Semion formalism for spin and qubit systems: Non-Markovian treatment

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Using semion substitution for spin variables we perform an *ab initio* derivation of effective action for an open quantum two-level system . For this purpose, we introduce, by using the Hubbard-Stratonovich transformation a two-time complex quantum field which average value plays the role of the Green's function for the spin variables. The field thus introduced allows us to develop a diagram technique in a standard way. The proposed formalism is used to study a spin embedded into an Ohmic reservoir as an example of the spin-boson model. Non-Markovian effects in this system are analyzed.

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I. INTRODUCTION

Quantum two-level systems have been the focus of considerable research interest. Recently, they have also attracted attention as a platform for quantum computing. Besides, two-level systems such as spin ensembles studied on the basis of the Heisenberg, Hubbard, t-J models, etc., are essential, especially in light of the rapid development of spintronics [1].

A new direction is analog quantum modeling, when a fairly complex quantum system is modeled or the behavior of a quantum model is studied with the help of a controlled quantum system [2–4]. Qubit systems often play the role of a quantum simulator [5,6]. From a technical point of view, the matter is complicated by the difference in the methods used to describe systems. Qubit systems are usually considered as systems of artificial atoms, and conventional quantum optics consideration is applied. It includes the techniques of the density matrix, Langevin equations, and Fokker-Planck-type equations for the probability of the coherent states and similar functions, such as the Q function, Wigner function, etc. (see, for example, Refs. [7,8]). At the same time, the theory of strongly correlated systems whose behavior is studied by analog quantum simulation uses the methods of path integral representation, Green's functions, renormalization group, and others that make up the tools of quantum field theory (see Ref. [9]). Thus, an important challenge is to construct an approach that can describe both simulated and simulating systems in a single way.

A significant obstacle to the application of the quantum field methods to the analysis of spin (qubit) systems is the non-Abelian nature of spin operators: the commutator of two spin operators is also spin operator, but not a c number, as it is in the case of bosons or fermions. Approaches to overcoming this difficulty can be divided into two areas. First, one can use a version of the diagram technique for spin systems (let us call it the spin diagram technique—SDT), within the framework of which all the main results of the Ising and Heisenberg models are reproduced and some new ones are obtained [10–12]. The problem with using this technique is that it is cumbersome and peculiar, which makes it practically not applicable outside the framework of the Ising and Heisenberg models. Second, there are several transformations of spin variables into fermionic and bosonic quantities: the Jordan-Wigner transform [13], the Agranovich-Toshich formula [14], the Holstein-Primakoff substitution [15], the Bravyi-Kitaev transformation [16], the Majorana fermions [17], Abrikosov fermions [18], the Popov-Fedotov semion substitution [19]. Recently, variants of Majorana fermions have been used [20–22]. Some interesting results have been obtained in this technique. In this paper we use semion substitution. This substitution is not widely used and only a few results were obtained in this way. This is the work of Popov and Fedotov [19], where this substitution was introduced and it is shown how this approach can be applied for the analysis of the Dicke, Ising, and Heisenberg models. In addition, Kiselev and coauthors expanded the list of the this substitution applications to a number of other models: the Hubbard model, the t-J model, the Kondo problem, etc. [23–25].

Our work is also based on semion substitution. We introduce a two-time complex quantum field, which average value plays, in fact, the role of the Green's function for the spin variable. We use the path integral approach and a Hubbard-Stratonovich transformation [26,27] as the most natural way to construct this two-time object. The field thus introduced allows us to develop a diagram technique in a standard way and apply a well-known powerful method of summing infinite sets of diagrams. We illustrate the applicability of this general approach with an example of the spin-boson model [5,28] with the linear coupling between the thermal reservoir and the transverse component of the spin. This version of the spin-boson model was actively studied in quantum optics and,

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on the other hand, it most easily describes the interaction of a qubit with a microwave resonator. Also, this version of the spin-boson model was considered in Ref. [22] as an example of the application of the Majorana fermion technique, and therefore it seems interesting to compare the two approaches within the same model.

The effect of the environment on a spin system is a problem of great interest, and many techniques have been proposed. We do not list all of them here but mention the most popular ones such as an assumption of temporal locality and using Lindblad master equations [29], which is typical for quantum optics. Another idea is to use Holstein-Primakoff substitution of spin operators for bosonic ones [30]. In general, the assumptions and approximations of such a model may by difficult to overcome and can lead to insufficient accuracy when applied to a particular physical realization, such as developing fast quantum computer gates with high fidelity. Our approach potentially allows us to solve similar problems by using standard quantum field theory methods with controlled accuracy.

The central idea of our study is to integrate out bosonic degrees of freedom and obtain effective dissipative action for a spin. This idea was developed by Caldeira and Leggett [28,31], but the origin of the non-Gaussian behavior of the effective action is different. In the original work, the action of the system under consideration was non-Gaussian while its interaction with the bath was described by a tunneling term which is linear either in the system operators or in the operators of the bath. In contrast to this problem, in our case the action related to the free spin is Gaussian but the interaction of the spin with the environment includes three operators and, thus, results in the non-Gaussian character of the effective action.

It is rather natural to use Schwinger-Keldysh functional-integral formalism to study the problem. Non-Markovian dynamics of a quantum system, either bosonic or fermionic, coupled to a reservoir has already been studied with the use of this approach [32]. But a similar problem for a spin system remains unsolved.

We show below how to calculate the characteristics of the equilibrium spin coupled to the bosonic reservoir and to obtain an equation governing the dynamics of the spin driven from the state of thermodynamic equilibrium. We argue that this equation has a substantially non-Markovian form. The transition to conventional equations of quantum optics is discussed.

II. POPOV-FEDOTOV SEMIONS

In this section, we briefly consider the Popov-Fedotov semion substitution and introduce the notation used later.

The Hamiltonian of the isolated spin has the form

$$\hat{H}_0 = \varepsilon \hat{S}^z, \tag{1}$$

where $\hat{S}^z = \hat{\sigma}^z/2$ is the projection operator of the spin on the axis z, and $\pm\varepsilon/2$ are the energies of the spin oriented in the direction against and along the magnetic field. We use the standard notation for Pauli matrices:

$$\hat{\sigma}^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}^y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \hat{\sigma}^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

Abrikosov [18] introduced two fermions, described by destruction operators $\hat{a}$ and $\hat{b}$, and made the substitution

$$\hat{\sigma}^z = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}, \quad \hat{\sigma}^x = \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}, \quad \hat{\sigma}^y = i\hat{a}^\dagger\hat{a} - i\hat{b}^\dagger\hat{b}. \tag{3}$$

The transformed Hamiltonian

$$\hat{H}_0 = \frac{\varepsilon}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) \tag{4}$$

has four eigenstates $|00\rangle$, $|11\rangle$, $|01\rangle$, and $|10\rangle$, but only the states $|01\rangle$ and $|10\rangle$ relate to a spin system. Two nonphysical states $|00\rangle$ and $|11\rangle$ with the eigenvalue $\varepsilon = 0$ do not contribute to the average of physical operators but break their normalization due to a nonzero contribution to a value of the statistical sum. To overcome this difficulty, Popov and Fedotov [19] introduced the artificial imaginary chemical potential $\mu = i\pi/2\beta$, where $\beta = 1/k_B T$, T is the temperature, so that the calculation of the partition function of a grand-canonical ensemble should be produced with the operator

$$\hat{K}_0 = \hat{H}_0 - \mu\hat{N} = \frac{\varepsilon}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) - \frac{i\pi}{2\beta}(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}). \tag{5}$$

For the partition function one gets

$$\begin{aligned} \mathrm{tr}\, e^{-\beta\hat{K}_0} &= \mathrm{tr}\, e^{\frac{-\beta\varepsilon}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) + \frac{i\pi}{2}(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})} \\ &= \langle 00|\cdots|00\rangle + \langle 11|\cdots|11\rangle + \langle 01|\cdots|01\rangle \\ &\quad + \langle 10|\cdots|10\rangle \\ &= e^0 + e^{i\pi} + e^{\frac{\beta\varepsilon}{2} + \frac{i\pi}{2}} + e^{-\frac{\beta\varepsilon}{2} + \frac{i\pi}{2}} = i\big(e^{\frac{\beta\varepsilon}{2}} + e^{-\frac{\beta\varepsilon}{2}}\big). \end{aligned} \tag{6}$$

This calculation demonstrates the cancellation of contributions of nonphysical states, which is achieved by the introduction of the imaginary chemical potential. Note the coefficient i acquired by the statistical sum.

Analogously, we have

$$\mathrm{tr}\{e^{-\beta\hat{K}_0}\hat{S}^z\} = \frac{i}{2}\big(-e^{\frac{\beta\varepsilon}{2}} + e^{-\frac{\beta\varepsilon}{2}}\big) \tag{7}$$

and for the spin $\langle\hat{S}^z\rangle_T^{(0)}$ at temperature T obtain

$$\langle\hat{S}^z\rangle_T^{(0)} = \frac{\mathrm{tr}\{e^{-\beta\hat{K}_0}\hat{S}^z\}}{\mathrm{tr}\{e^{-\beta\hat{K}_0}\}} = -\frac{1}{2}\tanh\frac{\beta\varepsilon}{2}. \tag{8}$$

This calculation shows how the nonphysical coefficient i disappears from a physically observed value.

