Relativistic paradox exposing the ubiquity of hidden momentum

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The tight connection between mass and energy unveiled by special relativity, summarized by the iconic formula $E = mc^2$, has revolutionized our understanding of nature and even shaped our political world over the past century through its military application. It is certainly one of the most exhaustively tested and well-known equations of modern science. Although we have become used to its most obvious implication—mass-energy equivalence —it is surprising that one of its subtle—yet, inevitable—consequences is still a matter of confusion: the so-called hidden momentum. Often considered as a peculiar feature of specific systems or as an artifact to avoid paradoxical situations, here we present a relativistic "paradox" which exposes the true nature and ubiquity of hidden momentum. We also show that hidden momentum can be forced to reveal itself through observable effects, hopefully putting an end to decades of controversy about its reality.

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I. INTRODUCTION

Einstein's iconic mass-energy relation, $E = mc^2$, is arguably the most famous formula in modern science. It expresses the equivalence between total mass m and total energy E of a system (c being the speed of light in vacuum), with wide-ranging consequences: from the unattainability of the speed of light for massive objects to particle production in high-energy accelerators, and from the origin of the energy of stars—less than 0.1% of the star's mass, converted into radiation over its entire existence—to violent bursts of gravitational waves from merging black holes—some of them sourced by several solar masses converted into energy in a fraction of a second. Given the importance and generality of mass-energy equivalence, it may strike one as a surprise that one of its subtle—but inevitable—consequences is still a matter of confusion: the concept of *hidden momentum* (HM) [1]—here generalized as the (purely relativistic) part of total momentum which is not encoded in the motion of the center of mass-energy (CME) of the system.

In Newtonian mechanics, the total momentum $\mathbf{P}$ of a *closed* mechanical system—one which does not exchange *matter* with "the rest of the universe"— is always given by $\mathbf{P} = M\mathbf{V}_{\text{c.m.}}$, where M is the total mass of the system and $\mathbf{V}_{\text{c.m.}}$ is the velocity of its center of mass. This result, known as the center-of-mass theorem, holds true regardless of whether the *Newtonian* system is subject to external forces or not. In contrast to that, a variety of *relativistic* systems possessing nonzero total momentum in the rest frame of their CME has been identified over the past decades (see, e.g., Refs. [1–15]). Here, the term "relativistic" does not necessarily mean that large velocities are involved, but rather that different inertial-frame descriptions are supposed to be Lorentz covariant instead of Galilean covariant. This covariance constraint leads to "unfamiliar" results (i.e., results nonexistent in Newtonian mechanics) even in the rest frame of the system—such as nonzero total momentum. Such rest-frame momentum has been termed *hidden momentum* [1], which now seems to be somewhat unfortunate because apparently this has misled many to interpret its nature as somehow distinct from "regular" momentum—as we argue here, from the relativity-theory perspective, it is not. Adding confusion to the story, *all* systems in which HM had been identified, until now, involved interaction with electromagnetic fields—where it even bears an interesting connection with the difference between canonical and kinematic electromagnetic momenta [15]—and/or moving inner parts subject to some external force field. This masked the generic nature of HM as if it were an exotic feature—undesired by some—of peculiar interaction laws or specific systems.

Here, we present a relativistic "paradox" which shows that this view is limited and that HM is ubiquitous in a relativistic world (i.e., a world supposed to be covariant under Lorentz transformations). Moreover, the general definition of HM we propose, freeing its computation from the rest frame of the system, leads to a formula which explicitly shows its relation to asymmetric exchange (and hence flow) of energy. Finally, in order to conclusively show that HM is as real as it could be, in the end we discuss an observational consequence of its existence. The present analysis is intended to put an end to decades of confusion about the nature and reality of the so-called HM.

The paper is organized as follows. In Sec. II, we present the relativistic paradox involving a heat-conducting bar (HCB) analyzed from different inertial-frame perspectives. In order to make the presentation clearer, textbook-level relativistic calculations which support statements made in this section are described in the Appendix. In Sec. III, we put the HCB paradox in context with other known pseudoparadoxes, stressing their origin in our intuition based on space and time as separate entities rather than in inconsistencies with known theories. We also argue that the HCB paradox is a close thermal analog of another relativistic pseudoparadox known as

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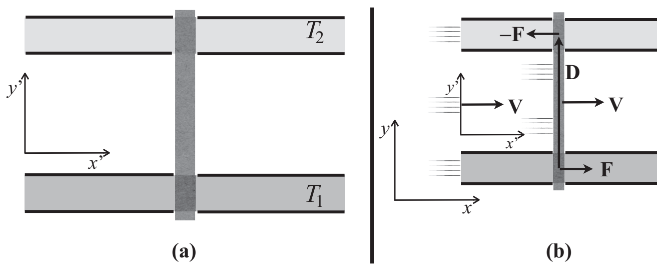

FIG. 1. Heat-conducting-bar paradox. (a) A bar stands still in an inertial frame, connecting two thermal reservoirs. In the stationary regime, the bar absorbs heat from the thermal reservoir at temperature T_1, at a rate W, and delivers heat at the same rate to the thermal reservoir at temperature $T_2 \leqslant T_1$. There are no net forces between the bar and the reservoirs. (b) The same situation observed from an inertial frame with respect to which the bar moves with velocity $\mathbf{V}$ perpendicular to itself. According to observers static in this latter frame, the reservoirs apply opposite forces $\pm\mathbf{F} = \pm W\mathbf{V}/c^2$ on the bar. Therefore, in this frame there exists a torque $\mathbf{T} = W(\mathbf{V}\times\mathbf{D})/c^2$ on the heat-conducting moving bar, where $\mathbf{D}$ is the spatial vector depicted in the figure.

