Magnetic-moment probability distribution of a quantum charged particle in thermodynamic equilibrium

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We consider a quantum charged particle moving in the xy plane under the action of a uniform perpendicular constant magnetic field, in the presence of a parabolic binding potential. The magnetic-moment operator has a continuous spectrum, and its eigenfunctions in the momentum representation are expressed in terms of modified Bessel functions. The probability distribution of the magnetic moment in the thermodynamic equilibrium state is calculated. At zero temperature, it has a simple exponential form in the diamagnetic region, with a sharp jump to zero at the origin. With an increase of temperature, a paramagnetic wing of the distribution becomes more and more pronounced. In the high-temperature regime, the diamagnetic and paramagnetic wings of the distribution have almost identical forms, described with a high precision by simple exponential functions with very large extensions. Therefore, diamagnetic and paramagnetic contributions almost cancel each other. The remaining non-zero diamagnetic value is due to a small asymmetry of the distribution nearby the origin, where some nonexponential fine structure is observed. The total width of the distribution function strongly depends on the strength of the binding potential. Strong fluctuations of the magnetic moment (described in terms of the variance) are discovered in all temperature regimes.

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I. INTRODUCTION

The magnetic properties of (quasi-)free-charged particles in the thermodynamic equilibrium state have attracted the attention of many researchers for more than a century, since Bohr understood that the diamagnetic effect due to the circular motion of classical particles can be totally compensated by the surface currents [1]. A more rigorous proof of the absence of magnetism in classical thermodynamically equilibrium states was given in the frameworks of the Hamiltonian mechanics in the famous work by van Leeuwen [2]. The first calculations resulting in a nonzero mean magnetic moment in the quantum equilibrium state of a free charged particle in a homogeneous magnetic field were performed in the famous paper by Landau [3]. However, he had used some "trick," in order to obtain the finite value of the statistical sum, despite the infinite degeneracy of the quantized energy levels. Also, he had confined himself with the case of a weak magnetic field. A more rigorous derivation was given by Darwin [4], who considered the charged particle confined by an isotropic harmonic potential. Then, the infinite degeneracy is removed and the motion of the particle becomes finite, so that the statistical sum can be calculated without problems. After this has been done, the limit transition to zero frequency of the binding potential yields the following formula for the mean magnetic moment of a spinless particle with mass M and charge e in the homogeneous magnetic field B (we use the Gauss system of units),

$$\mathcal{M} = \mu\left[(\mu B\beta)^{-1} - \coth(\mu B\beta)\right], \tag{1}$$

where $\mu = e\hbar/(2Mc)$ is the Bohr magneton and β the inverse temperature. In particular, $\mathcal{M} = -\mu$ in the low-temperature limit $\mu B\beta \gg 1$, whereas $\mathcal{M} = -\mu^2 B\beta/3$ in the high-temperature limit $\mu B\beta \ll 1$, which was the result obtained by Landau.

The Landau-Darwin problem continued to attract the attention of numerous researchers during subsequent decades. For example, Sondheimer and Wilson [5] showed how to calculate the equilibrium statistical sum without any previous knowledge of the energy spectrum, using the density matrix formalism. The Wigner function was used for this purpose in Refs. [6,7]. Feldman and Kahn [8] derived formula (1), using the continuous basis of coherent states in a uniform magnetic field, introduced by Malkin and Man'ko [9]. Other applications of coherent states in a uniform magnetic field were considered in Ref. [10]. For further developments of the model of free electrons, bounded by a harmonic potential and obeying the Fermi-Dirac statistics, one can consult, e.g., Refs. [11–13] and references therein. The relation between the Fisher information and Landau diamagnetism was established in Ref. [14]. Recently, a comparison of classical and quantum approaches was made in Refs. [15,16].

The authors of the cited papers calculated the mean magnetization only, having in mind extrapolations to macroscopic species, where fluctuations of thermodynamic quantities can be neglected, as soon as the relative fluctuations of additive quantities decay as $N^{-1/2}$ in ideal gases consisting of N particles. However, fluctuations can be rather strong in

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mesoscopic systems. This fact was noticed and investigated, e.g., in Refs. [11,17–19].

We consider a single-particle system, having in mind that the properties of such systems can be verified nowadays experimentally in various electron or ion traps. Our initial goal was to find an answer to the following question: Why is the mean magnetic moment so small in the high-temperature regime? The known thermodynamical answer is as follows. At zero temperature, when the quantum system is pure, the mean magnetic moment equals $\mathcal{M} = -\partial E/\partial B$, where E is the mean energy. Hence, the lowest Landau energy μB yields $\mathcal{M} = -\mu$. But the high-temperature quantum state is highly mixed, with a huge entropy. In this case, one must calculate the derivative over the magnetic field not of the energy E, but of the *free energy* $F = E - S/\beta$, so that the big value of the product of entropy S by the absolute temperature β^{-1} almost cancels the contribution of the total energy E.

