambda_j) = 2\delta_{ij}$, one can always express a quantum state as

$$\rho_{\mathbf{r}} = \frac{1}{d}(\mathbb{1} + C_d \mathbf{r} \cdot \mathbf{\Lambda}), \tag{11}$$

where $\mathbf{r} = \sum_{i=1}^{d^2-1} r_i \mathbf{e}_i$, $\mathbf{\Lambda} = \sum_{i=1}^{d^2-1} \Lambda_i \mathbf{e}_i$, $\{\mathbf{e}_i\}_{i=1}^{d^2-1}$ is an orthonormal basis in $\mathbb{R}^{d^2-1}$, and $C_d = \sqrt{d(d-1)/2}$. Through the above parametrization, any quantum state is represented by the vector $\mathbf{r}$ in a $(d^2 - 1)$-dimensional real ball $B(\mathbb{R}^{d^2-1})$. Projection operators admit the description

$$X_j = \frac{1}{d}(\mathbb{1} + C_d \mathbf{x}_j \cdot \mathbf{\Lambda}), \tag{12}$$

with $\sum_j \mathbf{x}_j = \mathbf{0}$ and $\mathbf{x}_i \cdot \mathbf{x}_j = (\delta_{ij} d - 1)/(d - 1)$, which follow from $\sum_j X_j = \mathbb{1}$ and $\text{Tr}(X_i X_j) = \delta_{ij}$. From the algebra induced by Λ_i and pertinent vector products, one may prove that $\text{Tr}[(\mathbf{r}_1 \cdot \mathbf{\Lambda})(\mathbf{r}_2 \cdot \mathbf{\Lambda})] = 2(\mathbf{r}_1 \cdot \mathbf{r}_2)$, with $\mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^{d^2-1}$. This simple formula allows one to show that $||\rho_{\mathbf{r}}||^2 = \frac{1}{d}[1 + (d-1)r^2]$, with $r^2 = \mathbf{r} \cdot \mathbf{r}$, and $p_{x_i} = \text{Tr}(X_i \rho) = \frac{1}{d}[1 + (d-1)\mathbf{x}_i \cdot \mathbf{r}]$. In addition, one shows that $r_i = \frac{d}{2C_d}\text{Tr}(\Lambda_i \rho)$. With this formalism, we find

$$\Phi_X(\rho_{\mathbf{r}}) = \frac{1}{d}(\mathbb{1} + C_d \mathbf{u} \cdot \mathbf{\Lambda}), \quad \mathbf{u} = \frac{d-1}{d}\sum_{j=1}^{d}(\mathbf{x}_j \cdot \mathbf{r})\mathbf{x}_j, \tag{13a}$$

$$\Phi_{YX}(\rho_{\mathbf{r}}) = \frac{1}{d}(\mathbb{1} + C_d \mathbf{v} \cdot \mathbf{\Lambda}), \quad \mathbf{v} = \frac{d-1}{d}\sum_{k=1}^{d}(\mathbf{y}_k \cdot \mathbf{u})\mathbf{y}_k, \tag{13b}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{d^2-1}$ and $\mathbf{y}_i \cdot \mathbf{y}_j = (\delta_{ij} d - 1)/(d - 1)$. Traceless by hypothesis, the considered observables assume the form $X = \mathbf{x} \cdot \mathbf{\Lambda}$ and $Y = \mathbf{y} \cdot \mathbf{\Lambda}$, where $\mathbf{x} = (C_d/d)\sum_j x_j \mathbf{x}_j$ and $\mathbf{y} = (C_d/d)\sum_k y_k \mathbf{y}_k$. Relations (11)–(13) allow us to speak of the incompatibility

$$\mathscr{I}_{\mathbb{C}} = H\left(\frac{1 + (d-1)\mathbf{y}_k \cdot \mathbf{u}}{d}\right) - H\left(\frac{1 + (d-1)\mathbf{x}_j \cdot \mathbf{r}}{d}\right) \tag{14}$$

of the "geometrical context" $\mathbb{C} = \{\mathbf{r}, \mathbf{x}, \mathbf{y}\}$. In connection with the noisy state (6), we have $\mathcal{N}_\epsilon(\mathbb{C}) = \{\epsilon \mathbf{r}, \mathbf{x}, \mathbf{y}\}$, for which we find a particularly insightful result for the proportionality ratio,

$$\mathscr{R}_{\mathbb{C}} = \frac{||\mathbf{u} - \mathbf{v}||^2}{||\mathbf{u}||^2} = 1 - \frac{||\mathbf{v}||^2}{||\mathbf{u}||^2}. \tag{15}$$

To compute the measurement incompatibility we set $\rho = X_j$, which implies that $\mathbf{r} = \mathbf{x}_j = \mathbf{u}$ and, hence, $\mathscr{R}_{\mathbb{C}_j} = 1 - ||\mathbf{v}_j||^2$, with $\mathbf{v}_j = \frac{d-1}{d}\sum_{k=1}^{d}(\mathbf{y}_k \cdot \mathbf{x}_j)\mathbf{y}_k$. It then follows that

$$\mathscr{M}_{\{X,Y\}} = 1 - \frac{1}{d}\sum_{j=1}^{d}||\mathbf{v}_j||^2 = 1 - \frac{d-1}{d^2}\sum_{j,k=1}^{d}(\mathbf{x}_j \cdot \mathbf{y}_k)^2. \tag{16}$$