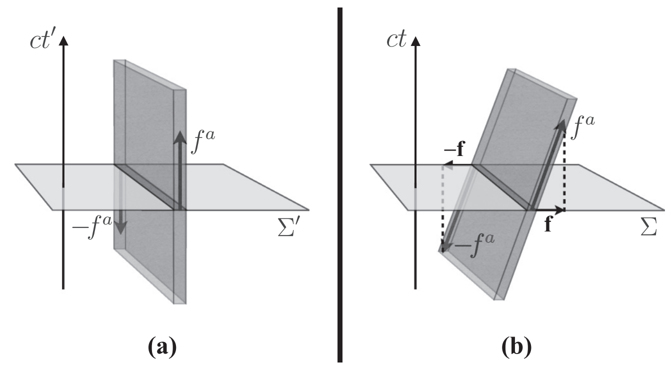

FIG. 2. Spacetime depiction of the world volume of the bar and the energy-momentum exchange (given by the four-force densities $\pm f^a$) between the bar and the reservoirs. (a) From the rest-frame perspective, f^a has only a component along the time direction, describing exchange of energy without net spatial forces. (b) From the moving-frame perspective, the *same* f^a clearly has nonvanishing spatial projection. Therefore, in this frame, there are force densities $\pm\mathbf{f}$ acting on the bar.

"Mansuripur's paradox" [6]. In Sec. IV, we argue that HM is just another inevitable consequence of Einstein's mass-energy relation $E = mc^2$, obtaining a general expression for HM in terms of the dipole moment of the energy-exchange rate (Sec. IV A) and then applying this general result to solve the HCB paradox (Sec. IV B). In Sec. V, we show that, contrary to widespread belief expressed in the literature, the existence of HM in a system can be objectively tested. Finally, in Sec. VI, we present our final comments and discussion. It is important to stress that (i) the HCB pseudoparadox (Sec. II) and (ii) the general deduction of the HM formula (Sec. IV A) are independent presentations; the former is discussed only because it evidentiates, in a concrete scenario, the generic nature of HM—which is the main point of this paper.

II. HEAT-CONDUCTING-BAR PARADOX

Consider the system depicted in Fig. 1(a), composed by a free bar connecting two thermal reservoirs at temperatures T_1 and T_2—with, say, $T_1 \geqslant T_2$—at rest in an inertial frame. In order to avoid unnecessary subtleties, we consider that (i) the stationary heat-flow regime has been established, (ii) thermal contact between the bar and each reservoir is symmetric (for instance, through the lateral surface of the bar), and (iii) the bar is coated with a thermal insulator all over the parts which are not in contact with the reservoirs. Conditions (i) and (ii) ensure that the CME of the bar stays at rest in the inertial rest frame of the reservoirs without the need for any mechanical constraint; the heat-conducting bar in the stationary regime is in static mechanical equilibrium. [Condition (iii) only serves to keep the system simple.] From the rest-frame perspective, the effect of the reservoirs on the bar is merely exchange of heat, with no net forces or torques being applied. Let $W > 0$ represent the (constant) rate at which heat is exchanged between the bar and the reservoirs—flowing into (out of) the bar from (to) the reservoir at temperature T_1 (T_2). (Side note: for any given heat-exchange rate W, we can consider the temperature difference $T_2 - T_1$ to be arbitrarily small by choosing bars with arbitrarily large thermal conductivities; therefore, although unnecessary, one can simplify further the setup considering the mass-energy and temperature distributions along the bar to be arbitrarily close to homogeneous.)

Now, let us analyze the same setup from the perspective of another inertial frame, with respect to which the bar (and the whole system) moves with velocity $\mathbf{V}$ perpendicular to itself—the "moving frame" for short. Although it may sound odd at first, it follows directly from Einstein's special relativity that, in this frame, the reservoirs apply opposite net forces $\pm\mathbf{F} = \pm W\mathbf{V}/c^2$ at the moving bar's ends [see Fig. 1(b)].

The proof of this fact is actually quite simple (a textbook-level exercise) and is explained in detail in the Appendix. In essence, due to Lorentz covariance, what is seen in the rest frame as a pure exchange of energy (i.e., a four-force density f^a with only a time component) corresponds to exchange of both energy and momentum according to the moving frame (see Fig. 2, which is the spacetime depiction of the bar in Fig. 1).

Once the reader is convinced of the existence of such forces, he/she promptly realizes that they lead to a torque on the heat-conducting moving bar, which (neglecting the spatial extension of the thermal contacts) is given by

$$\mathbf{T} = W(\mathbf{V}\times\mathbf{D})/c^2, \tag{1}$$

where $\mathbf{D}$ is the separation vector between the thermal contacts (see Fig. 1); although the opposite forces have no net effect on the total momentum of the bar as time passes, they do change the bar's angular momentum. If we apply our Newtonian intuition—as is customary when arriving at relativistic paradoxes—this torque with respect to the instantaneous CME position should try to rotate the bar. But this is obviously in conflict with the fact that in the reservoirs' rest frame the bar is in static mechanical equilibrium; there is absolutely no reason for rotation. We have stumbled on a relativistic paradox.

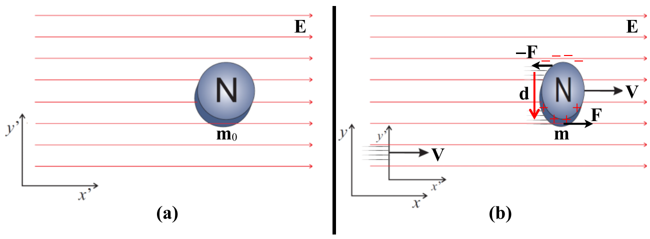

FIG. 3. Mansuripur's paradox. (a) A magnet stands still in an inertial frame, with magnetic dipole moment $\mathbf{m}_0$ perpendicular to a uniform electric field $\mathbf{E}$. (b) The same situation observed from an inertial frame with respect to which the magnet moves with velocity $\mathbf{V}$ parallel to the electric field. Now, the magnet also carries an electric dipole moment $\mathbf{d}$, upon which the electric field exerts a torque.

