Editors' Suggestion

Quantum-optical realization of an Ornstein-Uhlenbeck-type process via simultaneous action of white noise and feedback

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(Received 24 December 2019; accepted 17 August 2020; published 10 September 2020)

Based on a Heisenberg equation of motion description of the feedback dynamics in quantum optics, we discuss the impact of phase noise on the excitation dynamics and provide examples in which noise itself is not detrimental but supports and enhances typical features of quantum feedback such as self-stabilization of the electronic population. We furthermore establish a connection between coherent quantum feedback and an Ornstein-Uhlenbeck-type process in quantum optics in the presence of phase noise. The interfering and time-shifted amplitudes introduce a finite memory kernel which, convoluted with a white noise process, results in a resonance fluorescence dynamics of a damped random walk.

DOI: 10.1103/PhysRevA.102.033712

Coherent quantum feedback mechanisms have been proven to be a versatile strategy to steer and control systems noninvasively [1–10], since they are not related to measurement-based or invasive quantum feedback control [11–14]. The coherent and non-Markovian nature imposes interferences between present and past quantum states onto the dynamics and allows for interesting time-ordered two-photon processes [7,8,10,15–18], enhanced entanglement and nonclassical photon statistics [19], dimerization [20,21], and a stabilization of quantum coherence due to interference effects between incoming and outgoing probability waves between the emitter (system) and the mirror (structured reservoir) [22–26]. A typical paradigm for such processes is the formation of dark states and subsequently emerging population trapping [10,27–30]. An important parameter in all of these examples is the feedback phase φ, imprinted by the feedback on the original system. It is given by $\varphi = \omega_0\tau$, i.e., the product of the delay time τ of the feedback and the transition frequency ω_0 of the controlled electronic coherence $P(t)$. Here, we discuss the interplay of the feedback phase and additional external phase noise on the electron population trapping and the resonance fluorescence dynamics in feedback-induced radiative decay processes and extend the current paradigm of coherent quantum feedback to genuine irreversible and incoherent processes such as phase noise, which has remained so far elusive. Technically, the key element is to use the Heisenberg picture which allows one to consider incoherent processes in contrast to purely unitary models such as matrix-product states [7,15,20,21,30,31] or the full integration of the corresponding Schrödinger equation [8,19,32] and without tracing out higher-order contributions in memory-expensive time-ordered time-evolving block decimation techniques [33]. We demonstrate that white noise-based energy fluctuations are not necessarily detrimental to the control success of quantum feedback, as it counteracts destructive and unwanted interference effects between the incoming and outgoing photon emission processes and even supports population trapping for certain quantum feedback phase relations. Furthermore, we establish an important connection between the non-Markovian quantum feedback process and the generation of an Ornstein-Uhlenbeck-type of noise correlations. Interestingly, this result can be attributed to a mirror charge-induced dipole-dipole correlation due to the mirror, as has been derived by Zwanzig [34], only here retarded in time and of purely single quantum nature.

I. MODEL

In order to discuss the impact of white phase noise on the emission dynamics of a two-level system in front of a mirror, we introduce the following Hamiltonian of the system ($\hbar = 1$):

$$H = (\omega_0 + F_t)P^\dagger P + \int d\omega \left[r^\dagger_\omega \left(\frac{\omega r_\omega}{2} + g^*_\omega P \right) + \text{H.c.} \right], \quad (1)$$

where $P = |g\rangle\langle e|$ denotes the microscopic coherence operator from the excited state $|e\rangle$ to the ground state $|g\rangle$ of the two-level system with a transition energy of $\hbar\omega_0$. The radiative continuum is included via the photon creation and annihilation operators $r^{(\dagger)}_\omega$ for a photon in the mode $\omega = c|k|$ (c is the speed of light in the waveguide) with bosonic commutation relations: $[r_\omega, r^\dagger_{\omega'}] = \delta(\omega - \omega')$. The coupling between the emitter and the radiative continuum is denoted by $g_\omega = g_0 \sin(\omega\tau/2)$ [1–9,35] and includes the mirror-imposed boundary condition at a distance L between the mirror and an atom with a strength of g_0. The distance defines the feedback round-trip time with

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$\tau = 2L/c$. F_t describes a stochastic force acting upon the excited level of the two-level system and models, e.g., a spectral diffusion process [36–38]. Next, we solve this model in the Heisenberg picture [39].

The equation of the coherence operator in the Heisenberg picture reads in the corresponding rotating frame as follows:

$$P^\dagger(t) = iF_t P^\dagger(t) + i\int d\omega g_\omega^* e^{i(\omega-\omega_0)t} r_\omega^\dagger(t)[P(t), P^\dagger(t)]. \quad (2)$$

The coherence operator couples to the inversion and to the quantized light field. As we are interested only in the system's dynamics, we can integrate out the reservoir's degrees of freedom and obtain the corresponding Heisenberg-Langevin equation of motion. Within the one-electron assumption for the two-level system [40], the inversion operator can be written as $[P(t), P^\dagger(t)] = 1 - 2P^\dagger(t)P(t)$ and for the dynamics of the coherence operator it follows

$$\begin{aligned}\dot{P}^\dagger(t) = & -[\Gamma - iF_t]P^\dagger(t) + \Gamma e^{-i\varphi}P^\dagger(t-\tau)\theta(t-\tau) \\ & - 2\Gamma e^{-i\varphi}P^\dagger(t-\tau)P^\dagger(t)P(t)\theta(t-\tau) \\ & + ig_0 R^\dagger(t)[P(t), P^\dagger(t)], \end{aligned} \quad (3)$$

