Quantum speed limit for robust state characterization and engineering

Kohei Kobayashi and Naoki Yamamoto
Department of Applied Physics and Physico-Informatics, Keio University, Hiyoshi 3-14-1, Kohoku, Yokohama 223-8522, Japan

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In this paper we propose to use a quantum speed limit (QSL) as a measure of robustness of states, defining a state with a bigger QSL as more robust. From this perspective, it is important to have an explicitly computable QSL, because then we can formulate an engineering problem of the Hamiltonian that makes a target state robust against decoherence. Hence we derive an explicitly computable QSL that is applicable to general Markovian open quantum systems. This QSL is tighter than another explicitly computable QSL, in an important setup such that decoherence is small. Also, the Hamiltonian engineering problem with this QSL is a quadratic convex optimization problem, and thus it is efficiently solvable. The idea of robust state characterization and the Hamiltonian engineering, in terms of the QSL, is demonstrated with several examples.

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I. INTRODUCTION

The quantum speed limit (QSL) is a lower bound on the evolution time of a quantum system from an initial state to a final state. It has numerous applications in quantum computation [1,2], metrology [3,4], optimal control [5–9], and so on. The study of the QSL started on closed systems; Mandelstam and Tamm derived a QSL between orthogonal states, which is given by the variance of the Hamiltonian [10], and Margolus and Levitin derived another QSL represented by the mean energy [11]. Moreover, extensions to mixed states [12], nonorthogonal states [13,14], and time-dependent driven systems [15–18] were presented later. In recent years, several type of QSLs for open quantum systems [19–31] have been extensively investigated.

In this paper we exploit an application of the QSL, that is, we use the QSL to characterize robust quantum states of a given open quantum system. Typically, the QSL is used to characterize the potential for speeding up the time evolution toward a target state [23,27–31]. More precisely, let us consider the problem of transferring an initial state ρ_0 to a target final state ρ_f; if the QSL from ρ_0 to ρ_f of a system Σ is smaller than that of another system Σ', then Σ should be chosen to do this task. In contrast, in this paper we consider an undesired state evolution driven by decoherence. That is, we consider a QSL from ρ_0 to any state ρ_T such that the distance between ρ_0 and ρ_T is bigger than a certain fixed value. If this QSL is large, this means that the decoherence needs a long time to drive the state initialized at ρ_0 toward ρ_T; in other words, ρ_0 is not largely affected by the decoherence. From this perspective, therefore, ρ_0 with a large QSL is robust against the decoherence.

Based on the above-mentioned use of the QSL, we consider the following optimization problem: The goal is to engineer a system Hamiltonian that maximizes the QSL for a given ρ_0 and the decoherence. Note that, to make this optimization problem tractable, it is important that the QSL has an explicit expression in terms of the parameters, rather than an implicit one that needs, for instance, solving a differential equation. Actually, in this paper we derive an easy-to-compute QSL applicable to a general Markovian open quantum system and prove that it is tighter than another explicit QSL given in Ref. [20], in the setup where the decoherence strength and the distance are both small. Moreover, it is shown that the Hamiltonian engineering problem based on this QSL is a quadratic convex optimization problem, which is efficiently solvable.

II. EXPLICIT QUANTUM SPEED LIMIT

A. Setup and derivation

In this paper we consider the general open quantum system obeying the Markovian master equation

$$\frac{d\rho_t}{dt} = -i[H, \rho_t] + \mathcal{D}[M]\rho_t, \tag{1}$$

where H is the time-independent Hamiltonian and $\mathcal{D}[M]$ is the Lindblad superoperator defined by $\mathcal{D}[M]\rho = M\rho M^\dagger - M^\dagger M\rho/2 - \rho M^\dagger M/2$. Throughout the paper we assume that the initial state is pure, i.e., $\rho_0 = |\psi_0\rangle\langle\psi_0|$ [but see remark (vi) later in this section]. Next, following [20,23,25], we define the relative purity between ρ_0 and ρ_t as

$$\Theta_t = \arccos\{\mathrm{Tr}(\rho_0\rho_t)\}. \tag{2}$$

Clearly, $0 \leqslant \Theta_t \leqslant \pi/2$. This takes the maximum when ρ_t is orthogonal to ρ_0, and the minimum is achieved only when $\rho_t = \rho_0$. Hence, the relative purity can be interpreted as the distance between ρ_0 and ρ_t. Here we derive a lower bound of the time T, needed for the relative purity to evolve from $\Theta_0 = 0$ to a given $\Theta_T \in (0, \pi/2]$.

First, we find that the dynamics of Θ_t is given by

$$\begin{aligned}\frac{d\Theta_t}{dt} &= \frac{-1}{\sqrt{1-\mathrm{Tr}(\rho_0\rho_t)^2}}\mathrm{Tr}\left(\rho_0\frac{d\rho_t}{dt}\right)\\ &= \frac{1}{\sin\Theta_t}\mathrm{Tr}\{(i[\rho_0, H] - \mathcal{D}^\dagger[M]\rho_0)\rho_t\},\end{aligned} \tag{3}$$

where $\mathcal{D}^\dagger[M]\rho = M^\dagger\rho M - M^\dagger M\rho/2 - \rho M^\dagger M/2$. To have an upper bound of the rightmost side of Eq. (3), we use two inequalities. One is the Cauchy-Schwarz inequality for matrices X and Y,

$$|\mathrm{Tr}(X^\dagger Y)| \leqslant \|X\|_{\mathrm{F}}\|Y\|_{\mathrm{F}}, \tag{4}$$

where $\|X\|_{\mathrm{F}} = \sqrt{\mathrm{Tr}(X^\dagger X)}$ is the Frobenius norm. The other one is

$$\begin{aligned}\|\rho_t - \rho_0\|_{\mathrm{F}} &= \sqrt{\mathrm{Tr}[(\rho_t-\rho_0)^2]} = \sqrt{\mathrm{Tr}\big(\rho_t^2 - 2\rho_t\rho_0 + \rho_0^2\big)}\\ &\leqslant \sqrt{2 - 2\,\mathrm{Tr}(\rho_t\rho_0)} = \sqrt{2-2\cos\Theta_t},\end{aligned}$$

where $\mathrm{Tr}(\rho_t^2) \leqslant 1$ and $\mathrm{Tr}(\rho_0^2) = 1$ are used. Using these inequalities, the rightmost side of Eq. (3) is upper bounded by