In the following sections, we show the existence of another answer, which can be considered as complementary to the known ones. Namely, we demonstrate that the equilibrium state is characterized by a wide magnetic-moment probability distribution. This distribution is asymmetric with respect to zero value in the low-temperature case. However, it becomes extremely wide and almost symmetric in the high-temperature regime. Therefore, the mean value of the magnetic moment is small in the high-temperature case, because positive and negative values of the moment almost compensate each other. However, the fluctuations of the magnetic moment of a single particle are very strong. This is the second main result of our study.

II. EIGENSTATES OF THE HAMILTONIAN AND MAGNETIC-MOMENT OPERATOR

To avoid problems with the normalization and degeneracy of the energy levels, we assume, following Darwin [4], that the particle motion is confined by means of the isotropic harmonic potential $V(x, y) = Mg^2(x^2 + y^2)/2$. Therefore, in the presence of a magnetic field, directed along the z axis, the motion in the xy plane is governed by the Hamiltonian

$$\hat{H} = \hat{\boldsymbol{\pi}}^2/(2M) + Mg^2\left(x^2 + y^2\right)/2, \quad \boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}/c, \tag{2}$$

where $\boldsymbol{A}(x, y)$ is the vector potential of the magnetic field. We discard the motion along the magnetic field vector, since this motion is independent from the motion in the xy plane in the nonrelativistic approximation. In addition, we discard the effects of spin, since they are also independent from the orbital motion effects within the same approximation.

A. Energy spectrum and wave functions

The stationary Schrödinger equation $\hat{H}\psi = E\psi$ with Hamiltonian (2) and $\boldsymbol{A} = B(-y, x)/2$ was solved in polar coordinates by Fock [20],

$$\tilde{\psi}_{n_r m}(r, \varphi) = \sqrt{\frac{\tilde{\kappa}_g n_r!}{\pi(n_r + |m|)!}}\left(\tilde{\kappa}_g r^2\right)^{|m|/2} L_{n_r}^{(|m|)}\left(\tilde{\kappa}_g r^2\right) \times \exp\left(-\tilde{\kappa}_g r^2/2 + im\varphi\right), \tag{3}$$

$$E_{n_r m} = \hbar\omega_g(1 + |m| + 2n_r) - \hbar\omega m, \tag{4}$$

where $m = 0, \pm 1, \pm 2, \ldots, n_r = 0, 1, 2, \ldots,$

$$\omega_g = \sqrt{\omega^2 + g^2}, \quad \tilde{\kappa}_g = M\omega_g/\hbar.$$

Function $L_n^{(\alpha)}(z)$ is the generalized Laguerre polynomial, defined as [21,22]

$$L_n^{(\alpha)}(z) = \frac{1}{n!}e^z z^{-\alpha}\frac{d^n}{dz^n}(e^{-z}z^{\alpha+n}).$$

We shall need the energy eigenfunctions in the momentum space. The two-dimensional Fourier transform of function (3) can be calculated with the aid of the known integral representation of the Bessel function [21,22],

$$\int_0^{2\pi} dx \exp[-iz\cos(x) + imx] = (-i)^{|m|}2\pi J_{|m|}(z),$$

and the integral (7.422.4) from Ref. [22],

$$\int_0^{\infty} x^{\nu+1}e^{-\beta x^2}L_n^{(\nu)}(\alpha x^2)J_\nu(xy) = \frac{(\beta-\alpha)^n y^\nu}{(2\beta)^{\nu+1}\beta^n}e^{-y^2/(4\beta)}L_n^{(\nu)}\left(\frac{\alpha y^2}{4\beta(\alpha-\beta)}\right). \tag{5}$$

The result is similar to (3), except for the phase factor,

$$\psi_{n_r m}(p, \varphi) = (-i)^{|m|}(-1)^{n_r}\sqrt{\frac{\kappa_g n_r!}{\pi(n_r + |m|)!}}\left(\kappa_g p^2\right)^{|m|/2} \times L_{n_r}^{(|m|)}\left(\kappa_g p^2\right)\exp\left(-\kappa_g p^2/2 + im\varphi\right), \tag{6}$$

where $\kappa_g = (M\hbar\omega_g)^{-1}$ and $p = \sqrt{p_x^2 + p_y^2}$.

