Radiation-induced interaction potential of two qubits strongly coupled with a quantized electromagnetic field

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We investigate the interaction of two two-level qubits with a single-mode quantum field in a cavity without the rotating wave approximation and considering that qubits can be located at an arbitrary distance from each other. We demonstrate that there exists a radiation-induced interaction potential between atoms. We studied the properties of the system numerically and in addition constructed a simple analytical approximation. It is shown that the observable characteristics are substantially dependent on the distance between the qubits in the strong-coupling regime. This allows one to perform the quantum control of the qubits, which can be exploited for the recording and transmission of quantum information.

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I. INTRODUCTION

The quantum Rabi model (QRM) describes the interaction of a two-level atom with a resonant single-mode quantum field in a cavity [1,2]. This model plays fundamental role in the radiation-matter interaction in cavity quantum electrodynamics [3–5], quantum optics [6], quantum information [7], and the physics of condensed matter [8]. Recently, the model attracted attention due to the fact that, in many systems, it is possible [9,10] to control the interaction strength over a wide range, including the so-called ultrastrong coupling (USC) regime, which corresponds to the variation interval from 0.3 to 1.0 of a dimensionless coupling constant f between an atom (qubit) and a field. The systems with the coupling constant from the USC range were lately realized experimentally [9,11]. As a result, these achievements are crucial for control of an interaction of quantum emitters with individual photons that is an important part of recording and transmission of quantum information.

Another related direction is the generalization of the QRM to systems containing multiple qubits, in particular the two qubits interacting with a resonance quantum field—the Tavis-Cummings model (TCM) [12,13]. This model is based on two approximations: The first is the rotating wave approximation (RWA) applicable for small values of the detuning between frequencies of the field and the resonant atomic transition and small values of the coupling constant f of the atom-field interaction. The second approximation assumes that the distance $\rho = |\boldsymbol{R}_1 - \boldsymbol{R}_2|$ between the qubits is small in comparison with the wavelength λ of the resonance field, i.e., $\rho \ll \lambda$. In addition, in works [4,14] the TCM was investigated beyond the RWA, however, still under the assumption $\rho \ll \lambda$.

At the same time, the systems containing two qubits located at the distance $\rho \approx \lambda$ and interacting with a resonant quantum field were recently realized experimentally. Moreover, it was possible to control the positions of qubits in a broad range with the use of tightly focused optical tweezers [15,16]. Similar experiments were recently performed for systems of Rydberg atoms located inside a cavity [15,17]. As a result, in our work we theoretically investigate the spectrum and dynamics of the system in the USC regime beyond the RWA and as a function of the distance ρ between qubits. We demonstrate that the dependence on ρ becomes important in the USC regime where the RWA in not applicable.

In our work we employ a dipole approximation for an individual qubit interacting with a quantum field. This approximation is consistent with the assumption that $\rho \approx \lambda$ due to the fact that, for optical frequencies of the resonance field, the distance ρ between qubits is much larger than the characteristic qubit size a_0, therefore $\rho \gg a_0$, allowing us to work in the dipole approximation for each qubit. In addition, the large qubit mass allows us to employ the adiabatic approximation in an analogy with the Born-Oppenheimer approximation of molecular physics. As a result, we treat the operator of kinetic energy of qubits as a perturbation. We show that the energy levels of the systems form potential surfaces as a function of the distance ρ, which define the radiation-induced potential of the interaction between qubits. The form of this potential is defined by the coupling constant f. Moreover, the distance between qubits can be considered as an additional parameter to be used to control the system's characteristics for the recording and the transmission of quantum information.

In addition, the dependence of observable characteristics of the system of two qubits on the distance between them adds

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an additional possibility to control the system. In particular, this allows one to control the location of the peak in the scattering cross section of the resonance radiation [18]; to vary the degree of entanglement when the transmission of quantum information is happening by two emitters (qubits) [19]; to change the population of states of two two-level systems to control the probability of a spontaneous emission [20]; and to obtain a periodic structure in the system of N atoms—the extended Dicke model [21].

II. CONSTRUCTION OF A MODEL HAMILTONIAN

The Hamiltonian of two identical two-level atoms (qubits) with the mass M, located at positions $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ in the dipole approximation for the interaction of atoms with the field, written in natural units ($\hbar = c = 1$) reads [22,23]

$$\begin{aligned}\hat{H} =& -\frac{1}{2M}(\Delta_{\boldsymbol{R}_1} + \Delta_{\boldsymbol{R}_2}) + \frac{\epsilon}{2}\left(\sigma_3^1 + \sigma_3^2\right) \\ &+ \omega f\left[(\hat{a}e^{i\boldsymbol{k}\cdot\boldsymbol{R}_1} + \hat{a}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{R}_1})\sigma_1^1 + (\hat{a}e^{i\boldsymbol{k}\cdot\boldsymbol{R}_2} + \hat{a}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{R}_2})\sigma_1^2\right] \\ &+ \omega\hat{a}^\dagger\hat{a} + V_a(\boldsymbol{R}_1 - \boldsymbol{R}_2), \end{aligned} \tag{1}$$