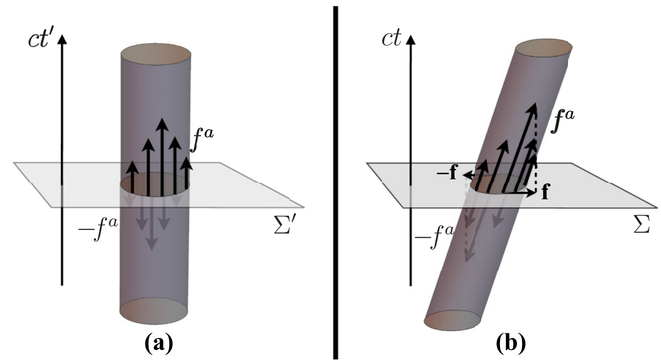

FIG. 4. Four-dimensional representation of (the cross section $z = 0$ of) the system depicted in Fig. 3: (a) privileging the magnet's rest frame and (b) privileging the moving frame. The Lorentz four-force density is future directed (f^a) where the electric field favors the current density ($\mathbf{j} \cdot \mathbf{E} > 0$) and past directed otherwise ($\mathbf{j} \cdot \mathbf{E} < 0$). Note that it has null projection on Σ'—therefore, no forces according to the magnet's rest frame— while being nonzero and circulating on Σ—therefore, applying a torque on the magnet according to the moving frame.

III. NO REAL PARADOX

Relativistic paradoxes—more precisely, situations the descriptions of which from different inertial perspectives *seem* paradoxical when compared to each other—are numerous and even serve as teaching tools in relativity. Rather than pointing to inconsistencies in fundamental theories, they reveal how our Newtonian perception of space and time as separate entities, instead of interwoven in an absolute four-dimensional spacetime, can be deceiving. Their nature can be loosely classified as kinematical—those which involve only time-interval and spatial-distance measurements—and dynamical—those which involve forces. The twins', barn-pole, and Bell's spaceship paradoxes are well-known textbook samples of the kinematical type—see, e.g., Ref. [16]—whereas the Trouton-Noble [17], right-angle-lever [18], and submarine [19–22] paradoxes are representative of the dynamical type. The HCB paradox presented above clearly fits into this latter class. Contrary to kinematical paradoxes, the dynamical ones are rarely addressed in relativity textbooks and introductory courses. This may explain why many of them are unknown to nonrelativists or, when known, concepts involved in their resolution are seen with suspicion.

In 2012, Mansuripur [6] analyzed in detail an ingenious dynamical paradox—previously discussed in Ref. [23]—which, in a simplified but equivalent version, can be realized by a neutral magnet at rest in an inertial frame, where there exists a uniform (external) electric field $\mathbf{E}$ perpendicular to the magnet's magnetic dipole moment $\mathbf{m}_0$ (see Fig. 3). In the magnet's rest frame [Fig. 3(a)], the magnet "seems" oblivious to the presence of the electric field—apart from induced polarization, which can be made negligible. However, looking at the same system from another inertial frame, with respect to which the magnet moves with velocity $\mathbf{V}$ along the electric field's direction [Fig. 3(b)], the magnet now also bears an electric dipole moment $\mathbf{d} = \mathbf{V} \times \mathbf{m}_0/c^2$—since $\mathbf{m}_0$ is ultimately due to electric currents, not pairs of magnetic monopoles [24]. Thus, according to this inertial frame, there must exist a torque $\mathbf{T} = \mathbf{d} \times \mathbf{E} = (\mathbf{V} \cdot \mathbf{E})\, \mathbf{m}_0/c^2$ acting on the magnet, which would supposedly make it spin—in gross contradiction with the fact that in its inertial rest frame the magnet stands still. Mansuripur concludes that this contradiction is an "incontrovertible theoretical evidence of the incompatibility of the Lorentz law [of force] with the fundamental tenets of special relativity" [6].

The HCB paradox presented earlier is a close thermal analog of Mansuripur's, with the thermal reservoirs playing the role of the external electric field and the heat-conducting bar substituting the magnet. Less obvious is the analog, in Mansuripur's setup, of the heat exchange rate W between the bar and the reservoirs. But recalling that magnetism in materials is ultimately due to current densities $\mathbf{j}$ (even if quantum mechanical in nature), one concludes that the magnet in its rest frame does exchange energy with the external electric field at a rate, per volume, $\mathbf{j} \cdot \mathbf{E}$: the magnet predominantly absorbs (delivers) energy from (to) the external field where $\mathbf{j} \cdot \mathbf{E} > 0$ ($\mathbf{j} \cdot \mathbf{E} < 0$), leading to a net flow of energy across the magnet. This completes the analogy between the HCB paradox and Mansuripur's (compare Fig. 4 with Fig. 2). Notwithstanding, there is one important difference: in the thermal analog, there is no specific "law of force" to blame for the apparent contradiction between different inertial-frame descriptions; the torque on the bar seen from the moving-frame perspective is *enforced* by Lorentz covariance and, particularly, by $E = mc^2$. Certainly, no one would hold that Einstein's mass-energy relation is "incompatible with the fundamental tenets of special relativity." Therefore, there is no logical reason for taking this stand regarding the Lorentz force.

