Theory of high-gain twin-beam generation in waveguides: From Maxwell's equations to efficient simulation

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We provide an efficient method for the calculation of high-gain, twin-beam generation in waveguides derived from a canonical treatment of Maxwell's equations. Equations of motion are derived that naturally accommodate photon generation via spontaneous parametric down-conversion (SPDC) or spontaneous four-wave mixing and, also, include the effects both of self-phase modulation of the pump and of cross-phase modulation of the twin beams by the pump. The equations we solve involve fields that evolve in space and are labeled by a frequency. We provide a proof that these fields satisfy bona fide commutation relations and that in the distant past and future they reduce to standard time-evolving Heisenberg operators. Having solved for the input-output relations of these Heisenberg operators we also show how to construct the ket describing the quantum state of the twin beams. Finally, we consider the example of high-gain SPDC in a waveguide with a flat nonlinearity profile, for which our approach provides an explicit solution that requires only a single matrix exponentiation.

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I. INTRODUCTION

The generation of twin beams is an important technique for the production of nonclassical light [1]. In early experiments, the twin beams were generated over a manifold of modes. This was because the nonlinear medium was pumped with a quasicontinuous-wave source. As pulsed sources were developed and mode engineering improved, it became possible to drastically reduce the number of spatiotemporal modes to essentially just one [2]. Furthermore, recent developments in photonics have allowed for the tight confinement of the traveling waves participating in the three- or four-wave mixing process necessary for the generation of twin beams [3–5]. These developments have moved the focus of theoretical descriptions of twin-beam generation from the perturbative regime to the nonperturbative regime.

Theoretical descriptions of twin-beam generation broadly follow three approaches, each of which can be identified by the space-time variables used to describe the propagation of states, Heisenberg operators, or their correlation functions. The first is a $(\vec{k}, t)$ approach [6–8], in which the amplitudes of expansion fields specified by wave vectors $\vec{k}$ are propagated in time. As the vectorial nature of $\vec{k}$ suggests, this strategy can be applied to propagation geometries in any number of dimensions. It has not yet been extended beyond the perturbative regime.

The second is a (z, t) approach, in which slowly varying envelope operators are propagated forward in time [9–11]. This strategy can accommodate dispersion, but it requires the calculation of the propagation of a sufficiently complete set of classical pulses undergoing the nonlinear dynamics of a stimulated experiment and then the use of this information to describe the spontaneous experiment. For certain limiting situations, no numerics are needed since the equations of motion admit an analytic solution [12].

The third strategy is a (z, ω) approach, where one deals with Fourier transforms of the (z, t) operators [13–17]. This approach has been heavily used since the early days of quantum nonlinear optics and has been justified, e.g., by Bergman [18], who argued that "[e]volution in time of an operator in the Heisenberg picture is given by its commutation with the Hamiltonian. Here the propagation distance, z, plays the role of time." However, Huttner *et al.* [19] pointed out that this approach "is not derived from a Lagrangian and therefore has not been justified in terms of a canonical scheme." As noted by Haus [20,21], the validity of the argument expressed by Bergman and used by many others arises physically because "the formalism implies the application to narrowband spectra within which such a frequency independence can be assumed and a group velocity defined." In even simpler terms: *If a group velocity v can be defined, then* time = position/v.

In this paper, we focus on the regime where such a simple link between space and time is provided by a group velocity. We provide a rigorous proof of the validity of the (z, ω) approach for twin-beam generation, connect it to canonical (Hamiltonian) schemes, and use it to study twin-beam generation via spontaneous parametric down-conversion (SPDC) or spontaneous four-wave mixing (SFWM) in the high-gain regime. We do this by showing that, even in the presence of a nonlinear medium, suitably defined field operators $a(z, \omega)$ satisfy correct commutation relations *if* the dispersion relation for the mode specified by the operator $a(z, \omega)$ is linear, $k(\omega) = k(\bar{\omega}) + (\omega - \bar{\omega})/v$, where $\bar{\omega}$ is some properly defined central frequency. Furthermore, we show that, if the relation between

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the wave vector and the frequency is *not* linear (in the simplest case quadratic, as, for example, considered by Caves and Crouch [22]), then the field operators $a(z,\omega)$ defined here for the twin beams have pathological commutation relations.

To derive these results, in Sec. II we provide a self-contained derivation of the equations of motion of the quantum operators that classically correspond to slowly varying envelope functions, starting from Maxwell's equations and a Hamiltonian canonical quantization procedure [19,23–29]. In Sec. III, we introduce the (z,ω) operators, which are the Fourier transforms of the (z,t) operators, and derive their equations of motion. These equations account for twin-beam generation via SPDC or SFWM and, also, include automatically phase-matched interactions such as self-phase modulation (SPM) of the pump and cross-phase modulation (XPM) of the generated twin beams by the pump. In Sec. IV we show that these equations, upon discretization, can be efficiently solved using matrix exponentiation and study some properties of their solution by the introduction of Schmidt modes. In Sec. VI, we use the techniques developed in the previous sections to study spontaneous twin-beam generation and, as an example, consider a homogeneous medium with a pump beam that does not undergo SPM. Under these circumstances, the solution of the equations of motion can be reduced to a single matrix exponentiation. In a companion paper [30] we use these techniques to validate a recent tomographical method for the characterization of two-mode squeezing in the high-gain regime. Finally, in Sec. VII we present some general conclusions and comment on the validity of the (z,ω) approach when the relation between k and ω cannot be approximated by a linear function; a detailed calculation is presented in Appendix G.

II. QUANTIZATION IN NONLINEAR MEDIA

In this section we quantize the electromagnetic field in a source-free nonlinear material, and obtain the Hamiltonian governing the generation of photons in twin beams via SPDC or SFWM, the self-phase modulation of the pump, and the XPM of the twin beams by the pump.

A. Quantization

We start by writing Maxwell's equations in a source-free medium:

$$\frac{\partial}{\partial t}\mathbf{B} = -\nabla\times\mathbf{E}, \tag{1a}$$

$$\frac{\partial}{\partial t}\mathbf{D} = \nabla\times(\mathbf{B}/\mu_0), \tag{1b}$$

$$\nabla\cdot\mathbf{D} = \nabla\cdot\mathbf{B} = 0. \tag{1c}$$

We take $\mathbf{B}$ and $\mathbf{D}$ as the fundamental fields [23–28] and write the polarization appearing in the constitutive relation,

$$\mathbf{P} = \mathbf{D} - \epsilon_0\mathbf{E}, \tag{2}$$

as a function of the *displacement field* $\mathbf{D}$,

$$\mathbf{P}(\mathbf{D}) = \Gamma^{(1)}\mathbf{D} + \Gamma^{(2)}\mathbf{D}^2 + \Gamma^{(3)}\mathbf{D}^3 + \dots . \tag{3}$$

The notation here is schematic but, of course, indicates the appropriate summation over Cartesian components; for the moment we neglect any material dispersion. Having expressed the macroscopic polarization in terms of $\mathbf{D}$, we can now write the energy density of the system as

$$\mathcal{H} = \int \mathbf{E}(\mathbf{D})\cdot d\mathbf{D} + \int \mathbf{H}(\mathbf{B})\cdot d\mathbf{B} \tag{4}$$

$$= \frac{\mathbf{B}^2}{2\mu_0} + \frac{1-\Gamma^{(1)}}{2\epsilon_0}\mathbf{D}^2 - \frac{\Gamma^{(2)}\mathbf{D}^3}{3\epsilon_0} - \frac{\Gamma^{(3)}\mathbf{D}^4}{4\epsilon_0} - \dots, \tag{5}$$

with the Hamiltonian H given by the integral over space of this density. The Heisenberg equations of motion, which for an arbitrary operator are

$$i\hbar\frac{d}{dt}O(t) = [O(t), H], \tag{6}$$

give precisely Maxwell's equations, (1a) and (1b), for the operators $\mathbf{D}$ and $\mathbf{B}$ if one uses the Hamiltonian H defined above and the commutation relations [23,31]

$$[D_j(\mathbf{r}), B_l(\mathbf{r}')] = i\hbar\varepsilon_{jlm}\frac{\partial}{\partial r_m}\delta(\mathbf{r}-\mathbf{r}'), \tag{7a}$$

$$[D_j(\mathbf{r}), D_l(\mathbf{r}')] = [B_j(\mathbf{r}), B_l(\mathbf{r}')] = 0. \tag{7b}$$

In Eq. (7) the indices j, l, m denote Cartesian components, ε_{jlm} is the Levi-Civita symbol, and $\delta(\mathbf{r})$ is the Dirac distribution. The divergence conditions, (1c), are satisfied by choosing a basis of modes that are divergenceless; see Eqs. (9) and (10) below. Note that if instead one quantized in terms $\mathbf{E}$ and $\mathbf{B}$ one would not obtain Maxwell's equations, (1a) and (1b), as the Heisenberg equations of motion for such fields [28]. Furthermore, note that $\mathbf{D}$ and $\mathbf{B}$ are transverse, unlike $\mathbf{E}$.

B. Linear field expansion

To introduce expansion fields for the displacement and magnetic fields we follow the approach of Bhat and Sipe [26]. This approach can be generalized to include material dispersion in the linear response of the medium; we simply sketch the results. We consider fields in the linear regime of the form $f(\mathbf{r},t) = f_{\mu k}(\mathbf{r})\exp(-i\omega_{\mu k}t) + \text{c.c.}$ They will satisfy the linear Maxwell equations if they satisfy the so-called master equation [32]

$$\nabla\times\left[\frac{\nabla\times\mathbf{B}_{\mu k}(\mathbf{r})}{n^2(x,y;\omega_{\mu k})}\right] = \left(\frac{\omega_{\mu k}}{c}\right)^2\mathbf{B}_{\mu k}(\mathbf{r}), \tag{8}$$

and also

$$\nabla\cdot\mathbf{B}_{\mu k}(\mathbf{r}) = 0, \tag{9}$$

$$\mathbf{D}_{\mu k}(\mathbf{r}) = \frac{i}{\mu_0\omega_{\mu k}}\nabla\times\mathbf{B}_{\mu k}(\mathbf{r}), \tag{10}$$

where $n(x,y;\omega)$ is the (local) position and frequency-dependent refractive index. In the nondispersive limit and for an isotropic material, the $\Gamma^{(1)}$ coefficient can be related to the more standard linear polarizability $\chi^{(1)}$ and the index of refraction n as follows:

$$1-\Gamma^{(1)} = \left(1+\chi^{(1)}\right)^{-1} = \frac{1}{n^2}. \tag{11}$$

We take the refractive index to be independent of z, the distance along a waveguide. Then the solution of the master equation will be of the form

$$\mathbf{D}_{\mu k}(\mathbf{r}) = \frac{\mathbf{d}_{\mu k}(x,y)}{\sqrt{2\pi}} e^{ikz}, \quad \mathbf{B}_{\mu k}(\mathbf{r}) = \frac{\mathbf{b}_{\mu k}(x,y)}{\sqrt{2\pi}} e^{ikz}, \tag{12}$$

where the label k is a wave vector, and we use the Greek label μ to identify which field we are describing, writing $\mu = p$ for the pump and $\mu = s, i$ for the twin-beam fields.

This is a convenient expansion basis for the field operators $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ even in the presence of material dispersion, under the assumption that at frequencies of interest there is no absorption; normalization must then be done according to

$$\int dxdy \frac{\mathbf{d}^*_{\mu k}(x,y)\cdot \mathbf{d}_{\mu k}(x,y)}{\epsilon_0 n^2(x,y;\omega_{\mu k})} \frac{v_{\mathrm{ph}}(x,y;\omega_{\mu k})}{v_{\mathrm{g}}(x,y;\omega_{\mu k})} = 1, \tag{13}$$

where $v_{\mathrm{ph}}(x, y; \omega)$ and $v_{\mathrm{g}}(x, y; \omega)$ are, respectively, the local phase and group velocities at each point in the waveguide [26].

A rough estimate of the magnitude of these coefficients can be obtained by assuming that the field has a transverse area A, giving

$$|\mathbf{d}| \approx \sqrt{\frac{\epsilon_0 n^2 v_g}{v_p A}}, \tag{14}$$

and we assume that the index of refraction and group and phase velocities are evaluated at some central frequency of interest. Using the fields in Eq. (12) as basis functions normalized according to Eq. (13), the displacement and magnetic fields can be written in the very symmetric form

$$\mathbf{B}(\mathbf{r}) = \sum_\mu \int dk \sqrt{\frac{\hbar\omega_{\mu k}}{2}} b_{\mu k}\mathbf{B}_{\mu k}(\mathbf{r}) + \mathrm{H.c.}, \tag{15}$$

$$\mathbf{D}(\mathbf{r}) = \sum_\mu \int dk \sqrt{\frac{\hbar\omega_{\mu k}}{2}} b_{\mu k}\mathbf{D}_{\mu k}(\mathbf{r}) + \mathrm{H.c.}, \tag{16}$$

and furthermore, the linear part of the Hamiltonian can then be written as

$$H_L = \int dk \sum_\mu \hbar\omega_{\mu k} b^\dagger_{\mu k} b_{\mu k}, \tag{17}$$

with the neglect of zero-point energy.