On the other hand, $\langle\hat{S}^z\rangle_T^{(0)}$ following (3) can be written as

$$\langle\hat{S}^z\rangle_T^{(0)} = \tfrac{1}{2}(\langle\hat{a}^\dagger\hat{a}\rangle_T - \langle\hat{b}^\dagger\hat{b}\rangle_T) = \tfrac{1}{2}(n_a - n_b). \tag{9}$$

From the relation (8), at first glance, it follows that

$$\begin{aligned} \tilde{n}_a &= \frac{e^{-\frac{\beta\varepsilon}{2}}}{e^{\frac{\beta\varepsilon}{2}} + e^{-\frac{\beta\varepsilon}{2}}} = \frac{1}{e^{\beta\varepsilon} + 1}, \\ \tilde{n}_b &= \frac{e^{\frac{\beta\varepsilon}{2}}}{e^{\frac{\beta\varepsilon}{2}} + e^{-\frac{\beta\varepsilon}{2}}} = \frac{1}{e^{-\beta\varepsilon} + 1}, \end{aligned} \tag{10}$$

i.e., we get the fermion distributions, but with the wrong energy, which violates the Gibbs distribution. To resolve this issue, we represent n_a and n_b as the sum

$$\frac{1}{\hbar\beta}\sum_{m=-\infty}^{\infty} G^{(0)}(i\omega_m) = n, \tag{11}$$

where $G^{(0)}(i\omega_m)$ is the Matsubara Green's function, and use the well-known formula

$$\lim_{\eta\to 0}\frac{1}{\hbar\beta}\sum_{m=-\infty}^{\infty}\frac{e^{i\omega_m\eta}}{i\omega_m-(\varepsilon-\mu)/\hbar}=\frac{1}{e^{\beta(\varepsilon-\mu)}+1}. \tag{12}$$

Here the summation is made over the Matsubara fermionic frequencies $\omega_m = 2\pi(m+1/2)/(\hbar\beta)$. This approach leads to the following expressions for the bare Matsubara Green's functions of the fermions a and b:

$$G_a^{(0)}(i\omega_m)=\frac{1}{i\omega_m-(\varepsilon/2-\mu)/\hbar},$$
$$G_b^{(0)}(i\omega_m)=\frac{1}{i\omega_m-(-\varepsilon/2-\mu)/\hbar}, \tag{13}$$

while their densities read

$$n_a=\frac{1}{-ie^{\beta\varepsilon/2}+1},\quad n_b=\frac{1}{-ie^{-\beta\varepsilon/2}+1}. \tag{14}$$

Here we use the above imaginary chemical potential $\mu = i\pi/2\beta$, which leads to the expression $e^{\beta(\varepsilon-\mu)/2} = -ie^{\beta\varepsilon/2}$. In expressions of type (13) we can include the imaginary chemical potential into the Matsubara frequency. The summation in Eq. (11) now should be carried out over frequencies $\omega_m = 2\pi(m+3/4)/(\hbar\beta)$, which are intermediate between bosonic and fermionic frequencies. For this reason, fermions with imaginary chemical potential were called semions.

It is easy to see that $(n_a+n_b)/2$ with n_a and n_b from Eq. (14) matches Eq. (8). In expressions (14), the correct energies are present, but the densities n_a and n_b are complex functions now. This illustrates the main difficulty of semion formalism, which relates to the problem of constructing the spin Green's functions directly by using fields a and b. In the next section, we present an approach that allows us to circumvent this difficulty, using, as an example, the spin-boson model.

III. SPIN-BOSON MODEL

A. Action of the system

We consider a variant of the spin-boson model describing a spin in a boson reservoir with the linear coupling, which is transverse relative to the eigenbasis of the perturbed Hamiltonian (1):

$$\hat{H}=\varepsilon\hat{S}^z+g\hat{X}\hat{S}^x+H_X, \tag{15}$$

where H_X is the Hamiltonian describing the boson reservoir and the coupling constant g is explicitly introduced. In the following calculations, we will derive series expansion with respect to g, so we assume that g is small where it is important.

The action corresponding to the Hamiltonian (15) can be represented as the sum

$$\mathcal{A}=\mathcal{A}_0+\mathcal{A}_{\rm int}+A_X, \tag{16}$$

where $\mathcal{A}_0$ is the action of the isolated spin which can be expressed in terms of semion fields as

$$\mathcal{A}_0=\int_0^{\hbar\beta}d\tau\{\bar{a}(\tau)(\hbar\partial_\tau+\varepsilon/2-\mu)a(\tau)$$
$$+\bar{b}(\tau)(\hbar\partial_\tau-\varepsilon/2-\mu)b(\tau)\}, \tag{17}$$

$\mathcal{A}_{\rm int}$ is the action of the interaction between the spin and the reservoir,

$$\mathcal{A}_{\rm int}=\int_0^{\hbar\beta}d\tau g\{\bar{a}(\tau)b(\tau)+\bar{b}(\tau)a(\tau)\}X(\tau), \tag{18}$$

and $\mathcal{A}_X$ is the action of the reservoir

$$\mathcal{A}_X=-\hbar\int_0^{\hbar\beta}d\tau d\tau' X(\tau)\Pi^{-1}(\tau-\tau')X(\tau'). \tag{19}$$

Here $\Pi^{-1}(\tau-\tau')$ is the matrix inverse to the reservoir correlation function $\Pi(\tau-\tau')=\langle T_\tau[X(\tau)X(\tau')]\rangle$. $X=\sum_j v_j(c_j+c_j^+)$ is the real operator. The reservoir correlation function in the Matsubara representation can be written as

$$\Pi(i\omega_m)=\int_{-\infty}^{\infty}\frac{dx}{\pi}\frac{\rho(|x|){\rm sgn}(x)}{x-i\omega_m}=\int_0^\infty\frac{dx}{\pi}\rho(x)\frac{2x}{x^2+\omega_m^2}. \tag{20}$$

The results of this section are valid for arbitrary reservoir spectral density $\rho(x)$.

One can formally integrate out the reservoir degree of freedom in a standard way so that the action becomes

$$\mathcal{A}=\mathcal{A}_0+\frac{g^2}{4\hbar}\int_0^{\hbar\beta}d\tau d\tau'\bar{b}(\tau)a(\tau)\Pi(\tau-\tau')\bar{a}(\tau')b(\tau'). \tag{21}$$

The rearrangement of the spin variables brings it to the form

$$\mathcal{A}=\mathcal{A}_0+\frac{g^2}{4\hbar}\int_0^{\hbar\beta}d\tau d\tau' b(\tau')\bar{b}(\tau)\Pi(\tau-\tau')\bar{a}(\tau')a(\tau). \tag{22}$$

The sign of the last term remains the same since an even number of permutations of Grassmann variables were made.

B. Two-time fields

To eliminate the nonlinear term (22), we introduce complex fields $\Phi^a(\tau,\tau')$ and $\Phi^b(\tau,\tau')$, which depend on two moments of imaginary time, and apply the Hubbard-Stratonovich transformation. As a result, we obtain

$$\mathcal{A}=\mathcal{A}_0+\hbar\int_0^{\hbar\beta}d\tau d\tau'\bigg\{-\frac{\hbar^2}{g^2}\frac{\Phi^a(\tau,\tau')\Phi^b(\tau',\tau)}{\Pi(\tau-\tau')}$$
$$+\bar{b}(\tau)b(\tau')\Phi^a(\tau,\tau')+\bar{a}(\tau)a(\tau')\Phi^b(\tau,\tau')\bigg\}. \tag{23}$$

Here we use the Hermiticity of the introduced fields by time arguments $\Phi^{a,b}(\tau,\tau')=\bar{\Phi}^{a,b}(\tau',\tau)$. They inherit this property from the original matrices $(\bar{a}a)_{\tau'\tau}$ and $(\bar{b}b)_{\tau'\tau}$, which obey the chain of equalities $(\bar{a}a)_{\tau'\tau}=\bar{a}(\tau')a(\tau)=\overline{\bar{a}(\tau)a(\tau')}=\overline{(\bar{a}a)}_{\tau\tau'}$. The consequence of the Hermiticity is that the introduced fields with matching time arguments are real numbers; for example, $\Phi^a(\tau,\tau)=\bar{\Phi}^a(\tau,\tau)$.

As a result, the exponent of the path integral takes the form

$$-\frac{\mathcal{A}}{\hbar}=\int_0^{\hbar\beta}d\tau d\tau'\bigg\{\frac{\hbar^2}{g^2}\frac{\Phi^a(\tau,\tau')\Phi^b(\tau',\tau)}{\Pi(\tau-\tau')}$$
$$+\bar{a}(\tau)\big[\big(G_a^{(0)}\big)^{-1}(\tau,\tau')-\Phi^b(\tau,\tau')\big]a(\tau')$$
$$+\bar{b}(\tau)\big[\big(G_b^{(0)}\big)^{-1}(\tau,\tau')-\Phi^a(\tau,\tau')\big]b(\tau')\bigg\}, \tag{24}$$

where

$$\big(G_a^{(0)}\big)^{-1}(\tau,\tau') = -\frac{1}{\hbar}(\hbar\partial_\tau + \varepsilon/2 - \mu)\delta(\tau-\tau'),$$
$$\big(G_b^{(0)}\big)^{-1}(\tau,\tau') = -\frac{1}{\hbar}(\hbar\partial_\tau - \varepsilon/2 - \mu)\delta(\tau-\tau'). \tag{25}$$

Integrating over the fields a, $\bar{a}$, b, and $\bar{b}$, we come to the effective action of the system

$$-\frac{\mathcal{A}_{\rm eff}}{\hbar} = \int_0^{\hbar\beta} d\tau d\tau' \frac{\hbar^2}{g^2}\frac{\Phi^a(\tau,\tau')\Phi^b(\tau',\tau)}{\Pi(\tau-\tau')} + \ln\det[(\mathbf{G}^{(0)})^{-1} - \mathbf{\Phi}]\mathbf{G}^{(0)}. \tag{26}$$

Here,

$$\mathbf{\Phi}_{\tau\tau'} = \begin{bmatrix}\Phi^b(\tau,\tau') & 0\\ 0 & \Phi^a(\tau,\tau')\end{bmatrix},$$
$$\mathbf{G}^{(0)}_{\tau\tau'} = \begin{bmatrix}G_a^{(0)}(\tau-\tau') & 0\\ 0 & G_b^{(0)}(\tau-\tau')\end{bmatrix}, \tag{27}$$

with

$$G^{(0)}_{a,b}(\tau-\tau') = \frac{1}{\hbar\beta}\sum_n e^{-i\omega_n(\tau-\tau')}G^{(0)}_{a,b}(i\omega_n), \tag{28}$$

where $G^{(0)}_{a,b}(i\omega_n) = \hbar(i\hbar\omega_n \mp \varepsilon/2 + \mu)^{-1}$.

Equation (26) illustrates the difference to the Caldeira-Leggett problem: in the last case the propagator of Hubbard-Stratonovich fields is determined by the properties of the system itself, but in our case, this propagator is determined by the bath. In addition, in the Caldeira-Leggett model, the self-energy term appears in the argument of $\ln\det\dots$ while in our problem this not the case, as one can see from Eq. (26).