B. Eigenstates of the magnetic-moment operator

To introduce the magnetic-moment operator, we use the definition of the classical magnetic moment [23,24]

$$\mathbf{M} = \frac{1}{2c}\int dV[\mathbf{r} \times \mathbf{j}]. \tag{7}$$

Then, using the expression for the quantum probability current density in the presence of a magnetic field,

$$\mathbf{j} = ie\hbar(\psi\nabla\psi^* - \psi^*\nabla\psi)/(2m) - e^2\mathbf{A}\psi^*\psi/(mc), \tag{8}$$

one can write the right-hand side of (7) as the mean value of the operator

$$\hat{\mathcal{M}} = (\hat{x}\hat{\pi}_y - \hat{y}\hat{\pi}_x)e/(2mc). \tag{9}$$

This form of the magnetic-moment operator was justified from different points of view in Refs. [7,11,25–30]. Using the "circular" gauge of the vector potential, $\boldsymbol{A} = B(-y, x)/2$, we can write

$$\hat{\mathcal{M}} = \left[\hat{x}\hat{p}_y - \hat{y}\hat{p}_x - M\omega\left(\hat{x}^2 + \hat{y}^2\right)\right]e/(2mc) \equiv \mu\hat{\Lambda}. \tag{10}$$

Here, $\omega = eB/(2Mc)$ is the Larmor frequency, so that $\mu B = \hbar\omega$. We assume that the direction of the magnetic field is chosen in such a way that $\omega > 0$. The spectrum of the dimensionless operator $\hat{\Lambda}$ is continuous. In order to avoid troubles with delta functions in the coordinate representation, it seems reasonable to look for eigenfunctions of $\hat{\Lambda}$ in the momentum representation. Using the polar coordinates in the momentum plane, $p_x = p\cos(\varphi)$, $p_y = p\sin(\varphi)$, we arrive at the equation

$$\left[-i\frac{\partial}{\partial\varphi} + M\hbar\omega\left(\frac{\partial^2}{\partial p^2} + \frac{1}{p}\frac{\partial}{\partial p} + \frac{1}{p^2}\frac{\partial^2}{\partial\varphi^2}\right)\right]\psi = \Lambda\psi,$$

whose solutions are obviously expressed in terms of the Bessel functions [21,22],

$$\Psi_{\Lambda,m}(p,\varphi) = (\kappa/4\pi)^{1/2}e^{im\varphi}J_{|m|}(\gamma_m p), \tag{11}$$

$$m = 0, \pm1, \pm2, \dots, \quad \gamma_m = \sqrt{\kappa(m-\Lambda)}, \quad \kappa = (M\hbar\omega)^{-1}.$$

For each value of the discrete parameter m, the continuous parameter Λ must satisfy the restriction $-\infty < \Lambda \leqslant m$. Otherwise, the Bessel function will turn into the modified Bessel function, which grows exponentially at $p \to \infty$, so that it cannot be normalized. Functions (11) satisfy the necessary continuous normalization condition

$$\int_0^{2\pi} d\varphi \int_0^\infty p\,dp\,\Psi^*_{\Lambda,m}\Psi_{\Lambda',m'} = \delta_{mm'}\delta(\Lambda-\Lambda'), \tag{12}$$

which can be verified (following the standard quantum mechanical scheme [31]) with the aid of the known asymptotics of the Bessel function [21,22],

$$J_m(\gamma p) \approx (\pi\gamma p/2)^{-1/2}\cos(\gamma p - m\pi/2 - \pi/4), \quad p \to \infty,$$

and the known formulas for the delta function,

$$\int_{-\infty}^{\infty}\exp[i(\gamma-\gamma')p]dp = 2\pi\delta(\gamma-\gamma'),$$

$$\delta(ax) = |a|^{-1}\delta(x).$$

III. MAGNETIC-MOMENT PROBABILITY DISTRIBUTION IN THE THERMAL STATES

A. The case of zero temperature

At zero temperature, the particle is in the pure ground state with $n_r = m = 0$, described by the wave function $\tilde{\psi}_{00}(r) = \sqrt{\tilde{\kappa}_g/\pi}\exp(-\tilde{\kappa}_g r^2/2)$. The corresponding wave function in the momentum representation reads $\psi_{00}(p) = \sqrt{\kappa_g/\pi}\exp(-\kappa_g p^2/2)$. The only nonzero projection of this state on the magnetic-momentum eigenstates (11) exists for $m = 0$ and $\Lambda \leqslant 0$,

$$\mathcal{A}(\Lambda) = \langle\Psi_{\Lambda,0}|\psi_{00}\rangle = \sqrt{\kappa\kappa_g}\int_0^\infty p\,dp\,J_0(\gamma_0 p)e^{-\kappa_g p^2/2}$$
$$= \sqrt{q}\exp[-|\Lambda|q/2], \quad q = \omega_g/\omega.$$

Here, we used formula 7.7(24) from Ref. [21],

$$\int_0^\infty J_m(at)e^{-b^2t^2}t^{m+1}dt = \frac{a^m\exp[-a^2/(4b^2)]}{(2b^2)^{m+1}}. \tag{13}$$

Consequently, the magnetic-moment distribution function in the ground state equals

$$\mathcal{P}(\Lambda) \equiv |\mathcal{A}(\Lambda)|^2 = \begin{cases} q\exp(-|\Lambda|q), & \Lambda \leqslant 0, \\ 0, & \Lambda > 0. \end{cases} \tag{14}$$

This function is correctly normalized: $\int_{-\infty}^{\infty}\mathcal{P}(\Lambda)d\Lambda = 1$. Its mean value equals $\langle\Lambda\rangle = \int_{-\infty}^{\infty}\Lambda\mathcal{P}(\Lambda)d\Lambda = -\omega/\omega_g$, resulting in the magnetic-moment mean value $\mathcal{M} = -\mu$ in the limit case of $g \to 0$. The distribution width is characterized by the variance $\sigma_\Lambda \equiv \langle\Lambda^2\rangle - \langle\Lambda\rangle^2$. Since $\langle\Lambda^2\rangle = 2(\omega/\omega_g)^2$ for the distribution (14), the variance is rather big in this case: $\sigma_\Lambda = (\omega/\omega_g)^2 = \langle\Lambda\rangle^2$.