where $R^\dagger(t) = \int d\omega r_\omega^\dagger(0)\sin(\omega\tau/2)\exp[i(\omega-\omega_0)t]$ includes the quantum noise contribution to conserve the commutation for all times with $\Gamma = g_0^2\pi/2$ and $\varphi = \omega_0\tau$, the aforementioned feedback phase. Clearly, the signal $P^\dagger$ at the feedback delay time τ occurs in Eq. (3). In Appendix A it is shown that the second line and the last (third) line vanish in the case of a reservoir initially in the vacuum state and a system described by the Hamiltonian in Eq. (1). The differential equation of the matrix elements of the microscopic coherence operator P_{eg} is therefore given for all times t via

$$\dot{P}_{eg}^*(t) = (iF_t - \Gamma)P_{eg}^*(t) + \Gamma e^{-i\varphi}P_{eg}^*(t-\tau)\theta(t-\tau), \quad (4)$$

where we have taken the matrix elements of the coherence operator $P_{ij}^*(t) = \langle i, \text{vac}|P^\dagger(t)|j, \text{vac}\rangle$, with $|j, \text{vac}\rangle = |j\rangle_S|\text{vac}\rangle_R$ and j being either e or g for the system state and the reservoir in the vacuum state. In the following, we discuss results based on this equation of motion of the coherence operator, namely, in Sec. II the conditions for a nontrivial steady condition ($|P_{eg}^*(t)|^2 > 0$) in the absence of noise, in Sec. III the resonance fluorescence dynamics in the presence of phase noise, in Sec. IV the emergence of Ornstein-Uhlenbeck noise correlation due to feedback, and in Sec. V the stabilized population trapping due to noise in the case of $\varphi \mod 2\pi \neq 0$.

II. STEADY-STATE ANALYSIS WITHOUT NOISE ($F_t \equiv 0$)

The dynamics of the coherence operator can be analytically solved via the Laplace transformation for vanishing phase noise [1,26,35,41]. The solution is lengthy and given in Appendix A. For the steady-state analysis, we can assume $t \gg \tau$ and approximate the equation of motion in Eq. (4) via a Taylor-expansion:

$$\dot{P}_{eg}^*(t) \approx -\Gamma\alpha P_{eg}^*(t) - \tau\Gamma e^{-i\varphi}P_{eg}^*(t)\delta(t), \quad (5)$$

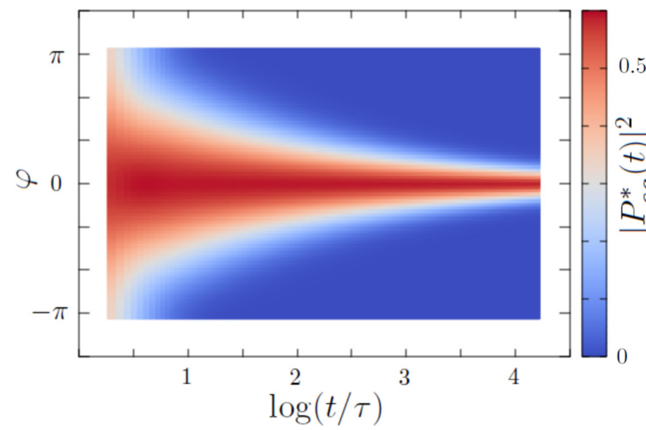

FIG. 1. The dynamics of the population without phase noise for an initially excited emitter and different phases $\varphi = \omega_0\tau$ from $-\pi$ to π and times natural logarithm $\log(t/\tau)$. In the long time limit, only the phase $\varphi = 2\pi n$ with n integer survives. For all other phases, the destructive interference leads to a trivial steady state of $|P_{eg}^*(t)|^2 = 0$, as discussed in the main text.

with $\alpha := (1 - e^{-i\varphi})/(1 + \Gamma\tau e^{-i\varphi})$. The solution is derived via integration and reads as follows:

$$|P_{eg}^*(t)|^2 = e^{-2\Gamma\text{Re}[\alpha]t}\left|\frac{P_{eg}^*(0)}{1 + \Gamma\tau e^{-i\varphi}}\right|^2. \quad (6)$$

Obviously, only for vanishing Re[α], i.e., a specific choice of the product of transition frequency ω_0 and τ, we find a finite, stationary occupation probability in the emitter in the long time limit: $t\Gamma \gg 1$. This condition reads explicitly as

$$\text{Re}[\alpha] = \frac{[1 - \Gamma\tau][1 - \cos(\varphi)]}{1 + 2\Gamma\tau\cos(\varphi) + \Gamma^2\tau^2} = 0. \quad (7)$$

To achieve a finite steady-state occupation Re[α] = 0, i.e., $\varphi \mod (2\pi) = 0$ must be valid. In all other cases, if no phase noise is present, the steady-state occupation is zero, as demonstrated in Fig. 1. We include now phase noise in our investigation and find indications pointing towards finite occupation probabilities for a much wider range of phase choices φ.

III. RESONANCE FLUORESENCE DYNAMICS WITH NONVANISHING PHASE NOISE ($F_t \neq 0$)

For nonvanishing noise $F_t \neq 0$, we iteratively solve the equation of motion via subsequent integration with respect to time (Appendix B), e.g. for $t \in [0, 3\tau]$:

$$\begin{aligned}P_{eg}^*(t) = & e^{-\Gamma t + i\phi(t,0)}[\theta(t) + \theta(t-\tau)\Gamma e^{-i\varphi+\Gamma\tau}N(t,\tau) \\ & + \theta(t-2\tau)(\Gamma e^{-i\varphi+\Gamma\tau})^2 M(t, 2\tau)],\end{aligned} \quad (8)$$

with $\phi(b, a) = \int_a^b F_{t'}dt'$ and the memory kernel amplitudes with definitions

$$N(t,\tau) := \int_\tau^t dt_1 e^{-i\phi(t_1, t_1-\tau)},$$

$$M(t, 2\tau) := \int_{2\tau}^t dt_1 e^{-i\phi(t_1,t_1-\tau)}\int_\tau^{t_1-\tau} dt_2 e^{-i\phi(t_2,t_2-\tau)},$$

which recover in the limit of $\phi(b,a) \to 0$ the analytical solution without noise, i.e., $N(t,\tau) = (t-\tau)$ and $M(t,2\tau) = (t-2\tau)^2/2$. This step by step solution of Eq. (8), ordered in time intervals, is necessary as we cannot perform a straightforward solution via a Laplace transform due to the time-dependent and implicitly given stochastic function F_t, rendering the solution via the Lambert W function inaccessible [1,26,35,41].