Next, one can write

$$\begin{aligned}\ln\det&[(\mathbf{G}^{(0)})^{-1} - \mathbf{\Phi}]\mathbf{G}^{(0)}\\ &= \mathrm{tr}\ln(\mathbf{I} - \mathbf{\Phi}\mathbf{G}^{(0)})\\ &= -\sum_{k=1}^{\infty}\frac{1}{k}\big[\mathrm{tr}\big(\Phi^b G_a^{(0)}\big)^k + \mathrm{tr}\big(\Phi^a G_b^{(0)}\big)^k\big],\end{aligned} \tag{29}$$

where we assume that

$$\mathrm{tr}K^k = \int dz_1\cdots dz_k K(z_1,z_2)K(z_2,z_3)\cdots K(z_k,z_1). \tag{30}$$

Now we find the relationship between the spin S^z and its corresponding field Φ^z. To do this, we add a term with the source $\eta(\tau)$ to the action (21):

$$\begin{aligned}\mathcal{A}[\eta] &= \mathcal{A} - \hbar\int_0^{\hbar\beta} d\tau\,\eta(\tau)S^z(\tau)\\ &= \mathcal{A} - \hbar\int_0^{\hbar\beta} d\tau \frac{\eta(\tau)}{2}[\bar{a}(\tau)a(\tau) - \bar{b}(\tau)b(\tau)].\end{aligned} \tag{31}$$

The average spin value can now be found as a variational derivative

$$\langle S^z(\tau)\rangle = \frac{\delta}{\delta\eta(\tau)}\frac{\int D\Xi e^{-\mathcal{A}[\eta]/\hbar}}{\int D\Xi e^{-\mathcal{A}/\hbar}}\bigg|_{\eta=0}, \tag{32}$$

while the Matsubara Green's function is

$$\langle T_\tau S^z(\tau)S^z(\tau')\rangle = \frac{\delta^2}{\delta\eta(\tau)\delta\eta(\tau')}\frac{\int D\Xi e^{-\mathcal{A}[\eta]/\hbar}}{\int D\Xi e^{-\mathcal{A}/\hbar}}\bigg|_{\eta=0}. \tag{33}$$

Here $\int D\Xi$ denotes the path integral taken over all fields present in the action.

The action (26) is the main relation for the analysis of the spin-boson model using the Green's functions $G_a^{(0)}$ and $G_b^{(0)}$ introduced in Ref. [19]. We can go further and introduce variables which use the symmetry of the functions $G_a^{(0)}$ and $G_b^{(0)}$:

$$G_b^{(0)}(i\omega_n) = -\bar{G}_a^{(0)}(i\omega_n), \tag{34}$$

which leads to

$$G_b^{(0)}(\tau-\tau') = -\bar{G}_a^{(0)}(\tau'-\tau) = -\big(G_a^{(0)}\big)^*(\tau-\tau'). \tag{35}$$

Here, the asterisk stands for Hermitian conjugation. This symmetry allows us to use further only the function $G_a^{(0)} = G^{(0)}$ and to replace initial fields Φ^a and Φ^b by the fields Φ^z and $\bar{\Phi}^z$ according to the relations

$$\Phi^a(\tau,\tau') = \Phi^z(\tau,\tau'), \quad \Phi^b(\tau,\tau') = -\bar{\Phi}^z(\tau,\tau'). \tag{36}$$

The action (26) acquires the form

$$-\frac{\mathcal{A}_{\rm eff}}{\hbar} = -\int_0^{\hbar\beta} d\tau d\tau' \frac{\hbar^2}{g^2}\frac{\bar{\Phi}^z(\tau,\tau')\Phi^z(\tau',\tau)}{\Pi(\tau-\tau')} + \mathrm{tr}\ln(\mathbf{I} - \mathbf{\Phi}\mathbf{G}^{(0)}), \tag{37}$$

where

$$\mathbf{\Phi}_{\tau\tau'} = \begin{bmatrix}-\bar{\Phi}^z(\tau,\tau') & 0\\ 0 & \Phi^z(\tau,\tau')\end{bmatrix},$$
$$\mathbf{G}^{(0)}_{\tau\tau'} = \begin{bmatrix}G^{(0)}(\tau-\tau') & 0\\ 0 & -(G^{(0)})^*(\tau-\tau')\end{bmatrix}. \tag{38}$$

Now, we represent the field $\Phi^z(\tau,\tau')$ as a sum of the average value and quantum fluctuations with zero average:

$$\Phi^z(\tau,\tau') = \langle\Phi^z(\tau,\tau')\rangle + \phi^z(\tau,\tau'), \quad \langle\phi^z(\tau,\tau')\rangle = 0. \tag{39}$$

For the average spin we obtain

$$\langle S^z(\tau)\rangle = \langle S^z\rangle_T = \frac{\hbar^2}{2g^2}\frac{1}{\Pi(0)}[\langle\Phi^z(\tau,\tau)\rangle + \langle\bar{\Phi}^z(\tau,\tau)\rangle]. \tag{40}$$

Detailed calculations are given in Appendix A.

The average spin $\langle\hat{S}^z\rangle_T$ can be found as a power series in the coupling constant g^2 as follows: In the action (37), the series expansion of $\mathrm{tr}\ln(\dots)$ according to Eq. (29) contains the linear term ($k=1$)

$$\int_0^{\hbar\beta} d\tau_1 d\tau_2[\bar{\Phi}^z(\tau_1,\tau_2)G^{(0)}(\tau_2,\tau_1) + \Phi^z(\tau_1,\tau_2)\bar{G}^{(0)}(\tau_2,\tau_1)]. \tag{41}$$

This term can be omitted by a shift in the fields Φ^z and $\bar{\Phi}^z$. Often we make such a shift, completing a square in an action. In our case, the procedure is iterative. As a result, the series expansion for $\langle\Phi^z(\tau,\tau')\rangle$ is obtained. The calculation process is presented in Appendix C.

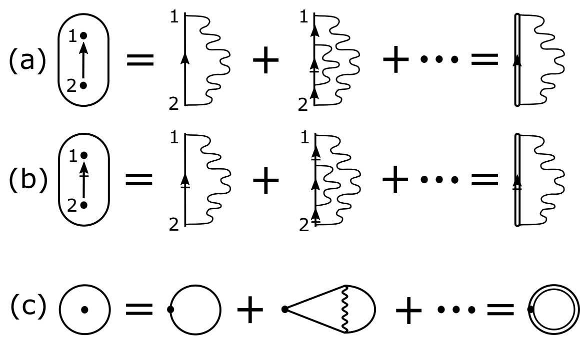

FIG. 1. Power series in g^2 for functions (a) $\langle\Phi^z(\tau_1,\tau_2)\rangle$, (b) $\langle\bar{\Phi}^z(\tau_1,\tau_2)\rangle$, (c) $\langle S^z(\tau)\rangle$.

To depict the resulting series expansion, we use the following rules: The Green's function $G^{(0)}(\tau_1-\tau_2)$ is represented by a solid line, the Green's function $\bar{G}^{(0)}(\tau_1-\tau_2)$ is shown as a crossed-out solid line, the reservoir correlator $\Pi(\tau_1-\tau_2)$ is represented by a wavy line. The average values of the two-time fields $\langle\Phi^z(\tau_1,\tau_2)\rangle$ and $\langle\bar{\Phi}^z(\tau_1,\tau_2)\rangle$ are depicted by vertically oriented ovals with arrows (normal and crossed-out, respectively). The arrows are directed from the second argument to the first argument (see Fig. 1).

As follows from formula (40), the average spin relates to the sum of the propagator $\langle\Phi^z(\tau_1,\tau_2)\rangle$ and its Hermitian conjugate $\langle\bar{\Phi}^z(\tau_1,\tau_2)\rangle$, taken at a same time $\tau_1=\tau_2=\tau$, so that the average spin always remains a real number. For brevity, it is convenient to omit the type of arrows assuming that a Hermitian conjugate must be added to each diagram. We come to the following rule to pass from the diagrams obtained for fields a and b to the spin: one should consider either a or b diagrams, connect the outer ends of each diagram, and also discard the connecting wavy line, which, in turn, transforms each into a closed loop equal to $g^2\Pi(0)/\hbar^2$. The final step is to take a real part of the expression obtained. The resulting series for the average spin is represented in Fig. 1(c).

The spin Matsubara Green's function is connected with the field $\Phi^z(\tau,\tau')$ through the relation

$$\langle T_\tau S^z(\tau)S^z(\tau')\rangle-\langle S^z\rangle_T^2=-\frac{\hbar^2}{2g^2}\frac{1}{\Pi(0)}\delta^2(\tau-\tau')+\frac{\hbar^4}{4g^4}\frac{1}{\Pi^2(0)}\langle T_\tau[\phi^z(\tau,\tau)+\bar{\phi}^z(\tau,\tau)][\phi^z(\tau',\tau')+\bar{\phi}^z(\tau',\tau')]\rangle. \tag{42}$$

To calculate $\langle\hat{S}^z(\tau)S^z(\tau')\rangle$, we leave the action in only the quadratic terms with respect to fluctuations. The effective action can be written in the matrix form

$$-\frac{\mathcal{A}_{\text{eff}}^{(II)}}{\hbar}=-\int_0^{\hbar\beta}d\tau_1d\tau_2d\tau_3d\tau_4[\bar{\phi}^z(\tau_1,\tau_2),\phi^z(\tau_3,\tau_4)]\cdot\mathbf{G}^{-1}\cdot\begin{bmatrix}\phi^z(\tau_1,\tau_2)\\ \bar{\phi}^z(\tau_3,\tau_4)\end{bmatrix}. \tag{43}$$

Here, $\mathbf{G}^{-1}=\mathbf{G}_0^{-1}-\mathbf{\Sigma}$, where

$$\left(\mathbf{G}_0^{-1}\right)_{11}=\left(\mathbf{G}_0^{-1}\right)_{22}=\frac{\hbar^2}{2g^2}\frac{\delta(\tau_2-\tau_3)\delta(\tau_1-\tau_4)}{\Pi(\tau_1-\tau_2)},\quad \left(\mathbf{G}_0^{-1}\right)_{12}=\left(\mathbf{G}_0^{-1}\right)_{21}=0, \tag{44}$$

and

$$(\mathbf{\Sigma})_{11}=(\mathbf{\Sigma})_{22}=0,\quad (\mathbf{\Sigma})_{12}=(\mathbf{\Sigma})_{21}^*=-\tfrac{1}{2}G^{(0)}(\tau_2-\tau_3)G^{(0)}(\tau_4-\tau_1). \tag{45}$$