$$\begin{aligned}&\mathrm{Tr}\{(i[\rho_0, H] - \mathcal{D}^\dagger[M]\rho_0)\rho_t\}\\ &= \mathrm{Tr}\{(i[\rho_0, H] - \mathcal{D}^\dagger[M]\rho_0)(\rho_t - \rho_0)\} - \mathrm{Tr}(\rho_0\mathcal{D}^\dagger[M]\rho_0)\\ &= \mathrm{Tr}\{(i[H, \rho_0] + \mathcal{D}^\dagger[M]\rho_0)(\rho_0 - \rho_t)\} + \mathrm{Tr}(M^\dagger M\rho_0) - \mathrm{Tr}(M^\dagger\rho_0 M\rho_0)\\ &\leqslant \|i[H,\rho_0] + \mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}}\|\rho_t - \rho_0\|_{\mathrm{F}} + \|M|\psi_0\rangle\|^2 - |\langle\psi_0|M|\psi_0\rangle|^2\\ &\leqslant \sqrt{2}\|i[H,\rho_0] + \mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}}\sqrt{1-\cos\Theta_t} + \|M|\psi_0\rangle\|^2 - |\langle\psi_0|M|\psi_0\rangle|^2,\end{aligned} \tag{5}$$

where $\||\psi_0\rangle\|^2 = \langle\psi_0|\psi_0\rangle$ is the Euclidean norm. From Eqs. (3) and (5) we have

$$\frac{d\Theta_t}{dt} \leqslant \frac{1}{\sin\Theta_t}(\mathcal{A}\sqrt{1-\cos\Theta_t} + \mathcal{E}), \tag{6}$$

where

$$\mathcal{A} = \sqrt{2}\|i[H,\rho_0] + \mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}},$$
$$\mathcal{E} = \|M|\psi_0\rangle\|^2 - |\langle\psi_0|M|\psi_0\rangle|^2.$$

Then, by integrating the inequality (6) from 0 to T, we end up with

$$T \geqslant T_*(\rho_0) := \frac{2\lambda}{\mathcal{A}} + \frac{2\mathcal{E}}{\mathcal{A}^2}\ln\left(\frac{\mathcal{E}}{\mathcal{E}+\mathcal{A}\lambda}\right), \tag{7}$$

where $\lambda = \sqrt{1-\cos\Theta_T}$. We often write simply T_* rather than $T_*(\rho_0)$. This T_* is our QSL, giving a lower bound on the evolution time T for the state ρ_t to evolve from ρ_0 to any state ρ_T satisfying $\Theta_T = \arccos\{\mathrm{Tr}(\rho_0\rho_T)\}$ for a given value of Θ_T.

Here we list some properties of T_*.

(i) T_* is explicitly represented in terms of (ρ_0, H, M, Θ_T) and thus it is readily computable once those parameters are specified. There is no need to solve any equation.

(ii) T_* is monotonically decreasing with respect to the magnitude of M (see Appendix A for the proof). This implies that, as the decoherence becomes bigger, the dynamical change of state can become faster.

(iii) T_* is monotonically decreasing with respect to $\mathcal{A}$ for a fixed $\mathcal{E}$ (see Appendix B for the proof). In general, a closed system with bigger Hamiltonian H evolves faster, but in the case of open quantum systems, this effect may be changed by the decoherence effect M. Intuitively, $\mathcal{A}$ corresponds to the amplitude of such an effective Hamiltonian. Later, in Sec. IV, we will see that the monotonically decreasing property of T_* with respect to $\mathcal{A}$ is used to formulate the Hamiltonian engineering problem for robust state generation.

(iv) Assuming that $M = M^\dagger$, $H = 0$, and $|\psi_0\rangle$ is an eigenstate of M, then $\mathcal{E} = 0$ and $\mathcal{A} = 0$ hold, and hence $T_* \to \infty$. This is consistent with the fact that, under this assumption, $\rho_0 = |\psi_0\rangle\langle\psi_0|$ is a steady state of the master Eq. (1), meaning that the state never reaches any $\rho_T \neq \rho_0$. In Secs. III A and III B we will see an example realizing this situation.

(v) It is straightforward to extend the result to the case where the system is subjected to multiple decoherence channels and Hamiltonians. In this case, T_* is given by Eq. (7) with

$$\mathcal{A} = \sqrt{2}\left\|\sum_j i[H_j, \rho_0] + \sum_j \mathcal{D}^\dagger[M_j]\rho_0\right\|_{\mathrm{F}},$$
$$\mathcal{E} = \sum_j(\|M_j|\psi_0\rangle\|^2 - |\langle\psi_0|M_j|\psi_0\rangle|^2).$$

(vi) This result can be generalized to the case where the initial state ρ_0 is a mixed state. By extending the relative purity (2) to $\Theta_t' = \arccos\{\mathrm{Tr}(\rho_0\rho_t)/\mathrm{Tr}(\rho_0^2)\}$, we can derive the generalized QSL

$$T \geqslant T_*'(\rho_0) := \frac{2(\lambda_T' - \lambda_0')}{\mathcal{A}'} + \frac{2\mathcal{E}'}{\mathcal{A}'^2}\ln\left[\frac{\mathcal{E}' + \mathcal{A}'\lambda_0'}{\mathcal{E}' + \mathcal{A}'\lambda_T'}\right], \tag{8}$$

where

$$\mathcal{A}' = \sqrt{2}\|i[H,\rho_0] + \mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}},$$
$$\mathcal{E}' = \mathrm{Tr}(M^\dagger M\rho_0) - \mathrm{Tr}(M^\dagger\rho_0 M\rho_0),$$
$$\lambda_t' = \sqrt{1 - 2\mathrm{Tr}\big(\rho_0^2\big)\cos\Theta_t' + \mathrm{Tr}\big(\rho_0^2\big)}/\sqrt{2}.$$

Note that, if ρ_0 is pure, then the lower bound (8) is equivalent to (7). The derivation of Eq. (8) is given in Appendix C.