The creation and destruction operators $b^\dagger_{\mu k}$ and $b_{\mu k}$ satisfy the bosonic commutation relations [26]

$$[b_{\mu k}, b_{\mu' k'}] = [b^\dagger_{\mu k}, b^\dagger_{\mu' k'}] = 0, \tag{18}$$

$$[b_{\mu k}, b^\dagger_{\mu' k'}] = \delta_{\mu\mu'}\delta(k-k'); \tag{19}$$

recall that we use the Greek label $\mu \in \{p, s, i\}$ to refer to the three fields of interest pump, signal, and idler.

At this point the index μ is superfluous if the pump, signal, and idler expansion fields are associated with the same transverse profile function in the xy plane. This is often true for SFWM but not for SPDC. We henceforth redefine the index μ to indicate both the different ranges of k associated with the pump, signal, and idler and their transverse profile functions. We now introduce *field* operators

$$\psi_\mu(z) = \int \frac{dk}{\sqrt{2\pi}} b_{\mu k} e^{i(k-\bar{k}_\mu)z}, \tag{20}$$

which are quantum operators analogous to the slowly varying envelope functions in space, since we have removed a central wave vector $\bar{k}_\mu$ associated with the central frequency $\bar{\omega}_\mu$. In the limit where group velocity dispersion in each field can be neglected, the dispersion relation for each field, with group velocity v_μ, can be written as

$$k - \bar{k}_\mu = (\omega - \bar{\omega}_\mu)/v_\mu. \tag{21}$$

The Schrödinger picture field operators satisfy the commutation relations

$$[\psi_\mu(z), \psi_{\mu'}(z')] = [\psi^\dagger_\mu(z), \psi^\dagger_{\mu'}(z')] = 0, \tag{22}$$

$$[\psi_\mu(z), \psi^\dagger_{\mu'}(z')] = \delta_{\mu,\mu'}\delta(z-z'), \tag{23}$$

again, under the assumptions that the pump, signal, and idler fields span different wave-vector and frequency ranges and, thus, that for each field operator, (20), we can formally let k range from $-\infty$ to ∞ when evaluating the commutation relations.

Now we assume that group velocity does not vary significantly over the bandwidths of interest, ignoring group velocity dispersion. Then the linear part of the Hamiltonian, given in Eq. (17), can be written as

$$H_{\mathrm{L}} = \sum_\mu \hbar\bar{\omega}_\mu \int dz\, \psi^\dagger_\mu(z)\psi_\mu(z) + \frac{i}{2}\sum_\mu \hbar v_\mu \int dz \left(\frac{\partial \psi^\dagger_\mu(z)}{\partial z}\psi_\mu(z) - \psi^\dagger_\mu(z)\frac{\partial\psi_\mu(z)}{\partial z}\right) \tag{24}$$

(see Appendix A). The second term in the last equation accounts for the linear dependence of the frequency on the momentum in reciprocal space, which in real space acts as a derivative on the field operator.

We can write the displacement field $\mathbf{D}(\mathbf{r})$, (16), in terms of the field operators as

$$\mathbf{D}(\mathbf{r}) \approx \sum_\mu e^{i\bar{k}_\mu z}\left[\sqrt{\frac{\hbar\omega_\mu}{2}}\mathbf{d}_{\mu k_\mu}(x,y)\right]_{k_\mu=\bar{k}_\mu} \psi_\mu(z) + \mathrm{H.c.}, \tag{25}$$

where we have performed a Taylor expansion of the terms inside the integral around $k_\mu = \bar{k}_\mu$ and assumed any variation in the transverse mode profiles $\mathbf{d}_{\mu k_\mu}$ and the frequencies ω_μ to be negligible; the magnetic field $\mathbf{B}(\mathbf{r})$, (15), can be written in a similar way.

C. The nonlinear interaction

We now turn to the nonlinear part of the Hamiltonian, which is given by the integral over space of the third and fourth terms on the right-hand side of (5). Explicitly indicating Cartesian components and with the usual Einstein summation convention, we have

$$H_{\mathrm{NL}} = -\frac{1}{3\epsilon_0}\int d\mathbf{r}\, \Gamma_2^{ijl}(\mathbf{r})\, D^i(\mathbf{r})D^j(\mathbf{r})D^l(\mathbf{r}) - \frac{1}{4\epsilon_0}\int d\mathbf{r}\, \Gamma_3^{ijlm}(\mathbf{r})\, D^i(\mathbf{r})D^j(\mathbf{r})D^l(\mathbf{r})D^m(\mathbf{r}). \tag{26}$$

In terms of the usual nonlinear susceptibilities $\chi_2^{ijl}(x,y,z)$ and $\chi_3^{ijlm}(x,y,z)$ characterizing the second- and third-order optical responses, we have

$$\Gamma_2^{ijl}(x,y,z) = \frac{\chi_2^{ijl}(x,y,z)}{\epsilon_0 n_o^6(x,y)}, \tag{27}$$

$$\Gamma_3^{ijlm}(x,y,z) = \frac{\chi_3^{ijlm}(x,y,z)}{\epsilon_0^2 n_o^8(x,y)} - \sum_q \frac{2\chi_2^{ijq}(x,y,z)\chi_2^{qlm}(x,y,z)/n_q^2(x,y)}{\epsilon_0^2 n_o^8(x,y)}, \tag{28}$$

where we neglect the effects of material dispersion on the nonlinear Hamiltonian and take $n_o(x,y)$ to be an index of refraction at some "typical" wavelength [31]. We can now write the nonlinear Hamiltonian, Eq. (26), in terms of the field operators $\psi_\mu(z)$, considering processes in which three beams, labeled pump (p), signal (s), and idler (i), are coupled by the nonlinear interaction. We assume that we can choose our center frequencies $\bar{\omega}_\mu$ and the associated wave vectors $\bar{k}_\mu$ such that either

$$2\bar{\omega}_p - \bar{\omega}_s - \bar{\omega}_i = 0, \tag{29a}$$

$$2\bar{k}_p - \bar{k}_s - \bar{k}_i = 0 \tag{29b}$$

or

$$\bar{\omega}_p - \bar{\omega}_s - \bar{\omega}_i = 0, \tag{30a}$$

$$\bar{k}_p - \bar{k}_s - \bar{k}_i = 0. \tag{30b}$$

The first condition will allow for the creation of twin beams via spontaneous four-wave mixing and the second condition will allow for their creation via spontaneous parametric down-conversion. Note that both conditions cannot be satisfied at the same time. Yet even if only the SPDC process is phase-matched, other third-order nonlinear processes, such as self- and cross-phase modulation, are still phase-matched and can modify the properties of the photons generated in SPDC. Of course, this will also happen if SFWM is used to generate photons instead of SPDC. Note that if *quasi*-phase-matching is used for a second-order process, the right-hand side of Eq. (30b) should be changed to $\pm 2\pi/\Lambda_{\text{pol}}$, where Λ_{pol} is the poling period.

Under these assumptions we can write the nonlinear part of the Hamiltonian as

$$H_{\text{NL}} = -\hbar \int dz \bigg\{ \frac{1}{2}\, \zeta_p(z)\psi_p^\dagger(z)\psi_p^\dagger(z)\psi_p(z)\psi_p(z) \tag{31a}$$

$$+ \zeta_i(z)\psi_p^\dagger(z)\psi_p(z)\psi_i^\dagger(z)\psi_i(z) \tag{31b}$$

$$+ \zeta_s(z)\psi_p^\dagger(z)\psi_p(z)\psi_s^\dagger(z)\psi_s(z) \tag{31c}$$

$$+ (\xi_\delta(z)\psi_s^\dagger(z)\psi_i^\dagger(z)(\psi_p(z))^\delta + \text{H.c.}) \bigg\}, \tag{31d}$$

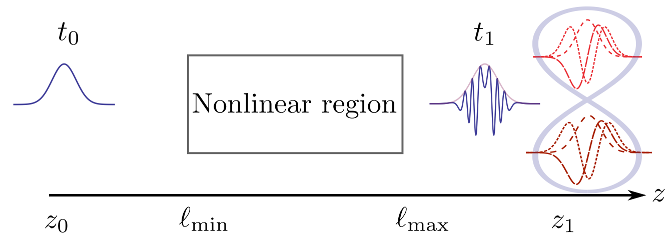

FIG. 1. Propagation geometry. A pump field localized around z_0 is directed towards the nonlinear region, where $z \in [\ell_{\text{min}}, \ell_{\text{max}}]$. After the pump field has left the nonlinear region it has undergone self-phase modulation and has created twin beams in a set of Schmidt modes, indicated by the dashed waveforms on the right.

where we have assumed full permutation symmetry of the Γ tensors in their Cartesian indices and kept the terms that are energy- and phase-matched consistent with Eqs. (29) and (30); we have also introduced the quantities ζ_p, ζ_i, ζ_s, and ξ_δ, defined in detail in Appendix B, which capture the strength of the nonlinear interactions corresponding to the SPM of the pump, (31a), XPM between the pump and the idler, (31b), XPM between the pump and the signal, (31c), and twin-beam generation via either SPDC ($\delta = 1$) or SFWM ($\delta = 2$), (31d), respectively. We take these quantities to be nonzero only in the region $\ell_{\text{min}} \leqslant z \leqslant \ell_{\text{max}}$ where the nonlinear coupling occurs; this is schematically represented in Fig. 1. Note that in the last set of equations we have only included SPM of the pump, since the intensities of the signal and idler field are typically small enough for SPM to be negligible.

III. DYNAMICS OF THE FIELDS

With the full Hamiltonian of the system in place we can write the Schrödinger equation satisfied by the evolution operator,

$$i\hbar \frac{d}{dt}\hat{\mathcal{U}}(t,t_0) = (H_{\text{L}} + H_{\text{NL}})\hat{\mathcal{U}}(t,t_0), \tag{32}$$

where t_0 is conventionally the time at which the Schrödinger and Heisenberg pictures coincide; $\hat{\mathcal{U}}(t_0,t_0) = \mathbb{I}$, where $\mathbb{I}$ is the identity operator. We take this time to be long before the pump beam enters the nonlinear region. Once the unitary evolution operator is obtained one can propagate the operators, for example,

$$b_{\mu k}(t_1) = \hat{\mathcal{U}}^\dagger(t_1,t_0) b_{\mu k}(t_0) \hat{\mathcal{U}}(t_1,t_0) = \mathcal{F}[b_{\mu' k'}(t_0), b_{\mu' k'}^\dagger(t_0)]. \tag{33}$$

In the last equation we use $\mathcal{F}$ to indicate that the quantities on the left-hand side, the operators at time t_1, are functions of all the operators at t_0.

The main objective of the next sections is to provide a detailed derivation of the mapping connecting time-evolving operators at some $t = t_0$ in the distant past with operators at $t = t_1$ in the *distant future*, long after the pump pulse has exited the nonlinear region. Henceforth we assume that t_0 and t_1 are so chosen.