So, we obtain

$$\mathbf{G}=-\left\langle\begin{bmatrix}\phi^z(\tau_1,\tau_2)\\ \bar{\phi}^z(\tau_3,\tau_4)\end{bmatrix}\cdot[\bar{\phi}^z(\tau_1,\tau_2),\phi^z(\tau_3,\tau_4)]\right\rangle. \tag{46}$$

The expression (43) allows one to obtain the components of the Green's function $\mathbf{G}$ as a perturbation series. To do this, we add to the action $\mathcal{A}_{\text{eff}}^{(II)}$ the sources $J(\tau,\tau')$ and $\bar{J}(\tau,\tau')$:

$$\mathcal{A}_{\text{eff}}^{(II)}[J,\bar{J}]=\mathcal{A}_{\text{eff}}^{(II)}-\hbar\int_0^{\hbar\beta}d\tau d\tau'[\bar{J}(\tau,\tau')\phi^z(\tau,\tau')+J(\tau,\tau')\bar{\phi}^z(\tau,\tau')]. \tag{47}$$

We can obtain the Green's functions of the field $\phi^z(\tau,\tau')$ by taking the second variational derivative of the generating functional with respect to the sources $J,\bar{J}$:

$$\langle\phi^z(\tau_1,\tau_2)\bar{\phi}^z(\tau_3,\tau_4)\rangle=\frac{\delta^2}{\delta\bar{J}(\tau_1,\tau_2)\delta J(\tau_3,\tau_4)}\left.\frac{\int D\Xi e^{-\mathcal{A}_{\text{eff}}^{(II)}[J,\bar{J}]/\hbar}}{\int D\Xi e^{-\mathcal{A}_{\text{eff}}^{(II)}/\hbar}}\right|_{J,\bar{J}=0}, \tag{48}$$

and analogously for $\langle\bar{\phi}^z\phi^z\rangle$, $\langle\phi^z\phi^z\rangle$, and $\langle\bar{\phi}^z\bar{\phi}^z\rangle$. We calculate the Green's function to second order in the coupling constant g in Appendix B.

To graphically represent the result, we relate functions of the type $\langle\phi^z(\tau_1,\tau_2)\bar{\phi}^z(\tau_3,\tau_4)\rangle$ to a rounded rectangle that contains two vertical lines inside. The diagram shown in Fig. 2 depicts the sum $\langle\phi^z(\tau_1,\tau_2)\bar{\phi}^z(\tau_3,\tau_4)\rangle+\langle\bar{\phi}^z(\tau_1,\tau_2)\phi^z(\tau_3,\tau_4)\rangle$ in accordance with the convention proposed above. As we can see, this sum is given by the series of the ladder type. We introduced an effective interaction (heavy wavy line), for

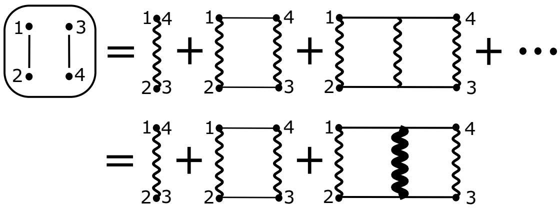

FIG. 2. Power series in g^2 for the function $\langle \phi^z(\tau_1,\tau_2)\bar{\phi}^z(\tau_3,\tau_4)\rangle + \langle \bar{\phi}^z(\tau_1,\tau_2)\phi^z(\tau_3,\tau_4)\rangle$.

which the Bethe-Salpeter equation can be written in the standard way. Note that the resulting diagram series, of course, is not complete, because we used the quadratic action (43), not the full one (37).

The series expansion for the spin Green's function $\langle T_\tau S^z(\tau_1)S^z(\tau_2)\rangle - \langle S^z\rangle_T^2$ can be obtained in accordance with formula (42). It prescribes in each diagram to combine vertically arranged external ends and discard arising wavy loops. Also, the first term of the series must be discarded. As a result, we obtain for the function $\langle T_\tau S^z(\tau_1)S^z(\tau_2)\rangle - \langle S^z\rangle_T^2$ the diagrams shown in Fig. 3. We dressed the Green's functions according to Fig. 1.

Note that the circle with the point in the center shown in Fig. 1 and the horizontally arranged oval with two points inside shown in Fig. 3 are graphic elements of SDT [10–12]. Thus, we believe that the graphical elements, which we introduced in Figs. 1 and 2, help to visualize the transition from the conventional fermionic diagram technique to the spin one.

C. Spin embedded into an Ohmic reservoir

The Ohmic reservoir is characterized by a linear spectral density

$$\rho(|x|) = \lambda_0 |x| \Theta(\omega_D - |x|). \tag{49}$$

Here ω_D is the maximal frequency of the reservoir modes, for example, the Debye frequency of phonons. It is convenient to introduce the dimensionless constant $\tilde{\lambda} = g^2\lambda_0$.

The lowest-order correction to the average spin is

$$\langle \hat{S}^z\rangle_T^{(1)} \approx \tilde{\lambda}\frac{\beta\varepsilon + \sinh\beta\varepsilon}{4\cosh^2\frac{\beta\varepsilon}{2}}\ln\omega_D. \tag{50}$$

We have calculated the correction to the retarded Green's function, $\Sigma^{(R)}(\omega)$, in the second order of the perturbation theory (see Appendix D). As it is described in Appendix D, one can split $\Sigma^{(R)}(\omega)$ into a sum of two terms. The first one is "thermal" and vanishes at low temperatures. This term has a trivial form of a δ function, so we will consider in detail only the "quantum" one, which survives only at the low-temperature limit and has less trivial behavior. In this limit,

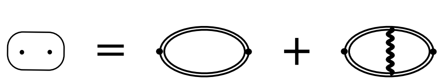

FIG. 3. Power series for the function $\langle T_\tau S^z(\tau_1)S^z(\tau_2)\rangle - \langle S^z\rangle_T^2$.

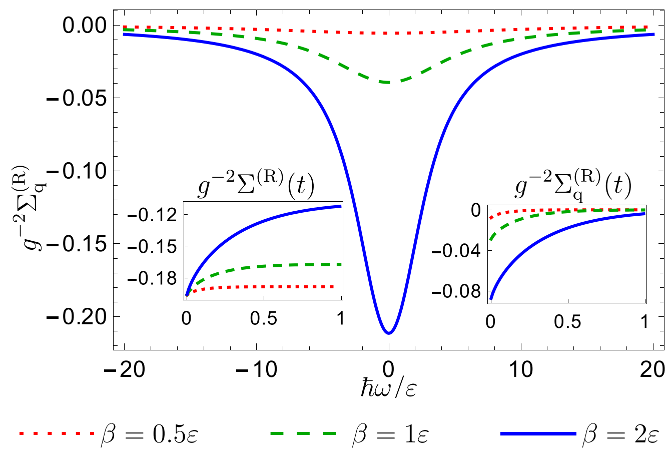

FIG. 4. Function $\Sigma_q^{(R)}(\omega)/g^2$ for $\beta = 0.5\varepsilon$ (red dotted line), $\beta = \varepsilon$ (green dashed line), $\beta = 2\varepsilon$ (blue solid line); ω is given in units of $\varepsilon/\hbar$. Inset shows functions $\Sigma^{(R)}(t)/g^2$ and $\Sigma_q^{(R)}(t)/g^2$ for the same relations between β and ε, t is given in units of $\hbar/\varepsilon$. One can see the increase of a constant vertical shift $\Sigma^{(R)}(t)/g^2$ as the temperature increases. This shift results from the "thermal" part of the self-energy.

quantum term prevails:

$$\Sigma_q^{(R)}(\omega) = \frac{\tilde{\lambda}}{\omega^2}\tanh\left(\frac{\beta\varepsilon}{2}\right)\text{Re}\left\{2\varepsilon\psi^{(1)}\left(\frac{i\beta\varepsilon}{2\pi}\right) - (\varepsilon+\hbar\omega)\psi^{(1)}\left(\frac{i\beta(\varepsilon+\hbar\omega)}{2\pi}\right) - (\varepsilon-\hbar\omega)\psi^{(1)}\left(\frac{i\beta(\varepsilon-\hbar\omega)}{2\pi}\right)\right\}, \tag{51}$$

where $\psi^{(n)}(z)$ is the nth logarithmic derivative of the Gamma function:

$$\psi^{(n)}(z) = \frac{d^n}{dz^n}\ln[\Gamma(z)], \quad \Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{52}$$

The function $\Sigma_q^{(R)}(\omega)$ for several values of the ratio ε/β is shown in Fig. 4. Functions $\Sigma^{(R)}(t)$ and $\Sigma_q^{(R)}(t)$ are obtained by the Fourier transformation: $\Sigma^{(R)}(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\Sigma^{(R)}(\omega)d\omega/(2\pi)$. In the vicinity of $\omega = 0$, the function $\Sigma_q^{(R)}(\omega)$ can be approximated by the Lorentzian, and the Fourier transform integral can be calculated by using residues. As a result, we obtain an expression asymptotically exact for large t:

$$\Sigma_q^{(R)}(t) = -\frac{\tilde{\lambda}}{2}\sqrt{\frac{m^3}{n}}\exp\left(-\sqrt{\frac{m}{n}}t\right), \tag{53}$$

where

$$m = \beta\tanh\left(\frac{\beta\varepsilon}{2}\right)\text{Re}\left\{\frac{i}{2\pi}\psi^{(1)}\left(\frac{i\beta\varepsilon}{2\pi}\right) - \frac{\beta\varepsilon}{8\pi^2}\psi^{(2)}\left(\frac{i\beta\varepsilon}{2\pi}\right)\right\},$$

$$n = \beta^3\tanh\left(\frac{\beta\varepsilon}{2}\right)\text{Re}\left\{\frac{i}{48\pi^3}\psi^{(3)}\left(\frac{i\beta\varepsilon}{2\pi}\right) - \frac{\beta\varepsilon}{384\pi^4}\psi^{(4)}\left(\frac{i\beta\varepsilon}{2\pi}\right)\right\}. \tag{54}$$