B. Nonzero temperatures

The scalar product $\langle\Psi_{\Lambda,m'}|\psi_{n_r m}\rangle$ is nonzero if $m' = m$. The corresponding integral again has the form (5). Thus, we obtain the following magnetic-moment probability distribution in the general Fock state,

$$\mathcal{P}_{n_r m}(\Lambda) = \frac{n_r!q}{(n_r+|m|)!}\xi_m^{|m|}e^{-\xi_m}\left[L_{n_r}^{(|m|)}(\xi_m)\right]^2, \tag{15}$$

where $\xi_m = q(m-\Lambda) \geqslant 0$. If $\Lambda > m$, then $\mathcal{P}_{n_r m}(\Lambda) = 0$. The normalization $\int_{-\infty}^m \mathcal{P}_{n_r m}(\Lambda)d\Lambda = 1$ is merely the standard normalization of the Laguerre polynomials: See formula (8.980) in Ref. [22]. The magnetic-moment probability density in the equilibrium state can be calculated as

$$\mathcal{P}(\Lambda) = \sum_{n_r,m}\mathcal{P}_{n_r m}(\Lambda)\exp(-\beta E_{n_r m})/\mathcal{Z}(\beta), \tag{16}$$

where $\mathcal{Z}(\beta) = \sum_{n_r,m}e^{-\beta E_{n_r m}}$ is the statistical sum, which can be easily calculated [4], due to the linear nature of the spectrum (4) (see also Refs. [7,32] for other approaches). Useful formulas are

$$(2\mathcal{Z})^{-1} = \cosh(\eta_g) - \cosh(\eta) = 2\sinh(\eta_+)\sinh(\eta_-), \tag{17}$$

$$\eta = \hbar\beta\omega, \quad \eta_g = q\eta, \quad \eta_\pm = \eta(q\pm1)/2. \tag{18}$$

The summation over n_r in (16) can be performed with the aid of formula (8.976.1) from Ref. [22], connecting the Laguerre polynomials with the modified Bessel function $I_\alpha(z)$,

$$\sum_{n=0}^{\infty}\frac{n!z^n L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)}{\Gamma(n+\alpha+1)}$$
$$= \frac{(xyz)^{-\alpha/2}}{1-z}\exp\left[-z\frac{x+y}{1-z}\right]I_\alpha\left(\frac{2\sqrt{xyz}}{1-z}\right). \tag{19}$$

The result is

$$\mathcal{P}(\Lambda) = \sum_{m>\Lambda}\mathcal{P}_m(\Lambda), \tag{20}$$

$$\mathcal{P}_m(\Lambda) = qG\exp[m\eta - \xi_m\coth(\eta_g)]I_{|m|}\left[\frac{\xi_m}{\sinh(\eta_g)}\right], \tag{21}$$

$$G = 2\sinh(\eta_+)\sinh(\eta_-)/\sinh(\eta_g). \tag{22}$$

Formula (21) holds for $m > \Lambda$, otherwise $\mathcal{P}_m(\Lambda) = 0$. The relative contribution of each term in the series (20) is given by the integral $\mathcal{P}_m = \int_{-\infty}^m \mathcal{P}_m(\Lambda)d\Lambda$, which can be calculated with the aid of formula (6.611.4) from Ref. [22],

$$A_0 = \int_0^\infty e^{-ax}I_\nu(bx)dx = \frac{\left(a-\sqrt{a^2-b^2}\right)^\nu}{b^\nu\sqrt{a^2-b^2}}, \tag{23}$$

so that

$$\mathcal{P}_m = G\exp[\eta(m-|m|q)]. \tag{24}$$

We see a great asymmetry with respect to positive and negative values of the canonical angular momentum m in the low-temperature case ($\eta \gg 1$), which almost disappears in the high-temperature case $\eta \ll 1$. The total contributions of positive and nonpositive values of m are given by simple sums,

$$\mathcal{P}_+ = \sum_{m=1}^{\infty} \mathcal{P}_m = \frac{\exp(\eta) - \exp(-\eta_g)}{2\sinh(\eta_g)}, \quad (25)$$

$$\mathcal{P}_- = \sum_{m=0}^{-\infty} \mathcal{P}_m = \frac{\exp(\eta_g) - \exp(\eta)}{2\sinh(\eta_g)}. \quad (26)$$