The noise increment $\phi(b,a)$ introduces an incoherent element and leads after averaging to a mixed state. This already indicates that phase noise is a process acting on the expectation value of the observable [36,42,43]. Since we are interested in the resonance fluorescence dynamics $\langle P^\dagger(t)P(t')\rangle$, we cannot take the average of the matrix element of the microscopic coherence operator before the expectation value is calculated because $\langle\langle |P^*_{eg}(t)|^2\rangle\rangle \neq |\langle\langle P^*_{eg}(t)\rangle\rangle|^2$. Due to this stochasticity of F_t in Eq. (8), we cannot replace $\phi(b,a)$ with a c-number, and a brute force numerical evaluation becomes conceptually difficult. However, given the analytical evolution of the matrix element of the Heisenberg operator, we can circumvent this problem and derive the dynamics recursively in the different time intervals. We find, for example, for the population dynamics, the following expression for $(2 \leqslant (t/\tau) \leqslant 3)$:

$$\langle\langle |P^*_{eg}(t)|^2\rangle\rangle = e^{-2\Gamma t}[1 + 2\Gamma e^{\Gamma\tau}\cos\varphi\langle\langle N(t,\tau)\rangle\rangle + \Gamma^2 e^{2\Gamma\tau}\langle\langle N(t,\tau)N^*(t,\tau)\rangle\rangle + 2\Gamma^2 e^{2\Gamma\tau}\cos(2\varphi)\langle\langle M(t,2\tau)\rangle\rangle + 2\Gamma^3 e^{3\Gamma\tau}\cos\varphi\langle\langle N^*(t,\tau)M(t,2\tau)\rangle\rangle + \Gamma^4 e^{4\Gamma\tau}\langle\langle M(t,2\tau)M^*(t,2\tau)\rangle\rangle]. \tag{9}$$

Evaluating the noise integrals involves up to four time-ordered integrals and their corresponding noise-noise correlations, cf. Appendix C. This yields lengthy expressions which are given explicitly in Appendix D. We like to point out that Eq. (9) nonperturbatively combines the non-Markovian feedback dynamics, the radiative decay, and stochastically averaged phase noise together and is the main result of this study. We hereby extend on-going efforts in the field of coherent quantum feedback and studies of differential-delay equations by including phase noise resulting in a pure dephasing, i.e., an incoherent process.

In the following, we discuss the population dynamics on the basis of Eq. (9) for two cases in the presence of finite phase noise with the vanishing average $\langle\langle F_t\rangle\rangle = 0$ and the δ-correlated correlation function $\langle\langle F_t F_s\rangle\rangle = \gamma\delta(t-s)$. First, in Sec. IV and for $1 \leqslant (t/\tau) \leqslant 2$, we show that the quantum feedback contribution leads to a Ornstein-Uhlenbeck (O-U) process for the population dynamics due to the assumed white noise correlation. Second, in Sec. V and for $0 \leqslant (t/\tau) \leqslant 3$, we discuss the impact of phase noise and show that it need not necessarily be detrimental to quantum feedback stabilization of quantum states, since disadvantageous destructive interferences, if they occur, are substantially suppressed. These time intervals have become recently accessible in the field of superconducting circuits where surface acoustic waves at two points lead to a significant internal time-delay and give rise to non-Markovian dynamics of exactly the discussed form [44,45].

IV. FEEDBACK-INDUCED ORNSTEIN-UHLENBECK PROCESS

We focus first on the special case of $\varphi \mod (2\pi) = 1$ and $1 \leqslant (t/\tau) \leqslant 2$ which implies $M(t,2\tau) \equiv 0$. The solution of the population dynamics then reads as follows:

$$\frac{\langle\langle |P^*_{eg}(t)|^2\rangle\rangle}{e^{-2\Gamma(t-\tau)}} = e^{-2\Gamma\tau} + \Gamma^2\frac{2}{\gamma^2}[\gamma(t-\tau) + e^{-\gamma(t-\tau)} - 1],$$

where we perform the stochastic averaging for a Gaussian random variable in Eq. (9). The detailed evaluation for the noise correlations is given in Appendix D. Since there are no length restrictions for the round-trip time τ, we can safely study the limits $2\Gamma\tau \gg 1$ and $\gamma\tau \gg 1$. We introduce a shifted time towards the first τ interval, i.e., $t' = t - \tau$ and $t' \in [0,\tau]$, and we find

$$\frac{\langle\langle |P^*_{eg}(t'+\tau)|^2\rangle\rangle}{e^{-2\Gamma t'}} = \frac{2\Gamma^2}{\gamma^2}(\gamma t' + e^{-\gamma t'} - 1). \tag{10}$$

Despite the trivial envelope of the exponential decay, this equation bears a clear resemblance to the O-U process with a noise correlation $\langle\langle F_t F_s\rangle\rangle|_{O\text{-}U} = \Gamma\exp[-\gamma|t-s|]$ and a resulting mean-squared displacement of a Brownian particle, using the Einstein-Smoluchowski relation [46–48] $\langle\langle [x(t)]^2\rangle\rangle = 6k_BT/(m\gamma^2)[\gamma t + e^{-\gamma t} - 1]$, where we can identify $3k_BT/m \to \Gamma^2$. Thus, Eq. (10) constitutes a quantum-optical realization of the O-U process. In other words, for nonvanishing noise $F_t \neq 0$, the initial, nonconvoluted white noise contribution in the emission process takes the form of an Ornstein-Uhlenbeck-type process. Here, however, it results from the interfering time-shifted amplitudes of the system's past and their phase relation. These interfering parts depend on their own time and the time difference to the conjugated amplitude and yield therefore a time-integrated Ornstein-Uhlenbeck correlation. This has interesting implications for the interpretation and application of the U-O process and its role in quantum mechanics and quantum optics in particular [12,36,42,49–51], if non-Markovianity and irreversibility are equally assumed [13,14,42,52–56].