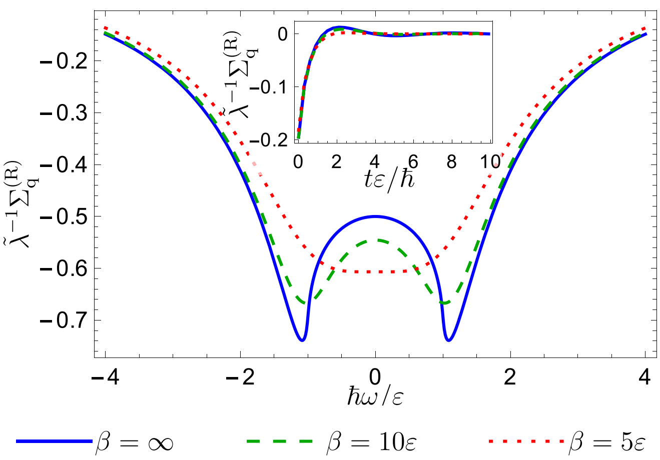

FIG. 5. Function $\Sigma_q^{(R)}(\omega)/\tilde\lambda$ for different values of β; ω is given in units of $\varepsilon/\hbar$. Inset shows the function $\Sigma_q^{(R)}(t)/\tilde\lambda$ for the same values of β; t is given in units of $\hbar/\varepsilon$. In this temperature limit $\Sigma^{(R)} \approx \Sigma_q^{(R)}$.

In the high-temperature limit when $\beta \to 0$, the decay time of the function $\Sigma_q^{(R)}(t)$ is given by $\sqrt{n/m} = \hbar\beta/(2\sqrt{15})$, and its maximal value is $\Sigma_q^{(R)}(0) \simeq -0.144\tilde\lambda\beta\varepsilon\tanh(\beta\varepsilon/2)$.

At low temperatures $\beta \gg \varepsilon$, the function $\Sigma_q^{(R)}(\omega)$ has two symmetric peaks at $\omega \approx \pm\varepsilon$ (see Fig. 5). In the limit $\beta \to \infty$ the function $\Sigma_q^{(R)}(\omega)$ is given by the simple expression (blue line in Fig. 5)

$$\Sigma_q^{(R)}(\omega) = -\frac{\tilde\lambda\varepsilon}{2\omega^2}\left\{\left(1-\frac{\omega}{\varepsilon}\right)\ln\left(\left|1-\frac{\omega}{\varepsilon}\right|\right) + \left(1+\frac{\omega}{\varepsilon}\right)\ln\left(\left|1+\frac{\omega}{\varepsilon}\right|\right)\right\}. \tag{55}$$

The inset in Fig. 5 illustrates that the Fourier transform of the Eq. (55) exposes that the decay rate of the function $\Sigma_q^{(R)}(t)$ is of order ε as well as its damped oscillation frequency.

Our consideration shows that, at a higher temperature, the function $\Sigma_q^{(R)}(\omega)$ vanishes and only the "thermal" term survives. This term has a form of the δ-function, indicating that spin at different times does not correlate in this limit.

We can assume that the regimes of low and high temperatures are separated by the relation $\beta \sim 5\varepsilon$, at which $\partial^2\Sigma^{(R)}(0)/\partial\omega^2 = 0$ (red dotted line in Fig. 5).

Markovian spin dynamics in the high-temperature limit and non-Markovian behavior in the low-temperature limit correlates with the results obtained for a similar system in Ref. [33], where the dynamics of a two-level system coupled to an Ohmic bath was studied.

D. Mathematical aspects of non-Markovian behavior

In this section, we illustrate within the framework of a simple toy model how the results obtained above are reflected in the spin dynamics.

As we have shown in the previous two sections, for many physical systems the interaction with a thermal reservoir is described by the function $\Sigma^{(R)}(t-t')$, whose low-temperature behavior can be qualitatively described by the following expression:

$$\Sigma^{(R)}(t-t') = -a\, e^{-\lambda(t-t')}\sin\left[\omega(t-t')+\alpha\right], \tag{56}$$

where constants $a,\ \lambda > 0$ are determined by the parameters of the original system.

Following standard rules, one can write the quantum Langevin equation with random sources for the spin variable. There is a well-known trick when one substitutes these random sources by pumping at some probe frequency. We do not provide here any details for these calculations, but it is natural to assume that the dynamics of the spin will be determined by $\Sigma^{(R)}(t-t')$ through the equation, as it follows from Eq. (56):

$$s(t) = F(t) - a\int_{-\infty}^{t} e^{-\lambda(t-t')}\sin\left[\omega(t-t')+\alpha\right]s(t')dt', \tag{57}$$

where $F(t)$ is a pump that shifts the system from thermal equilibrium $s=0$. If the pump is constant in time, $F(t) = \mathrm{const}(t) = F_0$, then the steady-state solution reads

$$s_0 = F_0\frac{\lambda^2+\omega^2}{\lambda^2+\omega^2+a\,\omega\cos\alpha+a\lambda\sin\alpha}. \tag{58}$$

Suppose the pump was turned on at $t=-\infty$ so that the system reaches the solution (58) at $t=0$ when we turn off the pump. The system will migrate to a new stationary solution $s=0$ according to

$$s_1(t) = -\frac{a}{\Omega}F_0e^{-kt}\begin{cases}(b-ck)\sinh(\Omega t)+c\,\Omega\cosh(\Omega t) & \text{if } D>0\\ (b-ck)\sin(\Omega t)+c\,\Omega\cos(\Omega t) & \text{if } D<0,\end{cases} \tag{59}$$

where

$$k = \lambda+\frac{a}{2}\sin\alpha,\quad D = \frac{a^2}{4}-\left(\omega+\frac{a}{2}\cos\alpha\right)^2,$$
$$\Omega = \sqrt{|D|},\quad c = \frac{\lambda\sin\alpha+\omega\cos\alpha}{\lambda^2+\omega^2+a\,\omega\cos\alpha+a\lambda\sin\alpha},$$
$$b = \frac{(\lambda^2-\omega^2)\sin\alpha+2\lambda\omega\cos\alpha}{\lambda^2+\omega^2+a\,\omega\cos\alpha+a\lambda\sin\alpha}. \tag{60}$$

This result can be easily obtained by using the Laplace transformation. The expressions show that the oscillating kernel of the integral Eq. (57) does not always lead to an oscillating solution. As follows from the expression (60) for D, the oscillating regime occurs at

$$\omega > \frac{a}{2}(1-\cos\alpha). \tag{61}$$

It is interesting to note that this condition does not include λ.

The kernel (56) becomes monotonically decreasing $\Sigma^{(R)}(t-t') = -a\, e^{-\lambda(t-t')}$ at $\omega = 0$ and $\alpha = \pi/2$. This case relates to a spin at higher temperature when resonance peaks in $\Sigma^{(R)}(\omega)$ disappear (see Fig. 5). The solution (59) turns into

$$s_1(t) = -\frac{a}{\lambda} s_0 e^{-(\lambda+a)t}, \tag{62}$$

where $s_0 = \lambda F_0/(\lambda + a)$ is the steady-state solution under the constant pump F_0, and $t_r = (\lambda + a)^{-1}$ is the relaxation time of the system.

In this case, the integral Eq. (57) can be approximated by a differential one under an extra condition. By differentiating the Eq. (57) with the monotonically decreasing kernel at $t > 0$, i.e., with the pump turned off, we get

$$\frac{\partial s(t)}{\partial t} = -as(t) + a\lambda \int_{-\infty}^{t} e^{-\lambda(t-t')} s(t')dt'. \tag{63}$$

Substituting $s(t)$ for the function (62) turns this equation into an identity. When $\lambda \ll a$, the integral term is small and Eq. (63) transforms into

$$\frac{\partial s(t)}{\partial t} = -as(t). \tag{64}$$

In the opposite case $\lambda \gg a$, the decay time of the integral kernel is short, so the Markov approximation becomes valid

$$\int_{-\infty}^{t} e^{-\lambda(t-t')} s(t')dt' \approx s(t) \int_{-\infty}^{t} e^{-\lambda(t-t')}dt' = -\frac{1}{\lambda} s(t). \tag{65}$$

As a result, we arrive at the equation

$$\frac{\partial s(t)}{\partial t} = -2as(t). \tag{66}$$

Thus, the analysis performed shows that, in a wide range of values of the parameter α, the integral Eq. (57) can be effectively replaced by the differential one. However, this approach does not work when $a \approx \lambda$. In the case of the kernel (56), the situation seems much more complicated and requires further study in more detail.

We note in conclusion that our approach qualitatively reproduces one of the main features of the spin-boson model, namely, the crossover between coherent oscillations and incoherent relaxation [28].

E. Kinetics of the spin

In this section, we consider how the relations obtained correlate with the equations conventionally used in quantum optics in the analysis of two-level systems in an external field.

Let us introduce the notation

$$\tilde{S}^z(\tau) = S^z(\tau) - \langle S^z \rangle_T. \tag{67}$$

We can write the spin Green's function $G_S(\tau - \tau') = \langle \tilde{S}^z(\tau)\tilde{S}^z(\tau')\rangle = \langle S^z(\tau)S^z(\tau')\rangle - \langle S^z \rangle_T^2$ as the sum

$$G_S(\tau - \tau') = G_S^{(0)}(\tau - \tau') + G_S^{(I)}(\tau - \tau'), \tag{68}$$

where $G_S^{(0)}(\tau - \tau')$ is the bare Green's function and $G_S^{(I)}(\tau - \tau')$ is the total correction to the bare Green's function arising due to the interaction with the reservoir. The first term can be easily calculated and is equal to

$$G_S^{(0)}(\tau - \tau') = 1 - 4\big(\langle S^z \rangle_T^{(0)}\big)^2 = \frac{1}{\cosh^2(\beta\varepsilon/2)}. \tag{69}$$

The corresponding Matsubara components are related by

$$G_S(i\omega_n) = G_S^{(0)}(i\omega_n) + \Sigma(i\omega_n). \tag{70}$$

For the inverse Green's function, we can write

$$G_S^{-1}(i\omega_n) = \big[G_S^{(0)}(i\omega_n) + \Sigma(i\omega_n)\big]^{-1} = \big(G_S^{(0)}\big)^{-1}(i\omega_n) - \Sigma_S(i\omega_n), \tag{71}$$

where

$$\Sigma_S = \frac{\Sigma}{G_S^{(0)}\big(G_S^{(0)} + \Sigma\big)} \approx \frac{1}{\big(G_S^{(0)}\big)^2}\Sigma. \tag{72}$$

In Appendix D, we calculate the function Σ in the second order of the perturbation theory for the coupling constant g. An explicit expression (51) is given.