B. Quantum speed limit as a measure of robustness

Let us consider the situation where an initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$ and a value of $\lambda = \sqrt{1-\cos\Theta_T}$ are given. This means that we are given a region $\mathcal{R}_\lambda(\rho_0)$, which is the set of all states whose distance from ρ_0 is less than λ, that is, λ can be interpreted as the radius of a circle region $\mathcal{R}_\lambda(\rho_0)$ (see Fig. 1). Then the transition time T of Θ_t for evolving from $\Theta_0 = 0$ to Θ_T has the meaning of the escape time that the state first exits from $\mathcal{R}_\lambda(\rho_0)$. Therefore, if T is large for a given λ,

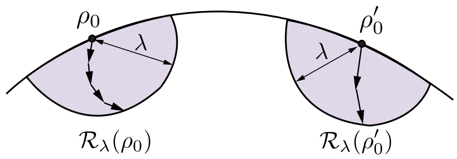

FIG. 1. Robustness of the quantum state. Shown on the left, when $T_*(\rho_0)$ is large, the quantum state ρ_t must take a long time to exit from the region $\mathcal{R}_\lambda(\rho_0)$ with fixed radius λ. This implies that ρ_0 is robust. Shown on the right, when $T_*(\rho_0')$ is small for the same λ, then ρ_t may quickly exit from $\mathcal{R}_\lambda(\rho_0')$, meaning that ρ_0' is fragile compared to ρ_0.

this means that the state ρ_t starting from the initial state ρ_0 takes a long time to exit from $\mathcal{R}_\lambda(\rho_0)$. In this case, we can say that ρ_0 is robust against the decoherence M. In contrast, if we take another initial state ρ_0' and find that the transition time T' is smaller than T for the same value of λ, this means that the state quickly escapes from $\mathcal{R}_\lambda(\rho_0')$; that is, as illustrated in Fig. 1, this is the case where the state ρ_0' is largely affected by the decoherence and can be easily changed. Thus, ρ_0' is fragile.

Hence it is clear that the QSL $T_*(\rho_0)$ can be used to characterize a state ρ_0 that is robust against a given decoherence M. That is, for a given λ, the state initialized to ρ_0 with a large value of $T_*(\rho_0)$ is guaranteed to take a long time T to escape from $\mathcal{R}_\lambda(\rho_0)$, hence it is robust against M. Also, in this paper we define that, if $T_*(\rho_0) > T_*(\rho_0')$, then ρ_0 is more robust than ρ_0', although this does not always lead to $T(\rho_0) > T(\rho_0')$. Moreover, for a given ρ_0 and (a relatively small value of) λ, it makes sense to appropriately design the system operators that maximize $T_*(\rho_0)$, to protect ρ_0 against the decoherence; in Sec. IV we discuss this problem, in particular in the case where H is the design object.

We finally remark that in the literature there are many studies on robustness of quantum states, particularly that of entangled states, in several noise models such as local decoherence, global decoherence, and particle loss [32–39]; related case studies will be given in Secs. III B and III C.

C. Comparison to the QSL derived in Ref. [20]

Applying the Cauchy-Schwarz inequality (4) to the right-hand side of Eq. (3), we have

$$\frac{d\Theta_t}{dt} \leqslant \frac{1}{\sin\Theta_t}\|i[\rho_0, H] - \mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}}\|\rho_t\|_{\mathrm{F}} \leqslant \frac{\mathcal{A}}{\sqrt{2}\sin\Theta_t}.$$

Then, by integrating both sides of this inequality from 0 to T, we find that the transition time T for Θ_t evolving from $\Theta_0 = 0$ to a given Θ_T is lower bounded by [20]

$$T \geqslant T_{\mathrm{DC}} := \frac{\sqrt{2}(1-\cos\Theta_T)}{\mathcal{A}} = \frac{\sqrt{2}\lambda^2}{\mathcal{A}}. \tag{9}$$

Similar to T_*, T_{DC} is also explicitly represented in terms of (ρ_0, H, M, Θ_T), which is indeed the key point for engineering a system having a robust state ρ_0 in the sense of the QSL as described in Sec. II B. Note that the only explicit form of QSL for open quantum systems that has been developed is for T_* and T_{DC}.

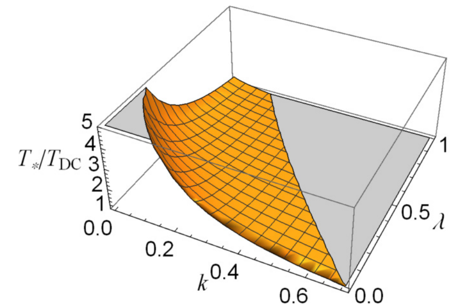

FIG. 2. Three-dimensional plot of the ratio of bounds T_*/T_{DC} as a function of $k \in [0, 1/\sqrt{2}]$ and $\lambda \in [0, 1]$.

Therefore, it is important to compare T_* and T_{DC}. We study the quantity

$$\begin{aligned}\frac{T_*}{T_{\mathrm{DC}}} &= \left\{\frac{2\lambda}{\mathcal{A}} + \frac{2\mathcal{E}}{\mathcal{A}^2}\ln\left(\frac{\mathcal{E}}{\mathcal{E}+\mathcal{A}\lambda}\right)\right\}\frac{\mathcal{A}}{\sqrt{2}\lambda^2}\\ &= \frac{\sqrt{2}}{\lambda} + \frac{\sqrt{2}k}{\lambda^2}\ln\left(\frac{k}{k+\lambda}\right),\end{aligned} \tag{10}$$

where $k = \mathcal{E}/\mathcal{A}$. Note that again from Eq. (4) we have

$$\begin{aligned}\mathcal{A} &= \sqrt{2}\|i[H,\rho_0] + \mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}}\|\rho_0\|_{\mathrm{F}}\\ &\geqslant \sqrt{2}|\mathrm{Tr}\{(i[H,\rho_0] + \mathcal{D}^\dagger[M]\rho_0)\rho_0\}| = \sqrt{2}\mathcal{E};\end{aligned}$$

hence $0 \leqslant k \leqslant 1/\sqrt{2}$. First, T_*/T_{DC} is a monotonically decreasing function with respect to k, because

$$\begin{aligned}\frac{\partial}{\partial k}\left(\frac{T_*}{T_{\mathrm{DC}}}\right) &= \frac{\sqrt{2}}{\lambda}\left\{-\frac{1}{\lambda}\ln\left(1+\frac{\lambda}{k}\right) + \frac{1}{k+\lambda}\right\}\\ &\leqslant \frac{\sqrt{2}}{\lambda}\left(-\frac{1}{\lambda}\frac{\lambda}{k+\lambda} + \frac{1}{k+\lambda}\right) = 0,\end{aligned}$$

where we used $\ln(1+x) \geqslant x/(1+x)$ for $x \geqslant 0$. Now, when $k = 0$ or equivalently when the system is closed (i.e., $\mathcal{E} = 0$), then $T_*/T_{\mathrm{DC}} \geqslant \sqrt{2}/\lambda > 1$. Together with the above monotonically decreasing property of T_*/T_{DC} with respect to k, hence T_* is tighter than T_{DC} if the decoherence is small.