$$f = e_0\omega\Delta d\sqrt{\frac{4\pi}{\omega^3 V}}. \tag{2}$$

Here V is the volume of the cavity, f is the dimensionless coupling constant of an atom-field interaction, ϵ is the resonance transition energy between two qubit states $\chi_\uparrow$, $\chi_\downarrow$ with the dipole transition matrix element d; e_0, m_0 are the electron charge and mass respectively; $\hat{a}^\dagger$, $\hat{a}$ are the creation and annihilation operators of the resonant quantum field with frequency ω and the wave vector $\boldsymbol{k}$, $V_a(\boldsymbol{R}_1 - \boldsymbol{R}_2)$ is the atom-atom interaction potential due to the exchange and dipole-dipole interactions, Δ is the Laplace operator and $\boldsymbol{\sigma}^1$, $\boldsymbol{\sigma}^2$ are Pauli matrices for qubits one and two, respectively. The limit of $M \to \infty$ corresponds to the situation of two immovable qubits.

Let us switch to the center-of-mass reference system in Eq. (1),

$$\boldsymbol{R} = \frac{\boldsymbol{R}_1 + \boldsymbol{R}_2}{2}, \quad \boldsymbol{\rho} = \boldsymbol{R}_1 - \boldsymbol{R}_2, \tag{3}$$

in which the Hamiltonian (1) transforms into

$$\begin{aligned}\hat{H} =& -\frac{1}{4M}\Delta_{\boldsymbol{R}} - \frac{1}{M}\Delta_{\boldsymbol{\rho}} + \frac{\epsilon}{2}\left(\sigma_3^1 + \sigma_3^2\right) \\ &+ \omega f\left[(\hat{a}e^{i\boldsymbol{k}\cdot(\boldsymbol{R}+\boldsymbol{\rho}/2)} + \hat{a}^\dagger e^{-i\boldsymbol{k}\cdot(\boldsymbol{R}+\boldsymbol{\rho}/2)})\sigma_1^1\right. \\ &\left. + (\hat{a}e^{i\boldsymbol{k}\cdot(\boldsymbol{R}-\boldsymbol{\rho}/2)} + \hat{a}^\dagger e^{-i\boldsymbol{k}\cdot(\boldsymbol{R}-\boldsymbol{\rho}/2)})\sigma_1^2\right] \\ &+ \omega\hat{a}^\dagger\hat{a} + V_a(\boldsymbol{\rho}). \end{aligned} \tag{4}$$

The system possesses translational invariance with respect to the center-of-mass coordinate $\boldsymbol{R}$. Therefore, in analogy with the polaron problem we perform the Lee-Low-Pines transformation [24]

$$\hat{H}' = \hat{L}^{-1}\hat{H}\hat{L}, \quad \hat{L} = e^{i(\boldsymbol{P} - \boldsymbol{k}\hat{a}^\dagger\hat{a})\cdot\boldsymbol{R}}, \tag{5}$$

where $\boldsymbol{P}$ is the total momentum of the system, which in this case is an integral of motion. As a result, we find the following expression for the Hamiltonian of the system

$$\begin{aligned}\hat{H}' =& \frac{1}{4M}(\boldsymbol{P} - \boldsymbol{k}\hat{a}^\dagger\hat{a})^2 - \frac{1}{M}\Delta_{\boldsymbol{\rho}} + \frac{\epsilon}{2}\left(\sigma_3^1 + \sigma_3^2\right) \\ &+ \omega f\left[(\hat{a}e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}/2} + \hat{a}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}/2})\sigma_1^1\right. \\ &\left. + (\hat{a}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}/2} + \hat{a}^\dagger e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}/2})\sigma_1^2\right] + \omega\hat{a}^\dagger\hat{a} + V_a(\boldsymbol{\rho}). \end{aligned} \tag{6}$$

The characteristic scale with respect to the coordinate of a relative motion $\boldsymbol{\rho}$ is defined by the wavelength of the radiation $\lambda = 2\pi/k$ and, for the optical frequencies, substantially exceeds the size of the atom, a_0, such that $a_0/\lambda \approx 10^{-3}$. At these distances the contribution from the exchange interaction into the potential $V_a(\boldsymbol{\rho})$ is exponentially suppressed. The dipole-dipole interaction potential (van der Waals) of an atom-atom interaction is proportional [25] to $\kappa(a_0/\rho)^6 \sim \kappa(a_0/\lambda)^6$, with a_0 being the characteristic atomic size and κ being the dimensional constant. Consequently, in the dipole approximation ($a_0/\lambda \ll 1$) for the interaction of an individual atom with the electromagnetic field, the potential $V_a(\boldsymbol{\rho})$ should be neglected in the Hamiltonian $\hat{H}'$ (6) for consistency. In addition, we can also neglect a recoil in the operator of kinetic energy since we are working in the nonrelativistic limit. Consequently, one can write

$$\left\langle\frac{1}{2M}(\boldsymbol{P}\boldsymbol{k}\hat{a}^\dagger\hat{a})\right\rangle \approx \frac{P}{2M}\omega \ll \omega. \tag{7}$$