Returning to the time domain, we can write down the fictitious action for the spin quantum field

$$-\frac{A_S}{\hbar} = \int_0^{\hbar\beta} d\tau d\tau' \tilde{S}^z(\tau)\big[\big(G_S^{(0)}\big)^{-1}(\tau - \tau') - \Sigma_S(\tau - \tau')\big] \times \tilde{S}^z(\tau'). \tag{73}$$

We called the action fictitious because it does not allow us to construct a series expansion for the Green's function G_S. By varying the action with respect to the field $\tilde{S}^z(\tau)$ we obtain the equation of motion

$$\tilde{S}^z(\tau) = \cosh^2(\beta\varepsilon/2) \int_0^{\hbar\beta} d\tau' \Sigma_S(\tau - \tau')\tilde{S}^z(\tau'). \tag{74}$$

The corresponding equation for the quantities analytically continued to real time has the form

$$\tilde{S}^z(t) = \cosh^2(\beta\varepsilon/2) \int_{-\infty}^{t} dt' \Sigma_S^{(R)}(t - t')\tilde{S}^z(t') + \text{noise}. \tag{75}$$

Here we wrote the term "noise" to emphasize that the correct equation for the fluctuating quantum field $\tilde{S}^z(t)$ must be of the Langevin Eq. type [26]. Such an equation can be obtained by considering the action (26) on the Schwinger-Keldysh contour. The problem, however, is that at the moment the Keldysh technique for two-time fields $\Phi(t, t')$ has not been developed. On the other hand, we cannot consider the action (73) on the Schwinger-Keldysh contour due to nontrivial statistics of the field $\tilde{S}^z$. We only note that, according to the fluctuation-dissipation theorem, there is the relation between the Keldysh Green's function $\Sigma_S^{(K)}(t, t')$ and the retarded Green's function $\Sigma_S^{(R)}(t, t')$,

$$\Sigma_S^{(K)}(t, t') = \langle S^z \rangle_T \Sigma_S^{(R)}(t, t'). \tag{76}$$

It is interesting that the Eq. (75) obtained for the spin is integral, and not integrodifferential, as for many other systems. It suggests that the spin does not have its own dynamics. Without interaction with the reservoir, the spin retains its initial value, unlike, say, photons, which obey the Maxwell equations for vacuum, i.e., have their own dynamics. The spin dynamics is governed by the nonlocal in time integral term in (75), which is consistent with the hysteresis behavior inherent in magnetism. Nonlocality in time means the non-Markovian nature of the system behavior, and the absence of intrinsic dynamics makes this behavior substantially non-Markovian, i.e., not reducible to the Markovian by dropping small terms.

From the technical point of view, this result is a consequence of the fact that the function $(G_S^{(0)})^{-1}(\tau-\tau')$ in the action (73) does not have the form of the differential operator. The same bare Green's function arises when applying SDT to Ising and Heisenberg models [10–12]. We conclude that the approaches developed in the theory of strongly correlated systems lead to the integral equations of motion, and hence to essentially non-Markovian behavior of spin systems. This conclusion contradicts the form of equations conventionally used in considering quantum two-level systems coupled to a thermal reservoir. The rest of the section is devoted to a discussion of how to overcome this contradiction.

Note that a similar equation of the spin dynamics was obtained in Ref. [34] for the spin-boson model with the linear coupling between the thermal reservoir and the longitudinal component of the spin.

We now differentiate Eq. (75):

$$\frac{1}{\cosh^2(\beta\varepsilon/2)}\frac{\partial\langle\tilde{S}^z(t)\rangle}{\partial t}=\Sigma_S^{(\mathrm{R})}(0)\langle\tilde{S}^z(t)\rangle+\int_{-\infty}^{t}dt'\frac{\partial\Sigma_S^{(\mathrm{R})}(t-t')}{\partial t}\langle\tilde{S}^z(t')\rangle. \tag{77}$$

As the previous consideration of the model problem shows, if $t_s^{-1}\ll\Sigma_S^{(\mathrm{R})}(0)$, where t_s is the decay time of the function $\Sigma_S^{(\mathrm{R})}(t)$, we can neglect the integral term and, substituting $-\cosh^2(\beta\varepsilon/2)\Sigma_S^{(\mathrm{R})}(0)$ by γ, we obtain the equation

$$\frac{\partial\langle\tilde{S}^z\rangle}{\partial t}=-\gamma\langle\tilde{S}^z\rangle. \tag{78}$$

This formula can be rewritten as a typical equation for two-level systems in an external field

$$\frac{\partial\langle S^z\rangle}{\partial t}=-\gamma\langle S^z\rangle-\kappa. \tag{79}$$

Here, $\kappa=-\gamma\langle S^z\rangle_T$. The Eq. (78) can be associated with the equation for the Green's function,

$$\left(\frac{\partial}{\partial t}+\gamma\right)G_s(t,t')=\delta(t-t'). \tag{80}$$

For the frequency dependence of the imaginary part of the retarded Green's function we obtain in the standard way

$$\mathrm{Im}G_s^{(\mathrm{R})}(\omega)=\frac{\gamma}{\omega^2+\gamma^2}. \tag{81}$$

The Green's function $\langle\tilde{S}^z(t)\tilde{S}^z(t')\rangle(\omega)$ for the isolated spin has the form

$$\langle\tilde{S}^z(t)\tilde{S}^z(t')\rangle(\omega)=\left(1-4\langle S^z\rangle_T^2\right)\pi\delta(\omega). \tag{82}$$

In the case when the spin is coupled with the reservoir, this function also obeys Eq. (78), so for the retarded Green's function we get

$$\mathrm{Im}\langle\tilde{S}^z(t)\tilde{S}^z(t')\rangle^{(\mathrm{R})}(\omega)=\langle\{\tilde{S}^z(t),\tilde{S}^z(t')\}\rangle(\omega)=\left(1-4\langle S^z\rangle_T^2\right)\frac{\gamma}{\omega^2+\gamma^2}, \tag{83}$$

the relation turning into Eq. (82) in the limit $\gamma\to 0$. This expression can be rewritten in the form

$$2\langle\{S^z(t),S^z(t')\}\rangle(\omega)=4\langle S^z\rangle_T^2 2\pi\delta(\omega)+\left(1-4\langle S^z\rangle_T^2\right)\frac{2\gamma}{\omega^2+\gamma^2}. \tag{84}$$

The obtained relation reproduces the known Bloch-Redfield result [35,36]. Bloch and Redfield got it for an Ohmic reservoir; however, our consideration shows that the relation is universal in the low-frequency limit. Only the coefficient γ depends on the type of reservoir. As we see, in our derivation, the Bloch-Redfield formula is an elementary consequence of Eq. (79), used in the theory of two-level systems, and the expression (69) for the bare spin Green's function used in the theory of magnetism. Thus, the combination of the two theories made it possible to easily obtain this classical result.

In conclusion, we note that, for the coefficient γ, the value $(\tilde{\lambda}\varepsilon/2)\coth(\beta\varepsilon/2)$ is most often used. This value can be obtained if we make two assumptions [8]: (1) the interaction between the system and the reservoir does not depend on the state of the system, and (2) the system interacts only with the resonant mode of the reservoir, $\omega_0=\varepsilon$. As the discussion in Sec. III C shows, both of these assumptions are far from reality.

IV. DISCUSSION

We proposed a different approach to the analysis of open quantum two-level systems by using the Popov-Fedotov substitution. With the help of the Hubbard-Stratonovich transformation, we introduced a two-time complex quantum field. We showed how the average spin and its correlators are related to it. This approach, at least in the spin-boson model that we have analyzed, does not lead to the appearance of artifacts and ambiguities, in contrast to the approach using Majorana fermions [22].

We showed how to modify the standard diagram technique for fermions to obtain SDT.

A possible transition to equations conventionally used in quantum optics is discussed.

We obtained the Bloch-Redfield formula for the spin correlator in the bosonic reservoir as an easy consequence of the combination of two theories, the quantum-optical theory of two-level systems and the theory of magnetism. Also, we showed that this formula is universal for the entire class of bosonic reservoirs, and the type of reservoir determines only the damping rate present in the formula.

We examined a spin coupled to an Ohmic reservoir as an example described by the spin-boson model. It was found that the spin dynamics should be considered as nonlocal at times of scale $t\ll\hbar\beta$, which is 10^{-11} s for helium temperatures. With the advent of experimental femtosecond spectroscopy, consideration of quantum systems at such times is relevant [1]. We showed that, at such times, the spin dynamics in the bosonic reservoir is purely non-Markovian, i.e., the current state of the spin is completely determined by the history of its states, and the longer is the history, the lower is the temperature of the system.

The developed technique allows us to consider other systems. For example, the Ising model, where in addition to the interspin interaction there is the interaction of each spin with the phonon subsystem. Another example is an array of superconducting qubits connected to a resonator, and one goes to take into account both the interaction of qubits through the resonator and the direct interaction of qubits (so-called cross-talk). Further presentations will be devoted to these questions.