Next T_*/T_{DC} decreases with respect to λ, because

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(\frac{T_*}{T_{\mathrm{DC}}}\right) &= -\frac{\sqrt{2}}{\lambda^2} + \frac{2\sqrt{2}k}{\lambda^3}\ln\left(1+\frac{\lambda}{k}\right) - \frac{\sqrt{2}k}{\lambda^2}\frac{1}{k+\lambda}\\ &\leqslant \frac{\sqrt{2}}{\lambda^2}\left\{-1 + \frac{2k+\lambda}{k+\lambda} - \frac{k}{k+\lambda}\right\} = 0,\end{aligned}$$

where we used $\ln(1+x) \leqslant x(2+x)/2(1+x)$ for $x \geqslant 0$. This means that T_* will work as a tighter bound than T_{DC} in the region $\mathcal{R}_\lambda(\rho_0)$ with small radius λ.

The above observations can be quantitatively seen in Fig. 2, which plots Eq. (10) as a function of $k \in [0, 1/\sqrt{2}]$ and $\lambda \in [0, 1]$. The yellow-colored region shows the set of parameters λ and k such that $T_* > T_{\mathrm{DC}}$. Notably, when the decoherence is weak (i.e., k is small) and $\mathcal{R}_\lambda(\rho_0)$ is small (i.e., λ is small), T_* functions as a much bigger lower bound for the escape time T than T_{DC}.

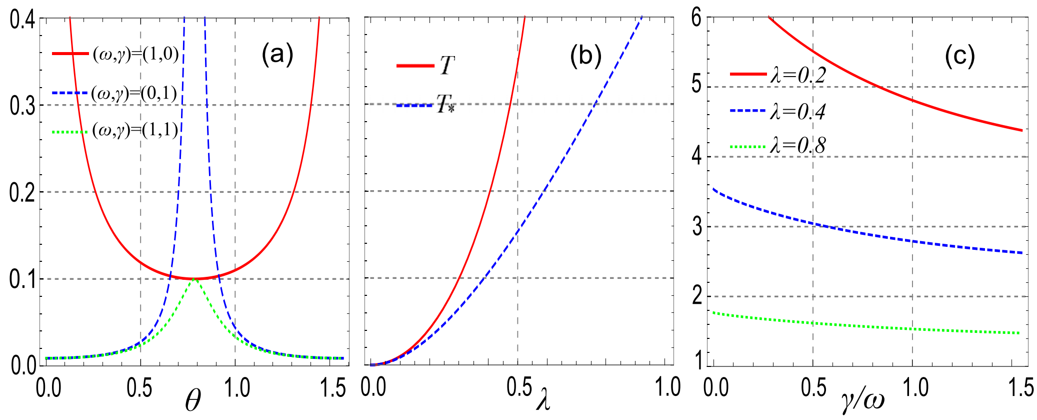

FIG. 3. (a) Lower bound T_* as a function of θ, for several values of (ω, γ). For all cases, $\lambda = 0.1$. (b) Comparison of the exact escape time T and its lower bound T_*, as a function of λ, where $\gamma = 1$ is fixed. (c) Ratio $T_*/T_{\rm DC}$, as a function of γ/ω, for several values of λ.

III. EXAMPLES

A. Two-level atom

The first example is a two-level atom consisting of the excited state $|0\rangle = (1, 0)^\top$ and the ground state $|1\rangle = (0, 1)^\top$. Let the initial state $|\psi_0\rangle$ be

$$|\psi_0\rangle = \begin{pmatrix} \cos\theta \\ e^{i\varphi}\sin\theta \end{pmatrix}, \quad 0 \leqslant \theta < \pi/2,\ 0 \leqslant \varphi < 2\pi.$$

We consider the system operators

$$H = \omega\sigma_z, \quad M = \sqrt{\gamma}\sigma_x, \tag{11}$$

where $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_y = i(|1\rangle\langle 0| - |0\rangle\langle 1|)$, and $\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$ are the Pauli matrices. In addition, H rotates the state vector along the z axis with frequency $\omega > 0$ and M represents the dephasing noise with decay rate $\gamma > 0$. In this setting the QSL is given by Eq. (7) with

$$\mathcal{A}^2/4 = \gamma^2(\cos^2 2\theta + \sin^2 2\theta \sin^2\varphi) + \omega^2\sin^2 2\theta + \omega\gamma\sin^2 2\theta\sin 2\varphi,$$
$$\mathcal{E} = \gamma - \gamma\sin^2 2\theta\cos^2\varphi.$$

Figure 3(a) shows T_* for the initial state with $\varphi = 0$, as a function of θ, for a fixed value $\lambda = 0.1$. That is, this figure shows the lower bound of the escape time at which the state initialized at $|\psi_0\rangle = (\cos\theta, \sin\theta)^\top$ first exits from the region $\mathcal{R}_\lambda(\rho_0)$. If $\gamma = 0$, $T_* = \lambda/\omega|\sin 2\theta|$, which is plotted with the red solid line; in this case the state simply rotates along z axis, and hence, if $|\psi_0\rangle$ is nearly $|0\rangle$ or $|1\rangle$, the state remains inside $\mathcal{R}_\lambda(\rho_0)$ for all time, resulting in $T_* \to \infty$. Also, T_* takes the minimum at $\theta = \pi/4$, simply because the state on the equator of the Bloch sphere changes the most; hence $|+\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$ is the most fragile state in our definition.

