Maximal-value condition of coherence measures holds for mixed states if and only if it does for pure states

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While various coherence measures based on the framework for quantifying coherence [T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014)] have been proposed, it is often asked whether a coherence measure fulfills the maximal value condition that only maximally coherent states should achieve the maximal value of a coherence measure [Y. Peng, Y. Jiang, and H. Fan, Phys. Rev. A **93**, 032326 (2016)]. Although it may be easy to rule out the measures that violate the condition by giving a counterexample, it is usually complicated to confirm the validity of the condition for a general coherence measure due to the deal with all mixed states. In this paper, we prove that the maximal value condition of coherence measures holds for mixed states if and only if it holds for a special subset of pure states. Our finding can greatly reduce the examination of whether a coherence measure fulfills the maximal value condition since one only needs to consider a subset of pure states, avoiding the work of considering mixed states.

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I. INTRODUCTION

Quantum coherence is a fundamental property of quantum mechanics and describes the capability of a quantum state to exhibit quantum interference phenomena. It provides an important resource for various quantum information processing tasks, such as quantum algorithms, quantum cryptography [1], nanoscale thermodynamics [2], quantum metrology [3], and quantum biology [4,5]. The resource theory of coherence has attracted a growing interest due to the rapid development of quantum information science [6–14].

By following the approach that has been established for an entanglement resource [15,16], Baumgratz *et al.* proposed a rigorous framework for quantifying coherence [8]. The framework consists of four conditions, i.e., the coherence being zero (positive) for incoherent states (all other states), the monotonicity of coherence under incoherent operations, the monotonicity of coherence under selective measurements on average, and the nonincreasing of coherence under mixing of quantum states. Based on this framework, a number of coherence measures [6,8,17–29], such as the l_1 norm of coherence, the relative entropy of coherence, the coherence of formation, and the robustness of coherence, have been proposed. With these measures, various topics of quantum coherence [18,20,30–53], such as the dynamics of quantum coherence [31–33], the distillation of quantum coherence [20,34–36], and the relations between quantum coherence and other quantum resources [18,38–51], have been addressed. There is no doubt that more coherence measures with interesting properties can be found along with the development of research.

A maximally coherent state of a quantum system is the state that can be transformed into any other state of the quantum system under incoherent operations. Although a coherence measure satisfying the above four conditions must reach its maximal value for maximally coherent states, the states that make a coherence measure take the maximal value may not be the maximally coherent states. To reconcile maximally coherent states with the states that take the maximal value of a coherence measure, Peng *et al.* suggested that a valid coherence measure should satisfy a fifth condition, which says that only maximally coherent states achieve the maximal value of a coherence measure [54]. We refer to it as the maximal value condition for simplicity.

The coherence measures that fulfill the maximal value condition have some particular properties, which are not held for coherence measures only satisfying the four conditions. For example, unitary incoherent operations preserve the coherence of a quantum system, i.e., they do not change the value of any coherence measure, but the incoherent operations that do not change the value of the coherence measure may not be unitary incoherent operations. However, the two subsets of incoherent operations are exactly equivalent for the coherence measures satisfying the maximal value condition. Due to these merits, such as the consistency of the maximally coherent states and the states with the maximal coherence value, and the equivalence of the two subsets of incoherent operations mentioned above, the coherence measures satisfying the maximal value condition look more interesting in some sense. While a coherence measure is proposed based on the above four conditions, it is often asked whether it fulfills the maximal value condition. For instance, Refs. [21,22] proved the robustness of coherence C_R, Ref. [24] proved the coherence concurrence C_C, and Ref. [29] proved a family of functionals, denoted as $\mathscr{C}$ by its presenters, to be the coherence measures satisfying the maximal value condition, while Ref. [28]

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indicated that the modified trace distance of coherence C'_{tr} does not fulfill the maximal value condition. Besides, Ref. [55] proved that the trace distance of coherence C_{tr}, a previous candidate of coherence measures, fulfills the maximal value condition, although it was lately proved in Ref. [23] that C_{tr} violates the monotonicity of coherence under selective measurements on average.

Although it may be easy to rule out the measures that violate the maximal value condition by giving a counterexample, such as the modified trace distance of coherence [23,28] and the rank measure of coherence [27], it is usually complicated to confirm the validity of the condition for a general coherence measure due to the deal with all mixed states, especially for those that only admit a closed-form expression for pure states but lack a closed-form expression for mixed states, such as the robustness of coherence measure [21] and the convex roof coherence measures [19,26]. In this paper, we prove that the maximal value condition holds for mixed states if and only if it holds for a special subset of pure states. Our finding can greatly reduce the examination of whether a coherence measure fulfills the maximal value condition, since one only needs to consider a subset of pure states, avoiding the work of considering mixed states. Our paper is organized as follows. In Sec. II, we review some notions related to coherence measures. In Sec. III, we present our main result as a lemma, a theorem, and a corollary. In Sec. IV, we apply our theorem to several coherence measures. Section V is our summary.