V. SUPPRESSION OF DESTRUCTIVE INTERFERENCES

Furthermore, the Heisenberg equation of motion formulation of coherent quantum feedback in Eq. (9) allows us to investigate the role of phase noise in typical radiative decay processes in the presence of delay and its impact on the population trapping conditions derived before for vanishing phase noise. In Sec. II, we have shown, that only for the case of $\varphi \mod (2\pi) = 0$ does stabilization of the emitter population occur. We show now that we find first indications that this condition is relaxed in the presence of phase noise due to the phase fluctuations F_t. Since it is often the goal of quantum feedback to stabilize electronic populations due to destructive

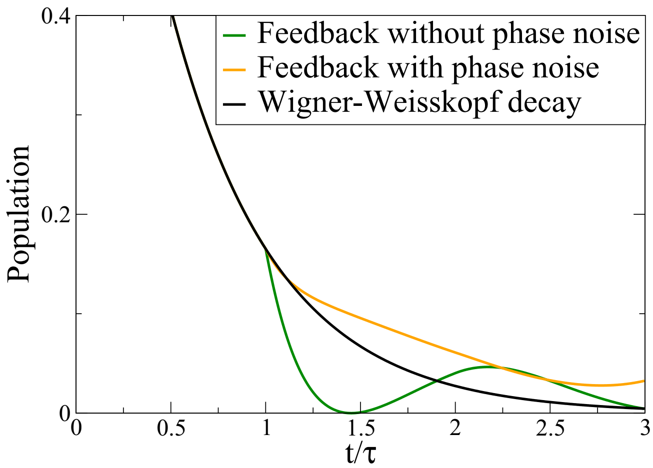

FIG. 2. The population dynamics in the interval $t/\tau \in [0, 3]$ and for the phase $\varphi = \omega_0\tau = 3.3$ in the case of the Wigner-Weisskopf decay without feedback (black line), with feedback but without phase noise (green line), and with feedback and phase noise (orange line). Interestingly, phase noise suppresses detrimental oscillations occurring due to the phase choice and remains well-above the Wigner-Weisskopf case.

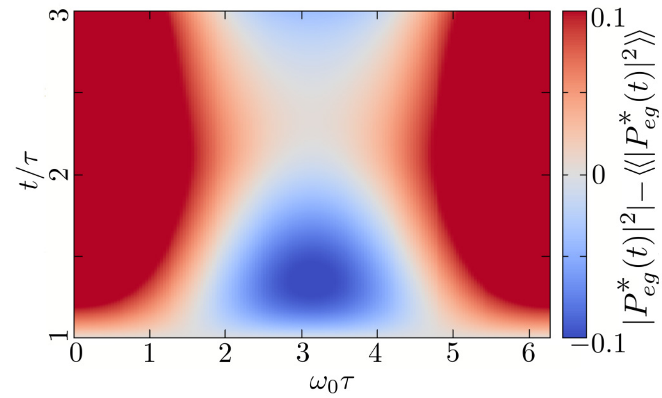

FIG. 3. The difference between the population dynamics in the interval $t/\tau \in [1, 3]$ with $\langle\langle|P^*_{eg}(t)|^2\rangle\rangle$ and without phase noise $|P^*_{eg}(t)|^2$ for different phase choices $\varphi = \omega_0\tau$. Phase noise is advantageous for population trapping if the phase value is in the interval $[\pi/2, 3\pi/2]$.

interference between absorption and reemission events [7,26], this is an important and positive result for the field of coherent quantum control.

In Fig. 2, the dynamics of the population is depicted for the case without phase noise and without feedback (Wigner-Weisskopf case, black line), without phase noise but with feedback (green line), and with feedback and with noise (orange line) for a phase of $\varphi = 3.3$. As can be seen, phase noise helps to suppress the destructively interfering parts of the solution in Eq. (9) proportional to $\cos(n\varphi)$ with n integer, cf. Appendix E. These contributions enforce damped oscillations of the population, leading eventually to a complete decay of the electronic excitation into the reservoir with zero excitation left in the emitter (green line), since $\varphi \mod (2\pi) \neq 0$. However, these contributions are strongly affected by the phase noise (orange line). Here, in the transient regime, noise helps to slow down the decay of the electronic population and prevents it from decaying rapidly to zero (orange line). In Fig. 2, the population in the emitter in the case of finite F_t is larger or for a short time ($t/\tau \approx 2.4$) equal or slightly less compared to the case with vanishing phase noise. As a comparison, we plot the dynamics imposed by just the Wigner-Weisskopf case (black line). The case with phase noise always stabilizes a population larger than the Wigner-Weisskopf dynamics, whereas the case with feedback and no phase noise oscillates following the decay of the Wigner-Weisskopf solution, indicating an inevitable complete decay for undisturbed feedback. Interestingly, the solution with feedback shows a nonmonotonic behavior to the end of the third τ interval where population is gained. Of course, these results depend on the choice of the feedback phase $\varphi = \omega_0\tau$. This indicates that the choice of the delay time provides another control parameter to optimize the phase noise action on the population number: In Fig. 3, we plot the difference between the population with and without noise: $|P^*_{eg}(t)|^2 - \langle\langle|P^*_{eg}(t)|^2\rangle\rangle$. We clearly see that phase noise reduces the amount of population which can be trapped in the emitter in the vicinity of the ideal phase choice of $\varphi = 0$, as this phase choice renders the destructive interference terms already maximally unimportant, i.e., for $\omega_0\tau \in (0, \pi/2)$ and $\omega_0\tau \in (3\pi/2, 2\pi)$. However, for phases in between $(\pi/2, 3\pi/2)$, the population is enhanced due to noise. We conclude that for $\varphi \neq 2\pi$, phase noise still allows for important feedback effects relying on the population trapping mechanism.