When $\gamma > 0$, the dependence of T_* on θ remarkably changes, as shown with the blue dashed and green dotted lines in Fig. 3(a). Again, $\lambda = 0.1$ is chosen. When $\omega = 0$ and $\gamma = 1$, $T_* \to \infty$ at $\theta = \pi/4$; that is, $|+\rangle$ is the most robust. Actually, in this case, $|+\rangle$ is a steady state of the master equation $d\rho_t/dt = \mathcal{D}[M]\rho_t$, meaning that $|+\rangle$ does not change under the influence of this decoherence and the state around $|+\rangle$ remains in $\mathcal{R}_\lambda(\rho_0)$ for all time. On the other hand, when $\omega = 1$ and $\gamma = 1$, T_* takes a finite time for all θ, which implies that the state may escape from $\mathcal{R}_\lambda(\rho_0)$ at a certain time for any ρ_0.

Recall that T_* is a lower bound of the exact escape time T. Hence, it is worth comparing these quantities to see the tightness of T_*. For this purpose, here we set $\omega = 0$ and choose the initial state $|\psi_0\rangle = |0\rangle$. In this case the master equation $d\rho_t/dt = \mathcal{D}[M]\rho_t$ yields a simple solution $\cos\Theta_T = (1 + e^{-2\gamma T})/2$. As a result, we obtain T and T_* as follows:

$$T = -\frac{1}{2\gamma}\ln(1 - 2\lambda^2), \quad T_* = \frac{\lambda}{\gamma} - \frac{1}{2\gamma}\ln(2\lambda + 1).$$

Figure 3(b) shows the plots of T and T_* with $\gamma = 1$, as a function of λ, where the range of the vertical axis is the same as that of Fig. 3(a). This shows that both T and T_* are close to zero when λ is small, which is reasonable because the state will take a short time to escape from a small region $\mathcal{R}_\lambda(\rho_0)$. However, the gap between T and T_* quickly diverges, as λ becomes large. This fact suggests that we use T_*, only when λ is small.

Finally, let us see the ratio $T_*/T_{\rm DC}$ discussed in Sec. II C, particularly in the setup

$$H = \omega\sigma_z, \quad M = \sqrt{\gamma}\sigma_- = \sqrt{\gamma}|1\rangle\langle 0|,$$

where M represents the energy decay of the two-level atom with decay rate $\gamma > 0$. The initial state is set to the superposition $|\psi_0\rangle = |+\rangle$. In fact, this example demonstrates the difference of the two lower bounds more drastically than the setting (11). We now have $\mathcal{A} = \sqrt{48\omega^2 + 11\gamma^2}/4$ and $\mathcal{E} = \gamma/16$. Then $T_*/T_{\rm DC}$ given in Eq. (10) depends on only γ/ω and $\lambda = \sqrt{1 - \cos\Theta_T}$. Figure 3(c) shows the plot of $T_*/T_{\rm DC}$, as a function of γ/ω, for several values of λ. As expected from the discussion in Sec. II C, $T_*/T_{\rm DC}$ increases as γ/ω becomes small for all λ and also it becomes bigger for smaller λ. That is, T_* is a tighter bound than $T_{\rm DC}$ if the decoherence is relatively small and the region $\mathcal{R}_\lambda(\rho_0)$ is small.

B. Bell states

Next we study the Bell states defined by

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle),$$
$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle),$$

which are maximally entangled states. Which state is the most robust under a given decoherence? As seen in the previous example, comparing T_* of these states provides an answer to this natural question. Here we take the collective noise modeled by $M=\sqrt{\gamma}(\sigma_-\otimes I+I\otimes\sigma_-)$. Further, for simplicity, we assume that $H=0$. Then, for the same λ, the QSLs are obtained as

$$T_*(|\Phi^\pm\rangle)=\frac{2\lambda}{\sqrt{5}\gamma}-\frac{2}{5\gamma}\ln(1+\lambda),$$
$$T_*(|\Psi^+\rangle)=\frac{\lambda}{2\gamma}-\frac{1}{4\gamma}\ln(1+2\lambda).$$

Also we find $T_*(|\Psi^-\rangle)\to\infty$ because of $\mathcal{A}=0$, which is equivalent to $|\Psi^-\rangle$ being identical to an eigenstate of M. Thus, $|\Psi^-\rangle$ is the most robust Bell state in our definition. Moreover, $T_*(|\Phi^\pm\rangle)>T_*(|\Psi^+\rangle)$ always holds, and hence $|\Psi^+\rangle$ is the most fragile state. Note that, for the case of the noncollective (local) decoherence modeled by $M_1=\sqrt{\gamma}\sigma_-\otimes I$ and $M_2=\sqrt{\gamma}I\otimes\sigma_-$, we have $\mathcal{A}=\sqrt{5}\gamma$ and $\mathcal{E}=\gamma$ for all Bell states. That is, in this case there is no difference of states in robustness.

C. Atomic ensemble

Next let us consider an ensemble composed of N identical atoms. As typical states, we consider the product state of superposition $|+\rangle^{\otimes N}=(|0\rangle/\sqrt{2}+|1\rangle/\sqrt{2})^{\otimes N}$ and the GHZ state $|\mathrm{GHZ}\rangle=(|0\rangle^{\otimes N}+|1\rangle^{\otimes N})/\sqrt{2}$; the latter is a powerful resource in quantum metrology such as the frequency standard. In fact, $|\mathrm{GHZ}\rangle$ enables us to estimate the frequency with error (standard deviation) of the order $1/N$, while $1/\sqrt{N}$ is the best order in the case of $|+\rangle^{\otimes N}$ [40]. However, in a realistic situation, the system is always subjected to noise, typically the dephasing noise $M=\sqrt{\gamma}\sum_{j=1}^N\sigma_z^{(j)}$, where $\sigma_z^{(j)}$ acts on the jth atom, which eliminates the quantum advantage unless a specific control is applied [41].

To understand this undesired effect brought by the dephasing noise in the language of the QSL, let us examine T_* of those two states. For simplicity, we assume that the magnitude of the system Hamiltonian H is much smaller than γ. Then we have

$$(\mathcal{A},\mathcal{E})(|+\rangle^{\otimes N})\approx(\gamma\sqrt{6N^2-2N},\gamma N),$$
$$(\mathcal{A},\mathcal{E})(|\mathrm{GHZ}\rangle)\approx(2\gamma N^2,\gamma N^2).$$

These expressions lead to $T_*(|+\rangle^{\otimes N})\sim O(1/N)$ and $T_*(|\mathrm{GHZ}\rangle)\sim O(1/N^2)$ for fixed λ and γ. Therefore, although $|\mathrm{GHZ}\rangle$ can ideally improve the estimation accuracy, it is more fragile than $|+\rangle^{\otimes N}$.