II. PRELIMINARIES

We recapitulate some notions related to coherence measures, such as incoherent states, incoherent operations, and the framework of quantifying coherence.

Let $\mathcal{H}$ represent the Hilbert space of a d-dimensional quantum system. A particular basis of $\mathcal{H}$ is denoted as $\{|\alpha\rangle, \alpha = 0, 1, \dots, d-1\}$, which is chosen according to the physical problem under consideration. The coherence of a state is then measured based on the basis chosen. We use $\rho = \sum_{\alpha\beta} \rho_{\alpha\beta}|\alpha\rangle\langle\beta|$ to denote a general density operator in the basis, where $\rho_{\alpha\beta}$ are the elements of the density matrix. A state is called an incoherent state if its density operator is diagonal in the basis, and the set of all incoherent states is denoted by $\mathcal{I}$. It follows that a density operator ρ belonging to $\mathcal{I}$ is of the form $\rho = \sum_{\alpha=0}^{d-1} \rho_{\alpha\alpha}|\alpha\rangle\langle\alpha|$. All other states, which cannot be written as diagonal matrices in the basis, are called coherent states. A special family of coherent states is the maximally coherent states of the form $\rho = |\Psi\rangle\langle\Psi|$, where

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{\alpha=0}^{d-1} e^{i\theta_\alpha}|\alpha\rangle \tag{1}$$

with the phase parameters θ_α being real numbers. Maximally coherent states have the particular property that they allow for the deterministic generation of all other quantum states of the same dimension by means of incoherent operations. A general pure state is denoted by $|\psi\rangle = \sum_{\alpha=0}^{d-1} c_\alpha|\alpha\rangle$ with c_α being the coefficients, corresponding to density operator $\rho = |\psi\rangle\langle\psi|$.

An incoherent operation is defined by a completely positive and trace preserving (CPTP) map, $\Lambda(\rho) = \sum_n K_n\rho K_n^\dagger$ with the Kraus operators fulfilling not only $\sum_n K_n^\dagger K_n = I$ but also $K_n\mathcal{I}K_n^\dagger \subset \mathcal{I}$, i.e., each K_n maps an incoherent state to an incoherent state.

A functional C can be taken as a coherence measure, if it satisfies the following four conditions [8]:

(C1) $C(\rho) \geqslant 0$, and $C(\rho) = 0$ if and only if $\rho \in \mathcal{I}$.

(C2) Monotonicity under incoherent operations, $C(\rho) \geqslant C(\Lambda(\rho))$ if Λ is an incoherent operation.

(C3) Monotonicity under selective measurements on average, $C(\rho) \geqslant \sum_n p_n C(\rho_n)$, where $p_n = \mathrm{Tr}(K_n\rho K_n^\dagger)$, $\rho_n = K_n\rho K_n^\dagger/p_n$, and $\Lambda(\rho) = \sum_n K_n\rho K_n^\dagger$ is an incoherent operation.

(C4) Nonincreasing under mixing of quantum states, i.e., convexity, $\sum_n q_n C(\rho_n) \geqslant C(\sum_n q_n\rho_n)$ for any set of states $\{\rho_n\}$ and any probability distribution $\{q_n\}$.

Based on the four-condition framework of quantifying coherence, various coherence measures have been proposed. For any coherence measure satisfying the four conditions, $C(\rho)$ must take its maximal value when ρ is a maximally coherent state, but the states that can achieve the maximal value may not be limited to maximally coherent ones. As stated in the introduction section, since maximally coherent states have the particular property of allowing for the deterministic generation of all other states with incoherent operations regardless of any specific coherence measure, one more condition for a desirable coherence measure has been suggested in Ref. [54], denoted as the fifth condition (criterion), i.e.,

(C5) A valid coherence measure should only assign a maximal value to the maximally coherent states.

III. MAIN RESULTS

With these preliminaries, we may now present our main results as the following lemma, theorem, and corollary. For simplicity, we use $C(|\psi\rangle)$ to represent $C(|\psi\rangle\langle\psi|)$, and $\{|\Psi\rangle\langle\Psi|\}$ or simply $\{|\Psi\rangle\}$ to represent the set of all the maximally coherent states.