VI. CONCLUSION

We have studied the impact of white noise on the radiative decay dynamics of an atom in front of a mirror providing radiative feedback. Our method in the Heisenberg picture allows one to consider incoherent processes. As an example, we have shown that the non-Markovian feedback acts essentially as a low-pass filter for the initially uncorrelated white noise [57,58] and leads due to its intrinsic time-ordering to a Ornstein-Uhlenbeck-type process within the time interval $1 \leqslant (t/\tau) \leqslant 2$, where τ can be chosen arbitrarily to be long. Furthermore, we showed that the impact of phase noise on the population trapping dynamics is advantageous for a wide range of phase choices.

ACKNOWLEDGMENTS

A.C. and A.K. gratefully acknowledge support from the Deutsche Forschungsgemeinschaft (DFG) through Project 163436311 (SFB 910) and from the European Union's Horizon 2020 research and innovation program under SONAR Grant No. 734690.

APPENDIX A: DERIVATION OF THE HEISENBERG EQUATION OF MOTION FOR THE COHERENCE OPERATOR

The equation of the Heisenberg operator $P^\dagger(t) = U^\dagger(t)P^\dagger U(t)$ with $U(t) = \exp[-iHt]$ using $\dot{P}^\dagger(t) =$

$i[H, P^\dagger(t)]$ reads in the rotating-frame as follows:

$$\dot{P}^\dagger(t) = iF_t P^\dagger(t) + i\int d\omega g^*_\omega e^{i(\omega-\omega_0)t} r^\dagger_\omega(t)[P(t), P^\dagger(t)]. \tag{A1}$$

The coherence operator couples to the inversion and to the quantized light field. Starting with an initial condition at $t = 0$, the goal is to solve for the quantized light field exactly by integrating out the equation of motion of the photon creation operator:

$$r^\dagger_\omega(t) = r^\dagger_\omega(0) + ig_\omega \int_0^t dt_1 e^{-i(\omega-\omega_0)t_1} P^\dagger(t_1). \tag{A2}$$

This equation allows us to write down the Heisenberg-Langevin equation of motion. Within the one-electron assumption for the two-level system, the inversion operator can be written as $[P(t), P^\dagger(t)] = 1 - 2P^\dagger(t)P(t)$ and for the dynamics of the coherence operator it follows

$$\begin{aligned}\dot{P}^\dagger(t) =& -[\Gamma - iF_t]P^\dagger(t) + \Gamma e^{-i\omega_0\tau} P^\dagger(t-\tau)\theta(t-\tau)\\ &- 2\Gamma e^{-i\omega_0\tau} P^\dagger(t-\tau)P^\dagger(t)P(t)\theta(t-\tau)\\ &+ ig_0 R^\dagger(t)[P(t), P^\dagger(t)],\end{aligned} \tag{A3}$$

where $R^\dagger(t) = \int d\omega r^\dagger_\omega(0)\sin(\omega\tau/2)\exp[i(\omega-\omega_0)t]$ includes the quantum noise contribution to conserve the commutation for all times with $\Gamma = g_0^2\pi/2$. Clearly, the signal P at the feedback delay time τ occurs in Eq. (A3).

In the following, we show that the second line and the last (third) line vanish in the case of a reservoir initially in the vacuum state and a system described by the Hamiltonian in Eq. (1).

The solution of Eq. (A3) is derived for every τ interval iteratively [1,26,35]. For the time interval $t \in [0, \tau]$, we evaluate the matrix element of the coherence operator $P^*_{ij}(t) = \langle i, \mathrm{vac}|P^\dagger(t)|j, \mathrm{vac}\rangle$ with $|j, \mathrm{vac}\rangle = |j\rangle_S|\mathrm{vac}\rangle_R$ and j being either e or g for the system state and the reservoir in the vacuum state. The dynamics of the polarization reduces to

$$\dot{P}^*_{ij}(t) = -[\Gamma - iF_t]P^*_{ij}(t), \tag{A4}$$

$$P^*_{ij}(t) = e^{-\Gamma t + i\phi(t,0)} P^*_{ij}(0), \tag{A5}$$

contributing only for $i = e$ and $j = g$ and $\phi(b, a) := \int_a^b F_{t'}dt'$. Note that the matrix element does not represent the expectation value. However, the expectation value can be fully expressed by its corresponding matrix elements, e.g., $\langle P^\dagger(t)P(t)\rangle = |P^*_{eg}(t)|^2$. For the second interval, $t \in [\tau, 2\tau]$, the dynamics of the matrix element reads as follows:

$$\begin{aligned}\dot{P}^*_{ij}(t) =& -\Gamma P^*_{ij}(t) + \Gamma e^{-i\omega_0\tau} P^*_{ij}(t-\tau)\\ &- 2\Gamma e^{-i\omega_0\tau}\langle i, \mathrm{vac}|P^\dagger(t-\tau)P^\dagger(t)P(t)|j, \mathrm{vac}\rangle.\end{aligned} \tag{A6}$$