Therefore, we have the correct normalization

$$\mathcal{P}_+ + \mathcal{P}_- = \int_{-\infty}^{\infty} \mathcal{P}(\Lambda)d\Lambda = 1. \quad (27)$$

IV. MEAN VALUES AND VARIANCES

To calculate the mean value $\langle\Lambda\rangle = \int_{-\infty}^{\infty} \Lambda\mathcal{P}(\Lambda)d\Lambda$, we need the integral

$$A_1 = \int_0^{\infty} xe^{-ax}I_\nu(bx)dx = -\partial A_0/\partial a$$
$$= \frac{\left(a - \sqrt{a^2 - b^2}\right)^\nu}{b^\nu (a^2 - b^2)^{3/2}}\left(a + \nu\sqrt{a^2 - b^2}\right). \quad (28)$$

Again, we can write $\langle\Lambda\rangle = \sum_{m=-\infty}^{\infty} \langle\Lambda\rangle_m$, with

$$\langle\Lambda\rangle_m = \mathcal{P}_m\{m - r[|m| + \coth(\eta_g)]\}, \quad r = \omega/\omega_g. \quad (29)$$

Hence,

$$\langle\Lambda\rangle = \frac{\omega_- \sinh(\eta_+)}{\omega_g \sinh(\eta_-)\sinh(\eta_g)} - \frac{\omega}{\omega_g}\coth(\eta_g) - \frac{\omega_+ \sinh(\eta_-)}{\omega_g \sinh(\eta_+)\sinh(\eta_g)}, \quad (30)$$

where $\omega_\pm = (\omega_g \pm \omega)/2$. This expression is another form of the result obtained in Refs. [7,26,32] with the aid of the density matrix formalism. It goes to the Darwin formula (1), if $g \to 0$.

To calculate the second-order mean value,

$$\langle\Lambda^2\rangle = \int_{-\infty}^{\infty} \Lambda^2\mathcal{P}(\Lambda)d\Lambda = \sum_{m=-\infty}^{\infty} \langle\Lambda^2\rangle_m, \quad (31)$$

we need the integral

$$A_2 = \int_0^{\infty} x^2e^{-ax}I_\nu(bx)dx = -\frac{\partial A_1}{\partial a} = \frac{\left(a - \sqrt{a^2 - b^2}\right)^\nu}{b^\nu (a^2 - b^2)^{5/2}}$$
$$\times\left[3a\left(a + \nu\sqrt{a^2 - b^2}\right) + \left(\nu^2 - 1\right)\left(a^2 - b^2\right)\right]. \quad (32)$$

Then,

$$\langle\Lambda^2\rangle_m = \mathcal{P}_m\left\{(m - r|m|)^2 + r\coth(\eta_g)(3r|m| - 2m) + r^2\left[3\coth^2(\eta_g) - 1\right]\right\}. \quad (33)$$

Performing separate summations in (31) over positive and negative values of m, we obtain

$$\langle\Lambda^2\rangle = r^2\left[3\coth^2(\eta_g) - 1\right] + [2\sinh(\eta_g)]^{-1}\left\{\frac{(1-r)^2}{\sinh^2(\eta_-)}D\cosh(\eta_-)\sinh(\eta_+) + \frac{(1+r)^2}{\sinh^2(\eta_+)}\cosh(\eta_+)\sinh(\eta_-) + r\coth(\eta_g)\left[(3r+2)\frac{\sinh(\eta_-)}{\sinh(\eta_+)} + (3r-2)\frac{\sinh(\eta_+)}{\sinh(\eta_-)}\right]\right\}. \quad (34)$$

The last term in this expression diverges for $g \to 0$, when $\eta_- \approx \eta g^2/(4\omega^2) \to 0$. Therefore, we arrive at the formula

$$\langle\Lambda^2\rangle_{g\to 0} = 3\coth^2(\eta) - 1 + \frac{2}{\eta^2} + \frac{2\omega^2\coth(\eta)}{g^2\eta}. \quad (35)$$

The variance σ_Λ equals in this limit

$$\sigma_\Lambda = 2\coth^2(\eta) - 1 + \frac{2}{\eta^2} + \frac{2\coth(\eta)}{\eta}\left(1 + \frac{\omega^2}{g^2}\right). \quad (36)$$

At zero temperature ($\eta = \infty$), this formula yields the value $\sigma_\Lambda = 1$, in accordance with Sec. III A. Note, however, that the simple relation $\eta \gg 1$ is not sufficient to say that the system has achieved the zero-temperature regime: A stronger inequality $\eta \gg (\omega/g)^2$ must be satisfied. But the variance becomes extremely large in the high-temperature limit,

$$\sigma_\Lambda \approx 2\omega^2/(g^2\eta^2), \quad \eta \ll 1. \quad (37)$$

V. NUMERICAL STUDY OF THE DISTRIBUTION FUNCTION

Unfortunately, it seems that the series (20) cannot be calculated analytically. Therefore, we had to perform numerical calculations. Figures 1–3 show the function $\mathcal{P}(\Lambda)$ in the logarithmic scale (with details in the usual scale) for $\eta = 10, 1, 1/10$, with $\omega/g = 5$ and $\omega/g = 9$.