Lemma 1. Let $C(\rho)$ be a functional of ρ satisfying either the nonincreasing under mixing of quantum states or the monotonicity under incoherent operations. Then, $C(\rho) < C(|\Psi\rangle)$ for $\rho \notin \{|\Psi\rangle\}$ if and only if $C(|\psi\rangle) < C(|\Psi\rangle)$ for $|\psi\rangle \notin \{|\Psi\rangle\}$.

We prove the lemma. It is obvious that the validity of the inequality, $C(\rho) < C(|\Psi\rangle)$ for $\rho \notin \{|\Psi\rangle\}$, certainly implies $C(|\psi\rangle) < C(|\Psi\rangle)$ for $|\psi\rangle \notin \{|\Psi\rangle\}$ since the set of pure states is a subset of all the states. Hence, we only need to prove that the validity of $C(|\psi\rangle) < C(|\Psi\rangle)$ for $|\psi\rangle \notin \{|\Psi\rangle\}$ necessarily leads to $C(\rho) < C(|\Psi\rangle)$ for $\rho \notin \{|\Psi\rangle\}$.

We first prove that the lemma is valid for $C(\rho)$ satisfying the nonincreasing under mixing of quantum states [i.e., condition (C4)]. To this end, we first consider the family of convex roof functionals. It will be easy to extend the result for convex roof functionals to that for other functionals satisfying the nonincreasing under mixing of quantum states.

A convex roof functional is defined by extending a functional C_f acting only on pure states to mixed states via the standard convex roof construction [19,26]. It can be generally expressed as

$$C_f(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_f(|\psi_i\rangle), \tag{2}$$

where the minimum is taken over all possible ensemble decompositions, $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ with $p_i \geqslant 0$ and $\sum_i p_i = 1$. We will prove Lemma 1 by contradiction.

If the lemma was invalid for C_f, there should exist a density operator ρ that is not a maximally coherent state but has $C(\rho) = C_M$ even if $C(|\psi\rangle) < C_M$ is satisfied for all $|\psi\rangle \notin \{|\Psi\rangle\}$. Here, C_M represents the maximal value of C_f. We assume that $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$ is the optimal decomposition of ρ, which makes the right-hand side of Eq. (2) achieve the minimum value, i.e., $C_f(\rho) = \sum_i p_i C_f(|\psi_i\rangle)$. Since $C(\rho) = C_M$, there must be $\sum_i p_i C_f(|\psi_i\rangle) = C_M$, which necessarily lead to $C_f(|\psi_i\rangle) = C_M$. This implies that all the pure states in the optimal decomposition are maximally coherent states, i.e.,

$$|\psi_i\rangle = \frac{1}{\sqrt{d}}\sum_{\alpha=0}^{d-1} e^{i\theta_\alpha^i}|\alpha\rangle. \tag{3}$$

We use U to represent an arbitrary unitary matrix with the dimensions being the same as the number of nonzero terms in the optimal decomposition, of which the elements are denoted as U_{ij}, and let

$$|\bar{\psi}_i\rangle = \frac{1}{\sqrt{\bar{p}_i}}\sum_j \sqrt{p_j}U_{ij}|\psi_j\rangle \tag{4}$$

with

$$\bar{p}_i = \sum_{jk}\sqrt{p_j p_k}U_{ij}^* U_{ik}\langle\psi_j|\psi_k\rangle. \tag{5}$$

Then, $\{\bar{p}_i, |\bar{\psi}_i\rangle\}$ must be an ensemble decomposition of ρ as long as $\{p_i, |\psi_i\rangle\}$ is an ensemble decomposition, i.e.,

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i| = \sum_i \bar{p}_i|\bar{\psi}_i\rangle\langle\bar{\psi}_i|. \tag{6}$$

Since we have assumed that $C_f(\rho) = C_M$, all the vectors $|\bar{\psi}_i\rangle$ must be maximally coherent states too, regardless of any unitary matrix U.