A. Pump dynamics

We first look at the Heisenberg equation of motion for the pump field, which follows from using the commutation relations, (22), with a Hamiltonian that is the sum of the linear, (24), and nonlinear, (31), contributions,

$$\left(\frac{\partial}{\partial t} + v_p \frac{\partial}{\partial z} + i\bar{\omega}_p\right)\psi_p(z,t) = i\zeta_p(z)\psi_p^\dagger(z,t)\psi_p(z,t)\psi_p(z,t) + \text{back-action terms}, \tag{34}$$

where the "back-action terms" are contributions that contain the operators $\psi_s(z,t)$ and $\psi_i(z,t)$. We assume that the pump field is prepared in a strong coherent state with a large number of photons, and we assume that this number remains unchanged during the SFWM or SPDC process; we may then ignore the back-action terms, which are all proportional to the first power of $\psi_p(z,t)$ and second powers of $\psi_s(z,t)$ and $\psi_i(z,t)$, and have a much smaller effect than the self-phase modulation term appearing on the right-hand side of Eq. (34). Furthermore, because of the undepleted-classical pump approximation just described we replace $\psi_p(z,t) \to \langle\psi_p(z,t)\rangle$. The solution to the equation of motion for the pump mean field is

$$\langle\psi_p(z,t)\rangle = \Lambda[z - v_p(t-t_0)]e^{-i\bar{\omega}_p(t-t_0)+i\varphi(z,t)}, \tag{35}$$

where the phase accumulated due to SPM is

$$\varphi(z,t) = |\Lambda[z - v_p(t-t_0)]|^2 \int_{t_0}^{t} dt' \zeta_p[z - v_p(t-t')], \tag{36}$$

and where we have introduced

$$\langle\psi_p(z,t_0)\rangle = \Lambda(z). \tag{37}$$

The mean number of photons in the pump pulse is

$$N_p = \int dz |\langle\psi_p(z,t)\rangle|^2 = \int dz |\Lambda(z)|^2 \gg 1 \tag{38}$$

and its energy is simply $\mathcal{E}_p = \hbar\bar{\omega}_p N_p$. The spatial distribution of the energy in the field will not be affected by SPM,

$$|\langle\psi_p(z,t)\rangle|^2 = |\Lambda[z - v_p(t-t_0)]|^2, \tag{39}$$

and thus the spectral content (i.e., the Fourier transform) of $|\langle\psi_p(z,t)\rangle|^2$ remains unchanged under propagation; see Appendix C for details.

B. Twin-beam dynamics

We can now calculate the Heisenberg equations of motion for the signal and idler field operators ψ_s, $\psi_i^\dagger$,

$$\left(\frac{\partial}{\partial t} + v_s \frac{\partial}{\partial z} + i\bar{\omega}_s\right)\psi_s(z,t) = i\xi_\delta(z)\langle\psi_p(z,t)\rangle^\delta \psi_i^\dagger(z,t) + i\zeta_s(z)|\langle\psi_p(z,t)\rangle|^2\psi_s(z,t), \tag{40a}$$

$$\left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial z} - i\bar{\omega}_i\right)\psi_i^\dagger(z,t) = -i\xi_\delta^*(z)\langle\psi_p^\dagger(z,t)\rangle^\delta \psi_s(z,t) - i\zeta_i(z)|\langle\psi_p(z,t)\rangle|^2\psi_i^\dagger(z,t). \tag{40b}$$

The right-hand sides of Eqs. (40) and (40b) for ψ_s and $\psi_i^\dagger$ account for photon generation via either SPDC ($\delta = 1$) or SFWM ($\delta = 2$) and for XPM of the pump on the signal and idler fields. The left-hand side of Eqs. (40) accounts for propagation at group velocity v_j and oscillation at frequency $\bar{\omega}_j$. If group velocity dispersion were included within the bandwidth of each field, then further terms proportional to $\partial^2\psi_{s,i}/\partial z^2$ would also be present.

Henceforth we neglect group velocity dispersion within each of the pump, signal, and idler bandwidths and introduce the operators for the signal and idler fields,

$$a_j(z,\omega) = \int \frac{dt}{\sqrt{2\pi/v_j}} e^{i(\omega t - z(\omega-\bar{\omega}_j)/v_p)}\psi_j(z,t), \tag{41}$$

$$\psi_j(z,t) = \int \frac{d\omega}{\sqrt{2\pi v_j}} e^{-i(\omega t - z(\omega-\bar{\omega}_j)/v_p)} a_j(z,\omega), \tag{42}$$

where in the last set of equations we used the Latin label $j \in \{s,i\}$ exclusively to refer to the twin beams, signal and idler, and omitting the pump. The fields $a_j(z,\omega)$ are the (t,ω) Fourier transforms of the slowly varying envelope field operators in a moving frame at the group velocity of the pump field v_p [33]. The equations for the spatial evolution of the $a_j(z,\omega)$ are then found to be (see Appendix C for a derivation)

$$\frac{\partial}{\partial z}a_s(z,\omega) = i\Delta k_s(\omega)a_s(z,\omega) + i\frac{\gamma_{\text{XPM},s}h_s(z)}{2\pi} \times \int d\omega' \mathcal{E}_p(\omega-\omega')a_s(z,\omega') + i\frac{\gamma_\delta g(z)}{\sqrt{2\pi}}\int d\omega' \beta_p(z,\omega+\omega')a_i^\dagger(z,\omega'), \tag{43a}$$

$$\frac{\partial}{\partial z}a_i^\dagger(z,\omega) = -i\Delta k_i(\omega)a_i(z,\omega) - i\frac{\gamma_{\text{XPM},i}h_i(z)}{2\pi} \times \int d\omega' \mathcal{E}_p^*(\omega-\omega')a_i^\dagger(z,\omega') - i\frac{\gamma_\delta^* g(z)}{\sqrt{2\pi}}\int d\omega' \beta_p^*(z,\omega+\omega')a_s(z,\omega'). \tag{43b}$$

The first term on the right-hand side of these equations describes the pulse walk-off between the pump and the signal or idler; we have defined

$$\Delta k_j(\omega) = \left(\frac{1}{v_j} - \frac{1}{v_p}\right)(\omega - \bar{\omega}_j). \tag{44}$$

The second term, accounting for cross-phase modulation, contains a coupling strength profile,

$$\gamma_{\text{XPM},j}h_j(z) = \frac{\zeta_j(z)}{v_p v_j \hbar\bar{\omega}_p}, \tag{45}$$

where we take $h_j(z) = 1, 0$, respectively, in the region where the nonlinearity is present or absent, and the (t,ω) Fourier transform of the energy distribution of the field in the moving frame is

$$\mathcal{E}_p(\omega) = \mathcal{E}_p^*(-\omega) = e^{i\omega t_0}\hbar\bar{\omega}_p \int dz |\Lambda(z)|^2 e^{-i\omega z/v_p}. \tag{46}$$

The last term is responsible for twin-beam generation and contains a coupling strength profile,

$$\gamma_\delta g(z) = \frac{\xi_\delta(z)}{\sqrt{v_p v_s v_i (\hbar\bar{\omega}_p)^\delta}}, \tag{47}$$

with $g(z) = 0$ where the nonlinearity is absent and either 1 or -1 (the latter to describe quasi-phase-matching) where the nonlinearity is present, and the (t, ω) Fourier transform of the pump amplitude in the moving frame is

$$\begin{aligned}\beta_p(z,\omega) &= \frac{(\hbar\bar{\omega}_p)^{\delta/2}}{\sqrt{2\pi/v_p}} \int dt\, e^{i(\omega t - z(\omega - \delta\bar{\omega}_p)/v_p)} \langle \psi_p(z,t)\rangle^\delta \\ &= e^{i\omega t_0} \frac{(\hbar\bar{\omega}_p)^{\delta/2}}{\sqrt{2\pi v_p}} \int dz' e^{-iz'\frac{(\omega - \bar{\omega}_p \delta)}{v_p}} [\Lambda(z')]^\delta e^{i\delta\,\theta[z,z']},\end{aligned} \tag{48}$$

with a nonlinear phase,

$$\theta(z,z') \equiv \varphi\left(z, t_0 + \frac{z - z'}{v_p}\right) = |\Lambda(z')|^2 \int_{z'}^{z} \frac{dz''}{v_p} \zeta_p(z''). \tag{49}$$

In the limit of negligible SPM of the pump, $\theta(z,q) \ll 1$, the pump spectral function $\beta_p(z,\omega)$ becomes independent of z and the right-hand side of the equations of motion, (43), depends only on z via the prefactors $\gamma_{\text{SPM},s}$, $\gamma_{\text{SPM},i}$, and γ_δ. However, as soon as SPM becomes important this simple translational dependence is lost. Also, note that the SPDC pump spectral function ($\delta = 1$) in Eq. (48) satisfies

$$\begin{aligned}\int d\omega |\beta_p(z,\omega)|^2 &= \mathcal{E}_p(0) = \hbar\bar{\omega}_p \int dz |\Lambda(z)|^2 \\ &= \hbar\bar{\omega}_p N_p = E_p,\end{aligned} \tag{50}$$

where E_p is the energy contained in the pump pulse. Having introduced the operators $a_j(z,\omega)$ and their equations of motion, we would like to study their equal z commutation relation. For example,

$$\begin{aligned}[a_j(z,\omega), a_j^\dagger(z,\omega')] &= \frac{v_j}{2\pi} \int dt dt' e^{i\omega(t - z/v_p) - i\omega'(t' - z/v_p)} \\ &\quad \times [\psi_j(z,t), \psi_j^\dagger(z,t')],\end{aligned} \tag{51}$$

which shows that to know the equal position commutator of the $a_j(z,\omega)$ it is necessary to know the *unequal* time commutator $[\psi_j(z,t), \psi_j^\dagger(z,t')]$. To know this commutator requires, in principle, knowledge of the dynamics of the field operators $\psi_j(z,t)$ for all times, as given in Eq. (40). Despite this difficulty, partial progress can be made for positions $z_m = z_0 < \ell_{\min}$ or $z_m = z_1 > \ell_{\max}$ before or after the nonlinear region, where one can use the identity

$$\psi_j(z_m, t) = e^{-i\bar{\omega}_j(t - t_m)} \psi_j[z_m - v_j(t - t_m), t_m], \tag{52}$$

where $t_m = t_0$ and $t_m = t_1$ are times chosen, respectively, before and after there is any nonlinear coupling, to show that

$$\begin{aligned}&[\psi_j(z_m,t), \psi_j^\dagger(z_m,t')] \\ &= [\psi_j(z_m - v_j(t - t_m), t_m), \psi_j^\dagger(z_m - v_j(t' - t_m), t_m)] \\ &= \delta[v_j(t - t')]\end{aligned} \tag{53}$$

and use that result to show that for positions *outside* the nonlinear region

$$[a_j(z_m,\omega), a_{j'}^\dagger(z_m,\omega')] = \delta_{j,j'}\delta(\omega - \omega'), \tag{54a}$$

$$[a_j(z_m,\omega), a_{j'}(z_m,\omega')] = 0. \tag{54b}$$

In the next section we return to this question and show that, indeed, the commutation relations, Eqs. (54) and (54b), hold for all z, both inside and outside the nonlinear region. These allow us to interpret quantities such as

$$a_j^\dagger(z,\omega) a_j(z,\omega) \tag{55}$$

as a photon frequency density at position z, in such a way that the total number of photons passing through a plane cutting the waveguide at z is precisely $\int d\omega\, a_j^\dagger(z,\omega) a_j(z,\omega)$.

IV. SOLVING THE EQUATIONS OF MOTION

For computational purposes and notational simplicity we discretize the operators $a_j(z,\omega)$ on a grid of N points according to $\omega_n = \omega_0 + n\Delta\omega|_{n=0}^{N-1}$. We introduce the column vectors $\mathbf{u}$ and $\mathbf{v}^\dagger$ with components

$$\mathbf{u}_n(z) = a_s(z,\omega_n), \tag{56}$$

$$\mathbf{v}_n^\dagger(z) = a_i^\dagger(z,\omega_n), \tag{57}$$

and then using Eq. (43) we can write

$$\frac{\partial}{\partial z}\begin{pmatrix}\mathbf{u}(z) \\ \mathbf{v}^\dagger(z)\end{pmatrix} = i \underbrace{\left[\begin{array}{c|c}\mathbf{G}(z) & \mathbf{F}(z) \\ \hline -\mathbf{F}^\dagger(z) & -\mathbf{H}^\dagger(z)\end{array}\right]}_{:=\mathbf{Q}(z)} \begin{pmatrix}\mathbf{u}(z) \\ \mathbf{v}^\dagger(z)\end{pmatrix}, \tag{58}$$

where we have defined the matrices

$$\mathbf{F}_{n,m}(z) = \frac{\gamma_\delta g(z)}{\sqrt{2\pi}} \beta_p(z, \omega_n + \omega_m)\Delta\omega, \tag{59a}$$

$$\mathbf{G}_{n,m}(z) = \Delta k_s(\omega_n)\delta_{m,n} + \frac{\gamma_{\text{XPM},s} h_s(z)}{2\pi} \mathcal{E}_p(\omega_n - \omega_m)\Delta\omega, \tag{59b}$$

$$\mathbf{H}_{n,m}(z) = \Delta k_i(\omega_n)\delta_{m,n} + \frac{\gamma_{\text{XPM},i} h_i(z)}{2\pi} \mathcal{E}_p^*(\omega_n - \omega_m)\Delta\omega. \tag{59c}$$

We can now formally integrate the discretized equations of motion and obtain

$$\begin{pmatrix}\mathbf{u}(z) \\ \mathbf{v}^\dagger(z)\end{pmatrix} = \mathbf{U}(z,z_0) \begin{pmatrix}\mathbf{u}(z_0) \\ \mathbf{v}^\dagger(z_0)\end{pmatrix} \tag{60}$$

$$= \left[\begin{array}{c|c}\mathbf{U}^{s,s}(z,z_0) & \mathbf{U}^{s,i}(z,z_0) \\ \hline [\mathbf{U}^{i,s}(z,z_0)]^* & [\mathbf{U}^{i,i}(z,z_0)]^*\end{array}\right] \begin{pmatrix}\mathbf{u}(z_0) \\ \mathbf{v}^\dagger(z_0)\end{pmatrix}, \tag{61}$$

where the propagator $\mathbf{U}(z,z_0)$ is defined by the limit

$$\mathbf{U}(z,z_0) = \lim_{n\to\infty} \prod_{p=1}^{n} \exp[i\Delta z \mathbf{Q}(z_p)], \tag{62}$$

and $\Delta z = (z - z_0)/n$ and $z_p = z_0 + p\Delta z$. The intuition behind the Trotter-Suzuki expansion used in the last equation is that for sufficiently thin "slices" of propagation in z one can approximate the matrix $\mathbf{Q}(z)$ as a constant in that region; thus one can simply compound the propagation over all the small regions to get the net propagator. Finally, note that if $\mathbf{Q}$ is independent of z, then

$$\mathbf{U}(z, z_0) = \lim_{n\to\infty} \prod_{p=1}^{n} \exp(i\Delta z\mathbf{Q}) = \exp[i(z - z_0)\mathbf{Q}]. \quad (63)$$

This will always be the case for a uniform waveguide in the limit where the SPM of the pump is negligible.