As a result, we arrive to the final expression for the Hamiltonian, which describes the interaction of two two-level atoms within the above-described approximations:

$$\begin{aligned}\hat{H}' =& \frac{P^2}{4M} - \frac{1}{M}\Delta_{\boldsymbol{\rho}} + \frac{\epsilon}{2}\left(\sigma_3^1 + \sigma_3^2\right) + \omega\hat{a}^\dagger\hat{a} \\ &+ \omega f\left[(\hat{a}e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}/2} + \hat{a}^\dagger e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}/2})\sigma_1^1\right. \\ &\left. + (\hat{a}e^{-i\boldsymbol{k}\cdot\boldsymbol{\rho}/2} + \hat{a}^\dagger e^{i\boldsymbol{k}\cdot\boldsymbol{\rho}/2})\sigma_1^2\right]. \end{aligned} \tag{8}$$

III. THE INTERACTION POTENTIAL OF TWO QUBITS

Let us investigate the Schrödinger equation with the Hamiltonian (8). For this purpose we choose a coordinate system in which the x axis is directed along the $\boldsymbol{k}$ and seek a solution in the form $\Lambda(\boldsymbol{\rho}) = \exp\{-i(p_y y + p_z z)\}\Psi(x)$:

$$\begin{aligned}E\Psi(x) =& \left\{\frac{P^2}{4M} + \frac{\left(p_y^2 + p_z^2\right)}{M} - \frac{1}{M}\frac{d^2}{dx^2} + \frac{\epsilon}{2}\left(\sigma_3^1 + \sigma_3^2\right)\right. \\ &+ \omega\hat{a}^\dagger\hat{a} + \omega f\left[(\hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi})\sigma_1^1\right. \\ &\left.\left. + (\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi})\sigma_1^2\right]\right\}\Psi(x), \end{aligned} \tag{9}$$

where $\phi = \pi x/\lambda$.

The case of $x = 0$ corresponds to the situation that atoms are unified and form a system with different parameters. Therefore, we assume that, in the operator (9), the coordinate x is varying in the range $|x| > x_0$, where the quantity $x_0 \sim a \ll \lambda$. In the general case the vector $\boldsymbol{\rho}$ can be directed under an arbitrary angle with respect to the vector $\boldsymbol{k}$. However, we consider that the conditions $y \approx z \approx x_0$ are fulfilled for the projections of $\boldsymbol{\rho}$ on the direction perpendicular to the $\boldsymbol{k}$ vector.

As explained in the introduction, the operator of kinetic energy of a qubit is a small correction in comparison with

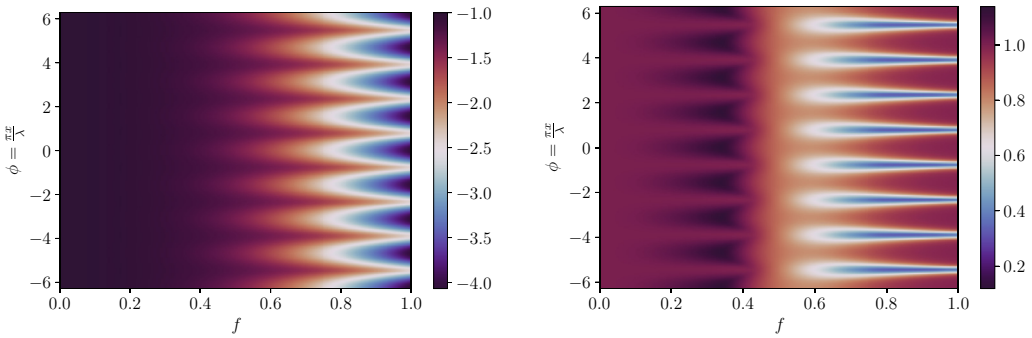

FIG. 1. (left panel) The potential surface $u_0(\phi)$ of the ground state as a function of the dimensionless coupling constant f and the coordinate $\phi = \pi x/\lambda$. The parameter $\delta = 1$. When the system of two qubits is in the USC regime, the potential wells are the deepest. (right panel) The transition frequency $\Omega_{\nu_1,\nu_2}(\phi)$ from the ground state to the third excited state as a function of the dimensionless coupling constant f and the coordinate ϕ. The parameter $\delta = 1$.

the operator of the interaction of a qubit and a resonance field. Therefore, we employ an adiabatic approximation and separate variables in Eq. (9):

$$\Psi(x) \approx \Phi(x)\psi_\nu, \tag{10}$$

$$U_\nu(x)\psi_\nu = \left\{\frac{\epsilon}{2}\left(\sigma_3^1 + \sigma_3^2\right) + \omega\hat{a}^\dagger\hat{a} + \omega f[(\hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi})\sigma_1^1 + (\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi})\sigma_1^2]\right\}\psi_\nu, \tag{11}$$

$$E\,\Phi(x) = \left\{\frac{P^2}{4M} + \frac{\left(p_y^2 + p_z^2\right)}{M} - \frac{1}{M}\frac{d^2}{dx^2} + U_\nu(x)\right\}\Phi(x). \tag{12}$$

In these equations the index ν denotes a set of quantum numbers of qubit-photon system and the terms $U_\nu(x)$ define the radiation-induced interaction potential of two qubits in full analogy with molecular physics. The solution of the Schrödinger equation (12) determines the relative motion of qubits induced by the potential $U_\nu(x)$.