To educe a logical contradiction, we consider two alternative decompositions defined by $U = U_1$ and $U = U_2$. Noting that ρ is not a maximally coherent state, there are at least two nonzero terms in the optimal decomposition $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. Without loss of generality, we assume that p_1 and p_2 are nonzero. We take

$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\bigoplus I \tag{7}$$

and

$$U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\bigoplus I. \tag{8}$$

For $U = U_1$, we have

$$\begin{aligned}|\bar{\psi}_1\rangle &= \frac{1}{\sqrt{M_+}}(\sqrt{p_1}|\psi_1\rangle + \sqrt{p_2}|\psi_2\rangle),\\ |\bar{\psi}_2\rangle &= \frac{1}{\sqrt{M_-}}(\sqrt{p_1}|\psi_1\rangle - \sqrt{p_2}|\psi_2\rangle),\\ |\bar{\psi}_i\rangle &= |\psi_i\rangle, \qquad i \neq 1, 2,\end{aligned} \tag{9}$$

where $M_\pm = p_1 + p_2 \pm 2\sqrt{p_1 p_2}\,\mathrm{Re}\langle\psi_1|\psi_2\rangle$. Substituting Eq. (3) into Eq. (9), we obtain that

$$\begin{aligned}|\bar{\psi}_1\rangle &= \frac{1}{\sqrt{dM_+}}\sum_{\alpha=0}^{d-1}\big(\sqrt{p_1}e^{i\theta_\alpha^1} + \sqrt{p_2}e^{i\theta_\alpha^2}\big)|\alpha\rangle,\\ |\bar{\psi}_2\rangle &= \frac{1}{\sqrt{dM_-}}\sum_{\alpha=0}^{d-1}\big(\sqrt{p_1}e^{i\theta_\alpha^1} - \sqrt{p_2}e^{i\theta_\alpha^2}\big)|\alpha\rangle.\end{aligned} \tag{10}$$

To make sure that $|\bar{\psi}_1\rangle$ and $|\bar{\psi}_2\rangle$ are maximally coherent states, we need to require $|\sqrt{p_1}e^{i\theta_\alpha^1} + \sqrt{p_2}e^{i\theta_\alpha^2}|/\sqrt{M_+} = 1$ and $|\sqrt{p_1}e^{i\theta_\alpha^1} - \sqrt{p_2}e^{i\theta_\alpha^2}|/\sqrt{M_-} = 1$, which lead to

$$\cos\big(\theta_\alpha^2 - \theta_\alpha^1\big) = \mathrm{Re}\langle\psi_1|\psi_2\rangle. \tag{11}$$

For $U = U_2$, we have

$$\begin{aligned}|\bar{\psi}_1\rangle &= \frac{1}{\sqrt{N_+}}(\sqrt{p_1}|\psi_1\rangle - i\sqrt{p_2}|\psi_2\rangle),\\ |\bar{\psi}_2\rangle &= \frac{1}{\sqrt{N_-}}(-i\sqrt{p_1}|\psi_1\rangle + \sqrt{p_2}|\psi_2\rangle),\\ |\bar{\psi}_i\rangle &= |\psi_i\rangle, \quad i \neq 1, 2,\end{aligned} \tag{12}$$

where $N_\pm = p_1 + p_2 \pm 2\sqrt{p_1 p_2}\,\mathrm{Im}\langle\psi_1|\psi_2\rangle$. Substituting Eq. (3) into Eq. (12), we obtain that

$$\begin{aligned}|\bar{\psi}_1\rangle &= \frac{1}{\sqrt{dN_+}}\sum_{\alpha=0}^{d-1}\big(\sqrt{p_1}e^{i\theta_\alpha^1} - i\sqrt{p_2}e^{i\theta_\alpha^2}\big)|\alpha\rangle,\\ |\bar{\psi}_2\rangle &= \frac{1}{\sqrt{dN_-}}\sum_{\alpha=0}^{d-1}\big(-i\sqrt{p_1}e^{i\theta_\alpha^1} + \sqrt{p_2}e^{i\theta_\alpha^2}\big)|\alpha\rangle.\end{aligned} \tag{13}$$

To make sure that $|\bar{\psi}_1\rangle$ and $|\bar{\psi}_2\rangle$ are maximally coherent states, we need to require $|\sqrt{p_1}e^{i\theta_\alpha^1} - i\sqrt{p_2}e^{i\theta_\alpha^2}|/\sqrt{N_+} = 1$ and $|\sqrt{p_1}e^{i\theta_\alpha^1} + i\sqrt{p_2}e^{i\theta_\alpha^2}|/\sqrt{N_-} = 1$, which lead to

$$\sin\big(\theta_\alpha^2 - \theta_\alpha^1\big) = \mathrm{Im}\langle\psi_1|\psi_2\rangle. \tag{14}$$

It is obvious that Eqs. (11) and (14) cannot be satisfied at the same time. The contradiction indicates that such a density operator ρ that is not a maximally coherent state but has $C_f(\rho) = C_M$ does not exist in the case of $C_f(|\psi\rangle) < C_M$ for $|\psi\rangle \notin \{|\Psi\rangle\}$. Therefore, Lemma 1 is valid for convex roof functionals.