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APPENDIX A

To calculate the average spin, we apply the relation (32) with the action $\mathcal{A}[\eta]$ from Eq. (31), where for $\mathcal{A}$ we take Eq. (37). As a result, we again obtain the expression (37) with the only difference being that now $\mathbf{\Phi}$ is meant by

$$\mathbf{\Phi}_{\tau\tau'} = \begin{bmatrix} -\bar{\Phi}(\tau,\tau') - \eta(\tau,\tau')/2 & 0 \\ 0 & \Phi(\tau,\tau') + \eta(\tau,\tau')/2 \end{bmatrix}, \tag{A1}$$

where $\eta(\tau,\tau') = \eta(\tau)\delta(\tau-\tau')$. We perform a shift in the variables

$$\Phi(\tau,\tau') + \eta(\tau,\tau')/2 \to \Phi(\tau,\tau'), \quad \bar{\Phi}(\tau,\tau') + \eta(\tau,\tau')/2 \to \bar{\Phi}(\tau,\tau'), \tag{A2}$$

and come to the action

$$-\frac{\mathcal{A}_{\text{eff}}[\eta]}{\hbar} = \int_0^{\hbar\beta} d\tau d\tau' \frac{\hbar^2}{g^2} \frac{1}{\Pi(\tau-\tau')} \left[\Phi(\tau,\tau')\bar{\Phi}(\tau',\tau) + \frac{\eta(\tau,\tau')}{2}(\Phi(\tau,\tau') + \bar{\Phi}(\tau',\tau)) - \frac{\eta^2(\tau,\tau')}{4} \right] + \ln\det\left(((\mathbf{G}^{(0)})^{-1} - \mathbf{\Phi})\mathbf{G}^{(0)}\right), \tag{A3}$$

where $\mathbf{\Phi}$ is meant by the initial matrix (38).

Up to second-order terms in η, we can write

$$e^{-\mathcal{A}_{\text{eff}}[\eta]/\hbar} = e^{-\mathcal{A}_{\text{eff}}/\hbar} \Bigg\{ 1 + \frac{\hbar^2}{g^2} \int_0^{\hbar\beta} d\tau_1 d\tau_2 \frac{1}{\Pi(\tau_1-\tau_2)} \left[\frac{\eta(\tau_1,\tau_2)}{2} [\Phi(\tau_1,\tau_2) + \bar{\Phi}(\tau_2,\tau_1)] - \frac{\eta^2(\tau_1,\tau_2)}{4} \right] + \frac{\hbar^4}{2g^4} \int_0^{\hbar\beta} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \frac{1}{\Pi(\tau_1-\tau_2)\Pi(\tau_3-\tau_4)} \frac{\eta(\tau_1,\tau_2)\,\eta(\tau_3,\tau_4)}{4} \times [\Phi(\tau_1,\tau_2) + \bar{\Phi}(\tau_2,\tau_1)][\Phi(\tau_3,\tau_4) + \bar{\Phi}(\tau_4,\tau_3)] \Bigg\}. \tag{A4}$$

Using this expansion, we obtain for the average spin the expression (40). The relation (42) is obtained in a similar way.

APPENDIX B

We obtain the bare Green's function, leaving in the action $\mathcal{A}_{\text{eff}}^{(\text{II})}$ only the term $\mathcal{A}^{(0)}$:

$$\langle \phi^z(\tau_1,\tau_2)\bar{\phi}^z(\tau_3,\tau_4)\rangle^{(0)} = \frac{\delta^2}{\delta\bar{J}(\tau_1,\tau_2)\delta J(\tau_3,\tau_4)} \frac{\int D\Xi e^{-\int_0^{\hbar\beta} d\tau d\tau' \left[\bar{\phi}^z(\tau,\tau')\frac{\hbar^2}{g^2\Pi(\tau-\tau')}\phi^z(\tau',\tau) + \bar{J}(\tau,\tau')\phi^z(\tau,\tau') + J(\tau,\tau')\bar{\phi}^z(\tau,\tau')\right]}}{\int D\Xi e^{-\mathcal{A}^{(0)}/\hbar}}\Bigg|_{J,\bar{J}=0} = \frac{\delta^2}{\delta J(\tau_1,\tau_2)\delta J(\tau_3,\tau_4)} \frac{\int D\Xi e^{\int_0^{\hbar\beta} d\tau d\tau' \bar{J}(\tau,\tau')\frac{g^2\Pi(\tau-\tau')}{\hbar^2}J(\tau',\tau)}}{\int D\Xi e^{-\mathcal{A}^{(0)}/\hbar}}\Bigg|_{J,\bar{J}=0} = \frac{g^2\Pi(\tau_1-\tau_2)}{\hbar^2}\delta(\tau_1-\tau_4)\delta(\tau_2-\tau_3). \tag{B1}$$

Similarly, we obtain

$$\begin{aligned} \langle \bar{\phi}^z(\tau_1,\tau_2)\phi^z(\tau_3,\tau_4)\rangle^{(0)} &= \langle \phi^z(\tau_1,\tau_2)\bar{\phi}^z(\tau_3,\tau_4)\rangle^{(0)}, \\ \langle \phi^z(\tau_1,\tau_2)\phi^z(\tau_3,\tau_4)\rangle^{(0)} &= \langle \bar{\phi}^z(\tau_1,\tau_2)\bar{\phi}^z(\tau_3,\tau_4)\rangle^{(0)} = 0. \end{aligned} \tag{B2}$$

We find corrections to the bare Green's function by representing $e^{-\mathcal{A}_{\rm int}/\hbar}$ as the series expansion

$$e^{-\mathcal{A}_{\rm int}/\hbar} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (\mathcal{A}_{\rm int}/\hbar)^k \tag{B3}$$

and replacing $\mathcal{A}_{\rm int}$ by

$$\mathcal{A}_{\rm int}\left[\frac{\delta}{\delta J}, \frac{\delta}{\delta \bar{J}}\right] = \hbar \int_0^{\hbar\beta} d\tau_1 d\tau_2 d\tau_3 d\tau_4 \left\{ \frac{\delta}{\delta \bar{J}(\tau_1, \tau_2)} \frac{1}{2} G(\tau_2 - \tau_2) G(\tau_4 - \tau_1) \frac{\delta}{\delta \bar{J}(\tau_3, \tau_4)} + \frac{\delta}{\delta J(\tau_1, \tau_2)} \frac{1}{2} \bar{G}(\tau_3 - \tau_2) \bar{G}(\tau_1 - \tau_4) \frac{\delta}{\delta J(\tau_3, \tau_4)} \right\}. \tag{B4}$$

For example,

$$\langle \phi^z(\tau_1, \tau_2) \phi^z(\tau_3, \tau_4) \rangle^{(1)} = -\frac{\delta^2}{\delta J_{12} \delta J_{34}} \frac{\int D\Xi \int_0^{\hbar\beta} d\tau_a d\tau_b d\tau_c d\tau_d \left\{ \frac{\delta}{\delta \bar{J}_{ab}} \frac{1}{2} G_{b-c} G_{d-a} \frac{\delta}{\delta \bar{J}_{cd}} + \frac{\delta}{\delta J_{ab}} \frac{1}{2} \bar{G}_{c-b} \bar{G}_{a-d} \frac{\delta}{\delta J_{cd}} \right\} e^{\int_0^{\hbar\beta} d\tau_5 d\tau_6 J_{56} \frac{g^2 \Pi_{5-6}}{\hbar^2} J_{65}}}{\int D\Xi e^{-\mathcal{A}_{\rm eff}^{(\rm II)}/\hbar}} \Bigg|_{J,\bar{J}=0} = -\frac{g^4}{\hbar^4} \Pi(\tau_2 - \tau_1) \Pi(\tau_4 - \tau_3) G(\tau_1 - \tau_4) G(\tau_3 - \tau_2). \tag{B5}$$

Here, for brevity, we replaced time arguments with indexes. Similarly,

$$\langle \bar{\phi}^z(\tau_1, \tau_2) \bar{\phi}^z(\tau_3, \tau_4) \rangle^{(1)} = -\frac{g^4}{\hbar^4} \Pi(\tau_2 - \tau_1) \Pi(\tau_4 - \tau_3) \bar{G}(\tau_4 - \tau_1) \bar{G}(\tau_2 - \tau_3),$$
$$\langle \phi^z(\tau_1, \tau_2) \bar{\phi}^z(\tau_3, \tau_4) \rangle^{(1)} = \langle \bar{\phi}^z(\tau_1, \tau_2) \phi^z(\tau_3, \tau_4) \rangle^{(1)} = 0. \tag{B6}$$

APPENDIX C

We take the action (37) and in the expansion of the logarithm (29) we leave for simplicity the first two terms. For brevity, we write the action in the matrix form

$$\frac{\mathcal{A}}{\hbar} = \mathrm{tr}\left\{ \frac{\bar{\Phi}^z \Phi^z}{\Pi} + (\bar{G}\Phi^z + \bar{\Phi}^z G) + \frac{1}{2}(\bar{\Phi}^z G \bar{\Phi}^z G + \bar{G}\Phi^z \bar{G}\Phi^z) \right\}. \tag{C1}$$

We included the factor $g^2/\hbar^2$ in Π. To exclude the linear terms, we perform the field shift by introducing the field $\Phi_1^z = \Phi^z + \Pi G$. The action, up to the members independent of Φ^z and $\bar{\Phi}^z$, takes the form

$$\mathrm{tr}\left\{ \frac{\bar{\Phi}_1^z \Phi_1^z}{\Pi} - \left(\bar{G}\Pi G \bar{G} \Phi_1^z + \bar{\Phi}_1^z G \bar{G} \Pi G\right) + \frac{1}{2}\left(\bar{\Phi}_1^z G \bar{\Phi}_1^z G + \bar{G}\Phi_1^z \bar{G}\Phi_1^z\right) \right\}. \tag{C2}$$

As we see, this expression reproduces the structure of the action (C1) up to the form of the linear terms. There is the factor Π in the linear terms; therefore, they have the order of smallness g^2 with respect to the original linear terms in Eq. (C1). We can continue the iteration and go to the field $\Phi_2^z = \Phi_1^z + \Pi G \bar{G} \Pi G$ and so on. In the limit of an infinite number of iterations, linear terms disappear and we obtain a series in powers of g^2 for the equilibrium spin $\langle S^z \rangle_T = \mathrm{tr}\{G + \bar{G} + G\bar{G}\Pi G + \bar{G}\Pi G \bar{G} + \cdots\}$, shown in Fig. 1(c). However, this series is incomplete. To obtain a complete series, one has to take into account other terms in the expansion (29), in addition to the first two.

APPENDIX D

In the second order of the perturbation theory with respect to the coupling constant g, the function $\Sigma^{(\rm R)}(\omega)$ is given by the two diagrams depicted in Fig. 6. We calculate here only the diagram a. Direct calculations show that the diagram b has the same magnitude.