We want to mention here that when the ϕ equals zero the operator in Eq. (11) coincides with the one obtained in works [4,14]. In this case, $U_\nu(0)$ corresponds to the spectrum found in Refs. [4,14].

To calculate the potential function $U_\nu(x)$, approximate methods can be employed [16]. In our work we have developed the analytical approximation for energy levels and in addition performed an exact numerical solution by diagonalizing the Hamiltonian matrix written in the basis of eigenstates of σ_3 and Fock states of the field. The details of the calculations are given in Appendix.

From the structure of Eq. (11) it follows that the Hamiltonian of the system is a periodic function with a period 2λ.

It is convenient to express the energies in the units of the photon frequency

$$U_\nu(x) = \omega u_\nu(x), \tag{13}$$

$$u_\nu(x)\psi_\nu = \left\{\frac{\delta}{2}\left(\sigma_3^1 + \sigma_3^2\right) + \hat{a}^\dagger\hat{a} + f[(\hat{a}e^{i\phi} + \hat{a}^\dagger e^{-i\phi})\sigma_1^1 + (\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi})\sigma_1^2]\right\}\psi_\nu, \tag{14}$$

where we introduced $\delta = \epsilon/\omega$.

We plot in Fig. 1 (left panel) the potential surface as a function of the coupling constant f for the ground state of the system when $\delta = 1$. The depth of the potential wells varies by two orders of magnitude when the coupling constant is changing in the range $[0, 1]$. When the system is in the USC regime and approaching $f = 1$, the depth of the wells is four times larger (i.e., 4δ) than the splitting δ between qubit energy levels. As a result, we will mainly be investigating this most interesting regime.

We pay attention to the fact that the period of oscillations of the potential equals $\lambda/2$, which is different from the periodicity 2λ of the Hamiltonian of the system. This symmetry arises also in the analytical approximation derived in Appendix (see also Fig. 5). The decrease of the period of the oscillations of the interaction potential is explained by the fact that the eigenvalues of the Hamiltonian in Eq. (14) are degenerate—the eigenvalues are invariant under the transformation $R_j^{-1}HR_j$ of the Hamiltonian, $j = 1, 2, 3$ with $R_1 = \exp\{i\pi\hat{a}^\dagger\hat{a}/2\}$, $R_2 = \exp\{-i\pi\hat{a}^\dagger\hat{a}/2\}$ and $R_3 = \exp\{i\pi\hat{a}^\dagger\hat{a}\}$.

We also note here that the dependence of the energy of the ground state on x is caused mainly by the counter-rotating terms in the Hamiltonian (8). This follows from the fact that, in the RWA, the exact ground state of the system is given by $\psi_0^{\mathrm{RWA}} = \chi_\downarrow^1\chi_\downarrow^2|0\rangle$ and by acting with $\hat{H}'$ on ψ_0^{RWA} one finds that $u_\nu^{\mathrm{RWA}}(x) = -\delta$ and is independent of x. Here $\chi_\downarrow^1$, $\chi_\downarrow^2$ are the ground states of the first and second qubits, respectively, and $|0\rangle$ is the vacuum state of the field.

IV. EXACT NUMERICAL SOLUTION

Let us present how we performed the exact numerical solution for the analysis of the system's dynamics. For this, we consider that both atoms in the initial moment of time occupy the down states $\chi_\downarrow^1$ and $\chi_\downarrow^2$ and the field is prepared in the coherent state with the amplitude $\alpha = \sqrt{\bar{n}}$ ($\bar{n}$ is the average number of photons):

$$|\Psi(0)\rangle = |\alpha\rangle\chi_\downarrow^1\chi_\downarrow^2. \tag{15}$$

The functions $\chi_\downarrow^1$, $\chi_\downarrow^2$ and $\chi_\uparrow^1$, $\chi_\uparrow^2$ are the eigenfunctions of σ_3. We also use the following convention and ordering for the

extended spin space unit base vectors:

$$|\chi_1\rangle \equiv \chi_\uparrow^1\chi_\uparrow^2 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |\chi_2\rangle \equiv \chi_\uparrow^1\chi_\downarrow^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

$$|\chi_3\rangle \equiv \chi_\downarrow^1\chi_\uparrow^2 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |\chi_4\rangle \equiv \chi_\downarrow^1\chi_\downarrow^2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{16}$$

First one needs to obtain the exact numerical solution of the eigenvalue problem. For this purpose we introduce the following matrix elements of the Hamiltonian (14) in the Fock field states and spin base vectors (16):

$$\begin{aligned} H_{nk} =& \left(\frac{\delta}{2}(\sigma_3\otimes \mathrm{I}_2 + \mathrm{I}_2\otimes\sigma_3) + n\mathrm{I}_4\right)\delta_{nk} \\ &+ f[(\sqrt{k}e^{i\phi}\delta_{n,k-1} + \sqrt{k+1}e^{-i\phi}\delta_{n,k+1})\sigma_1\otimes \mathrm{I}_2 \\ &+ (\sqrt{k}e^{-i\phi}\delta_{n,k-1} + \sqrt{k+1}e^{i\phi}\delta_{n,k+1})\mathrm{I}_2\otimes\sigma_1], \end{aligned} \tag{17}$$

where I_4 is a unit 4×4 matrix, I_2 is the unit 2×2 matrix.