After having proved the validity of the lemma for convex roof functionals, it is ready to demonstrate its validity for all other functionals satisfying the nonincreasing under mixing of quantum states. Let C be a functional satisfying the nonincreasing under mixing of quantum states but not a convex roof functional. From C, we can always construct a convex roof functional C_f by taking $C_f(|\psi\rangle) = C(|\psi\rangle)$. As $C(\rho)$ satisfies the nonincreasing under mixing of quantum states, i.e., condition (C4), there is always $C(\rho) \leqslant \sum_i p_i C(|\psi\rangle)$ for any decomposition $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. On the other hand, by definition, there is $C_f(\rho) = \min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i C_f(|\psi_i\rangle)$. We then obtain

$$C(\rho) \leqslant C_f(\rho), \tag{15}$$

which is valid for any state ρ. It means that $C(\rho) < C_M$ as long as $C_f(\rho) < C_M$. Therefore, the density operator ρ that is not a maximally coherent state but has $C(\rho) = C_M$ does not exist in the case of $C(|\psi\rangle) < C_M$ for $|\psi\rangle \notin \{|\Psi\rangle\}$, since such a state does not exist for $C_f(\rho)$, as proved above. Lemma 1 is valid for functional C.

This completes the proof of Lemma 1 for the functionals satisfying the nonincreasing under mixing of quantum states.

We now prove that Lemma 1 is valid too for $C(\rho)$ satisfying the monotonicity under incoherent operations [i.e., condition $(C2)$]. To this end, we consider a general mixed state ρ, but not a maximally coherent state. We only need to show that $C(\rho)$ must be less than $C(\Psi)$ if $C(|\psi\rangle) < C(|\Psi\rangle)$ for $|\psi\rangle \notin \{|\Psi\rangle\}$.

We use $\{p_i, |\psi_i\rangle\}$ to represent an arbitrary ensemble decomposition of the mixed state,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{16}$$

with $|\psi_i\rangle = \sum_{\alpha=0}^{d-1} c_\alpha^i |\alpha\rangle$. Based on the decomposition, we can define an auxiliary pure state,

$$|\Phi\rangle = \sum_{\alpha=0}^{d-1} c_\alpha |\alpha\rangle, \tag{17}$$

where c_α is only required to satisfy $|c_\alpha|^2 = \sum_i p_i |c_\alpha^{i\downarrow}|^2$. Here, the number sequence $(c_0^{i\downarrow}, c_1^{i\downarrow}, \dots, c_{d-1}^{i\downarrow})$ is a rearrangement of $(c_0^i, c_1^i, \dots, c_{d-1}^i)$ in the decreasing order of $|c_\alpha^i|$. It is easy to verify that

$$\begin{aligned} C_{f_l}(|\Phi\rangle) &= \sum_{\alpha=l}^{d-1} |c_\alpha|^2 \\ &= \sum_{\alpha=l}^{d-1} \left(\sum_i p_i |c_\alpha^{i\downarrow}|^2 \right) \\ &= \sum_i p_i \left(\sum_{\alpha=l}^{d-1} |c_\alpha^{i\downarrow}|^2 \right) \\ &= \sum_i p_i C_{f_l}(|\psi_i\rangle), \end{aligned} \tag{18}$$

where $C_{f_l}(\sum_{\alpha=0}^{d-1} c_\alpha|\alpha\rangle) = \sum_{\alpha=l}^{d-1} |c_\alpha^{\downarrow}|^2,\ l = 1, 2, \dots, d-1$. Equation (18) indicates that $|\Phi\rangle$ can be transformed into ρ under incoherent operations [56]. Therefore, if $C(\rho)$ satisfies the monotonicity under incoherent operations [i.e., condition (C2)], there must be

$$C(\rho) \leqslant C(|\Phi\rangle). \tag{19}$$

On the other hand, $\{\bar{p}_i, |\bar{\psi}_i\rangle\}$ defined by Eqs. (4) and (5) is also an ensemble decomposition of ρ if $\{p_i, |\psi_i\rangle\}$ is an ensemble decomposition of ρ. Then, we can always find a decomposition in which at least one pure state is not a maximally coherent state. In fact, if $\{p_i, |\psi_i\rangle\}$ is not such a decomposition, one of the two decompositions given by U_1 and U_2 in Eqs. (7) and (8) must be such a decomposition. This point has been implied in the previous discussions. Without loss of generality, we take $\{p_i, |\psi_i\rangle\}$ as such an ensemble decomposition in which at least one pure state is not a maximally coherent state. In this case, the auxiliary state $|\Phi\rangle$ defined by Eq. (17) must not be a maximally coherent state, i.e., $|\Phi\rangle \notin \{|\Psi\rangle\}$, and hence there is $C(|\Phi\rangle) < C(|\Psi\rangle)$. From the relations (19), we immediately obtain

$$C(\rho) < C(|\Psi\rangle). \tag{20}$$

This completes the proof of Lemma 1 for the functionals satisfying the monotonicity under incoherent operations.