IV. MASS-ENERGY EQUIVALENCE AND HIDDEN MOMENTUM

Relativistic thermodynamics has its own history of subtleties and controversies. The most emblematic of them is probably the question of how temperature transforms from one inertial frame to another. It took about 90 years for this to be recognized as an ill-posed question—hence, the conflicting answers given during this period (see Refs. [25,26] and references therein). Fortunately, none of these subtleties—not even temperature transformation—concerns us; the purpose of thermal reservoirs in the setup of Fig. 1 is only to guarantee an eventual stationary situation in the rest frame of the system.

As mentioned earlier, several systems with nonvanishing total momentum in their rest frames have been found and discussed in the literature (see, e.g., Refs. [1–5])—including Mansuripur's setup [7–11,15]. All such systems involved interaction with electromagnetic fields and/or moving inner parts subject to some external force field, which led many to view it as a feature of peculiar interaction laws or systems. Mansuripur, for instance, considered HM to be an *ad hoc* addition to materials interacting with electromagnetic fields, with no justification other than artificially avoiding paradoxical situations [12–14]. A better law of electromagnetic force, he reasoned, should be one which leads to no torque on the moving magnet in an electric field—hence, doing away with HM. The advantage of the HCB paradox we discuss in this paper is that it does not depend on the inner details of the system (the bar and the heat or energy flow) or of the interaction with "the rest of the universe" (the thermal or energy reservoirs).

The resolution of the HCB paradox—as well as Mansuripur's—consists in taking mass-energy equivalence to its ultimate consequences. As heat (i.e., energy) flows through the bar, it contributes to momentum in very much the same way as would a flow of matter. In fact, distinguishing contributions to the total momentum coming from "different forms" of energy flows is quite contrary to the spirit of relativity theory. Therefore, the total momentum of the bar in its *rest* frame [Fig. 1(a)] does not vanish—a purely relativistic effect.

For the same reason, according to the inertial frame with respect to which the bar moves with velocity $\mathbf{V}$ perpendicular to itself [Fig. 1(b)], there is a momentum contribution along the bar. Consequently, the bar's total momentum $\mathbf{P}$ and the CME velocity $\mathbf{V}$ are misaligned, and dragging momentum $\mathbf{P}$ along a spatial direction which is not aligned to it inevitably leads to a time-varying angular momentum $\mathbf{L}$ (with $d\mathbf{L}/dt = \mathbf{V} \times \mathbf{P}$) and, therefore, *demands* a torque—which, as we shall see below, is precisely the one supplied by the thermal reservoirs in the moving frame.

A. General treatment: Hidden momentum as the dipole moment of the energy-exchange rate

Mass-energy equivalence has a very straightforward consequence which, nonetheless, is overlooked when arriving at dynamical relativistic paradoxes: the distinction between *closed* and *isolated* systems, as usually made in Newtonian (i.e., Galilean-covariant) physics (including nonrelativistic thermodynamics), has no absolute meaning in relativistic physics. While in Newtonian physics a system can exchange energy without exchanging mass (i.e., it can be nonisolated but closed), this is obviously impossible if mass and energy are the same physical quantity. In this sense, the *relativistic* dynamics of the heat-conducting bar depicted in Fig. 1—which, in the Newtonian context, is a closed system for which the center-of-mass theorem would apply—is essentially the same as that of a pipe segment carrying a steady fluid or particle current, with fluid or particles entering the system at one end of the pipe and leaving at the other—which, in the Newtonian context, is an open system, for which the center-of-mass theorem does *not* apply. No one would object that the pipe segment carrying a particle or fluid current possesses nonzero momentum in the rest frame of its center of mass. Distinguishing this momentum from the one carried by the heat-conducting bar—or, for that matter, from the one carried by the magnet in Mansuripur's setup—is solely motivated by our Newtonian view of the world.

Although distinguishing HM from "regular" momentum is artificial as far as relativistic dynamics is concerned, it is useful, in order to demystify it further, to pin down the elements which compel our Newtonian intuition to make such a distinction. Basically, all instances of HM involve systems where there is a clear notion of a *velocity field* $\mathbf{v}$ of their "constituents" —usually taken to be particles or fluid elements—and an associated non-negative (not identically null) *number density* n which, together, satisfy the continuity equation $\partial_t n + \nabla \cdot (n\mathbf{v}) = 0$ in *all* inertial frames.

Put in spacetime language: these systems in which HM can be identified possess, associated to their constituents, a timelike future-directed four-vector field n^a—the four-current number density, the components of which in inertial Cartesian coordinates read $n^\mu = (nc, n\mathbf{v})$—satisfying the tensorial equation $\nabla_a n^a = 0$. The key point is that the existence of such a four-current number density allows us to extend to the relativistic context, in a consistent manner, the notion of "closed systems."

Definition: A system will be said to be *closed* if one of the following holds.

(a) The system is *isolated*—i.e., its *extended* (see below) stress-energy-momentum tensor $\bar{T}^{ab}$ satisfies $\nabla_a \bar{T}^{ab} \equiv 0$ (and goes to zero sufficiently fast at spatial infinity so that any flux vanishes).

(b) The system possesses a "natural" notion of four-current number density n^a (as defined above) the *extension* $\bar{n}^a$ of which (see below) satisfies $\nabla_a \bar{n}^a \equiv 0$ (and goes to zero sufficiently fast at spatial infinity so that any flux vanishes).

(c) The system itself is a collection of closed systems as defined in the previous items.

[Given a tensor field $\mathcal{T}^{ab\ldots}_{cd\ldots}$ with support supp($\mathcal{T}$), its extension $\bar{\mathcal{T}}^{ab\ldots}_{cd\ldots}$ is the tensor field defined over the *whole* spacetime which coincides with $\mathcal{T}^{ab\ldots}_{cd\ldots}$ in supp($\mathcal{T}$) and is zero otherwise. This is a mere technicality, useful when treating systems which are, themselves, part of larger ones. Due to possible discontinuities in $\bar{\mathcal{T}}^{ab\ldots}_{cd\ldots}$, the equations above are to be taken in the distributional sense.]