We summed numerically the terms $\mathcal{P}_m(\Lambda)$ for $m \leqslant 10^4$, which guaranteed the convergence of the sum. The modified Bessel functions were generated via recurrence relations either for $I_n(z)$ or $e^{-z}I_n(z)$ (when $z \gg 1$), using the Miller's algorithm to avoid overflows. For $\omega/g = 5, 9$ the absolute error in the normalization (27) was smaller than 10^{-4} and 7×10^{-3}, respectively. Another figure of merit is the relative error between the numeric value of $\langle\Lambda^2\rangle$ and Eq. (34), found to be smaller than 10^{-4} and 4×10^{-2} for $\omega/g = 5$ and 9, respectively.

We see that the upper parts of all three figures look quite similar. However, a significant difference is in the horizontal and vertical scales. A striking feature is the discontinuity at $\Lambda = 0$. But the explanation is quite simple: When one goes

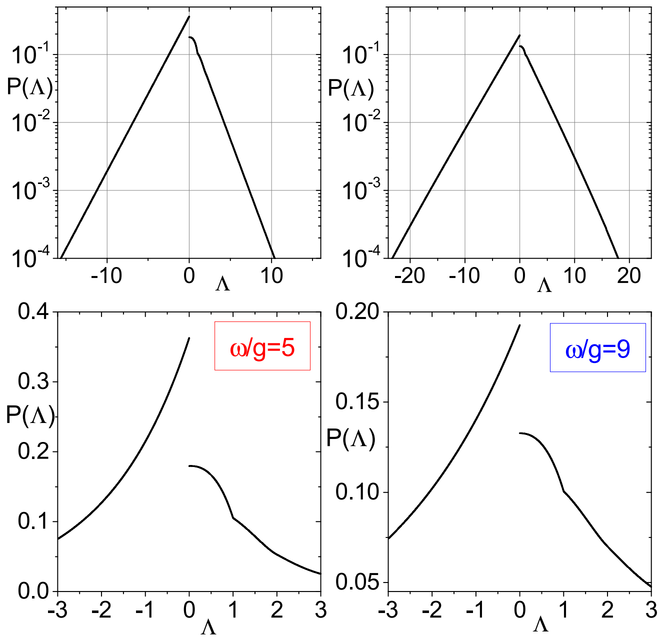

FIG. 1. Function $\mathcal{P}(\Lambda)$ for $\eta = 10$.

from $\Lambda = 0$ to $\Lambda > 0$ in series (20), then the term with $m = 0$ disappears. Therefore, the jump in the probability distribution equals exactly $\mathcal{P}_0(0) = qG$. The discontinuity of the derivative at $\Lambda = 1$ is connected with the disappearance of the contribution of the term with $m = 1$, which is proportional to the first modified Bessel function $I_1(z)$. The jumps of the derivatives at small integral values Λ are even more pronounced for small values of η, as one can see in Fig. 3 for $\eta = 1/10$.

All three figures show the linear exponential decrease of $\mathcal{P}(\Lambda)$, when $|\Lambda|$ is not too small. For big values of η, the

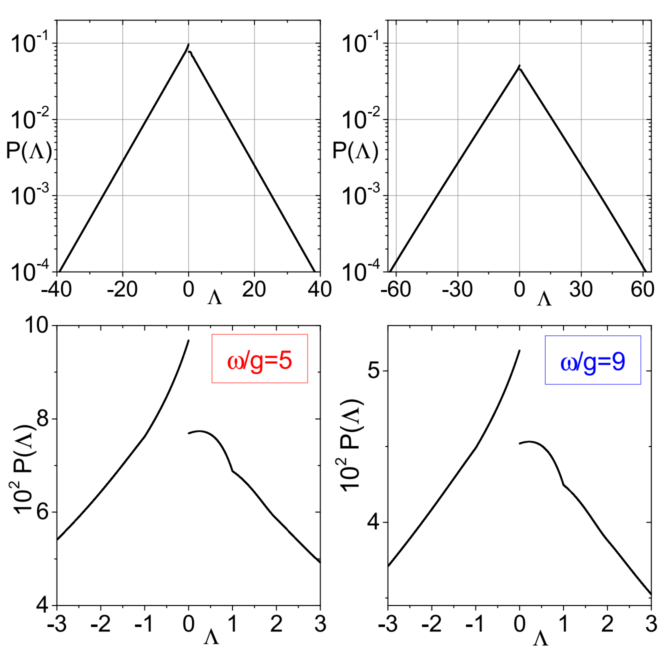

FIG. 2. Function $\mathcal{P}(\Lambda)$ for $\eta = 1$.