Therefore, the conclusion in Lemma 1 is valid for any functionals $C(\rho)$ satisfying either the nonincreasing under mixing of quantum states or the monotonicity under incoherent operations. Coherence measures are such functions satisfying these conditions, and therefore the lemma is applicable to coherence measures. Further, we use $\mathcal{S}$ to represent the set of the pure states in which all the coefficients c_α are non-negative numbers and satisfy $c_0 \geqslant c_1 \geqslant \cdots \geqslant c_{d-1}$, excluding the maximally coherent state with $c_0 = c_1 = \cdots = c_{d-1}$. By applying Lemma 1 to coherence measures, we can obtain the following theorem.

Theorem 1. A coherence measure satisfies $C(\rho) < C(|\Psi\rangle)$ for $\rho \notin \{|\Psi\rangle\}$ if and only if it satisfies $C(|\psi\rangle) < C(|\Psi\rangle)$ for $|\psi\rangle \in \mathcal{S}$.

Since a pure state with complex coefficients can be transformed into the pure state with non-negative coefficients by using unitary incoherent operations and all coherence measures are invariant under unitary incoherent operations, Theorem 1 can be directly derived from Lemma 1. From the theorem, we can further obtain the following corollary.

Corollary 1. Condition $(C5)$ does or does not hold for both coherence measures $C_1(\rho)$ and $C_2(\rho)$ at the same time, if a monotonically increasing function $f(\cdot)$ is such that $C_1(|\psi\rangle) = f(C_2(|\psi\rangle))$ for $|\psi\rangle \in \mathcal{S}$.

IV. APPLICATIONS

Theorem 1 indicates that the maximal value condition of coherence measures, i.e., condition $(C5)$, holds for mixed states if and only if it holds for a special subset of pure states. With the help of the theorem, to examine whether a coherence measure satisfies condition $(C5)$, one only needs to consider a subset of pure states. In this section, we will present some examples to show the usefulness of our theorem. The first example contains three coherence measures for which the validity of the maximal value condition have been examined before. By this example, we illustrate that our theorem and corollary make the examination more simple and effective. The second and third examples contain the coherence measures for which so far it is unknown whether the maximal value condition is valid. One will see that, by using our theorem and corollary, it is easy to prove these coherence measures fulfill the maximal value condition.

Example 1. We first consider three coherence measures: the l_1 norm of coherence C_{l_1} [8], the robustness of coherence C_R [21], and the coherence concurrence C_C [24], which are defined as

$$C_{l_1}(\rho) = \sum_{i\neq j} |\rho_{ij}|, \tag{21}$$

$$C_R(\rho) = \min_\tau \left\{ s \geqslant 0 \,\middle|\, \frac{\rho + s\tau}{1+s} \in \mathcal{I} \right\}, \tag{22}$$

and

$$C_C(\rho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i C_{l_1}(|\psi_i\rangle), \tag{23}$$

respectively. Here, the minimum in Eq. (22) is taken over all quantum states τ and the minimum in Eq. (23) is taken over all possible ensemble decompositions $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$.

Since $C_R(|\psi\rangle)=C_{l_1}(|\psi\rangle)$ [21] and $C_C(|\psi\rangle)=C_{l_1}(|\psi\rangle)$ [24], i.e.,

$$C_{l_1}(|\psi\rangle)=C_R(|\psi\rangle)=C_C(|\psi\rangle), \tag{24}$$

by using Corollary 1, the three coherence measures C_{l_1}, C_R, and C_C do or do not satisfy condition ($C5$) at the same time. Noting that C_{l_1} satisfies condition ($C5$), proved in Ref. [54], we immediately obtain that C_R and C_C satisfy condition ($C5$) too without the need of the extra proofs [21,24].