Due to the occurring time delay, we can use P_{ij} for $i = e$ and $j = g$ from Eq. (A5) in Eq. (A6) to evaluate the second line. For this, we insert now the unity relation $\mathbb{1} = \sum_{i=e,g} |i\rangle_{SS}\langle i| \otimes (|\mathrm{vac}\rangle_{RR}\langle \mathrm{vac}| + \int d\omega |1_\omega\rangle_{RR}\langle 1_\omega|)$ to evaluate the correlation between the "time-nonlocal" microscopic coherence and the time-local population density, and we take into account that only $\langle i, \mathrm{vac}|P^\dagger(t-\tau)|g, \mathrm{vac}\rangle$ can contribute nontrivially:

$$\begin{aligned}&\langle i, \mathrm{vac}|P^\dagger(t-\tau)P^\dagger(t)P(t)|j, \mathrm{vac}\rangle\\ &\quad = \langle i, \mathrm{vac}|P^\dagger(t-\tau)|g, \mathrm{vac}\rangle\langle g, \mathrm{vac}|P^\dagger(t)P(t)|j, \mathrm{vac}\rangle,\end{aligned} \tag{A7}$$

having reduced the problem to the matrix element $\langle g, \mathrm{vac}|P^\dagger(t)P(t)|j, \mathrm{vac}\rangle$. If we now again insert a unity operator between the operators $P^\dagger(t)P(t)$, we reduce this quantity again into further products of matrix elements. Since we know that only $P^*_{eg}(t)$ contributes initially in the first time interval, the quantity vanishes identically in the case of the Hamiltonian dynamics in Eq. (1) due to $\langle g, \mathrm{vac}|P^\dagger(t)|\phi\rangle = 0$ for arbitrary $|\phi\rangle$, and therefore we can conclude that, assuming an initially empty reservoir, the matrix elements of the microscopic coherence operator are governed by the dynamics for all times t:

$$\dot{P}^*_{eg}(t) = (iF_t - \Gamma)P^*_{eg}(t) + \Gamma e^{-i\omega_0\tau} P^*_{eg}(t-\tau)\theta(t-\tau).$$

In the case of $F_t \equiv 0$, this equation can be solved in the Laplace domain [1,8,26,27,41], yielding the following known dynamics valid for all t:

$$P^*_{eg}(t) = \sum_{n=0}^{\infty} \frac{e^{-\Gamma t}}{n!}[\Gamma e^{-i\omega_0\tau + \Gamma\tau}(t - n\tau)]^n \Theta(t - n\tau). \tag{A8}$$

APPENDIX B: EQUATION OF MOTION OF THE COHERENCE

The equation of motion for the microscopic coherence operator in the zeroth τ interval $t \in [0, \tau]$ reads as follows:

$$\dot{P}^\dagger(t) = (iF_t - \Gamma)P^\dagger(t) \rightarrow P^\dagger(t) = e^{-\Gamma t + i\phi(t,0)}P^\dagger(0),$$

with $\phi(t, 0) = \int_0^t F_s ds$. In the first τ interval, $t \in [\tau, 2\tau]$, the equation contains a feedback contribution:

$$\dot{P}^\dagger(t) = (iF_t - \Gamma)P^\dagger(t) + \Gamma e^{-i\omega_0\tau} P^\dagger(t-\tau)\theta(t-\tau).$$

After formal integration, we yield

$$\dot{P}^\dagger(t) = e^{-\Gamma t + i\phi(t,0)}\left(P^\dagger(0) + \Gamma e^{-i\omega_0\tau}\int_\tau^t e^{i\phi(t'-\tau,t')}dt'\right).$$

The solution up to the second τ interval $t \in [0, 3\tau]$ then reads as follows:

$$P_{eg}^{\dagger}(t) = e^{-\Gamma t + i\phi(t,0)}\bigg[\theta(t) + \theta(t-\tau)\Gamma e^{-i\omega_0\tau+\Gamma\tau}\int_{\tau}^{t} dt_1 e^{-i\phi(t_1,t_1-\tau)} + \theta(t-2\tau)(\Gamma e^{-i\omega_0\tau+\Gamma\tau})^2\int_{2\tau}^{t} dt_1 e^{-i\phi(t_1,t_1-\tau)}\int_{\tau}^{t_1-\tau} dt_2 e^{-i\phi(t_2,t_2-\tau)}\bigg]. \tag{B1}$$

In the limit $F(t) \to 0$, we yield the known noise-free solution.

APPENDIX C: EXAMPLES OF THE NOISE CORRELATIONS

In the following we assume Gaussian white noise, i.e., $\langle\langle \phi(a, b)\phi(c, d)\rangle\rangle = \gamma(\min[a, c] - \max[b, d])$, assuming a Gaussian random variable with $\langle\langle F(t_1)\rangle\rangle = 0$ and nonempty overlap between intervals $[a, b]$ and $[c, d]$. Given this noise correlation, we find

$$\begin{aligned}
\langle\langle e^{i\phi(s'+\tau,s')}\rangle\rangle &= \exp\left[-\frac{1}{2}\langle\langle \phi^2(s'+\tau, s')\rangle\rangle\right] = \exp\left[-\frac{\gamma}{2}\tau\right],\\
\langle\langle e^{-i\phi(s+\tau,s)+i\phi(s'+\tau,s')}\rangle\rangle &= \exp\left[-\frac{1}{2}\langle\langle(\phi(s+\tau, s) - \phi(s'+\tau, s')^2\rangle\rangle\right]\\
&= \exp\left[-\frac{1}{2}\langle\langle \phi^2(s'+\tau, s')\rangle\rangle\right]\exp\left[-\frac{1}{2}\langle\langle \phi^2(s+\tau, s)\rangle\rangle\right]\exp[\langle\langle \phi(s+\tau, s)\phi(s'+\tau, s')\rangle\rangle]\\
&= \exp\left[-\frac{\gamma}{2}\tau\right]\exp\left[-\frac{\gamma}{2}\tau\right]\exp\{\gamma[\min(s+\tau, s'+\tau) - \max(s, s')]\}\\
&= \exp\{-\gamma[\max(s, s') - \min(s, s')]\}.
\end{aligned}$$