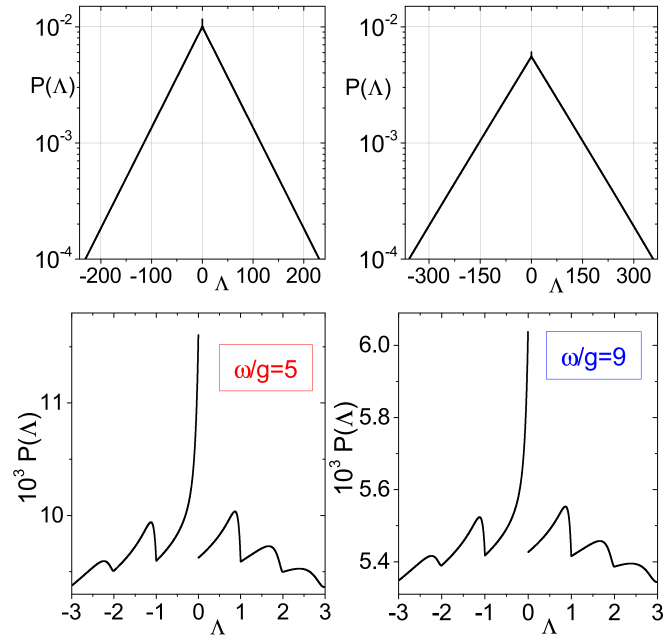

FIG. 3. Function $\mathcal{P}(\Lambda)$ for $\eta = 1/10$.

coefficients of the left and right exponential functions are different, as one can see clearly in Fig. 1. But if $\eta \ll 1$, then the distribution becomes practically symmetrical, and it can be described (outside the small initial region) by the approximate formula

$$\mathcal{P}_{ap}(\Lambda) = (b/2)\exp(-b|\Lambda|), \qquad b = g\eta/\omega, \tag{38}$$

where coefficient b is chosen in accordance with Eq. (37) for $\langle\Lambda^2\rangle$. Of course, the approximate formula (38) cannot explain the small nonzero mean value $\langle\Lambda\rangle$, which is determined by the "fine structure" of $\mathcal{P}(\Lambda)$ nearby the origin. But it shows that the width of the distribution is very large, being proportional to b^{-1}.

VI. CONCLUSION

Our results shed light on the properties of such an intricate physical quantity as the magnetic moment of a quasi-free-charged particle in a uniform magnetic field. We have calculated eigenfunctions of the magnetic-moment operator in the momentum representation, as well as their scalar products with the energy eigenstates of an isotropic harmonic oscillator, placed in the magnetic field, using the circular gauge of the vector potential. Then, using the Boltzmann weights of energy levels, we have succeeded to calculate the magnetic moment probability distribution in the thermal state. This distribution is totally asymmetric in the case of zero absolute temperature. However, the asymmetry becomes weaker and weaker with an increase in temperature, so that contributions of positive and negative values of the magnetic moment practically cancel each other in the high-temperature case, resulting in the small remaining mean magnetic moment, which is inversely proportional to the absolute temperature. At the same time, the contributions of the positive and negative wings of the probability distribution to the mean square of the magnetic moment do not cancel. This results in the huge variance of the magnetic moment in the high-temperature case. Such a huge value is a consequence of an extremely large extension of the distribution function. Moreover, this extension appears to be very sensitive to the strength of the confining potential (and its shape in the general case). This can be explained, qualitatively, by the nature of the magnetic-moment operator (10). It is not proportional to the canonical angular momentum, but it has the second part, describing the extension of the system in the coordinate space. In a certain sense, this division in two parts reflects the initial Bohr's ideas concerning opposite contributions of the "bulk" rotating currents (corresponding to the angular momentum part L_z) and "surface" currents (roughly described by the second term) to the total magnetic moment (see, e.g., Ref. [33] about the qualitative description of these two contributions).

In conclusion, let us rewrite formula (37) in terms of the variance of the magnetic moment in the high-temperature regime,

$$\sigma_{\mathcal{M}} = 2\mu^2\sigma_{\Lambda} = \tfrac{1}{2}(Mcg\beta/e)^{-2}. \qquad (39)$$

It does not contain the Planck's constant (nor the magnetic field B). This strong result causes us to conclude that the classical limit of the equilibrium magnetic properties of charged particles in the high-temperature case is not so simple as, probably, it was thought before. Namely, not all magnetism disappears in this limit, but its thermodynamic mean value only, while huge fluctuations of the magnetic moment survive even if $\hbar \to 0$. Perhaps it would be interesting to verify our results in experiments with single electrons or ions in traps. In addition, it could be interesting to generalize our results to the case of a Fermi-Dirac probability distribution (instead of the Boltzmann one) and to the case of anisotropic binding potentials.

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[1] N. Bohr, Studier over metallerties elektrontheori (Studies on the electron theory of metals), dissertation for the degree of doctor of philosophy, Copenhagen, 1911; reprinted and translated into English in *Niels Bohr Collected Works* (Elsevier, Amsterdam, 1971), Vol. 1, pp. 163–393.