IV. HAMILTONIAN ENGINEERING FOR ROBUST STATE PREPARATION

In the examples of Sec. III, we identified a robust state $\rho_0=|\psi_0\rangle\langle\psi_0|$, under given decoherence and system Hamiltonian. In this section, in contrast, we discuss designing an optimal Hamiltonian that maximizes T_*, given a fixed initial state ρ_0 and decoherence. That is, we aim to find H that protects ρ_0 against a given decoherence M by maximizing the lower bound of the escaping time of state from a region centered at ρ_0.

Now the problem of maximizing T_* with respect to H is equivalent to that of minimizing $\mathcal{A}$, because T_* is a monotonically decreasing function with respect to $\mathcal{A}$ (see Appendix B). In particular, we consider the cost function

$$F(H)=\mathrm{Tr}(H^2\rho_0)-\mathrm{Tr}(H\rho_0H\rho_0)+\mathrm{Tr}(i[\rho_0,\mathcal{D}^\dagger[M]\rho_0]H),$$

which is identical to $4\mathcal{A}^2-2\,\mathrm{Tr}\{(\mathcal{D}^\dagger[M]\rho_0)^2\}$. Note that, as can be clearly seen from the above expression, $F(H)$ is a convex quadratic function with respect to H. Hence, the optimal H_{opt} can be effectively determined. Note that this easy-to-handle problem can be formulated due to the explicit expression of the QSL; recall that this was the motivation to derive T_* and compare it to T_{DC}.

In order to have $H_{\mathrm{opt}}=\mathrm{argmin}_H F(H)$, let us take the derivative of $F(H)$ with respect to H,

$$\frac{\partial F}{\partial H}=(H\rho_0+\rho_0H)^\top-2(\rho_0H\rho_0)^\top+i([\rho_0,\mathcal{D}^\dagger[M]\rho_0])^\top,$$

where we have used the matrix formulas [42]

$$\frac{\partial}{\partial X}\mathrm{Tr}(XA)=A^\top,$$
$$\frac{\partial}{\partial X}\mathrm{Tr}(X^2A)=(XA+AX)^\top,$$
$$\frac{\partial}{\partial X}\mathrm{Tr}(AXAX)=2(AXA)^\top.$$

Therefore, H_{opt} satisfies

$$H_{\mathrm{opt}}\rho_0+\rho_0H_{\mathrm{opt}}-2\rho_0H_{\mathrm{opt}}\rho_0+i[\rho_0,\mathcal{D}^\dagger[M]\rho_0]=0. \quad (12)$$

This is a simple linear equation with respect to H, for a given ρ_0 and M. Thus, H_{opt} can be effectively computed by solving Eq. (12).

A. Qubit example

Let us again consider the qubit subjected to the decoherence $M=\sqrt{\gamma}\sigma_-$. Also we choose the initial state to be protected as $|\psi_0\rangle=(1/2,\sqrt{3}/2)$. We represent H_{opt} as $H_{\mathrm{opt}}=u_1\sigma_x+u_2\sigma_y+u_3\sigma_z$, where (u_1,u_2,u_3) are real parameters to be determined. Then by solving the linear equation (12) we have

$$u_2=-\frac{\sqrt{3}}{16}\gamma,\quad u_1+\sqrt{3}u_3=0.$$

The term $u_1\sigma_x+u_3\sigma_z$ always commutes with $\rho_0=|\psi_0\rangle\langle\psi_0|$ when $u_1+\sqrt{3}u_3=0$. Thus, only the $u_2\sigma_y$ term has an effect on the dynamics of Θ_t given in Eq. (3). Figure 4(a) shows the time evolution of $\cos\Theta_t$, in the following three cases: $H=H_{\mathrm{opt}}$, $H=0$ (i.e., the system is purely decohered), and $H=\sigma_z$. The decoherence strength is chosen as $\gamma=1$. Clearly, H_{opt} makes longer the time for the state escaping from $\mathcal{R}_\lambda(\rho_0)$ for any $\lambda=\sqrt{1-\cos\Theta_T}$ than the other two cases. In particular, when H_{opt} is applied, the state remains in the region $\mathcal{R}_\lambda(\rho_0)$ with radius $\lambda=\sqrt{1-0.9}$ for all time.

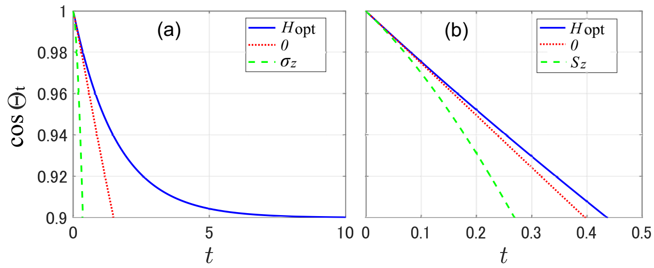

FIG. 4. Time evolution of $\cos\Theta_t$ in the case of (a) a qubit driven by $H = H_{\rm opt}$ (blue solid line), 0 (red dotted line), and σ_z (green dashed line) and (b) a qutrit driven by $H = H_{\rm opt}$ (blue solid line), 0 (red dotted line), and S_z (green dashed line).

B. Qutrit example

Finally, we study a qutrit system, composed of the three orthogonal states $|E\rangle = (1,0,0)^\top$, $|S\rangle = (0,1,0)^\top$, and $|G\rangle = (0,0,1)^\top$. We assume that the system is subject to the decoherence $M = \sqrt{\gamma}(|S\rangle\langle E| + |G\rangle\langle S|)$, which induces the ladder-type decay $|E\rangle \to |S\rangle \to |G\rangle$. The target initial state is chosen as $|\psi_0\rangle = (1/2, 1/\sqrt{2}, 1/2)^\top$. The control Hamiltonian to be determined can be parametrized as $H_{\rm opt} = \sum_{i=1}^8 u_i\Lambda_i$, where $\{\Lambda_i\}_{i=1}^8$ are the Gell-Mann matrices given in Appendix D. In this setting, using Eq. (12), we obtain the condition

$$\begin{aligned} 2\sqrt{2}u_1 + 5u_3 + 2u_4 - 2\sqrt{2}u_6 + \sqrt{3}u_8 &= 0,\\ 3u_3 + 2u_4 - \sqrt{3}u_8 &= 0,\\ 3\sqrt{2}u_2 + 2u_5 - \sqrt{2}u_7 &= -\gamma,\\ 8u_5 + 8\sqrt{2}u_7 &= -\gamma. \end{aligned}$$