By numerically solving the eigenvalue problem for Eq. (17), we obtain the set of eigenvalues $\{E_\varkappa\}$ and corresponding eigenvectors $\{|\psi_\varkappa\rangle\}$ (in the form of a list of expansion coefficients $\{C_{kq}^\varkappa\}$ for each eigenvector). As a result, one can construct the time-dependent wave vector as an expansion over the stationary states:

$$|\Psi(t)\rangle = \sum_\varkappa A_\varkappa|\psi_\varkappa\rangle e^{-iE_\varkappa t}, \tag{18}$$

where

$$|\psi_\varkappa\rangle = \sum_{k=0}^{\infty}\sum_{q=1}^{4} C_{nq}^\varkappa|k\rangle|\chi_q\rangle = \sum_{k=0}^{\infty}\begin{pmatrix}C_{k1}^\varkappa\\C_{k2}^\varkappa\\C_{k3}^\varkappa\\C_{k4}^\varkappa\end{pmatrix}|k\rangle. \tag{19}$$

The coefficients $A_\varkappa$ can be obtained from the initial condition:

$$A_\varkappa = \langle\psi_\varkappa|\Psi(0)\rangle = \sum_{k=0}^{\infty}\left(C_{k4}^\varkappa\right)^*\frac{\alpha^k}{\sqrt{k!}}e^{-\alpha^2/2}. \tag{20}$$

The density matrix of the system is

$$\hat\rho = |\Psi(t)\rangle\langle\Psi(t)|, \tag{21}$$

and the density matrix of the atomic subsystem can be calculated by tracing out the field degrees of freedom in Eq. (21):

$$\hat\rho_{\mathrm{TQ}} = \sum_{n=0}^{\infty}\langle n|\Psi(t)\rangle\langle\Psi(t)|n\rangle. \tag{22}$$

Finally, the probability of finding both atoms in the down state can be obtained as follows:

$$\begin{aligned} P_{-1}(t) &= \langle\chi_4|\hat\rho_{\mathrm{TQ}}|\chi_4\rangle \\ &= \sum_n\left|\sum_\varkappa A_\varkappa e^{-iE_\varkappa t}C_{n4}^\varkappa\right|^2. \end{aligned} \tag{23}$$

V. OBSERVABLE CHARACTERISTICS OF THE SYSTEM

One of the observable consequences of the dependence of the energy of the system on the distance between qubits is the variation of the transition frequencies with the variation of x, i.e.,

$$\Omega_{\nu_1\nu_2}(x) = \omega(u_{\nu_2}(x) - u_{\nu_1}(x)). \tag{24}$$

In Fig. 1 (right panel) we plot this example. The dependence of the transition frequency on the distance between qubits allows one to control the scattering cross section of the resonance radiation on the system [18]. At the same time the change of the population of quantum states of QRM with x (see Fig. 2) extends the possibility to control the transition probability and the linewidths of atomic transitions [20].

Another consequence of the dependence of the transition frequency on the distance is that, for different x, the period of the Rabi oscillations and the time evolution of the system is modified, as is demonstrated in Fig. 2. In this figure, instead of the time evolution of the ground state in the time domain $P_{-1}(t)$, we plot its Fourier transform $P_{-1}(\omega)$, which is an even function of the frequency when the field is prepared in the coherent state with the average number of photons $\bar n = 25$. As a result, we plotted only the region of positive frequencies ω. This quantity demonstrates the changes in the dynamics of the system with the change of the dimensionless coupling constant f. As such, for small values of the coupling constant [see Fig. 2 (left panel)] the spectrum possesses a strong maximum located at the frequency $\omega \approx 5\sqrt{\bar n} = 5f$, to which coincides a period of oscillations with only a single Rabi frequency. The contributions from other frequencies are highly suppressed. When the coupling constant increases we start to observe the increase of the amplitudes of other harmonics that are located on different frequencies. This signifies that the system transitions in the chaotic regime [26]. The spectrum is an even function of the frequency. Consequently, the appearance of the new maximum brings two new frequencies of an opposite sign (doubling). The amplitudes of new peaks in the spectrum increase in the region where the rotating wave approximation is not applicable. This region is defined by the dimensionless coupling constant $f \sim 1/\sqrt{\bar n}$ (see Ref. [27]). This behavior of the spectrum can be interpreted as bifurcations. As a result, in the strong-coupling regime [Fig. 2 (right panel)] the system possesses a spectrum with a broad range of frequencies. However, we pay attention to the fact that the spectrum changes with the distance between qubits.

In addition to the spectrum and dynamics of the system, we also investigate the correlation properties such as the average number of photons inside the cavity and the entanglement of the qubit states. The dependence of these quantities on the distance x between qubits allows one to perform their control. For this purpose let us investigate these quantities for the ground state of the system.