APPENDIX D: EQUATION OF MOTION OF THE NOISE-AVERAGED POPULATION

For the population dynamics, we find the following expression for $t \in [0, 3\tau]$:

$$\begin{aligned}
\langle\langle |P_{eg}(t)^{\dagger}|^2\rangle\rangle = e^{-2\Gamma t}[&1 + 2\Gamma e^{\Gamma\tau}\cos(\omega_0\tau)\langle\langle N(t,\tau)\rangle\rangle\Theta(t-\tau) + \Gamma^2 e^{2\Gamma\tau}\langle\langle N(t,\tau)N^*(t,\tau)\rangle\rangle\Theta(t-\tau)\\
&+ 2\Gamma^2 e^{2\Gamma\tau}\cos(2\omega_0\tau)\langle\langle M(t,2\tau)\rangle\rangle\Theta(t-2\tau) + 2\Gamma^3 e^{3\Gamma\tau}\cos(\omega_0\tau)\langle\langle N^*(t,\tau)M(t,2\tau)\rangle\rangle\Theta(t-2\tau)\\
&+ \Gamma^4 e^{4\Gamma\tau}\langle\langle M(t,2\tau)M^*(t,2\tau)\rangle\rangle\Theta(t-2\tau)],
\end{aligned}$$

with the definitions

$$N(t,\tau) := \int_{\tau}^{t} dt_1 e^{-i\phi(t_1,t_1-\tau)}, \quad M(t,2\tau) := \int_{2\tau}^{t} dt_1 e^{-i\phi(t_1,t_1-\tau)}\int_{\tau}^{t_1-\tau} dt_2 e^{-i\phi(t_2,t_2-\tau)}.$$

Evaluating the noise integrals, we have

$$\begin{aligned}
\langle\langle N(t,\tau)\rangle\rangle &= e^{-\frac{\gamma}{2}\tau}(t-\tau)\\
\langle\langle M(t,2\tau)\rangle\rangle &= \frac{1}{\gamma^2}[e^{-\gamma(3\tau-t)} - e^{-\gamma\tau} - \gamma(t-2\tau)e^{-\gamma\tau}]\\
\langle\langle N(t,\tau)N^*(t,\tau)\rangle\rangle &= \frac{2}{\gamma^2}[\gamma(t-\tau) - 1 + e^{-\gamma(t-\tau)}]\\
\langle\langle N(t,\tau)M^*(t,2\tau)\rangle\rangle &= \frac{1}{6}e^{-\frac{\gamma\tau}{2}}(t-2\tau)^2(t+\tau) + \frac{e^{-\frac{\gamma\tau}{2}}}{4\gamma^3}[1 - e^{-2\gamma(t-2\tau)} - 2\gamma(t-2\tau) + 2\gamma^2(t-2\tau)^2]\\
\langle\langle M(t,2\tau)M^*(t,2\tau)\rangle\rangle &= \frac{1}{\gamma^4}[e^{\gamma(t-2\tau)} - 1 - 3\gamma(t-2\tau)] + \frac{1}{\gamma^4}\left[\frac{1}{2}(e^{-2\gamma(t-2\tau)} - 1) - 3(e^{-\gamma(t-2\tau)} - 1)\right].
\end{aligned}$$

Approximating every quantity corresponding to its prefactor in γ, i.e., a quantity with a prefactor of γ^{-4} must be expanded up to the fourth order, we yield the following for the noise expressions:

$$\langle\langle N(t,\tau)\rangle\rangle|_{\gamma\to 0} = (t-\tau), \tag{D1}$$

$$\langle\langle M(t,2\tau)\rangle\rangle|_{\gamma\to 0} = \tfrac{1}{2}(t-2\tau)^2, \tag{D2}$$

$$\langle\langle N(t,\tau)N^*(t,\tau)\rangle\rangle|_{\gamma\to 0} = (t-\tau)^2, \tag{D3}$$

$$\langle\langle N^*(t,\tau)M(t,2\tau)\rangle\rangle|_{\gamma\to 0} = \tfrac{1}{2}(t-2\tau)^2(t-\tau), \tag{D4}$$

$$\langle\langle M(t,2\tau)M^*(t,2\tau)\rangle\rangle|_{\gamma\to 0} = \tfrac{1}{4}(t-2\tau)^4. \tag{D5}$$

For the population dynamics with $\omega_0\tau = \pi/2$, we find the following expression:

$$\langle\langle |P_{eg}(t)^\dagger|^2\rangle\rangle| = e^{-2\Gamma t}[1 - 2\Gamma^2 e^{2\Gamma\tau}\langle\langle M(t,2\tau)\rangle\rangle + \Gamma^2 e^{2\Gamma\tau}\langle\langle N(t,\tau)N^*(t,\tau)\rangle\rangle + \Gamma^4 e^{4\Gamma\tau}\langle\langle M(t,2\tau)M^*(t,2\tau)\rangle\rangle],$$

and plugging in the expression derived before, in the limit of negligible noise, we get

$$\begin{aligned}
\langle\langle |P_{eg}(t)^\dagger|^2\rangle\rangle_{\gamma\to 0} &= e^{-2\Gamma t}\big[1 - \Gamma^2 e^{2\Gamma\tau}(t-2\tau)^2 + \Gamma^2 e^{2\Gamma\tau}(t-\tau)^2 + \Gamma^4 e^{4\Gamma\tau}\tfrac{1}{4}(t-2\tau)^4\big] \\
&= \big|e^{-\Gamma t}\big(1 + i\Gamma e^{\Gamma\tau}(t-\tau) - \tfrac{1}{2}\Gamma^2 e^{2\Gamma\tau}(t-2\tau)^2\big)\big|^2 \\
&= \big|e^{-\Gamma t}\big(1 + \Gamma e^{\Gamma\tau + i\pi/2}(t-\tau) + \tfrac{1}{2}\Gamma^2 e^{2\Gamma\tau + i\pi}(t-2\tau)^2\big)\big|^2 \\
&= \big|e^{-\Gamma t}\big(1 + \Gamma e^{\Gamma\tau + i\omega_0\tau}(t-\tau) + \tfrac{1}{2}\Gamma e^{\Gamma\tau + i\omega_0\tau}(t-2\tau)^2\big)\big|^2.
\end{aligned}$$

Herewith, we have recovered the case without noise.