Under this condition, the terms with coefficients $(u_1, u_3, u_4, u_6, u_8)$ in $H_{\rm opt}$ always commute with ρ_0 and thus they do not affect the dynamics of Θ_t as well as $\mathcal{A}$. Figure 4(b) shows the time evolution of $\cos\Theta_t$, in the cases of $H = H_{\rm opt}$ with the parameters $(u_2, u_5, u_7) = (-3\gamma/8\sqrt{2}, 0, -\gamma/8\sqrt{2})$ and compares it to the cases $H = 0$ and $S_z = |E\rangle\langle E| - |G\rangle\langle G|$. We find that certainly $H = H_{\rm opt}$ makes the escape time longer, though the advantage over the other two cases is not so big compared to the previous qubit example.

V. CONCLUSION

In this paper we posed an idea to use the QSL to characterize robust quantum states and, based on it, formulated the engineering problem of a Hamiltonian that makes a target state robust against a given decoherence. In this engineering problem it is important for the QSL to be explicitly computable; the QSL derived in this paper indeed satisfies this condition and further it is tighter than another known QSL in a setup where the robustness issue is critical [that is, the case where the decoherence is small and the region $\mathcal{R}_\lambda(\rho_0)$ is small]. In addition, the Hamiltonian engineering problem is proven to be a convex quadratic optimization problem, which is efficiently solvable. Several examples have been studied, in particular showing another view of the fragility of the GHZ state in quantum metrology. We hope that the results given in this paper will provide a new perspective of the QSL as a tool in quantum engineering.

APPENDIX A: T_* AS A FUNCTION OF THE DECOHERENCE STRENGTH

We here prove that T_* monotonically decreases with respect to the strength of the decoherence γ, which is defined through $M = \sqrt{\gamma}M'$ with fixed M'. In terms of γ, we can express $\mathcal{A}$ and $\mathcal{E}$ as $\mathcal{A} = \sqrt{a\gamma^2 + b\gamma + c}$ and $\mathcal{E} = d\gamma$, where (a, c, d) are non-negative constants and b is a constant. Then T_* can be written as

$$\begin{aligned} T_* &= \frac{2\lambda}{\sqrt{a\gamma^2 + b\gamma + c}}\\ &\quad + \frac{d\gamma}{a\gamma^2 + b\gamma + c}\ln\left(\frac{d\gamma}{d\gamma + \lambda\sqrt{a\gamma^2 + b\gamma + c}}\right) \end{aligned}$$

and $\partial T_*/\partial\gamma$ is calculated as

$$\begin{aligned} \frac{\partial T_*}{\partial\gamma} &= -\frac{(2a\gamma + b)\lambda}{\mathcal{A}^3} + \frac{2d(a\gamma^2 - c)}{\mathcal{A}^4}\ln\left(1 + \frac{\mathcal{A}\lambda}{\mathcal{E}}\right)\\ &\quad + \frac{d(b\gamma + 2c)\lambda}{\mathcal{A}^3(\mathcal{E} + \mathcal{A}\lambda)}. \end{aligned}$$

Our goal is to show $\partial T_*/\partial\gamma \leqslant 0$. The proof is divided into three cases: $a\gamma^2 > c$, $a\gamma^2 = c$, and $a\gamma^2 < c$. First, for the case $a\gamma^2 > c$ we have

$$\begin{aligned} \frac{\partial T_*}{\partial\gamma} &\leqslant -\frac{(2a\gamma + b)\lambda}{\mathcal{A}^3} + \frac{2d(a\gamma^2 - c)}{\mathcal{A}^4}\frac{\mathcal{A}\lambda(\mathcal{A}\lambda + 2\mathcal{E})}{2\mathcal{E}(\mathcal{A}\lambda + \mathcal{E})}\\ &\quad + \frac{d(b\gamma + 2c)\lambda}{\mathcal{A}^3(\mathcal{E} + \mathcal{A}\lambda)} = \frac{-\lambda^2}{\gamma(\mathcal{E} + \mathcal{A}\lambda)} \leqslant 0, \end{aligned}$$

where the inequality $\ln(1 + x) \leqslant x(x + 2)/2(x + 1)$ for $x \geqslant 0$ is used. Next, for the case $a\gamma^2 = c$,

$$\frac{\partial T_*}{\partial\gamma} = -\frac{\lambda^2}{\gamma(\mathcal{E} + \mathcal{A}\lambda)} \leqslant 0.$$

Finally, for the case $a\gamma^2 < c$, we have

$$\begin{aligned} \frac{\partial T_*}{\partial\gamma} &= -\frac{(2a\gamma + b)\lambda}{\mathcal{A}^3} + \frac{2d(a\gamma^2 - c)}{\mathcal{A}^5\lambda}\mathcal{A}\lambda\ln\left(\frac{\mathcal{E} + \mathcal{A}\lambda}{\mathcal{E}}\right)\\ &\quad + \frac{d(b\gamma + 2c)\lambda}{\mathcal{A}^3(\mathcal{E} + \mathcal{A}\lambda)}\\ &\leqslant -\frac{(2a\gamma + b)\lambda}{\mathcal{A}^3} + \frac{2d(a\gamma^2 - c)}{\mathcal{A}^5\lambda}\frac{2\mathcal{A}^2\lambda^2}{2\mathcal{E} + \mathcal{A}\lambda}\\ &\quad + \frac{d(b\gamma + 2c)\lambda}{\mathcal{A}^3(\mathcal{E} + \mathcal{A}\lambda)}\\ &= \frac{-\lambda^2}{\mathcal{A}^2(\mathcal{E} + \mathcal{A}\lambda)}\left(2a\gamma + b + \frac{2d(c - a\gamma^2)}{2\mathcal{E} + \mathcal{A}\lambda}\right), \end{aligned}$$

where the inequality $2(\alpha - \beta)^2/(\alpha + \beta) \leqslant (\alpha - \beta)\ln(\alpha/\beta)$ for $\alpha, \beta \geqslant 0$ is used. Now $2a\gamma + b > 0$ readily leads to $\partial T_*/\partial\gamma \leqslant 0$. If $2a\gamma + b \leqslant 0$, the above inequality can be further computed as