The undiscretized form of Eq. (60) yields the linear transformation of the continuous-frequency (z, ω) operators

$$a_s(z, \omega) = \int d\omega' U^{s,s}(\omega, \omega'; z, z_0) a_s(z_0, \omega') + \int d\omega' U^{s,i}(\omega, \omega'; z, z_0) a_i^\dagger(z_0, \omega'), \quad (64a)$$

$$a_i^\dagger(z, \omega) = \int d\omega' [U^{i,s}(\omega, \omega'; z, z_0)]^* a_s(z_0, \omega') + \int d\omega' [U^{i,i}(\omega, \omega'; z, z_0)]^* a_i^\dagger(z_0, \omega'), \quad (64b)$$

where the blocks of the propagator $\mathbf{U}(z, z_0)$ are related to the continuous-frequency transfer functions as follows:

$$U^{j,k}(\omega_n, \omega_m; z, z_0) = \mathbf{U}^{j,k}_{n,m}(z, z_0)/\Delta\omega. \quad (65)$$

For notational simplicity, we omit the spatial dependence when we write the transfer functions connecting the input and output operators in the distant past and future. Defining $U^{j,j}(\omega, \omega') = U^{j,j}(\omega, \omega', z_1, z_0) e^{i\Delta k_j(\omega')z_0 - i\Delta k_j(\omega)z_1}$, $U^{j,l}(\omega, \omega') = U^{j,l}(\omega, \omega', z_1, z_0) e^{-i\Delta k_j(\omega')z_0 - i\Delta k_l(\omega)z_1}$ $(j \neq l)$, and $a_l^{(\text{in/out})}(\omega) = e^{-i\Delta k_l(\omega)z_{0/1}} a_l(z_{0/1}, \omega)$, we write

$$a_s^{(\text{out})}(\omega) = \int d\omega' U^{s,s}(\omega, \omega')\, a_s^{(\text{in})}(\omega') + \int d\omega' U^{s,i}(\omega, \omega')\, a_i^{\dagger(\text{in})}(\omega'), \quad (66a)$$

$$a_i^{(\text{out})}(\omega) = \int d\omega' U^{i,i}(\omega, \omega')\, a_i^{(\text{in})}(\omega') + \int d\omega' U^{i,s}(\omega, \omega')\, a_s^{\dagger(\text{in})}(\omega'). \quad (66b)$$

The propagator $\mathbf{U}(z, z_0)$ allows us to write the operators in the spatial region after the nonlinear region, $a_j(z_1, \omega)$ and $a_j^\dagger(z_1, \omega)$, as linear combinations of the operators before the nonlinear region, $a_j(z_0, \omega')$ and $a_j^\dagger(z_0, \omega')$. This is not, however, a solution of Heisenberg's equations; the latter, as in Eq. (33), would allow us to write *time-evolving* operators in the distant future in terms of the operators in the distant past. However, using the results from Appendix E, one can show that

$$b_{jk}(t_0)\big|_{k=\bar{k}_j+(\omega-\bar{\omega}_j)/v_j} = \sqrt{v_j} e^{-i\omega t_0 - i\Delta k_j(\omega)z_0} a_j(z_0, \omega), \quad (67a)$$

$$b_{jk}(t_1)\big|_{k=\bar{k}_j+(\omega-\bar{\omega}_j)/v_j} = \sqrt{v_j} e^{-i\omega t_1 - i\Delta k_j(\omega)z_1} a_j(z_1, \omega), \quad (67b)$$

allowing us to link the (proper, evolving-in-time) Heisenberg operators $b_{jk}(t)$ with the operators $a_j(z, \omega)$ and showing that they are the same operators in the distant past and future (modulo some phases and constant prefactors). Upon realizing this identity, it is immediately recognizable that the Heisenberg equations of motion have been solved, since now we can write the Heisenberg operators $b_{jk}(t_1)$ in the future in terms of the Heisenberg operators $b_{jk}(t_0)$ in the past. This is easily seen by inverting the relations in Eqs. (67) and (67b) and using them to replace $a_j(z_0, \omega)$ and $a_j(z_1, \omega)$ with $b_{jk}(t_0)$ and $b_{jk}(t_1)$ on the right- and left-hand sides of Eqs. (64) and (64b) with $z = z_1$.

V. COMMUTATION RELATIONS AND MODAL STRUCTURE

We now return to the question posed at the end of Sec. III and analyze the equal z commutators of the fields inside the nonlinear medium, which, upon using the solutions in Eq. (64) and the initial position commutators in Eq. (54), we find to be

$$[a_s(z, \omega), a_s^\dagger(z, \omega')] = \int d\omega'' U^{s,s}(\omega, \omega''; z, z_0)[U^{s,s}(\omega', \omega''; z, z_0)]^* - \int d\omega'' U^{s,i}(\omega, \omega''; z, z_0)[U^{s,i}(\omega', \omega''; z, z_0)]^*, \quad (68a)$$

$$[a_i(z, \omega), a_i^\dagger(z, \omega')] = \int d\omega'' U^{i,i}(\omega, \omega''; z, z_0)[U^{i,i}(\omega', \omega''; z, z_0)]^* - \int d\omega'' U^{i,s}(\omega, \omega''; z, z_0)[U^{i,s}(\omega', \omega''; z, z_0)]^*, \quad (68b)$$

$$[a_s(z, \omega), a_i(z, \omega')] = \int d\omega'' U^{s,s}(\omega, \omega''; z, z_0) U^{i,s}(\omega', \omega''; z, z_0) - \int d\omega'' U^{s,i}(\omega, \omega''; z, z_0) U^{i,i}(\omega', \omega''; z, z_0), \quad (68c)$$

and $[a_s(z, \omega), a_i^\dagger(z, \omega')] = 0$. To show that the right-hand sides of Eqs. (68), (68b), and (68c) are $\delta(\omega - \omega')$, $\delta(\omega - \omega')$, and 0, respectively, we note that the matrix discretized versions of these putative commutations relations would be

$$\mathbf{U}^{s,s}(z, z_0)[\mathbf{U}^{s,s}(z, z_0)]^\dagger - \mathbf{U}^{s,i}(z, z_0)[\mathbf{U}^{s,i}(z, z_0)]^\dagger = \mathbb{I}_N, \quad (69a)$$

$$\mathbf{U}^{i,i}(z, z_0)[\mathbf{U}^{i,i}(z, z_0)]^\dagger - \mathbf{U}^{i,s}(z, z_0)[\mathbf{U}^{i,s}(z, z_0)]^\dagger = \mathbb{I}_N, \quad (69b)$$

$$\mathbf{U}^{s,s}(z, z_0)[\mathbf{U}^{i,s}(z, z_0)]^T - \mathbf{U}^{s,i}(z, z_0)[\mathbf{U}^{i,i}(z, z_0)]^T = 0, \quad (69c)$$

with $\mathbb{I}_N$ being the N-dimensional identity matrix. Note that the last set of equations can be written more compactly in terms of the following equation for the propagator $\mathbf{U}(z, z_0)$:

$$\mathbf{U}(z, z_0)\, \mathbf{S}\, \mathbf{U}^\dagger(z, z_0) = \mathbf{S}, \quad (70)$$

with

$$\mathbf{S} = \begin{bmatrix} \mathbb{I}_N & 0 \\ 0 & -\mathbb{I}_N \end{bmatrix}. \quad (71)$$

Mathematically, Eq. (70) states that $\mathbf{U}(z, z_0)$ is an element of the SU(1,1) Lie group (cf. Appendix 11.1.4. of Klimov and Chumakov [34]). To show that $\mathbf{U}(z, z_0) \in \mathrm{SU}(1,1)$ it is sufficient to show that its generators, the matrices $\mathbf{Q}(z)$, belong to the algebra of this group, $\mathfrak{su}(1,1)$, thus they need to satisfy

$$\mathbf{Q}(z)\,\mathbf{S} = \mathbf{S}\,\mathbf{Q}^\dagger(z). \tag{72}$$

But this is trivial to show using the Hermiticity of the matrices $\mathbf{G}$ and $\mathbf{H}$, which, together with $\mathbf{F}$, define $\mathbf{Q}$ in Eq. (58). Thus, the bona fide commutation relations of the $a_j(z, \omega)$ are guaranteed by the algebraic structure of the equations of motion they satisfy, together with the initial conditions for the commutators derived [Eq. (54)]. Because of the Lie group constraints, the transfer functions can be *jointly* decomposed as

$$U^{s,s}(\omega, \omega'; z, z_0) = \sum_l \cosh(r_l)\big[\rho_s^{(l)}(\omega)\big]\big[\tau_s^{(l)}(\omega')\big]^*, \tag{73a}$$

$$U^{s,i}(\omega, \omega'; z, z_0) = \sum_l \sinh(r_l)\big[\rho_s^{(l)}(\omega)\big]\big[\tau_i^{(l)}(\omega')\big], \tag{73b}$$

$$[U^{i,i}(\omega, \omega'; z, z_0)]^* = \sum_l \cosh(r_l)\big[\rho_i^{(l)}(\omega)\big]^*\big[\tau_i^{(l)}(\omega')\big], \tag{73c}$$

$$[U^{i,s}(\omega, \omega'; z, z_0)]^* = \sum_l \sinh(r_l)\big[\rho_i^{(l)}(\omega)\big]^*\big[\tau_s^{(l)}(\omega')\big]^*, \tag{73d}$$

where the quantities r_l are the squeezing parameter of the Schmidt mode l and the sets of functions $\{\rho_{s,i}^{(l)}\}$, $\{\tau_{s,i}^{(l)}\}$ are complete and orthonormal, and thus, for example,

$$\int d\omega\, \rho_s^{(l)}(\omega)\, \big[\rho_s^{(l')}(\omega)\big]^* = \delta_{l,l'}, \tag{74a}$$

$$\sum_l \rho_s^{(l)}(\omega)\, \big[\rho_s^{(l)}(\omega')\big]^* = \delta(\omega - \omega'). \tag{74b}$$

VI. SOLVING THE SPONTANEOUS PROBLEM

Given the linearity of the input-output relations on the operators, the state generated when these are applied in vacuum must be Gaussian. In particular, in the distant future it will have the form

$$|\mathrm{TMSV}\rangle = \exp\left(\int d\omega d\omega' J(\omega, \omega') a_s^{(\mathrm{in})\dagger}(\omega) a_i^{(\mathrm{in})\dagger}(\omega') - \mathrm{H.c.}\right)|\mathrm{vac}\rangle. \tag{75}$$

This squeezed state is described univocally by its first and second moments. These are easily constructed once the scattering matrix $\mathbf{U}$ is known. For the sake of illustration, the covariance between signal and idler annihilation operators is

$$\begin{aligned} M(\omega, \omega') &= \langle \mathrm{vac}| a_s^{(\mathrm{out})}(\omega) a_i^{(\mathrm{out})}(\omega') |\mathrm{vac}\rangle \\ &= \int d\omega'' U^{i,i}(\omega, \omega'') U^{s,i}(\omega', \omega'') \\ &= \sum_l \frac{\sinh(2r_l)}{2}\, \rho_s^{(l)}(\omega) \rho_i^{(l)}(\omega'), \end{aligned} \tag{76}$$

where $|\mathrm{vac}\rangle$ is the vacuum state which is annihilated by the distant past (input) operators

$$a_j^{(\mathrm{in})}(\omega)|\mathrm{vac}\rangle = a_j(z_0, \omega)|\mathrm{vac}\rangle = b_{jk}(t_0)|\mathrm{vac}\rangle = 0. \tag{77}$$

From the moments M, one easily reconstructs the JSA in terms of the Schmidt modes and squeezing parameters of the scattering matrix $\mathbf{U}$, finding

$$J(\omega, \omega') = \sum_l r_l\, \rho_s^{(l)}(\omega) \rho_i^{(l)}(\omega'). \tag{78}$$

Note that in the low-gain regime $r_l \ll 1$ one can approximate $\sinh(2r_l)/2 \approx r_l$ and thus $M(\omega, \omega') \approx J(\omega, \omega')$, but in the high-gain regime the relation between the two functions is more complicated

We can use these results to study what is perhaps the simplest case of twin-beam generation: a $\chi^{(2)}$ process in which the nonlinearity has a flat top-hat profile and we ignore any effect of cross- and self-phase modulation. For a use of the theory presented here in the characterization of parametric down-conversion sources involving the aforementioned $\chi^{(3)}$ effects, see the companion paper [30].