The definition above clearly recovers, in the Newtonian regime, the notion of closed systems as those which do not exchange matter or mass, for in this case the mass density itself satisfies the continuity equation and can be taken to be n up to a multiplicative constant. This fact will be used later when we restrict attention to closed systems.

Let $\mathcal{S}$ be a system with a stress-energy-momentum tensor the components of which in inertial Cartesian coordinates $\{(ct, \mathbf{x})\}$ are given by $T^{\mu\nu}_{\mathcal{S}}$. We assume that, at each instant t, $T^{\mu\nu}_{\mathcal{S}}$ goes to zero sufficiently fast at spatial infinity so that the manipulations and integrals which follow below are well defined. The CME of $\mathcal{S}$ is given by

$$\mathbf{X}_{\mathrm{CME}} := \frac{1}{Mc^2} \int d^3x \, \bar{T}^{00}_{\mathcal{S}} \, \mathbf{x}, \qquad (2)$$

where $M = \int d^3x\, \bar{T}_S^{00}/c^2$ is the (possibly time-dependent) total mass of the system. (All integrations are carried over the entire spatial section $t = \text{const}$, hence the use of $\bar{T}_S^{\mu\nu}$ instead of $T_S^{\mu\nu}$.) Multiplying Eq. (2) by M and taking the time derivative, we get

$$M\mathbf{V}_{\text{CME}} = -\frac{dM}{dt}\mathbf{X}_{\text{CME}} + \frac{1}{c}\int d^3x\, \partial_0 \bar{T}_S^{00}\, \mathbf{x} = \frac{1}{c}\int d^3x\, \partial_0 \bar{T}_S^{00}\, (\mathbf{x} - \mathbf{X}_{\text{CME}}), \tag{3}$$

where $\mathbf{V}_{\text{CME}} := d\mathbf{X}_{\text{CME}}/dt$ is the velocity of the CME of $\mathcal{S}$.

The fact that system $\mathcal{S}$ is not necessarily isolated means that $\partial_\mu \bar{T}_S^{\mu\nu} = f^\nu$, where f^μ is the four-force density acting on $\mathcal{S}$—in particular, f^0 is related to energy exchange rate W through $d^3x\, f^0 = dW/c$. Substituting $\partial_0 \bar{T}_S^{00} = f^0 - \partial_j \bar{T}_S^{j0}$ into Eq. (3) leads to

$$\begin{aligned} M\mathbf{V}_{\text{CME}} &- \frac{1}{c}\int d^3x\, f^0\, (\mathbf{x} - \mathbf{X}_{\text{CME}}) \\ &= -\frac{1}{c}\int d^3x\, \partial_j \bar{T}_S^{j0}\, (\mathbf{x} - \mathbf{X}_{\text{CME}}) \\ &= -\frac{1}{c}\int d^3x\, \partial_j [\bar{T}_S^{j0}\, (\mathbf{x} - \mathbf{X}_{\text{CME}})] + \frac{1}{c}\int d^3x\, \bar{T}_S^{j0}\, \partial_j \mathbf{x} \\ &= \int d^3x\, \mathbf{p} = \mathbf{P}, \end{aligned} \tag{4}$$

where $(\mathbf{p})^j = \bar{T}_S^{j0}/c$ are the components of the momentum density of the system.

So far, very little has been imposed on the system $\mathcal{S}$. In fact, the only assumptions are that the integrals above converge and that the surface term coming from the first integral in the right-hand side of the third line of Eq. (4) vanishes at spatial infinity. But now we restrict attention to closed systems, as defined earlier. The reason is that for closed systems the second term in the left-hand side of Eq. (4) is purely relativistic, since in the Newtonian regime n is proportional to mass density and, therefore, $f^0/c \propto \nabla_a \bar{n}^a \equiv 0$. This expresses the well-known fact that in Newtonian mechanics the total momentum of an arbitrary closed system (isolated or not) is completely encoded in the motion of its center of mass and its total mass—the center-of-mass theorem. In relativity theory, on the other hand, we see that asymmetric (with respect to the CME) exchanges of energy between $\mathcal{S}$ and the rest of the universe contribute to the total momentum of the system; now, $\mathbf{P}$ cannot be assessed only by keeping track of the system's mass-energy distribution. This motivates us to define the "hidden" part of the total momentum of the system as $\mathbf{P}_\text{h} := \mathbf{P} - M\mathbf{V}_{\text{CME}}$, which can then be calculated by

$$\mathbf{P}_\text{h} = -\frac{1}{c}\int d^3x\, f^0\, (\mathbf{x} - \mathbf{X}_{\text{CME}}) = -\frac{1}{c^2}\int dW (\mathbf{x} - \mathbf{X}_{\text{CME}}). \tag{5}$$

In words, the hidden momentum of a closed system is given by (minus $1/c^2$ times) the dipole moment (with respect to $\mathbf{X}_{\text{CME}}$) of its energy-exchange rate. Notice that our definition not only frees HM from being identified only in the rest frame of the system (where $\mathbf{V}_{\text{CME}} = \mathbf{0}$), but also shows that HM does not depend on the inner details of the system; it does not depend on the nature of T_S^{ab} (electromagnetic, thermal, mechanical, etc.), but only on how it fails to be conserved (see "Note added" section).