$$\begin{aligned}\frac{\partial T_*}{\partial \gamma} &\leqslant \frac{-\lambda^2(2a\gamma^2+b\gamma)(2\mathcal{E}+\mathcal{A}\lambda)}{\gamma\mathcal{A}^2(\mathcal{E}+\mathcal{A}\lambda)(2\mathcal{E}+\mathcal{A}\lambda)}\\ &\quad+\frac{2\lambda^2(a\gamma^2-c)}{\gamma\mathcal{A}^2(\mathcal{E}+\mathcal{A}\lambda)(2\mathcal{E}+\mathcal{A}\lambda)}\\ &=\frac{-\lambda^2\{2\mathcal{E}\mathcal{A}^2-(4\mathcal{E}+\mathcal{A}\lambda)(2a\gamma^2+b\gamma)\}}{\gamma\mathcal{A}^2(\mathcal{E}+\mathcal{A}\lambda)(2\mathcal{E}+\mathcal{A}\lambda)}\leqslant 0.\end{aligned}$$

APPENDIX B: T_* AS A FUNCTION OF $\mathcal{A}$

We here prove that T_* is a monotonically decreasing function with respect to $\mathcal{A}$, as follows. First,

$$\begin{aligned}\frac{\partial T_*}{\partial \mathcal{A}} &= -\frac{2}{\mathcal{A}^2}\left\{\lambda+\frac{\mathcal{E}}{\mathcal{A}}\ln\left(\frac{\mathcal{E}}{\mathcal{E}+\mathcal{A}\lambda}\right)\right\}\\ &\quad-\frac{2\mathcal{E}}{\mathcal{A}^2}\left\{\frac{1}{\mathcal{A}}\ln\left(\frac{\mathcal{E}}{\mathcal{E}+\mathcal{A}\lambda}\right)+\frac{\lambda}{\mathcal{E}+\mathcal{A}\lambda}\right\}\\ &=-\frac{2}{\mathcal{A}^2}\left\{\lambda+\frac{k\lambda}{k+\lambda}-2k\ln\left(1+\frac{\lambda}{k}\right)\right\},\end{aligned}$$

where $k=\mathcal{E}/\mathcal{A}$. Then, from the inequality $\ln(1+x)\leqslant x(2+x)/2(1+x)$ for $x\geqslant 0$, we have

$$\frac{\partial T_*}{\partial \mathcal{A}}\leqslant -\frac{2}{\mathcal{A}^2}\left\{\lambda+\frac{k\lambda}{k+\lambda}-2k\frac{(\lambda/k)(2+\lambda/k)}{2(1+\lambda/k)}\right\}=0.$$

APPENDIX C: QSL WITH A MIXED INITIAL STATE

Here we derive the QSL (8), where the initial state ρ_0 is a mixed state. First, the relative purity (2) is redefined as

$$\Theta'_t=\arccos\left\{\frac{\mathrm{Tr}(\rho_0\rho_t)}{\mathrm{Tr}(\rho_0^2)}\right\}. \tag{C1}$$

Then the following inequality holds:

$$\begin{aligned}\|\rho_0-\rho_t\|_{\mathrm{F}} &= \sqrt{\mathrm{Tr}[(\rho_0-\rho_t)^2]}\\ &=\sqrt{\mathrm{Tr}(\rho_t^2)-2\,\mathrm{Tr}(\rho_t\rho_0)+\mathrm{Tr}(\rho_0^2)}\\ &\leqslant\sqrt{1-2\,\mathrm{Tr}(\rho_0^2)\cos\Theta'_t+\mathrm{Tr}(\rho_0^2)}\\ &=:\sqrt{2}\lambda'_t.\end{aligned}$$

Now the time derivative of Θ'_t is upper bounded by

$$\begin{aligned}\frac{d\Theta'_t}{dt} &= \frac{-1}{\sqrt{1-\left\{\frac{\mathrm{Tr}(\rho_0\rho_t)}{\mathrm{Tr}(\rho_0^2)}\right\}^2}}\frac{\mathrm{Tr}\left\{\rho_0\frac{d\rho_t}{dt}\right\}}{\mathrm{Tr}\left(\rho_0^2\right)}\\ &=\frac{-1}{\sin\Theta_t\mathrm{Tr}\left(\rho_0^2\right)}\mathrm{Tr}\{\rho_0(-i[H,\rho_t]+\mathcal{D}[M]\rho_t)\}\\ &=\frac{1}{\sin\Theta_t\mathrm{Tr}\left(\rho_0^2\right)}(\mathrm{Tr}\{(i[H,\rho_0]+\mathcal{D}^\dagger[M]\rho_0)(\rho_0-\rho_t)\}\\ &\quad+\mathrm{Tr}(M^\dagger M\rho_0)-\mathrm{Tr}(M^\dagger\rho_0 M\rho_0))\\ &\leqslant\frac{1}{\sin\Theta_t\mathrm{Tr}\left(\rho_0^2\right)}(\|i[H,\rho_0]\\ &\quad+\mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}}\|\rho_0-\rho_t\|_{\mathrm{F}}+\mathcal{E}')\\ &\leqslant\frac{1}{\sin\Theta_t\mathrm{Tr}\left(\rho_0^2\right)}(\mathcal{A}'\lambda'_t+\mathcal{E}'),\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}'&=\sqrt{2}\|i[H,\rho_0]+\mathcal{D}^\dagger[M]\rho_0\|_{\mathrm{F}},\\ \mathcal{E}'&=\mathrm{Tr}(M^\dagger M\rho_0)-\mathrm{Tr}(M^\dagger\rho_0 M\rho_0).\end{aligned}$$

Integrating both sides of the above inequality from 0 to T, we obtain Eq. (8).

APPENDIX D: GELL-MANN MATRICES

The Gell-Mann matrices $\{\Lambda_i\}_{i=1}^8$ are defined as

$$\Lambda_1=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad \Lambda_2=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix},$$

$$\Lambda_3=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$

$$\Lambda_4=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\quad \Lambda_5=\begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix},$$

$$\Lambda_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

$$\Lambda_7=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},\quad \Lambda_8=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

Note that these matrices form an orthonormal basis set in SU(3).

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