[2] H.-J. van Leeuwen, Problèmes de la théorie électronique du magnétisme, J. Phys. Radium **2**, 361 (1921).

[3] L. Landau, Diamagnetismus der Metalle, Z. Phys. **64**, 629 (1930).

[4] C. G. Darwin, The diamagnetism of the free electron, Math. Proc. Cambridge Philos. Soc. **27**, 86 (1931).

[5] E. H. Sondheimer and A. H. Wilson, The diamagnetism of free electrons, Proc. R. Soc. London, Ser. A **210**, 173 (1951).

[6] R. Kubo, Wigner representation of quantum operators and its applications to electrons in a magnetic field, J. Phys. Soc. Jpn. **19**, 2127 (1964).

[7] B. U. Felderhof and S. P. Raval, Diamagnetism of a confined electron gas, Physica A **82**, 151 (1976).

[8] A. Feldman and A. H. Kahn, Landau diamagnetism from the coherent states of an electron in a uniform magnetic field, Phys. Rev. B **1**, 4584 (1970).

[9] I. A. Malkin and V. I. Man'ko, Coherent states of a charged particle in a magnetic field, Sov. Phys. JETP **28**, 527 (1969).

[10] S. T. Pavlov and A. V. Prokhorov, Oscillation effects in metals in a magnetic field and macroscopic quantum interference, Sov. Phys. JETP **73**, 280 (1991).

[11] Y. Ishikawa and H. Fukuyama, Orbital magnetism and current distribution of two-dimensional electrons under confining potential, J. Phys. Soc. Jpn. **68**, 2405 (1999).

[12] S. Biswas, S. Sen, and D. Jana, A complete theory for the magnetism of an ideal gas of electrons, Phys. Plasmas **20**, 052503 (2013).

[13] G. Faussurier, Equation of state of the free electron gas in a magnetic field at arbitrary degeneracy, Phys. Plasmas **27**, 022709 (2020).

[14] S. Curilef, F. Pennini, and A. Plastino, Fisher information, Wehrl entropy, and Landau diamagnetism, Phys. Rev. B **71**, 024420 (2005).

[15] T. H. Boyer, Understanding the Planck blackbody spectrum and Landau diamagnetism within classical electromagnetism, Eur. J. Phys. **37**, 065102 (2016).

[16] T. H. Boyer, Diamagnetic behavior in random classical radiation, Am. J. Phys. **87**, 915 (2019).

[17] H. Fukuyama, Fluctuations of Landau diamagnetism in mesoscopic systems, J. Phys. Soc. Jpn. **58**, 47 (1989).

[18] S. Oh, A. Yu. Zyuzin, and R. A. Serota, Orbital magnetism of mesoscopic systems, Phys. Rev. B **44**, 8858 (1991).

[19] D. Yoshioka and H. Fukuyama, Orbital magnetism of two-dimensional electrons in confining potentials, J. Phys. Soc. Jpn. **61**, 2368 (1992).

[20] V. Fock, Bemerkung zur Quantelung des harmonischen Oszillators im Magnetfeld, Z. Phys. **47**, 446 (1928).

[21] *Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 2.

[22] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. (Academic, Amsterdam, 2007).

[23] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, U.K., 1971).

[24] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999).

[25] M. H. Johnson and B. A. Lippmann, Motion in a constant magnetic field, Phys. Rev. **76**, 828 (1949).

[26] N. H. March and M. P. Tosi, Thermally averaged orbital diamagnetism of a localized Wigner oscillator, Nuovo Cimento D **6**, 521 (1985).

[27] A. M. Stewart, General gauge independence of diamagnetism plus paramagnetism, Aust. J. Phys. **53**, 613 (2000).

[28] K. Y. Bliokh, P. Schattschneider, J. Verbeeck, and F. Nori, Electron Vortex Beams in a Magnetic Field: A New Twist on Landau Levels and Aharonov-Bohm States, Phys. Rev. X **2**, 041011 (2012).

[29] C. R. Greenshields, R. L. Stamps, and S. Franke-Arnold, Parallel axis theorem for free-space electron wavefunctions, New J. Phys. **17**, 093015 (2015).

[30] M. Wakamatsu, Y. Kitadono, and P.-M. Zhang, The issue of gauge choice in the Landau problem and the physics of

canonical and mechanical orbital angular momenta, Ann. Phys. (NY) **392**, 287 (2018).

[31] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, UK, 1965).

[32] V. V. Dodonov and V. I. Man'ko, Evolution of multidimensional systems. Magnetic properties of ideal gases of charged particles, in *Invariants and the Evolution of Nonstationary Quantum Systems*, edited by M. A. Markov, Proceedings of the P. N. Lebedev Physical Institute Vol. 183 (Nova Science, New York, 1989), pp. 263–414.

[33] R. Peierls, *Surprises in Theoretical Physics* (Princeton University Press, Princeton, NJ, 1979), Sec. 4.3.