B. Hidden momentum in a heat-conducting bar

Applying the definition given in Eq. (5), or its discrete version

$$\mathbf{P}_\text{h} = -\frac{1}{c^2}\sum_j (\mathbf{x}_j - \mathbf{X}_{\text{CME}}) W_j, \tag{6}$$

to the system depicted in Fig. 1, we promptly obtain

$$\mathbf{P}_\text{h} = W\mathbf{D}/c^2. \tag{7}$$

Therefore, in the situation depicted in Fig. 1(b), dragging the total momentum $\mathbf{P} = M\mathbf{V} + \mathbf{P}_\text{h}$ of the bar at a constant velocity $\mathbf{V}$ leads to an angular momentum $\mathbf{L}$ which changes at a rate

$$\frac{d\mathbf{L}}{dt} = \mathbf{V} \times \mathbf{P} = \mathbf{V} \times \mathbf{P}_\text{h} = W\, \mathbf{V} \times \mathbf{D}/c^2.$$

Comparing this result with the torque given in Eq. (1), provided by the forces $\pm\mathbf{F}$, we see that everything fits perfectly: the torque provided by the forces seen from the moving frame is exactly the one needed to keep the spinless bar in uniform motion.

Obviously, Eq. (5) can also be applied to the magnet depicted in Fig. 3 [7]. Recalling that $\mathbf{j} = \nabla \times \mathbf{M}$, where $\mathbf{M}$ is the magnet's magnetization, we have

$$\begin{aligned} \mathbf{P}_\text{h} &= -\frac{1}{c^2}\int d^3x\, (\mathbf{j} \cdot \mathbf{E})\, (\mathbf{x} - \mathbf{X}_{\text{CME}}) \\ &= -\frac{1}{c^2}\int d^3x\, [\nabla \cdot (\mathbf{M} \times \mathbf{E})]\, (\mathbf{x} - \mathbf{X}_{\text{CME}}) \\ &= \frac{1}{c^2}\int d^3x\, (\mathbf{M} \times \mathbf{E}) = \frac{1}{c^2}\left(\int d^3x\, \mathbf{M}\right) \times \mathbf{E} \\ &= \frac{1}{c^2}\mathbf{m}_0 \times \mathbf{E}. \end{aligned} \tag{8}$$

Therefore, in the situation depicted in Fig. 3(b), the total angular momentum $\mathbf{L}$ of the magnet changes at a rate

$$\frac{d\mathbf{L}}{dt} = \mathbf{V} \times \mathbf{P} = \mathbf{V} \times \mathbf{P}_\text{h} = (\mathbf{V} \cdot \mathbf{E})\mathbf{m}_0/c^2,$$

which matches exactly the torque applied on the magnet according to the moving frame—see Sec. III.

V. HIDDEN BUT REAL AND OBSERVABLE

By now, we hope we have convinced the reader that HM, far from being a peculiar property of specific systems or interaction laws, is simply a legitimate relativistic contribution to total momentum coming from energy flows in a closed, nonisolated system—a distinction motivated solely by our Newtonian intuition. In this sense, not only is it ubiquitous in a relativistic world, but also it is as real as any other form of momentum. In fact, we present below a final conclusive evidence which corroborates this view: we show that HM can be converted into "regular" momentum and, as such, its existence has observable consequences.

Consider the same system depicted in Fig. 1(a) in the stationary regime of heat flow. Suppose, now, that the thermal contacts with both reservoirs are suddenly interrupted simultaneously in the bar's frame (e.g., by some clever preprogrammed mechanism attached to the bar itself which shields the contacts with some thermal insulator). Since this interruption can be performed without any external forces applied on the bar, its total momentum will not change in the process. However, once the thermal contacts are interrupted, the bar is isolated and, as such, the center-of-mass theorem must hold. This implies that the bar's initial total momentum, which was completely hidden, now has to manifest itself as motion of the bar's CME:

$$\mathbf{V}_{\rm CME} = \frac{\mathbf{P}_{\rm h}}{M} = \frac{W\mathbf{D}}{Mc^2},$$

where M is the mass of the bar in the situation depicted in Fig. 1(a). [Note that if, in preparing the setup represented in Fig. 1(a), we had not held the bar fixed until the stationary heat-flow regime had been established, the bar might have acquired a velocity in the opposite direction in order to keep its total momentum zero—in case thermal contacts are established with no net external forces acting on the bar in its rest frame. This is the reason for assumption (i) made in Sec. II.] Although this velocity is probably too small in realistic situations, it constitutes evidence that HM is not some imaginary concept with no objective existence. (Obviously, HM contained in the rest of the universe will also be converted into motion of the CME of the latter, but since the rest-of-the-universe mass is supposedly much larger than that of the system of interest such effect may be completely neglected.)

The same conclusion holds for the magnet represented in Fig. 3: *if* the electric field is removed without exerting net forces on the magnet—which may be difficult due to possibly inhomogeneous magnetic fields generated in the process—the magnet would acquire a velocity

$$\mathbf{V} = \frac{\mathbf{P}_{\rm h}}{M} = \frac{\mathbf{m}_0 \times \mathbf{E}}{Mc^2},$$

with M being the magnet's mass. This may be seen as somewhat similar to the Richardson–Einstein–de Haas effect [27,28], but now for linear instead of angular velocity or momentum. (In this hypothetical scenario, it seems reasonable to conjecture that the opposite momentum of the rest of the universe may become manifest through emission of electromagnetic waves.) Since both M and $\mathbf{m}_0$, in ideal situations, scale with the magnet's volume, we can make an order-of-magnitude overestimation for $V := \|\mathbf{V}\|$ using Bohr's magneton, $m_0 := \|\mathbf{m}_0\| \sim \mu_B \sim 10^{-13}$ GeV/T, and the proton's mass, $M \sim M_p \sim 1$ GeV/c^2: $V \sim [E/(1\,\text{kV/m})] \times 10^{-1}$ nm/s, where $E := \|\mathbf{E}\|$. As expected, the typical velocities for real magnets in realistic external electric fields are extremely small. However, one might try to amplify this effect by using systems in which M scales with size slower than does m_0—as in solenoids, in the macroscopic scenario, or Rydberg atoms, in the atomic realm.