We represent the wave function of the system in the basis of Fock states $|n\rangle$ and $\chi_{s_1}^1$, $\chi_{s_2}^2$ the spin states of qubits,

$$\psi_0 = \sum_{n,s_1,s_2} C_{n,s_1,s_2}^0|n\rangle\chi_{s_1}^1\chi_{s_2}^2. \tag{25}$$

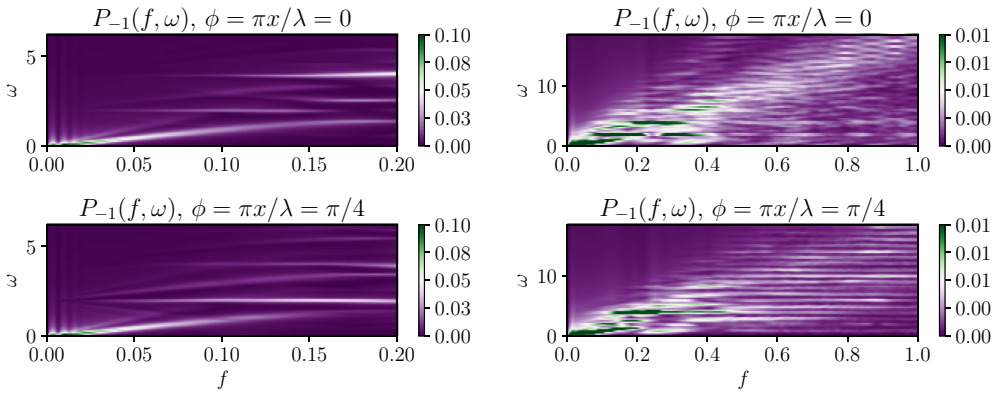

FIG. 2. (left panel) Fourier transform $P_{-1}(f,\omega)$ of the evolution of the population inversion as a function of the dimensionless coupling constant f plotted for two values of the distance between qubits, $\phi = 0$ and $\phi = \pi/4$. The parameter $\delta = 1$ and the field is prepared in the coherent state with $\bar{n} = 25$. When the coupling constant increases we observe the doubling of frequencies, which signals the transition to the chaotic behavior [26]. (right panel) Fourier transform $P_{-1}(f,\omega)$ of the evolution of the population inversion as a function of the dimensionless coupling constant f plotted for two values of the distance between qubits, $\phi = 0$ and $\phi = \pi/4$. The parameter $\delta = 1$ and the field is prepared in the coherent state with $\bar{n} = 25$. When the system transitions into the USC regime we observe chaotic behavior, when all frequencies appear in the spectrum.

The coefficients C^0_{n,s_1,s_2} are computed numerically by diagonalizing the Hamiltonian matrix in this basis. Consequently, with the help of this expansion we can compute the average number of photons inside the cavity as

$$\langle n(x,f)\rangle = \langle \hat{a}^\dagger \hat{a}\rangle = \sum_{n,s_1,s_2} n\left|C^0_{n,s_1,s_2}\right|^2, \tag{26}$$

which we plot in Fig. 3. This figure demonstrates that, in the USC regime, even when the system is in the ground state, the interaction between qubits and the quantum field leads to the excitation of three photons. However, this number depends significantly on the distance between qubits.

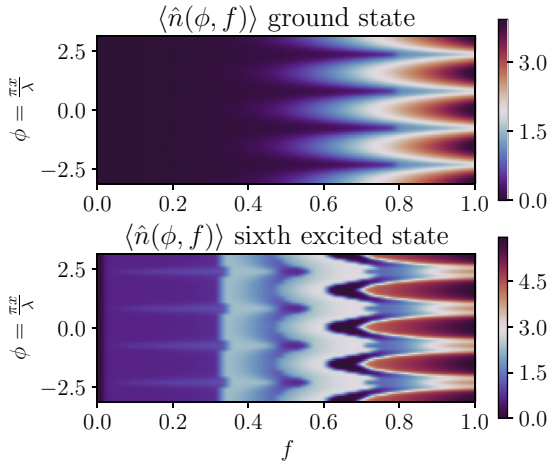

FIG. 3. The surface of the expectation value of the photon number operator $\langle n(x,f)\rangle = \langle \hat{a}^\dagger \hat{a}\rangle$ of the ground state and the sixth excited state as a function of the dimensionless coupling constant f and the coordinate ϕ. The parameter $\delta = 1$. When the system of two qubits is in the USC regime the expectation value is significantly different from zero.

To compute the entanglement $\mathcal{E}$ we use the definition [28]

$$\mathcal{E}(x,f) = -\sum_{s_1} p_{s_1}\log_2 p_{s_1}, \tag{27}$$

$$p_{s_1} = \sum_{n,s_2}\left|C^0_{n,s_1,s_2}\right|^2, \tag{28}$$

which is demonstrated in Fig. 4. As expected, for large values of the coupling constant f, the states of both qubits are entangled and the degree of entanglement depends on the distance between qubits. The control of the entanglement of different quantum states by the variation of the distance between qubits can be used for the encoding of quantum information by two two-level emitters [19]. In addition, an analytical approximation (A6) demonstrates that the spin part of the system is entangled with the field part.