The results (14)–(16) rephrase incompatibility in terms of the geometry defined by the vectors $\mathbf{r}, \mathbf{x}, \mathbf{y}$. Here, the free contexts (5) manifest themselves with respect to $\mathscr{I}_{\mathbb{C}}$ as, for instance, $\mathbb{C}_1^{\text{free}} = \{\mathbf{r}, \mathbf{x}, \mathbf{y}\}$ with $\mathbf{x}_j \cdot \mathbf{y}_k = \frac{d\delta_{jk}-1}{d-1}$ ("parallel operators," since one has $\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}\sum_i x_i y_i$) and $\mathbb{C}_2^{\text{free}} = \{\mathbf{y}_k, \mathbf{x}, \mathbf{y}\}$ with $\mathbf{x}_j \cdot \mathbf{y}_l = 0$ ("orthogonal operators," since $\mathbf{x} \cdot \mathbf{y} = 0$). Interestingly, as far as $\mathscr{M}_{X,Y}$ is concerned, we see that it vanishes for parallel (commuting) operators and reaches its maximum for orthogonal (MUB forming) operators, this being the heart of our geometrical interpretation.

The scenario becomes rather simple for generic qubit contexts. By setting $d = 2, C_d = 1, \mathbf{\Lambda} = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_{1,2,3}$ are the Pauli matrices, $\mathbf{x}_j = x_j \mathbf{x}$, and $\mathbf{y}_k = y_k \mathbf{y}$ in the precedent formulas we readily obtain

$$\mathscr{I}_{\mathbb{C}} = h\left(\frac{1 + (\mathbf{x} \cdot \mathbf{y})(\mathbf{x} \cdot \mathbf{r})}{2}\right) - h\left(\frac{1 + (\mathbf{x} \cdot \mathbf{r})}{2}\right), \tag{17a}$$

$$\mathscr{M}_{\{X,Y\}} = \mathscr{R}_{\mathbb{C}} = \mathscr{R}_{\mathbb{C}_j} = 1 - (\mathbf{x} \cdot \mathbf{y})^2, \tag{17b}$$

where $h(\nu) = -\nu \ln \nu - (1 - \nu) \ln(1 - \nu)$ is the binary Shannon entropy and $\mathbf{r}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. Some remarks are in order. First, $\mathscr{I}_{\mathbb{C}}$ is the only quantity that depends on the state ρ (via $\mathbf{r}$), this being the key aspect characterizing it as a context incompatibility. In particular, this ensures that $\mathscr{I}_{\mathbb{C}} \to 0$ as $|\mathbf{r}| \to 0$ (decoherence-induced classical limit). The "large mass" classical limit, on the other hand, effectively comes via $\mathbf{x} \cdot \mathbf{y} \cong 1$, which implements the nondisturbance scenario and implies, via $\Phi_{YX}(\rho) \cong \Phi_X(\rho)$, that $\mathscr{I}_{\mathbb{C}} \cong 0$ (see the Supplemental Material [35] for details). This regime is, of course, equivalent to the free context $\mathbb{C}_1^{\text{free}}$, where $\mathbf{x} = \mathbf{y}$ (implying parallel operators, that is, $[X, Y] = 0$). The context incompatibility vanishes also when $\mathbf{x} \cdot \mathbf{r} = 0$—the case in which the first measurement X is incompatible with the input state—and monotonically increases with the quantifiers given by Eq. (17b). Second, by taking $\frac{1}{4}||[X, Y]||^2 = |\mathbf{x} \times \mathbf{y}|^2$ as an estimate for the notion of noncommutativity, we see that the ratios $\mathscr{R}_{\mathbb{C}}$ and $\mathscr{R}_{\mathbb{C}_j}$, with $\mathbb{C}_j = \{X_j, X, Y\}$, and the measurement incompatibility $\mathscr{M}_{\{X,Y\}}$ are all indistinguishable concepts for qubit contexts. To a certain extent, this can be related to the bidirectional implication reported in Ref. [38] between the notions of nondisturbance and commutativity.

Conclusion. In this Rapid Communication, we have derived a notion of incompatibility from the fact that no information can be extracted from a premeasured state $\Phi_X(\rho)$ if a compatible observable Y is measured in sequence. A distinctive feature of our approach is that it makes reference to a physical context, $\mathbb{C} = \{\rho, X, Y\}$, composed not only of observables (measurements) but also of quantum states. Besides allowing one to describe the disappearance of incompatibility in classical regime, our results associate incompatibility with an information-based task in space-time, rather than an

algebraic construction in the Hilbert space. The proposed measure of context incompatibility is easily computable and yet admits a norm-based estimate. Remarkably, the context incompatibility is shown to be a resource, with particular application to a protocol devised to test information leakage, makes contact with the notion of measurement incompatibility, and admits a geometrical interpretation in a vector space of arbitrary dimension. Our results give rise to some noteworthy research lines. The first one concerns the extension of our approach to contexts involving more than two (eventually continuous) observables. The second refers to the use of our easily computable quantifier of measurement incompatibility for MUB searching, a long-lasting intricate problem in quantum physics.

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