Example 2. We consider a family of convex roof coherence measures, defined in Refs. [17,19],

$$C_{f_l}(\rho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i C_{f_l}(|\psi_i\rangle),\quad l=1,2,\dots,d-1 \tag{25}$$

with

$$C_{f_l}(|\psi\rangle)=\sum_{\alpha=l}^{d-1}\left|c_\alpha^{\downarrow}\right|^2 \tag{26}$$

for an arbitrary pure state $|\psi\rangle=\sum_{\alpha=0}^{d-1}c_\alpha|\alpha\rangle$, where the number sequence $(c_0^{\downarrow},c_1^{\downarrow},\dots,c_{d-1}^{\downarrow})$ is a rearrangement of $(c_0,c_1,\dots,c_{d-1})$ in the decreasing order of $|c_\alpha|$, where $|c_0^{\downarrow}|\geqslant|c_1^{\downarrow}|\geqslant\cdots\geqslant|c_{d-1}^{\downarrow}|$.

By using our theorem, we can easily prove that $C_{f_l}(\rho)$ fulfill condition (C5). By definition, $C_{f_l}(|\psi\rangle)=(d-l)/d$ for $|\psi\rangle\in\{|\Psi\rangle\}$. Therefore, according to our theorem, we only need to demonstrate that $C_{f_l}(|\psi\rangle)$ is less than $(d-l)/d$ for $|\psi\rangle=\sum_{\alpha=0}^{d-1}c_\alpha|\alpha\rangle\in\mathcal{S}$. Since $c_0\geqslant c_1\geqslant\cdots\geqslant c_{d-1}$ except for $c_0=c_1=\cdots=c_{d-1}$, some coefficients must be larger than others. The average of the first l numbers $(c_0^2,c_1^2,\dots,c_{l-1}^2)$ is larger than the average of the last $(d-l)$ numbers $(c_l^2,c_{l+1}^2,\dots,c_{d-1}^2)$, i.e., $(\sum_{\alpha=0}^{l-1}c_\alpha^2)/l>(\sum_{\alpha=l}^{d-1}c_\alpha^2)/(d-l)$. On the other hand, the average of all the d numbers $(c_0^2,c_1^2,\dots,c_{d-1}^2)$ is $1/d$. It implies that $(\sum_{\alpha=l}^{d-1}c_\alpha^2)/(d-l)<1/d$. We then obtain $C_{f_l}(|\psi\rangle)=\sum_{\alpha=l}^{d-1}c_\alpha^2<(d-l)/d$. This completes our proof of C_{f_l} satisfying condition ($C5$).

Example 3. We consider the geometric measure of coherence C_g [18] and the convex roof coherence measure C_F [25], both of which are based on the Uhlmann fidelity [57].

The geometric measure of coherence is defined as

$$C_g(\rho)=1-\max_{\delta\in\mathcal{I}}F(\rho,\delta) \tag{27}$$

with the Uhlmann fidelity $F(\rho,\delta)=(\mathrm{Tr}\sqrt{\sqrt{\rho}\delta\sqrt{\rho}})^2$, where the minimum in Eq. (27) is taken over all incoherent states δ.

The convex roof coherence measure, based on the same fidelity $F(\rho,\delta)$, is defined as

$$C_F(\rho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i C_F(|\psi\rangle) \tag{28}$$

with

$$C_F(|\psi\rangle)=\min_{\delta\in\mathcal{I}}\sqrt{1-F(|\psi_i\rangle,\delta)}, \tag{29}$$

where the minimum in Eq. (28) is taken over all the ensemble decomposition $\{p_i,|\psi_i\rangle\}$ of ρ, and $F(|\psi\rangle,\delta)=F(\rho,\delta)|_{\rho=|\psi\rangle\langle\psi|}$.

It is easy to see that $C_g(|\psi\rangle)=1-c_0^2$ and $C_F(|\psi\rangle)=\sqrt{1-c_0^2}$ for $|\psi\rangle=\sum_{\alpha=0}^{d-1}c_\alpha|\alpha\rangle\in\mathcal{S}$. Noting that $C_{f_1}(|\psi\rangle)=1-c_0^2$ [see Eq. (26), taking $l=1$], we then have

$$C_g(|\psi\rangle)=C_F^2(|\psi\rangle)=C_{f_1}(|\psi\rangle). \tag{30}$$

Therefore, by using Corollary 1, the coherence measures C_g and C_F satisfy condition ($C5$), as did C_{f_1} in example 2.