APPENDIX E: COHERENCE CORRELATION FUNCTION

The goal of this section is to calculate a two-time correlation up to $t = 2\tau$. Without feedback, we have

$$P_{\mathrm{eg}}^\dagger(t)P_{\mathrm{ge}}(t') = e^{-\Gamma(t+t')}. \tag{E1}$$

With feedback but without dephasing $F_t \equiv 0$, we have

$$P_{\mathrm{eg}}^\dagger(t)P_{\mathrm{ge}}^\dagger(t') = e^{-\Gamma(t+t')}[\theta(t) + \theta(t-\tau)\Gamma e^{-i\omega_0\tau+\Gamma\tau}(t-\tau)][\theta(t') + \theta(t'-\tau)\Gamma e^{i\omega_0\tau+\Gamma\tau}(t'-\tau)].$$

However, with dephasing, we have to calculate the noise correlation functions again. We have to take the average:

$$\begin{aligned}
\langle\langle P_{\mathrm{eg}}^\dagger(t)P_{\mathrm{ge}}(t')\rangle\rangle = e^{-\Gamma(t+t')}[&F_0(t,t') + \theta(t-\tau)\Gamma e^{-i\omega_0\tau+\Gamma\tau}F_1(t,t') + \theta(t'-\tau)\Gamma e^{i\omega_0\tau+\Gamma\tau}F_1(t',t) \\
&+ \theta(t'-\tau)\theta(t-\tau)\Gamma^2 e^{2\Gamma\tau}F_2(t,t')],
\end{aligned}$$

with the following abbreviations:

$$F_0(t,t') = \theta(t-t')e^{-\frac{\gamma}{2}(t-t')} + \theta(t'-t)e^{-\frac{\gamma}{2}(t'-t)}.$$

For $t \geqslant \tau$, we need the next correlation:

$$\begin{aligned}
F_1(t,t') = {}& \theta(t-t')e^{-(\gamma/2)(t-t'+\tau)}(t'-\tau) + \theta(t-t')\theta(\tau-t')e^{-(\gamma/2)(t-t'+\tau)}\frac{1}{\gamma}[e^{\gamma(t-t')} - e^{\gamma(\tau-t')}] \\
&+ \theta(t-t')\theta(t'-\tau)e^{-(\gamma/2)(t-t'+\tau)}\frac{1}{\gamma}[e^{\gamma(t-t')} - 1] + \theta(t'-t)e^{-(\gamma/2)(t'-t+\tau)}(t-\tau).
\end{aligned}$$

And for $t \geqslant \tau$ and $t' \geqslant \tau$,

$$\begin{aligned}
F_2(t,t') = {}& \theta(t-t')e^{-\frac{\gamma}{2}(t-t')}\frac{2}{\gamma^2}\left[\gamma(t'-\tau) + (e^{-\gamma(t'-\tau)} - 1)\left\{1 - \frac{\gamma}{2}(t-t')\right\}\right] \\
&+ \theta(t'-t)e^{-\frac{\gamma}{2}(t'-t)}\frac{2}{\gamma^2}\left[\gamma(t-\tau) + (e^{-\gamma(t-\tau)} - 1)\left\{1 - \frac{\gamma}{2}(t'-t)\right\}\right].
\end{aligned}$$

This results in a two-time correlation up to $t \leqslant 2\tau$:

$$\begin{aligned}
\langle\langle P_{\mathrm{eg}}^\dagger(t)P_{\mathrm{ge}}(t')\rangle\rangle = {}& \theta(t-t')\, e^{-\Gamma(t+t')-\gamma(t-t')/2} \\
&+ \theta(t-t')\theta(t-\tau)\, e^{-\Gamma(t+t'-\tau)-\gamma(t-t'+\tau)/2-i\omega_0\tau}\Gamma(t'-\tau)
\end{aligned}$$

$$+\ \theta(t-t')\theta(t'-\tau)\, e^{-\Gamma(t+t'-\tau)-\gamma(t-t'+\tau)/2+i\omega_0\tau}\,\Gamma(t'-\tau)$$
$$+\,\theta(t-t')\theta(t-\tau)\theta(\tau-t')\, e^{-\Gamma(t+t'-\tau)-\gamma(t-t'+\tau)/2-i\omega_0\tau}\,\frac{\Gamma}{\gamma}[e^{\gamma(t-t')}-e^{\gamma(\tau-t')}]$$
$$+\,\theta(t-t')\theta(t-\tau)\theta(t'-\tau)\, e^{-\Gamma(t+t'-\tau)-\gamma(t-t'+\tau)/2-i\omega_0\tau}\,\frac{\Gamma}{\gamma}[e^{\gamma(t-t')}-1]$$
$$+\,\theta(t-t')\theta(t-\tau)\theta(t'-\tau)\, e^{-\Gamma(t+t'-2\tau)-\gamma(t-t')/2}\,\frac{2\Gamma^2}{\gamma^2}\left\{\gamma(t'-\tau)+[e^{-\gamma(t'-\tau)}-1]\left[1-\frac{\gamma}{2}(t-t')\right]\right\}+\{t\leftrightarrow t'\}.$$

Herewith the results of the paper can be reproduced.

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