With the modification of the pump function by SPM neglected and the nonlinearity function $\xi_1(z)$ a top-hat function extending from $\ell_{\min} = -\frac{\ell}{2}$ to $\ell_{\max} = \frac{\ell}{2}$, matrix $\mathbf{Q}$ in Eq. (58) is independent of z in the region where the nonlinearity is present. Because of this we can write [recall Eq. (63)]

$$\mathbf{U}\left(-\frac{\ell}{2}, \frac{\ell}{2}\right) = \exp(i\mathbf{Q}\ell),$$

and the calculation of the matrix propagator $\mathbf{U}$ is reduced to a single exponentiation, which is one of the main advantages of working with the $a(z, \omega)$ operators instead of the $\psi(z, t)$ operators [33].

For illustration, we study a Gaussian pump,

$$\langle \psi_p(z, t_0)\rangle = \frac{\sqrt{N_p}}{\sqrt[4]{\pi (v_p/\sigma)^2}} \exp\left(-\frac{(z - z_0)^2}{2(v_p/\sigma)^2}\right),$$

localized around $z = z_0$ at time $t = t_0$ and with bandwidth σ and mean number of photons N_p. The low-gain JSA, in the limit where the spectral content of the pump is not modified, is simply

$$\begin{aligned} J(\omega, \omega') = &\frac{\xi_1^{(0)} \sqrt{N_p}}{\sqrt{2\pi v_s v_i v_p \sigma \sqrt{\pi}}} \exp\left(-\frac{(\omega + \omega' - \bar{\omega}_p)^2}{2\sigma^2}\right) \\ &\times \ell\, \mathrm{sinc}\left(\frac{\ell}{2}\{\Delta k_s(\omega) + \Delta k_i(\omega')\}\right) \end{aligned} \tag{79}$$

[35] (see Appendix F for a derivation), where $\xi_1^{(0)}$ is the nonzero value the nonlinearity function $\xi_1(z)$ takes in the the region $-\ell/2 < z < \ell/2$.

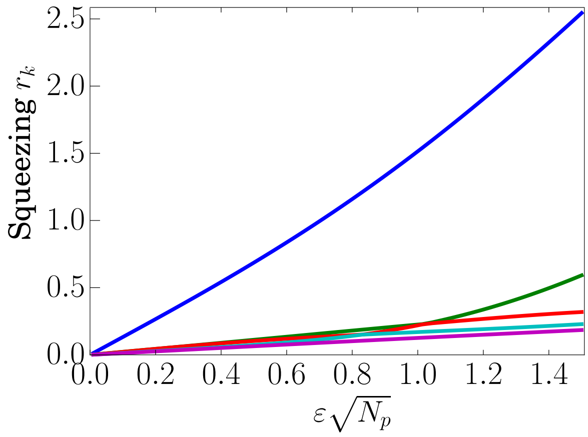

FIG. 2. Plot of the squeezing parameters $\{r_k\}$ of the four largest Schmidt modes. For low gain the squeezing parameters r_k are linear in $\sqrt{N_p}$. However, in the region where $\varepsilon\sqrt{N_p} \gtrsim 1$ the dependence of the two largest squeezing parameters on that variable deviates from linear.

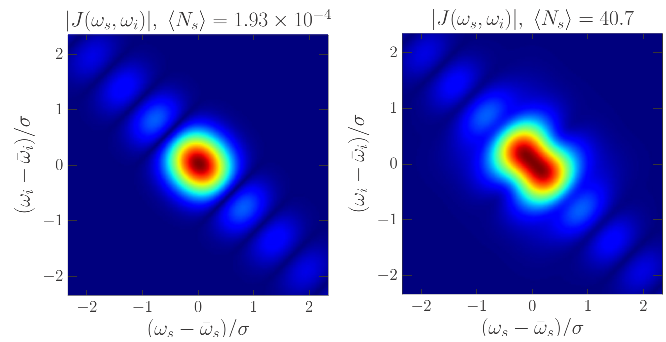

FIG. 3. Absolute values of the joint spectral amplitude (JSA) in the symmetric group-velocity-matched regime in the low-gain (left) and high-gain (right) regime. In the low-gain regime the JSA is simply the product of the pump function (Gaussian) and the phase-matching function (sinc). In the high-gain regime this is not the case because of so-called time-ordering corrections [37,38].

We work in the symmetric group-velocity-matched regime [36], where $(v_p^{-1} - v_s^{-1}) = -(v_p^{-1} - v_s^{-1}) = 2\kappa/\ell$, to obtain

$$\frac{\ell}{2}\{\Delta k_s(\omega) + \Delta k_i(\omega')\} = \kappa\{(\omega - \bar{\omega}_s) - (\omega' - \bar{\omega}_i)\} \quad (80)$$

and, furthermore, pick the parameter $\kappa = 1.61/(1.13\sigma)$ so as to maximize the separability of the low-gain JSA in Eq. (79) by matching the width of the sinc function and the Gaussian appearing appearing there [36].

In Fig. 2 we show the evolution of the squeezing parameters of the JSA from the low-gain regime to the high-gain regime as the pump energy N_p is increased. As predicted using the Magnus expansion [37,38], the time-ordering corrections will cause the squeezing parameters to behave in a nonlinear way as a function of $\sqrt{N_p}$. Note that this result will also be observed regardless of the shape of the pump function and the profile of the nonlinearity. In particular, these time-ordering corrections will also affect the optimal Gaussian PMF and Gaussian pump function combination that uniquely gives a fully factorable JSA in the low-gain regime [39,40].

In Fig. 3 we also show the JSA as defined in Eq. (78) in the low-gain regime, $\langle N_s\rangle = \langle N_i\rangle \ll 1$, and in the high-gain regime, where the mean number of photons in the signal and idler beams is $\langle N_s\rangle = \langle N_i\rangle = 41$ with

$$\langle N_j\rangle = \int d\omega \langle a_j^{(\text{out})\dagger}(\omega) a_j^{(\text{out})}(\omega)\rangle = \sum_l \sinh^2(r_l). \quad (81)$$

The computation for each JSA for a given value of N_p and for a grid of 600 frequencies takes seconds on a desktop computer using PYTHON's [41] SCIPY [42]; this time should be contrasted with the hours it takes with other methods and publicly available code [15,43] running in the same hardware and language or libraries.

VII. CONCLUSIONS AND OUTLOOK

We have presented a justification for the use of field operators in (z, ω) space in the study of twin-beam generation. These operators have been constructed rigorously, starting from a canonical formalism that has Maxwell's equations as the Heisenberg equations of motion. In the limit of a negligible group velocity dispersion, we showed that the $a_j(z, \omega)$ operators satisfy well-defined equal position commutator relations. Furthermore, we showed that for times and positions long before and after the pump has entered and exited the nonlinear region, these position-evolving operators indeed coincide with standard Heisenberg operators evolving in the standard Heisenberg picture in time. The solution to the equations these operators satisfy is easy to implement computationally and allows for the incorporation of many important processes that can alter the properties of the twin beams, such as poling inhomogeneities [via $\xi_1(z)$], self-phase modulation of the pump, and XPM of the pump on the twin beams. A thorough exploration of this mélange of wave mixings is presented in the companion paper [30].

The derivation presented for the (z, ω) operators is apparently not easily generalizable to include the presence of group velocity dispersion. Intuitively, if dispersion is important, position is not like time and wave vectors are not the same as frequencies. Mathematically, if the relation between frequencies and wave vectors is nonlinear, then one cannot obtain identities such as Eq. (52) and, thus, one cannot (at least in an obvious manner) prove the bona fide commutation relation of the $a_j(z, \omega)$ operators at equal positions. Indeed, we show in Appendix G that certain commutators of the z, ω operators that should be trivially 0 are nonzero when dispersion is included.

In principle, one can take dispersion into account by solving the dynamics of the Heisenberg operators that evolve *in time* by generalizing Eq. (40) to include a nonlinear dispersion relation [44]. Yet, for many applications in quantum nonlinear optics, it is sufficient and often necessary to work with narrow enough bandwidths, such that group velocities are well defined. This is especially true when generating twin beams with a small degree of frequency correlations [40].

Finally, we would like to point out that the methods presented here can easily be carried over to frequency conversion, where now the fields $a_s(z,\omega)$ will couple to $a_i(z,\omega')$ instead of $a_i^\dagger(z,\omega')$. In this case, the underlying group dictating the symmetry of the problem will be SU(2). The generalization of the techniques presented here should provide a useful tool to study highly mode-selective frequency conversion beyond the perturbative regime [15,45–47].

Note added. Recently, we became aware of related work by Sharapova *et al.* [48] where equations similar to the ones derived here are used to study the joint spectral amplitude of the transverse degrees of freedom of a bright squeezing source.

ACKNOWLEDGMENTS

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APPENDIX A: LINEAR HAMILTONIAN IN TERMS OF THE FIELD OPERATORS

Expanding $\omega_{\mu k}$ about $\omega_{\mu k_\mu} \equiv \bar{\omega}_\mu$ we can write the Hamiltonian, (17), as

$$H_L = \sum_\mu \hbar\bar{\omega}_\mu \int dk\, b_{\mu k}^\dagger b_{\mu k} + \sum_\mu \hbar v_\mu \int dk\,(k-k_\mu) b_{\mu k}^\dagger b_{\mu k} + \frac{1}{2}\sum_\mu \hbar v'_\mu \int dk\,(k-k_\mu)^2 b_{\mu k}^\dagger b_{\mu k} + \dots,$$

where

$$v_\mu = \left(\frac{d\omega_{\mu k}}{dk}\right)_{k_\mu}, \quad v'_\mu = \left(\frac{d^2\omega_{\mu k}}{dk^2}\right)_{k_\mu}. \tag{A1}$$

Since from (20) we can write

$$b_{\mu k}^\dagger b_{\mu k} = \int \frac{dzdz'}{2\pi} \psi_\mu^\dagger(z)\psi_\mu(z') e^{i(k-k_\mu)(z-z')}, \tag{A2}$$

we have

$$\int dk\, b_{\mu k}^\dagger b_{\mu k} = \int dz\, \psi_\mu^\dagger(z)\psi_\mu(z), \tag{A3}$$

while

$$\int dk\,(k-k_\mu) b_{\mu k}^\dagger b_{\mu k} = \frac{1}{2i}\int \frac{dkdzdz'}{2\pi}\psi_\mu^\dagger(z)\psi_\mu(z')\left[\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)e^{i(k-k_\mu)(z-z')}\right] \tag{A4}$$

$$= \frac{i}{2}\int \frac{dkdzdz'}{2\pi}\left[\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial z'}\right)\psi_\mu^\dagger(z)\psi_\mu(z')\right]e^{i(k-k_\mu)(z-z')} \tag{A5}$$

$$= \frac{i}{2}\int dz\left(\frac{\partial\psi_\mu^\dagger(z)}{\partial z}\psi_\mu(z) - \psi_\mu^\dagger(z)\frac{\partial\psi_\mu(z)}{\partial z}\right), \tag{A6}$$

and

$$\int dk\,(k-k_\mu)^2 b_{\mu k}^\dagger b_{\mu k} = \int \frac{dkdzdz'}{2\pi}\psi_\mu^\dagger(z)\psi_\mu(z')\left[\left(\frac{\partial}{\partial z}\frac{\partial}{\partial z'}\right)e^{i(k-k_\mu)(z-z')}\right] \tag{A7}$$

$$= \int \frac{dkdzdz'}{2\pi}\left[\left(\frac{\partial}{\partial z}\frac{\partial}{\partial z'}\right)\psi_\mu^\dagger(z)\psi_\mu(z')\right]e^{i(k-k_\mu)(z-z')} \tag{A8}$$

$$= \int dz \frac{\partial\psi_\mu^\dagger(z)}{\partial z}\frac{\partial\psi_\mu(z)}{\partial z}. \tag{A9}$$