VI. DISCUSSION

Although Mansuripur's speculation on alternative laws of electromagnetic force is a valid inquiry—which can only be definitely settled by experiments—the generic nature of the HCB paradox—with electromagnetism and moving inner parts playing no explicit essential role—shows that the existence of torques acting on spinless, uniformly moving objects is a ubiquitous feature of relativistic dynamics. As stated earlier, this torque ($\mathbf{T} = \mathbf{V}_{\rm CME} \times \mathbf{P}$) is responsible for translating the CME of the system (with velocity $\mathbf{V}_{\rm CME}$) along a direction which is not aligned to its total momentum ($\mathbf{P}$)—which exposes the existence of HM. The generalized definition of HM as

$$\mathbf{P}_{\rm h} := \mathbf{P} - M\mathbf{V}_{\rm CME},$$

proposed in Sec. IV A—which makes sense not only in the rest frame of the system (where $\mathbf{V}_{\rm CME} = \mathbf{0}$)—led to a formula relating HM with asymmetric (with respect to the system's CME) exchange of energy with the rest of the universe:

$$\mathbf{P}_{\rm h} = -\frac{1}{c^2}\sum_j (\mathbf{x}_j - \mathbf{X}_{\rm CME})W_j, \tag{9}$$

where $\mathbf{X}_{\rm CME}$ is the CME position and $\mathbf{x}_j$ is the position where energy exchange occurs at a rate W_j ($W_j > 0$, if energy enters the system; $W_j < 0$, if energy leaves the system). The interpretation is simple: this asymmetry leads to energy flows in the system which, regardless of their nature, contribute to momentum in very much the same way as do matter flows—thanks to mass-energy equivalence. As stressed earlier, distinguishing contributions to the total momentum coming from "different forms" of energy flows is quite contrary to the spirit of relativity theory—which is why a covariant, observer-independent definition of HM does not (and cannot) exist. Notwithstanding, although artificial and not strictly necessary, thinking in terms of HM may help our Newtonian intuition to spot effects which might pass unnoticed otherwise—as the one discussed in Sec. V.

Obviously, only experiments can decide on the correctness of candidate laws of nature. For instance, whether or not HM is in fact present in a magnet subject to an external electric field strongly depends on the ultimate origin of its magnetic dipole moment [15]. Here, in order to arrive at the well-known Eq. (8), we have adopted the standard view that any magnetic moment is ultimately due to electric currents, even if of a quantum nature. Notwithstanding, Eq. (9) is equally valid whatever is the microscopic modeling of magnetic dipoles. Be that as it may, the point is that aiming at substituting a law of force solely on the basis that it leads to HM is a misguided effort. As made explicit by the HCB paradox and Eq. (9), HM is simply an inevitable consequence of $E = mc^2$ when seen from an arbitrary inertial frame. Moreover, its existence can, in principle, be tested by forcing it to reveal itself as motion of the CME of the system—as shown in Sec. V. Looking at this the other way around, measuring the amount of HM in atomic-size "magnets" subject to external fields may end up being useful for investigating or confirming the ultimate nature of elementary magnetic moments.

Note added. Recently, Ref. [29] came to our attention, where "internal momentum," defined in a *similar* manner as HM here, is proposed to substitute for the latter. We essentially agree with the key points of Ref. [29], the attempt of which to demystify HM is aligned with ours. Notwithstanding, Ref. [29] still limits its discussion to an electrodynamic system, restricting energy exchanges to mechanical work: $f^0 = \mathbf{f} \cdot \mathbf{v}/c$. In particular, its definition of "internal momentum" [following from its Eq. (5)] would vanish for the scenario depicted in Fig. 1(a), in conflict with both our Eq. (7) and the observable effect predicted in Sec. V. The HCB paradox, the precise characterization of systems which bear HM, and the observational consequence of HM which we discuss in Sec. V constitute important ingredients to showing that HM (or "internal momentum") is a *generic* relativistic feature—as we had already claimed in Ref. [7].

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APPENDIX: ENERGY-MOMENTUM TRANSFER IN INELASTIC COLLISIONS AND FORCE EXCHANGE WITH THERMAL RESERVOIRS

Consider the symmetric process depicted in Fig. 5(a), where two identical particles with opposite momenta ($\mathbf{p}_2' = -\mathbf{p}_1'$) and vanishing total angular momentum collide with a surface at rest. We shall allow the collisions to be inelastic, each delivering an energy $\Delta E'/2$ into the surface. The symmetry of the setup makes it clear that, in this frame, the net momentum transferred to the surface is zero ($\Delta \mathbf{P}' = -(\Delta \mathbf{p}_1' + \Delta \mathbf{p}_2') = \mathbf{0}$). In Fig. 5(b), the *same* process is depicted as seen from an inertial frame with respect to which the surface moves with velocity $\mathbf{V}$. Obviously, the whole process is determined from its description above; all one has to do is to Lorentz transform the primed quantities to this frame. By doing so—which is a textbook exercise—, one realizes that the momentum exchanges between the particles and the surface are no longer symmetric ($\Delta \mathbf{p}_2 \neq -\Delta \mathbf{p}_1$) and that a net momentum $\Delta \mathbf{P} = \Delta E \mathbf{V}/c^2$ is transferred to the surface, where $\Delta E = \gamma \Delta E'$ is the net energy delivered into the surface in this frame (γ is the Lorentz factor). The implication is clear: inelastic collisions which are symmetric in the rest frame of the surface exert a *net force* on the surface when analyzed from inertial frames with respect to which the surface is moving.