The corresponding to diagram a expression is

$$\Sigma(i\omega, x) = \frac{g^2}{(\hbar\beta)^2} \sum_{(\omega_0, \omega_1)} \Pi(i\omega_0, x) G^{(0)}(i\omega_1) G^{(0)}(i\omega + i\omega_1) \times \bar{G}^{(0)}(i\omega + i\omega_1 - i\omega_0) \bar{G}^{(0)}(i\omega_1 - i\omega_0). \tag{D1}$$

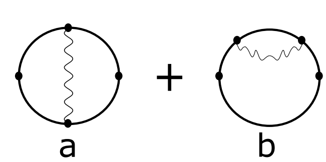

FIG. 6. Diagrammatic representation of the function $\Sigma^{(\rm R)}(\omega)$ in the second order of the perturbation theory with respect to the coupling constant g.

Using the relations (13) and (34), we arrive at the expression

$$\Sigma(i\omega, x) = \frac{g^2}{(\hbar\beta)^2} \sum_{(\omega_0,\omega_1)} \frac{1}{i\omega_0 - x} \frac{1}{i\omega_1 + (-\varepsilon/2 + \mu)/\hbar} \frac{1}{i\omega + i\omega_1 + (-\varepsilon/2 + \mu)/\hbar} \times \frac{1}{i\omega + i\omega_1 - i\omega_0 + (\varepsilon/2 + \mu)/\hbar} \frac{1}{i\omega_1 - i\omega_0 + (\varepsilon/2 + \mu)/\hbar}. \tag{D2}$$

We first perform the summation over fermionic frequencies ω_1. Decomposing fractions and summing up the resulting expression according to the formula (12), we arrive at the result

$$\Sigma(i\omega, x) = \frac{g^2}{\hbar\beta} \sum_{(\omega_0,\omega_1)} \frac{1}{i\omega_0 - x} \frac{1}{(i\omega_0 - \varepsilon)^2} \left(\frac{1}{i\omega_0 + i\omega - \varepsilon} + \frac{1}{i\omega_0 - i\omega - \varepsilon} \right) (n_a - n_b). \tag{D3}$$

First we find the sum

$$\sum_{\omega_0} \frac{1}{i\omega_0 - x} \frac{1}{(i\omega_0 - \varepsilon)^2} \frac{1}{i\omega_0 + i\omega - \varepsilon}. \tag{D4}$$

Decomposing fractions and summing over bosonic frequencies ω_0, we get

$$\begin{aligned} &-\frac{1}{i\omega} \frac{1}{x - \varepsilon} \left\{ \left(\frac{1}{i\omega + x - \varepsilon} - \frac{1}{x - \varepsilon} \right) [n_B(x) - n_B(\varepsilon)] + \sum_{\omega_0} \frac{1}{(i\omega_0 - \varepsilon)^2} \right\} \\ &= -\frac{1}{i\omega} \frac{1}{x - \varepsilon} \left\{ \left(\frac{1}{i\omega + x - \varepsilon} - \frac{1}{x - \varepsilon} \right) [n_B(x) - n_B(\varepsilon)] + \frac{1}{4\sinh^2(\beta\varepsilon/2)} \right\} \\ &= \frac{1}{(x - \varepsilon)^2} \frac{n_B(x) - n_B(\varepsilon)}{i\omega + x - \varepsilon} - \frac{1}{i\omega} \frac{1}{x - \varepsilon} \frac{1}{4\sinh^2(\beta\varepsilon/2)}. \end{aligned} \tag{D5}$$

To find the full sum, it is enough to add the same expression with the changed sign of ω. Finally, we get

$$\Sigma(i\omega, x) = g^2 \frac{1}{(x - \varepsilon)^2} \left(\frac{1}{i\omega + x - \varepsilon} - \frac{1}{i\omega - x + \varepsilon} \right) [n_B(x) - n_B(\varepsilon)](n_a - n_b). \tag{D6}$$

We obtain the retarded function $\Sigma^{(R)}(\omega, x)$, analytically continuing the expression (D6) on the real axis $i\omega \to \omega + i\gamma$. Here we introduced some effective dumping γ which relates to the reservoir:

$$\Sigma^{(R)}(\omega, x) = g^2 \frac{1}{(x - \varepsilon)^2} \left(\frac{1}{\omega + x - \varepsilon + i\gamma} - \frac{1}{\omega - x + \varepsilon + i\gamma} \right) [n_B(x) - n_B(\varepsilon)](n_a - n_b). \tag{D7}$$

As it was discussed in Sec. III B, we have to add the Hermitian conjugate diagram and obtain a real expression. Integrating it over the frequencies x of the Ohmic reservoir, we obtain the result

$$\Sigma^{(R)}(\omega) = \int_{-\infty}^{\infty} \frac{dx}{\pi} \rho(|x|)\mathrm{sgn}(x)\Sigma^{(R)}(\omega, x) = \int_{-\infty}^{\infty} \frac{dx}{\pi} \lambda_0 x \Sigma^{(R)}(\omega, x) = \Sigma_{\mathrm{q}}^{(R)}(\omega) + \Sigma_{\mathrm{T}}^{(R)}(\omega), \tag{D8}$$

$$\Sigma_{\mathrm{T}}^{(R)}(\omega) = \mathrm{Re}\left\{ \frac{\tilde{\lambda}}{(\hbar\omega + i\gamma)^2} \tanh\left(\frac{\beta\varepsilon}{2}\right) \left[i\pi(\varepsilon + \hbar\omega + i\gamma) \left(\coth\frac{\beta(\varepsilon + \hbar\omega + i\gamma)}{2} \tanh\frac{\beta\varepsilon}{2} - 1 \right) \right] \right\}, \tag{D9}$$

$$\Sigma_{\mathrm{q}}^{(R)}(\omega) = \mathrm{Re}\left\{ \frac{\tilde{\lambda}}{(\hbar\omega + i\gamma)^2} \tanh\left(\frac{\beta\varepsilon}{2}\right) \left[2\varepsilon\psi^{(1)}\left(\frac{i\beta\varepsilon}{2\pi}\right) - (i\gamma + \varepsilon + \hbar\omega)\psi^{(1)}\left(\frac{i\beta(\varepsilon + \hbar\omega + i\gamma)}{2\pi}\right) + (i\gamma - \varepsilon + \hbar\omega)\psi^{(1)}\left(\frac{i\beta(\varepsilon - \hbar\omega - i\gamma)}{2\pi}\right) \right] \right\}. \tag{D10}$$

Here $\psi^{(1)}(z)$ is the logarithmic derivative of the Gamma function (52). Typical behavior of $\Sigma_{\mathrm{q}}^{(R)}(\omega)$ and $\Sigma_{\mathrm{T}}^{(R)}(\omega)$ is shown in Fig. 7. The "thermal" term labeled as $\Sigma_{\mathrm{T}}^{(R)}(\omega)$ results from the residue of the expression in braces in the right-hand side of Eq. (D7). As one can see, $x = \omega$ is removable singularity and does not contribute to the integral. This term has form of a sharp peak with the width γ located at $\omega = 0$ and in the limit of high temperature this term reads

$$\Sigma_{\mathrm{T}}^{(R)}(\omega) \approx -\mathrm{Re}\frac{\pi\tilde{\lambda}}{\gamma - i\hbar\omega}, \quad \beta \to 0.$$

It is important to take it into account to ensure the condition $\langle S^z(t)S^z(t')\rangle \to \langle S^z\rangle^2$ as $|t - t'| \to \infty$ and the identity $\langle S^z(t)S^z(t)\rangle = 1/4$ for arbitrary t. This term vanishes at low temperature since, in the low-temperature limit, $\langle S^z\rangle \to -1/2$, and, hence, $\langle S^z(t)S^z(t')\rangle \to \langle S^z\rangle^2$ for any t and t'.

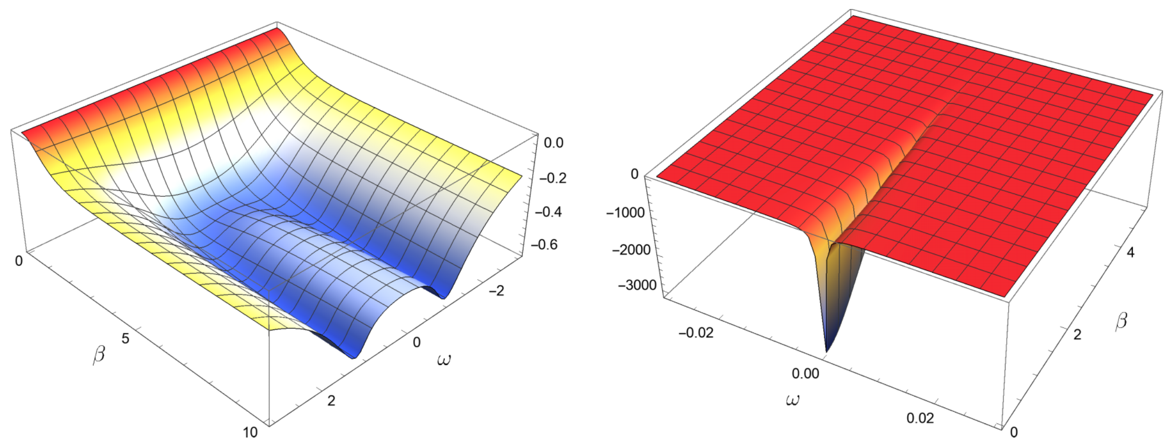

FIG. 7. $\Sigma_q^{(R)}(\omega)/\tilde{\lambda}$ (left panel) and $\Sigma_T^{(R)}(\omega)/\tilde{\lambda}$ (right panel) as functions of ω and β. ω and β are in units of ε. $\gamma = 0.001\varepsilon$.

The "quantum" term labeled as $\Sigma_q^{(R)}(\omega)$ results from poles of $n_B(x)$ and describes quantum correlations between the value of spin at different moments of time. This term disappears in high-temperature limit.

The behavior of $\Sigma_q^{(R)}(\omega)$ for different β is illustrated in Figs. 4 and 5. The function $\Sigma_q^{(R)}(t-t')$ can be obtained by the Fourier transformation.

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