So Hamiltonian (17) is

$$H_L = \sum_\mu \hbar\bar{\omega}_\mu \int dz\, \psi_\mu^\dagger(z)\psi_\mu(z) + \frac{i}{2}\sum_\mu \hbar v_\mu \int dz\left(\frac{\partial\psi_\mu^\dagger(z)}{\partial z}\psi_\mu(z) - \psi_\mu^\dagger(z)\frac{\partial\psi_\mu(z)}{\partial z}\right) + \frac{1}{2}\sum_\mu \hbar v'_\mu \int dz \frac{\partial\psi_\mu^\dagger(z)}{\partial z}\frac{\partial\psi_\mu(z)}{\partial z} + \dots. \tag{A10}$$

APPENDIX B: THE NONLINEAR COEFFICIENTS

The nonlinear coefficients describing the nonlinear interaction between the pump, signal, and idler modes are defined as follows:

$$\zeta_p(z) = \frac{3}{\epsilon_0\hbar}\left(\frac{\hbar\bar{\omega}_p}{2}\right)^2 \int dxdy\, \Gamma_3^{ijlm}(\mathbf{r})\left[d^i_{p\bar{k}_p}(x,y)\right]^*\left[d^j_{p\bar{k}_p}(x,y)\right]^* d^l_{p\bar{k}_p}(x,y) d^m_{p\bar{k}_p}(x,y) \tag{B1a}$$

$$= \frac{3}{\epsilon_0\hbar}\left(\frac{\hbar\bar{\omega}_p}{2}\right)^2 \int dxdy\, \frac{\chi_3^{ijlm}(\mathbf{r})}{\epsilon_0 n_i^2 n_j^2 n_l^2 n_m^2}\left[d^i_{p\bar{k}_p}(x,y)\right]^*\left[d^j_{p\bar{k}_p}(x,y)\right]^* d^l_{p\bar{k}_p}(x,y) d^m_{p\bar{k}_p}(x,y), \tag{B1b}$$

$$\zeta_{i/s}(z) = 2\frac{3}{\epsilon_0\hbar}\left(\frac{\hbar\bar{\omega}_{i/s}}{2}\right)\left(\frac{\hbar\bar{\omega}_p}{2}\right) \int dxdy\, \Gamma_3^{ijlm}(\mathbf{r})\left[d^i_{p\bar{k}_p}(x,y)\right]^*\left[d^j_{i,s\bar{k}_{i,s}}(x,y)\right]^* d^l_{i,s\bar{k}_{i,s}}(x,y) d^m_{p\bar{k}_p}(x,y) \tag{B1c}$$

$$= 2\frac{3}{\epsilon_0\hbar}\left(\frac{\hbar\bar{\omega}_{i/s}}{2}\right)\left(\frac{\hbar\bar{\omega}_p}{2}\right) \int dxdy\, \frac{\chi_3^{ijlm}(\mathbf{r})}{\epsilon_0 n_i^2 n_j^2 n_l^2 n_m^2}\left[d^i_{p\bar{k}_p}(x,y)\right]^*\left[d^j_{i,s\bar{k}_{i,s}}(x,y)\right]^* d^l_{i,s\bar{k}_{i,s}}(x,y) d^m_{p\bar{k}_p}(x,y), \tag{B1d}$$

$$\xi_2(z) = \frac{3}{\epsilon_0\hbar}\left(\frac{\hbar\sqrt{\bar{\omega}_s\bar{\omega}_i}}{2}\right)\left(\frac{\hbar\bar{\omega}_p}{2}\right) \int dxdy\, \Gamma_3^{ijlm}(\mathbf{r})\left[d^i_{s\bar{k}_s}(x,y)\right]^*\left[d^j_{i\bar{k}_i}(x,y)\right]^* d^l_{p\bar{k}_p}(x,y) d^m_{p\bar{k}_p}(x,y) \tag{B1e}$$

$$= \frac{3}{\epsilon_0\hbar}\left(\frac{\hbar\sqrt{\bar{\omega}_s\bar{\omega}_i}}{2}\right)\left(\frac{\hbar\bar{\omega}_p}{2}\right)\int dxdy\, \frac{\chi_3^{ijlm}(\mathbf{r})}{\epsilon_0 n_i^2 n_j^2 n_l^2 n_m^2}\left[d^i_{s\bar{k}_s}(x,y)\right]^*\left[d^j_{i\bar{k}_i}(x,y)\right]^* d^l_{p\bar{k}_p}(x,y)d^m_{p\bar{k}_p}(x,y), \tag{B1f}$$

$$\xi_1(z) = \frac{2}{\epsilon_0\hbar}\sqrt{\frac{\hbar^3\bar{\omega}_i\bar{\omega}_s\bar{\omega}_p}{(2)^3}}\int dxdy\, \Gamma_2^{ijl}(\mathbf{r})\left[d^i_{i\bar{k}_i}(x,y)\right]^*\left[d^j_{s\bar{k}_s}(x,y)\right]^* d^l_{p\bar{k}_p}(x,y) \tag{B1g}$$

$$= \frac{2}{\epsilon_0\hbar}\sqrt{\frac{\hbar^3\bar{\omega}_i\bar{\omega}_s\bar{\omega}_p}{(2)^3}}\int dxdy\, \frac{\chi_2^{ijl}(\mathbf{r})}{\epsilon_0^2 n_i^2 n_j^2 n_l^2}\left[d^i_{i\bar{k}_i}(x,y)\right]^*\left[d^j_{s\bar{k}_s}(x,y)\right]^* d^l_{p\bar{k}_p}(x,y). \tag{B1h}$$

Note the extra factor of 2 in the definition of $\zeta_{s/i}(z)$ that comes about because of the permutation symmetry of the Γ coefficients. In the last equations we ignored the χ_2 contributions to Γ_3, but they can be easily added.

APPENDIX C: CONNECTING THE ω, t, AND ω, z OPERATORS

We want to transform the equations of motion, Eq. (40), expressing them in the reciprocal frequency space and in a frame of reference that propagates at the pump group velocity. We begin by defining the (t,ω) Fourier transform of the field operators,

$$\tilde{\psi}_\mu(z,\omega) = \sqrt{v_\mu}\int \frac{dt}{\sqrt{2\pi}}e^{i\omega t}\psi_\mu(z,t), \tag{C1}$$

$$\psi_\mu(z,t) = \int \frac{d\omega}{\sqrt{2\pi v_\mu}}e^{-i\omega t}\,\tilde{\psi}_\mu(z,\omega). \tag{C2}$$

Here we consider SPDC as the interaction generating twin beams. Applying $\int \frac{dt}{\sqrt{2\pi}}e^{i\omega t}$ to both sides of Eq. (40) and substituting the z,t operators in terms of their Fourier transforms, we find the equivalent equation for the signal

$$\begin{aligned}\frac{\partial}{\partial z}\tilde{\psi}_s(z,\omega) &= i\left(\frac{\omega-\bar{\omega}_s}{v_s}\right)\tilde{\psi}_s(z,\omega)\\ &+i\int \frac{d\omega'}{\sqrt{2\pi v_p v_s v_i}}\xi_1(z)\,\langle\tilde{\psi}_p(z,\omega+\omega')\rangle\,\tilde{\psi}_i^\dagger(z,\omega')\\ &+i\int \frac{d\omega'}{\sqrt{2\pi}\,v_s}\zeta_s(z)\,I_0(z,\omega-\omega')\,\psi_s(z,\omega'),\end{aligned} \tag{C3}$$

where we have defined the (t,ω) Fourier transform of the energy density of the pump in z as

$$\begin{aligned}I_0(z,\omega) &= v_p\int dt\; e^{i\omega t}\;|\langle\psi_p(z,t)\rangle|^2\\ &= e^{i\omega t_0}e^{i\omega z/v_p}\int dz'\; e^{-i\omega z'/v_p}\;|\Lambda(z)|^2.\end{aligned} \tag{C4}$$

Now we make the following change of variables, moving to a frame of reference that propagates at the pump group velocity,

$$\tilde{\psi}_j(z,\omega) = e^{i\frac{\omega-\bar{\omega}_j}{v_p}z}a_j(z,\omega), \qquad j\in\{s,i\}, \tag{C5}$$

$$\langle\tilde{\psi}_p(z,\omega)\rangle = e^{i\frac{\omega-\bar{\omega}_p}{v_p}z}\beta_p(z,\omega). \tag{C6}$$

We can see that the pump amplitude in this frame of reference is z independent (in the absence of SPM) by applying the solution to the pump dynamics found in Eq. (35):

$$\begin{aligned}\beta_p(z,\omega) &= \sqrt{\hbar\bar{\omega}_p}e^{-i\frac{\omega-\bar{\omega}_p}{v_p}z}\int \frac{dt}{\sqrt{2\pi v_p}}e^{i\omega t}\;\langle\psi_p(z,t)\rangle\\ &= \sqrt{\hbar\bar{\omega}_p}e^{-i\frac{\omega-\bar{\omega}_p}{v_p}z}\int \frac{dt}{\sqrt{2\pi v_p}}e^{i\omega t}\;\Lambda[z-v_p(t-t_0)]\\ &\quad\times\; e^{-i\bar{\omega}_p(t-t_0)+i\varphi(z,t)},\end{aligned} \tag{C7}$$

which, upon making the change of variables $z' = z - v_p t$, yields

$$\beta_p(z,\omega) = \sqrt{\hbar\bar{\omega}_p}e^{i\bar{\omega}_p t_0}\int \frac{dz'}{\sqrt{2\pi/v_p}}e^{-i\frac{\omega-\bar{\omega}_p}{v_p}z'}\,\Lambda(z')\,e^{i\,\theta(z,z')}, \tag{C8}$$

where $\theta(z,z') = \varphi(z, t_0 + \frac{z-z'}{v_p})$. When SPM is negligible, the nonlinear phase $\varphi(z,t)$ is negligible, rendering Eq. (C8) independent of z.

The change of variables in Eq. (C5) yields the equation for the signal

$$\begin{aligned}\frac{\partial}{\partial z}a_s(z,\omega) &= i(\omega-\bar{\omega}_s)\left(\frac{1}{v_s}-\frac{1}{v_p}\right)a_s(z,\omega)\\ &+i\int \frac{d\omega'}{\sqrt{2\pi\,\hbar\bar{\omega}_p v_s v_i v_p}}\xi_1(z)e^{i\frac{\bar{\omega}_s+\bar{\omega}_i-\bar{\omega}_p}{v_p}z}\\ &\times\beta_p(z,\omega+\omega')a_i^\dagger(z,\omega')\\ &+i\int \frac{d\omega'}{2\pi\,\hbar\bar{\omega}_p v_s v_p}\zeta_s(z)\mathcal{E}_p\omega-\omega')a_s(z,\omega'),\end{aligned} \tag{C9}$$

where we have defined

$$\begin{aligned}\mathcal{E}_p(\omega) &= \hbar\bar{\omega}_p I_0(z,\omega)e^{-i\omega z/v_p}\\ &= \hbar\bar{\omega}_p e^{i\omega t_0}\int \frac{dz'}{\sqrt{2\pi}}\;e^{-i\omega z'}\;|\Lambda(z)|^2,\end{aligned} \tag{C10}$$

which is *always* independent of z, regardless of the SPM of the pump. Note that we can further simplify Eq. (C9) by noting that $\bar{\omega}_s + \bar{\omega}_i - \bar{\omega}_p = 0$.