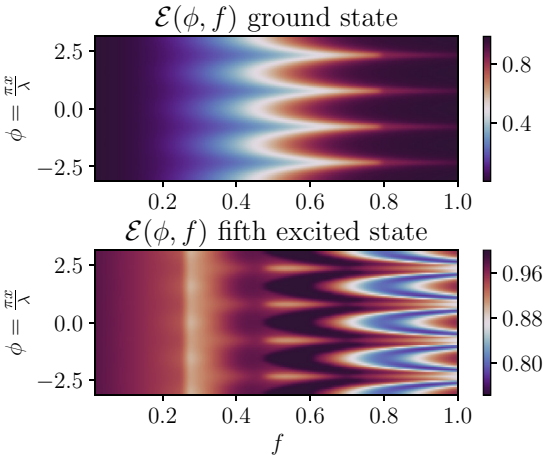

FIG. 4. The surface of the entanglement $\mathcal{E}(\phi,f)$ of the ground state and the fifth excited state as a function of the dimensionless coupling constant f and the coordinate ϕ. The parameter $\delta = 1$.

VI. CONCLUSION

In our work we have investigated a system of two qubits which interact with the quantum electromagnetic field inside a cavity. The qubits are located at distances of the order of the wavelength of the radiation, which is sufficiently larger than the corresponding qubit size. This allowed us to describe the individual qubit within the dipole approximation. In addition, we have employed the adiabatic approximation and considered the operator of the kinetic energy of the qubits as a perturbation. As a consequence, we have approximately separated the variables in the center-of-mass reference system of qubits. In a full analogy with molecular physics we have computed the radiation-induced potential (terms) of the system of two qubits. As a result, the observable characteristics of the system became functions of the distance between qubits and the coupling constant f of the qubits-field interaction.

We have identified the most interesting range of the coupling constant (USC regime) when the interaction between qubits and the field is the strongest. We have calculated the observable characteristics numerically and constructed an analytical approximation. Moreover, we have demonstrated that, by varying the distance between qubits, the observable characteristics are strongly changing. In addition, by changing the qubit positions one can perform a quantum control of the system, which is useful for the recording and transmission of quantum information. For example, this can be important in the following applications: scattering of the resonant radiation [18]; dynamics of an entanglement of two-level emitters [19]; controlling the characteristics of spontaneous emission in two-level systems [20]; appearance of the periodic structure in the interaction potential in a system of N atoms—the continuous Dicke model [21], which should be investigated beyond the rotating wave approximation.

APPENDIX: ANALYTICAL APPROXIMATION FOR ENERGY LEVELS

In this Appendix we will derive an analytical approximation for eigenvalues and eigenvectors of the system of two two-level atoms interacting with a single-mode quantum field.

Let us first consider the case when δ equals zero in the Hamiltonian of the system (14). In this case the problem becomes exactly solvable, since both spins are diagonalized by one of the following wave functions: $\chi_\uparrow^1\chi_\uparrow^2$, $\chi_\uparrow^1\chi_\downarrow^2$, $\chi_\downarrow^1\chi_\uparrow^2$, $\chi_\downarrow^1\chi_\downarrow^2$, where $\chi_\uparrow$, $\chi_\downarrow$ are eigenvectors of σ_1. The field part of the Hamiltonian contains only the first powers of operators. As is well known, in this case the field Hamiltonian is diagonalized by displacing the classical component from the field operators or in other words by performing a unitary transformation of the Hamiltonian with the operator of the coherent state $\hat{D}(u) = \exp(u\hat{a}^\dagger - u^*\hat{a})$ that transforms the field operators as

$$\hat{D}^\dagger(u)\hat{a}\hat{D}(u) = \hat{a} + u,$$
$$\hat{D}^\dagger(u)\hat{a}^\dagger\hat{D}(u) = \hat{a}^\dagger + u^*. \tag{A1}$$

The parameter u is then chosen from the condition that the first power of the operators vanishes. By performing the unitary transformation of the field part of the Hamiltonian we can find out the following expressions for each of the four spin wave functions listed above:

$$\hat{H}_{1f} = \hat{a}^\dagger\hat{a} + u_1(\hat{a} + \hat{a}^\dagger) + u_1^2 + 2f(\hat{a} + \hat{a}^\dagger)\cos\phi + 4u_1 f\cos\phi, \tag{A2}$$

$$\hat{H}_{2f} = \hat{a}^\dagger\hat{a} - iu_2(\hat{a} - \hat{a}^\dagger) + u_2^2 + 2if(\hat{a} - \hat{a}^\dagger)\sin\phi - 4u_2 f\sin\phi, \tag{A3}$$

$$\hat{H}_{3f} = \hat{a}^\dagger\hat{a} - iu_3(\hat{a} - \hat{a}^\dagger) + u_3^2 - 2if(\hat{a} - \hat{a}^\dagger)\sin\phi + 4u_3 f\sin\phi, \tag{A4}$$

$$\hat{H}_{4f} = \hat{a}^\dagger\hat{a} + u_4(\hat{a} + \hat{a}^\dagger) + u_4^2 - 2f(\hat{a} + \hat{a}^\dagger)\cos\phi - 4u_4 f\cos\phi. \tag{A5}$$

Here we considered that parameters of the coherent states for the cases 1, 4 are pure real and for the cases 2, 3 are pure imaginary numbers. From here we can conclude that $u_1 = 2f\cos\phi$, $u_4 = -2f\cos\phi$ and $u_2 = 2if\sin\phi$, $u_3 = -2if\sin\phi$.