Modeling a thermal reservoir as an isotropic bath of particles, the result above inevitably leads to the conclusion that an object static (and symmetrically immersed [31]) in a thermal reservoir is subject to a (purely relativistic) net force $\mathbf{F} = W\mathbf{V}/c^2$ when seen from an inertial frame with respect to which the whole system (object and reservoir) is moving with velocity $\mathbf{V}$, with W being the rate at which energy (heat) is absorbed by the object. Although this may sound odd at first, it becomes quite obvious when one realizes the *need* for an external force in order to keep the constant velocity of an object with increasing rest energy (i.e., rest mass). In fact, the existence of this relativistic force $\mathbf{F} = W\mathbf{V}/c^2$ can be inferred from this more general argument, independent of microscopic modeling of the reservoir (see Fig. 6). The importance of the microscopic collisional model is explicitly showing that the existence of such a force does *not* depend on the fate of the absorbed energy ΔE—for instance, whether it is accumulated

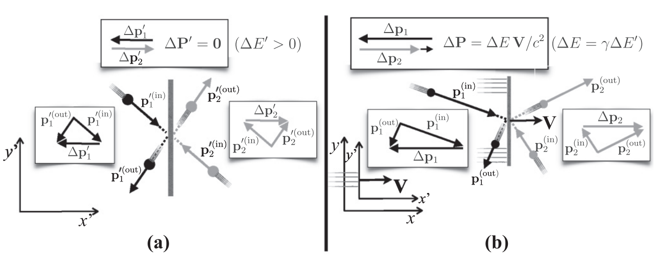

FIG. 5. (a) Symmetric inelastic collisions of two identical particles with a surface at rest. The symmetry of the setup leads to no net momentum transfer to the surface. (b) The same process analyzed from an inertial frame with respect to which the surface is moving with velocity $\mathbf{V}$. In this frame, the collisions are no longer symmetric and a net momentum $\Delta \mathbf{P} = \Delta E \mathbf{V}/c^2$ is transferred to the surface, where ΔE is the energy delivered into the surface during the process. Assuming processes like this occurring at a constant rate, a net force given by $\mathbf{F} = W\mathbf{V}/c^2$ would be exerted on the surface, where W is the rate at which energy is delivered into the surface. By modeling a thermal reservoir as an isotropic bath of particles, this implies that an object at rest in a thermal reservoir is subject to a net force $\mathbf{F} = W\mathbf{V}/c^2$ when seen from a reference frame with respect to which the whole system (object and reservoir) is moving—with W being the rate at which energy (heat) is absorbed by the object. Note that this force is exactly the one needed to keep an object with increasing mass, $dM/dt = W/c^2$, in uniform motion.

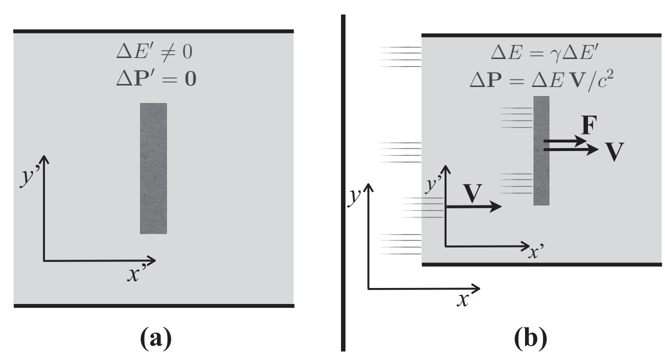

FIG. 6. Force applied on a moving object by a comoving thermal reservoir. (a) An object at rest in the reservoir's frame exchanges an amount $\Delta E'$ of energy in a time interval $\Delta t'$, with no momentum transfer (due to the symmetry of the reservoir in its rest frame). According to mass-energy equivalence, this corresponds to a (rest-)mass variation $\Delta M' = \Delta E'/c^2$. (b) The same situation seen from another reference frame: a variation $\Delta M = \gamma \Delta M'$ in the object's mass, at constant velocity $\mathbf{V}$, corresponds to a momentum transfer $\Delta \mathbf{P} = \Delta M \mathbf{V} = \gamma \Delta E' \mathbf{V}/c^2$ from the reservoir, in a time interval $\Delta t = \gamma \Delta t'$. Therefore, in this frame, the reservoir must (and does) apply a net force $\mathbf{F} = \Delta \mathbf{P}/\Delta t = \Delta E' \mathbf{V}/(c^2 \Delta t') = W\mathbf{V}/c^2$ on the object, where $W = \Delta E'/\Delta t' = \Delta E/\Delta t$ is the energy exchange rate. Causality and locality ensure that this final result cannot depend on whether the energy exchange $\Delta E'$ is accumulated in the object or if it is used to sustain a stationary heat flow, as in Fig. 1.

in the object or constantly drained to sustain a heat flow (as in Fig. 1) [32]. Therefore, although, strictly speaking, the situations depicted in Fig. 1 and in Figs. 5 and 6 represent different systems, the origin of the forces seen according to the moving frame is the same in all of them. For an interesting quantum microscopic scenario where this force is also needed to conciliate different inertial-frame descriptions, see Ref. [30].

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[31] If the immersion is not symmetric, there may appear an additional contribution related to the fact that the net force may not be zero in the rest frame of the system.
[32] The same conclusion can be reached in the more general argument presented in Fig. 6 by invoking locality and causality: the force exerted by the reservoir at the boundary of the object when the energy is exchanged cannot depend on the later use of this energy.