Finally, let us consider the case where the process is phase-matched for SFWM. In this case we define Fourier-transformed operators for the signal and idler fields as in Eq. (C1). However, for the pump we define

$$\begin{aligned}\phi_p(z,\omega) &= \sqrt{v_p}\int \frac{dt}{\sqrt{2\pi}}e^{i\omega t}\langle\psi_p(z,t)\rangle^2,\\ \langle\psi_p(z,t)\rangle^2 &= \int \frac{d\omega}{\sqrt{2\pi v_p}}e^{-i\omega t}\;\phi_p(z,\omega).\end{aligned} \tag{C11}$$

In terms of ϕ, the new equation of motion for $\tilde{\psi}_s$ has the same form as Eq. (C3) with the replacement $\xi_1(z)\langle\tilde{\psi}_p(z,\omega+\omega')\rangle \to \xi_2(z)\phi_p(z,\omega+\omega')$. We can shift to a frame moving at the pump group velocity as we did in Eq. (C5), but for the pump we define

$$\beta_p(z,\omega) = \hbar\bar{\omega}_p e^{-i\frac{\omega-2\bar{\omega}_p}{v_p}z}\int \frac{dt}{\sqrt{2\pi/v_p}}e^{i\omega t}\,\langle\psi_p(z,t)\rangle^2$$
$$= e^{i\omega t_0}\frac{(\hbar\bar{\omega}_p)}{\sqrt{2\pi v_p}}\int dz' e^{-iz'\frac{(\omega-2\bar{\omega}_p)}{v_p}}[\Lambda(z')]^2 e^{i2\theta(z,z')}. \tag{C12}$$

Note the factor of 2 multiplying $\bar{\omega}_p$ and exponentiating $\langle\psi_p(z,t)\rangle$. With these definitions we arrive at an equation analogous to Eq. (C9), but where we need to replace

$$\xi_1(z)\frac{1}{\sqrt{\hbar\bar{\omega}_p}}e^{i\frac{\bar{\omega}_s-\bar{\omega}_i-\bar{\omega}_p}{v_p}z} \to \frac{1}{\hbar\bar{\omega}_p}\xi_2(z)e^{i\frac{\bar{\omega}_s-\bar{\omega}_i-2\bar{\omega}_p}{v_p}z}. \tag{C13}$$

However, for SFWM one has $\bar{\omega}_s-\bar{\omega}_i-2\bar{\omega}_p=0$.

APPENDIX D: CONNECTING FREE OPERATORS IN SPACE AND TIME

We use the following definitions:

$$\psi_j(z,t) = \int\frac{d\omega}{\sqrt{2\pi v_j}\sqrt{v_j}}e^{i(\omega-\bar{\omega}_j)z/v_j}b_{jk_j(\omega)}(t), \quad k_j(\omega)\equiv\bar{k}_j+(\omega-\bar{\omega}_j), \tag{D1a}$$
$$= \int\frac{d\omega}{\sqrt{2\pi v_j}}e^{i(\omega-\bar{\omega}_j)z/v_p}e^{-i\omega t}c_j(z,\omega). \tag{D1b}$$

It is useful to label space-time coordinates (t_0,z_0) as "distant past," if t_0 is a time before the nonlinear interaction has effect and z_0 is a coordinate less than coordinates where the nonlinearity is present, and to label space-time coordinates (t_1,z_1) as "distant future," if t_1 is a time after the nonlinear interaction has effect and z_1 is a coordinate greater than coordinates where the nonlinearity is present. In Appendix E we show that for (t_n,z_n) in either the distant past or the distant future we have

$$\psi_j(z_n,t) = e^{-i\bar{\omega}_j(t-t_n)}\psi_j[z_n-v_j(t-t_n),t_n)]. \tag{D2}$$

Now we can use Eq. (D1b) for the left-hand side of the last equation and Eq. (D1a) for the right-hand side to find

$$\psi_j(z_n,t) = e^{-i\bar{\omega}_j(t-t_n)}\psi(z_n-v_j(t-t_n),t_n), \tag{D3}$$
$$\int\frac{d\omega}{\sqrt{2\pi v_j}}e^{i(\omega-\bar{\omega}_j)z_n/v_p}e^{-i\omega t}c_j(z_n,\omega) = e^{-i\bar{\omega}_j(t-t_n)}\int\frac{d\omega}{\sqrt{2\pi v_j}\sqrt{v_j}}b_{jk_j(\omega)}(t_n)e^{i(\omega-\bar{\omega}_j)(z_n-v(t-t_n))/v_j} \tag{D4}$$
$$= \int\frac{d\omega}{\sqrt{2\pi v_j}\sqrt{v_j}}e^{i(\omega-\bar{\omega}_j)z_n/v_j}e^{-i\omega t}e^{i\omega t_n}b_{jk_j(\omega)}(t_0). \tag{D5}$$

Comparing the quantities under the integral we see that

$$a_j(z_n,\omega)e^{-i\Delta k_j(\omega)z_n} = e^{i\omega t_n}b_{jk_j(\omega)}(t_n)/\sqrt{v_j}. \tag{D6}$$

APPENDIX E: FORMAL SOLUTION IN (*z*, *t*)

We construct an implicit solution of the (t,z) equations of motion, where we introduce space-time points (t_n,z_n) in the distant past ($n=0$) or the distant future ($n=1$), where these terms are defined in Appendix D. For $n=0$ or $n=1$ we can write a formal solution of the equations In either case we can write a formal solution of the propagation equation, (40a),

$$\bar{\psi}_s(z,t)$$
$$= \bar{\psi}_s[z-v_s(t-t_n),t_n]$$
$$+\frac{\theta(t-t_n)}{v_s}\int_{z_-}^{z}dz' f\left(z',t-\frac{z-z'}{v_s}\right)\bar{\psi}_i^\dagger\left(z',t-\frac{z-z'}{v_s}\right)$$
$$+\frac{\theta(t-t_n)}{v_s}\int_{z_-}^{z}dz' g\left(z',t-\frac{z-z'}{v_s}\right)\bar{\psi}_s\left(z',t-\frac{z-z'}{v_s}\right)$$
$$-\frac{\theta(t_n-t)}{v_s}\int_{z}^{z_-}dz' f\left(z',t-\frac{z-z'}{v_s}\right)\bar{\psi}_i^\dagger\left(z',t-\frac{z-z'}{v_s}\right)$$
$$-\frac{\theta(t_n-t)}{v_s}\int_{z}^{z_-}dz' g\left(z',t-\frac{z-z'}{v_s}\right)\bar{\psi}_s\left(z',t-\frac{z-z'}{v_s}\right), \tag{E1}$$

where we have defined

$$\bar{\psi}_j(z,t) = e^{i\bar{\omega}_j t}\psi_j(z,t), \tag{E2a}$$
$$z_- = z-v_s(t-t_n), \tag{E2b}$$
$$f(z,t) = \xi_\delta(z)\langle\psi_p(z,t)\rangle^\delta, \tag{E2c}$$
$$g(z,t) = \zeta_s(z)|\langle\bar{\psi}_p(z,t)\rangle|^2, \tag{E2d}$$

and $\theta(t)$ is the Heaviside step function, $\theta(t)=0$ if $t<0$, $\theta(t)=1$ if $t>0$, and $\theta(t)=1/2$ if $t=0$.

First, we investigate the distant past case (setting $n=0$). We can introduce new dummy integration variables for the integrals extending from z to z_- as

$$t' = t - \frac{z - z'}{v_s}, \tag{E3}$$

$$z' = z + v_s(t' - t), \tag{E4}$$

and when $z' = z$ we have $t' = t$ and when $z' = z_-$ we have $t' = t_0$. With this change of variables we have

$$\begin{aligned}
&\bar{\psi}_s(z_0, t)\\
&= \bar{\psi}_s[z_0 - v_s(t - t_0), t_0]\\
&+ \frac{\theta(t - t_0)}{v_s}\int_{z_-}^{z_0} f\left(z', t - \frac{z_0 - z'}{v_s}\right)\bar{\psi}_i^\dagger\left(z', t - \frac{z_0 - z'}{v_s}\right)dz'\\
&+ \frac{\theta(t - t_0)}{v_s}\int_{z_-}^{z_0} g\left(z', t - \frac{z_0 - z'}{v_s}\right)\bar{\psi}_s\left(z', t - \frac{z_0 - z'}{v_s}\right)dz'\\
&- \theta(t_0 - t)\int_t^{t_0} f[z_0 + v_s(t' - t), t']\bar{\psi}_i^\dagger[z + v_s(t' - t), t']dt'\\
&- \theta(t_0 - t)\int_t^{t_0} g[z_0 + v_s(t' - t), t']\bar{\psi}_s[z + v_s(t' - t), t']dt'.
\end{aligned} \tag{E5}$$

For (t_0, z_0) in the distant past the spatial extent of the classical pump $\bar{\psi}_p(z, t_0)$ has zero overlap with the nonlinear region, and z_0 is smaller than the values of the arguments for which the nonlinear coefficients $\zeta_j(z), \xi(z)$ are nonzero, so we have

$$\bar{\psi}_j(z_0, t) = \bar{\psi}_j[z_0 - v_j(t - t_0), t_0], \tag{E6}$$

$$\psi_j(z_0, t) = e^{-i\bar{\omega}_j(t - t_0)}\psi_j[z_0 - v_j(t - t_0), t_0]. \tag{E7}$$

This is readily established by noting that the first pair of integrals on the right-hand side of Eq. (E5) ranges over values of z' for which $f(z', t - \frac{z_o - z'}{v_s})$ and $g_s(z', t - \frac{z_o - z'}{v_s})$ will vanish (seen by examining the range of the first argument since $z' < z_0$), and the last pair of integrals on the right-hand side will range over values of t' for which $f(z_0 + v_s(t' - t), t')$ and $g(z_0 + v_s(t' - t), t')$ will vanish (seen by examining the range of the second argument since now $t' < t_0$).

Now let us study the distant future solution ($n = 1$). The formal solution corresponding to (E5) is then

$$\begin{aligned}
&\bar{\psi}_s(z_1, t) = \bar{\psi}_s[z_1 - v_s(t - t_1), t_1]\\
&+ \theta(t - t_1)\int_{t_1}^{t} dt' f[z_1 + v_s(t' - t), t']\bar{\psi}_i^\dagger[z_1 + v_s(t' - t), t']\\
&+ \theta(t - t_1)\int_{t_1}^{t} dt' g[z_1 + v_s(t' - t), t']\bar{\psi}_s[z_1 + v_s(t' - t), t']\\
&- \frac{\theta(t_1 - t)}{v_s}\int_{z_1}^{z_-} dz' f\left(z', t - \frac{z_1 - z'}{v_s}\right)\bar{\psi}_i^\dagger\left(z', t - \frac{z_1 - z'}{v_s}\right)\\
&- \frac{\theta(t_1 - t)}{v_s}\int_{z_1}^{z_-} dz' g\left(z', t - \frac{z_1 - z'}{v_s}\right)\bar{\psi}_s\left(z', t - \frac{z_1 - z'}{v_s}\right).
\end{aligned} \tag{E8}$$

Using arguments similar to those just made for distant past times ($n = 0$), we arrive at the corresponding results for distant future times ($n = 1$):

$$\bar{\psi}_j(z_1, t) = \bar{\psi}_j[z_1 - v_j(t - t_1), t_1], \tag{E9}$$

$$\psi_j(z_1, t) = e^{-i\bar{\omega}_j(t - t_1)}\psi_j[z_1 - v_j(t - t_1), t_1]. \tag{E10}$$

APPENDIX F: LOW-GAIN SOLUTIONS

We go back to Eq. (43) and solve these equations perturbatively for the case of SPDC (and assuming no XPM). First, we define the operators

$$c_j(z, \omega) = e^{i\Delta k_j(\omega)z}a_s(z, \omega). \tag{F1}$$

Using these definitions in Eq. (43) and integrating to first order we find

$$\begin{aligned}
c_s(z_1, \omega) = c_s(z_0, \omega) + i\int_{z_0}^{z_1} dz \int d\omega' \beta_p(z, \omega + \omega')\\
\times e^{-iz[\Delta k_s(\omega) + \Delta k_i(\omega')]}c_i^\dagger(z, \omega'),
\end{aligned} \tag{F2}$$

$$\begin{aligned}
c_i^\dagger(z_1, \omega) = c_i^\dagger(z_0, \omega) - i\int_{z_0}^{z_1} dz \int d\omega' \beta_p^*(z, \omega + \omega')\\
\times e^{iz[\Delta k_s(\omega) + \Delta k_i(\omega')]}c_s^\dagger(z, \omega').
\end{aligned} \tag{F3}$$