Now let us consider the general situation of $\delta \neq 0$. In this case the spins and the field are entangled and the approximate wave function of the system is expressed as a linear combination of the four previously found possibilities, namely,

$$|\Psi\rangle = A_1\chi_\uparrow^1\chi_\uparrow^2|n, 2f\cos\phi\rangle + A_2\chi_\uparrow^1\chi_\downarrow^2|n, -2if\sin\phi\rangle + A_3\chi_\downarrow^1\chi_\uparrow^2|n, 2if\sin\phi\rangle + A_4\chi_\downarrow^1\chi_\downarrow^2|n, -2f\cos\phi\rangle, \tag{A6}$$

where we introduced the notation $|n, u\rangle = \hat{D}(u)|n\rangle$. As a result, the energy of the system is given as the solution of the eigenvalue problem $\hat{H}|\Psi\rangle = E|\Psi\rangle$, with the Hamiltonian $\hat{H}$ defined by Eq. (14) in the finite basis, consistent with four wave functions. The solution of this eigenvalue problem leads to the desired coefficients A_i, $i = 1, \ldots, 4$

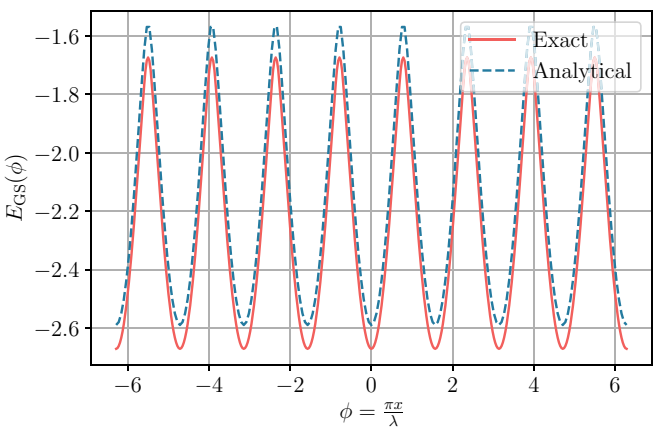

FIG. 5. The comparison of the exact numerical solution with the approximate analytical result of the ground-state energy E_{GS} as a function of the distance ϕ between two atoms. The parameter $\delta = 1$ and the USC regime is considered for $f = 0.8$.

and the eigenvalues. For example, for the first four states one finds the following expressions for the Hamiltonian matrix (here we employed the notation for the coherent states $|0,u\rangle = |u\rangle$):

$$H_{ij} = \begin{pmatrix} u_1^2 + 4fu_1\cos\phi & \frac{\delta}{2}\langle u_1|iu_2\rangle & \frac{\delta}{2}\langle u_1|iu_3\rangle & 0 \\ \frac{\delta}{2}\langle iu_2|u_1\rangle & \left(u_2^2 - 4fu_2\sin\phi\right) & 0 & \frac{\delta}{2}\langle iu_2|u_4\rangle \\ \frac{\delta}{2}\langle iu_3|u_1\rangle & 0 & \left(u_3^2 + 4fu_3\sin\phi\right) & \frac{\delta}{2}\langle iu_3|u_4\rangle \\ 0 & \frac{\delta}{2}\langle u_4|iu_2\rangle & \frac{\delta}{2}\langle u_4|iu_3\rangle & \left(u_4^2 - 4fu_4\cos\phi\right) \end{pmatrix}, \tag{A7}$$

where the overlapping integrals between different coherent states are defined as

$$\begin{aligned}\langle v|u\rangle &= \langle 0|e^{v^*\hat{a}-v\hat{a}^\dagger}e^{u\hat{a}^\dagger-u^*\hat{a}}|0\rangle \\ &= \langle 0|e^{(u-v)\hat{a}^\dagger-(u^*-v^*)\hat{a}}|0\rangle e^{1/2(uv^*-vu^*)} \\ &= e^{-1/2|u-v|^2}e^{1/2(uv^*-vu^*)}.\end{aligned}$$

By diagonalizing the matrix (A7) of the system one finds the following expressions for the energy levels of the first four states:

$$\begin{aligned}E_1 &= -2f^2 - \sqrt{4f^4\cos^2 2\phi + \delta^2 e^{-4f^2}}, \\ E_2 &= -4f^2\sin^2\phi, \\ E_3 &= -4f^2\cos^2\phi, \\ E_4 &= -2f^2 + \sqrt{4f^4\cos^2 2\phi + \delta^2 e^{-4f^2}}.\end{aligned}$$

Analogous formulas can be obtained for other states of the system.

Finally, we pay attention to the fact that the period of oscillations is equal to $\pi/2$, despite that fact that the Hamiltonian of the system has periodicity 2π in the dimensionless distance ϕ between qubits. In Fig. 5 we plot the slice of the potential surface for $f = 0.8$ and compare the numerical versus analytical energy of the ground state. As can be concluded from the figure, the analytical approximation well describes all the qualitative properties of the system and agrees with the exact numerical solution.

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Correction: An email address for the first author has been added, and the footnote indicator for the last author has been adjusted.