Now we assume that the nonlinear interaction is weak and replace $c_i(z, \omega') \approx c(z_0, \omega')$ on the right-hand side. Furthermore, we assume that the pump spectral amplitude is not modified by SPM and, thus, that there is no z dependence of β_p. We introduce the net phase mismatch

$$\begin{aligned}
\Delta k(\omega, \omega') &= \Delta k_s(\omega) + \Delta k_i(\omega)\\
&= \frac{\omega - \bar{\omega}_s}{v_s} + \frac{\omega' - \bar{\omega}_i}{v_i} - \frac{\omega + \omega' - \bar{\omega}_p}{v_p},
\end{aligned} \tag{F4}$$

and we can then write the transformation in Eq. (F2) as

$$\begin{aligned}
c_s(z_1, \omega) = \int \bar{U}^{s,s}(\omega, \omega'; z_1, z_0)c_s(z_0, \omega')\\
+ \int \bar{U}^{s,i}(\omega, \omega'; z_1, z_0)c_i^\dagger(z_0, \omega'),
\end{aligned} \tag{F5}$$

$$\begin{aligned}
c_i^\dagger(z_1, \omega) = \int (\bar{U}^{i,s}[\omega, \omega'; z_1, z_0)]^* c_s(z_0, \omega')\\
+ \int [\bar{U}^{i,i}(\omega, \omega'; z_1, z_0)]^* c_i^\dagger(z_0, \omega'),
\end{aligned} \tag{F6}$$

where the perturbative transfer functions can be jointly decomposed as follows:

$$\bar{U}^{s,s}(\omega, \omega'; z_1, z_0) = \sum_k \cosh(r_k)\big[\rho_s^{(k)}(\omega)\big]\big[\rho_s^{(k)}(\omega')\big]^*, \tag{F7}$$

$$\bar{U}^{s,i}(\omega, \omega'; z_1, z_0) = \sum_k \sinh(r_k)\big[\rho_s^{(k)}(\omega)\big]\big[\rho_i^{(k)}(\omega')\big]^*, \tag{F8}$$

$$(\bar{U}^{i,i}(\omega, \omega'; z_1, z_0))^* = \sum_k \cosh(r_k)\big[\rho_i^{(k)}(\omega)\big]^*\big[\rho_i^{(k)}(\omega')\big], \tag{F9}$$

$$[\bar{U}^{i,s}(\omega, \omega'; z_1, z_0)]^* = \sum_k \sinh(r_k)\big[\rho_i^{(k)}(\omega)\big]^*\big[\rho_s^{(k)}(\omega')\big]^*. \tag{F10}$$

Here the functions $\rho_{s/i}$ are complete and orthonormal, and are the Schmidt functions of the joint spectral amplitude

$$J(\omega,\omega') = \frac{1}{\sqrt{v_s v_i v_p}}\beta_p(\omega+\omega')\Phi[\Delta k(\omega,\omega')] = \sum_l r_l \rho_s^{(l)}(\omega)\rho_s^{(l)}(\omega), \quad \text{(F11)}$$

$$\Phi[\Delta k(\omega,\omega')] = \int_{z_0}^{z_1} \frac{dz}{\sqrt{2\pi}} e^{-iz\Delta k(\omega,\omega')}\xi_1(z) \quad \text{(F12)}$$

in the approximation that the squeezing parameters $r_l \ll 1$ and thus $\sinh(r_l) \approx r_l$ and $\cosh(r_l) \approx 1$. Note that in this limit we recover the well-known result that the JSA is simply the product of the pump and phase-matching function.

Comparing the results of this Appendix with the more general expression in Eq. (73) obtained for arbitrary gain, we see that in the low-gain regime $\tau_{s/i}(\omega) = \rho_{s/i}(\omega)$.

APPENDIX G: THE PROBLEM WITH GROUP VELOCITY DISPERSION

We investigate how group velocity dispersion modifies the conclusions drawn in this paper. In particular, we consider how equal position and different time commutators such as

$$[\bar{\psi}_i^\dagger(z,t), \bar{\psi}_s^\dagger(z,t_0)] \quad \text{(G1)}$$

are modified by the inclusion of group velocity dispersion. For the sake of concreteness we assume that one is only interested in SPDC and that XPM can be assumed to be unimportant. We can then write the generalized form of the equations of motion, (40a) and (40b), for the field operators as

$$\left(\frac{\partial}{\partial t} + v_s\frac{\partial}{\partial z} + i\frac{v_s'}{2}\frac{\partial^2}{\partial z^2}\right)\bar{\psi}_s(z,t) = if(z,t)\bar{\psi}_i^\dagger(z,t), \quad \text{(G2)}$$

$$\left(\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial z} + i\frac{v_i'}{2}\frac{\partial^2}{\partial z^2}\right)\bar{\psi}_i^\dagger(z,t) = -if^*(z,t)\bar{\psi}_s(z,t),$$

where $f(z,t) = \xi_1(z)\langle\bar{\psi}_p(z,t)\rangle$, and we have assumed that the group velocity dispersion v_s' and v_i' [see Eqs. (A1)] can be taken as independent of k. In the case of no nonlinearity one can write the formal solution of this problem in terms of a Green's function

$$\bar{\psi}_j(z,t) = \int dz' G_j(z-z';t-t-t_0)\bar{\psi}_j(z,t_0), \quad \text{(G3)}$$

where

$$G_j(z;t) = \frac{[1-i\,\mathrm{sgn}(v_j't)]}{\sqrt{4\pi|v_j't|}}e^{\left(\frac{i(z-v_jt)^2}{2v_j't}\right)}. \quad \text{(G4)}$$

Note that in the limit $v_j' \to 0$ the last equation collapses to

$$G_j(z;t) = \delta(z-v_jt). \quad \text{(G5)}$$

Using the Green's functions we can write a formal solution of the equations of motion including the nonlinearity as

$$\bar{\psi}_s(z,t) = \int G_s(z-z';t-t_o)\bar{\psi}_s(z',t_o)dz' + i\int G_s(z-z';t-t')\Theta(t-t',t_o-t') \times f(z',t')\bar{\psi}_i^\dagger(z',t')dz'dt', \quad \text{(G6)}$$

where

$$\Theta(t_2;t_1) \equiv \theta(t_2) - \theta(t_1), \quad \text{(G7)}$$

and a similar equation for $\bar{\psi}_i^\dagger(z,t)$.

Having constructed an implicit solution we can develop a perturbation theory in which on the right-hand side of the last equation we iteratively replace the "evolved" time fields $\bar{\psi}_j(z,t), t \neq t_0$ under the integral. To first order in the nonlinearity we find

$$\bar{\psi}_s(z,t) = \int G_s(z-z';t-t_o)\bar{\psi}_s(z',t_o)dz' \quad \text{(G8)}$$

$$+i\int G_s(z-z';t-t')\Theta(t-t';t'-t_o)f(z',t') \times G_i^*(z'-z'';t'-t_o)\bar{\psi}_i^\dagger(z'',t_o)dz'dz''dt' + \dots. \quad \text{(G9)}$$

Using the expansion for the fields we find that the commutator in Eq. (G1) is

$$[\bar{\psi}_i^\dagger(z,t), \bar{\psi}_s^\dagger(z_o,t_o)] \approx -i\int F(z,z_o;t,t_o,t')\Theta(t-t';t'-t_o)dt', \quad \text{(G10)}$$

$$F(z,z_o;t,t_o,t') = \int G_i^*(z-z';t-t')f^*(z',t')G_s(z'-z_o;t'-t_o)dz'. \quad \text{(G11)}$$

In the next sections we evaluate this quantity in two limits.

1. No group velocity dispersion

Using the results for the case of no group velocity dispersion we find

$$F(z,z_o;t,t_o,t') = \int \delta([z-z'-v_i(t-t')]g^*(z',t') \times \delta(z'-z_o-v_s(t'-t_o)dz' = \delta[z-z_o-v_s(t'-t_o)-v_i(t-t')] \times g^*[z_o+v_s(t'-t_o)].$$

Of particular interest for our Fourier transform variables is the equal position commutator

$$[\bar{\psi}_i^\dagger(z,t), \bar{\psi}_s^\dagger(z,t_o)] \approx \int F(z,z;t,t_o,t')\Theta(t-t';t'-t_o)dt' = \int \delta[-v_s(t'-t_o)-v_i(t-t')] \times g^*[z+v_s(t'-t_o)]\Theta(t-t';t'-t_o)dt'. \quad \text{(G12)}$$

But since v_s and v_i are both positive the Dirac delta function will only give a contribution at values of t' where the Θ function vanishes, and so we have

$$[\bar{\psi}_i^\dagger(z,t), \bar{\psi}_s^\dagger(z,t_o)] \approx 0,$$

and thus the equal position commutators in the presence of the pump are, at least to first order, equivalent to the equal position commutators in the absence of the pump. We saw in the text that, for no group velocity dispersion, this equivalence holds to all orders in the presence of the pump.

2. Finite group velocity dispersion

In this case after some lengthy algebra one finds

$$F(z,z;t,t_o,t') = \frac{e^{-\frac{2i\mathcal{A}^2}{D}}}{2\pi\sqrt{|\tau_1\tau_2 v_s' v_i'|}}\left(\frac{[1+i\mathrm{sgn}(D)]}{\sqrt{2}}\right)\left(\frac{[1-i\mathrm{sgn}(\frac{\tau_1\tau_2 v_s' v_i'}{D})]}{\sqrt{2}}\right)$$

$$\times \int e^{\frac{iD[\mathcal{D}-\frac{4\mathcal{A}A}{D}-2(z-z')]^2}{8\tau_1\tau_2 v_s' v_i'}} f^*(z',t')dz', \quad \text{(G13)}$$

where

$$\tau_1 = t - t', \quad \text{(G14)}$$

$$\tau_2 = t' - t_0, \quad \text{(G15)}$$

$$\mathcal{D} = \tau_1 v_i - \tau_2 v_s, \quad \text{(G16)}$$

$$\mathcal{A} = \tfrac{1}{2}(\tau_1 v_i + \tau_2 v_s), \quad \text{(G17)}$$

$$D = \tau_1 v_i' - \tau_2 v_s', \quad \text{(G18)}$$

$$A = \tfrac{1}{2}(\tau_1 v_i' + \tau_2 v_s'). \quad \text{(G19)}$$

The last integral can be evaluated asymptotically in the limit that the v_j' are "small" (cf. Sec. 2.9 of Ref. [49]) and one finds

$$F(z,z;t,t_o,t') \sim \left(\frac{[1+i\mathrm{sgn}(D)]}{\sqrt{2}}\right)\frac{1}{\sqrt{2\pi|D|}}e^{-\frac{2i\mathcal{A}^2}{D}}g^*(\bar{z},t'),$$

and then

$$[\bar{\psi}_i^\dagger(z,t),\bar{\psi}_s^\dagger(z,t_o)] \sim i\int\left(\frac{[1+i\mathrm{sgn}(D)]}{\sqrt{2}}\right)\frac{1}{\sqrt{2\pi|D|}}e^{-\frac{2i\mathcal{A}^2}{D}}f^*(\bar{z},t')\Theta(t-t';t'-t_o)dt'. \quad \text{(G20)}$$

Again, one can take the limit of vanishing group velocity dispersion by putting $|D| \to 0$, in which limit

$$\sqrt{\frac{2}{\pi|D|}}\left(\frac{1+i\mathrm{sgn}(D)}{\sqrt{2}}\right)e^{-\frac{2i\mathcal{A}^2}{D}} \to \delta(\mathcal{A}),$$

and using this in (G20) we have

$$\begin{aligned}
[\bar{\psi}_i^\dagger(z,t),\bar{\psi}_s^\dagger(z,t_o)] &\to \frac{1}{2}\int\delta(\mathcal{A})f^*\left(z-\frac{\mathcal{D}}{2},t'\right)\Theta(t-t';t'-t_o)dt' \\
&= \frac{1}{2}\int\delta\left(\frac{1}{2}(\tau_1 v_i+\tau_2 v_s)\right)f^*\left(z-\frac{\tau_1 v_i-\tau_2 v_s}{2}\right)\Theta(t-t';t'-t_o)dt' \\
&= \int\delta(\tau_1 v_i+\tau_2 v_s)f^*\left(z-\frac{\tau_1 v_i-\tau_2 v_s}{2}\right)\Theta(t-t';t'-t_o)dt' \\
&= \int\delta(\tau_1 v_i+\tau_2 v_s)f^*(z+v_s\tau_2)\Theta(t-t';t'-t_o)dt' \\
&= \int\delta(v_i(t-t')+v_s(t'-t_o))f^*[z+v_s(t'-t_o)]\Theta(t-t';t'-t_o)dt' = 0,
\end{aligned}$$

indeed in agreement with the limit, (G12), of vanishing group velocity dispersion, as expected. But for a finite group velocity dispersion (G20) indicates that we cannot expect this commutator to vanish.

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