In passing, we give two instances that fulfill conditions (C1)–(C4) but violate the maximal value condition. One is the modified trace distance of coherence, defined as $C'_{\mathrm{tr}}(\rho)=\min_{\lambda\geqslant0,\delta\in\mathcal{I}}\|\rho-\lambda\delta\|_{\mathrm{tr}}$ [23], and another is the rank measure of coherence, defined as $C_{\mathrm{rank}}(\rho)=\min_{\{p_i,|\psi_i\rangle\}}\sum_i p_i C_{\mathrm{rank}}(|\psi_i\rangle)$, where $C_{\mathrm{rank}}(|\psi_i\rangle)=\log_2[r_c(|\psi_i\rangle)]$ with the coherence rank $r_c(|\psi\rangle)$ being the number of nonzero coefficients in $|\psi\rangle=\sum_\alpha c_\alpha|\alpha\rangle$ [27]. By direct calculations, we have $C'_{\mathrm{tr}}(|\Psi\rangle)=C'_{\mathrm{tr}}(|\phi\rangle)=1$ and $C_{\mathrm{rank}}(|\Psi\rangle)=C_{\mathrm{rank}}(|\phi\rangle)=\log 3$ for both a three-dimensional maximally coherent state $|\Psi\rangle$ and a three-dimensional nonmaximally coherent state $|\phi\rangle=1/\sqrt{2}|0\rangle+1/\sqrt{3}|1\rangle+1/\sqrt{6}|2\rangle$. It means that the modified trace distance of coherence does not fulfill the maximal value condition, and neither does the rank measure of coherence.

Before going into the summary, we would like to stress that our theorem is applicable to all the functionals satisfying condition (C2), regardless of conditions (C3) and (C4). This point is implied in the second part of the proof of Lemma 1. As the convexity condition is not necessary in obtaining Theorem 1, our result can be applied to examine the maximal value condition for nonconvex coherence measures too. For example, the functional defined by $C(\rho)=\min_{\delta\in\mathcal{I}}D(\rho\|\delta)$ with $D(\rho\|\delta)=\min\{\lambda:\rho\leqslant2^\lambda\delta\}$ [58] is a nonconvex coherence measure. By using Theorem 1 and Corollary 1, we can easily prove that $C(\rho)$ satisfies the maximal value condition. Indeed, since $C(|\psi\rangle)$ can be expressed as $C(|\psi\rangle)=\log[1+C_R(|\psi\rangle)]$ [58] and $C_R(|\psi\rangle)$ satisfies the maximal value condition (see Example 1), $C(\rho)$ must satisfy the maximal value condition, according to Corollary 1.

V. SUMMARY

To examine whether a candidate of coherence measures fulfills the maximal value condition, we first present a lemma. It says that $C(\rho)<C(|\Psi\rangle)$ for $\rho\notin\{|\Psi\rangle\}$ if and only if $C(|\psi\rangle)<C(|\Psi\rangle)$ for $|\psi\rangle\notin\{|\Psi\rangle\}$, which is applicable to any functional $C(\rho)$ satisfying either the nonincreasing under mixing of quantum states or the monotonicity under incoherent operations. Here, $\{|\Psi\rangle\}$ represents the set of all the maximally coherent states.

By applying the lemma to coherence measures, we then obtain the theorem, i.e., a coherence measure satisfies $C(\rho)<C(|\Psi\rangle)$ for $\rho\notin\{|\Psi\rangle\}$ if and only if it satisfies $C(|\psi\rangle)<C(|\Psi\rangle)$ for $|\psi\rangle\in\mathcal{S}$. Here, $\mathcal{S}$ is a subset of pure states, defined as $|\psi\rangle=\sum_{\alpha=0}^{d-1}c_\alpha|\alpha\rangle$ with the coefficients being non-negative

and satisfying $c_0 \geqslant c_1 \geqslant \cdots \geqslant c_{d-1}$ but not $c_0 = c_1 = \cdots = c_{d-1}$.

From the theorem, we further derive a corollary. That is, condition ($C5$) does or does not hold for both coherence measures $C_1(\rho)$ and $C_2(\rho)$ at the same time, if a monotonically increasing function $f(\cdot)$ is such that $C_1(|\psi\rangle) = f(C_2(|\psi\rangle))$ for $|\psi\rangle \in \mathcal{S}$.

Our finding can greatly reduce the examination on whether a coherence measure fulfills the maximal value condition, since one only needs to consider a subset of pure states, avoiding the extra work of considering mixed states. To show the usefulness of our finding, we have given some examples.

We would like to point out that the above results give an illuminating insight into the resource theory of coherence in the sense that a problem involving all mixed states may be reduced to that only related to pure states. Such an idea may be useful to other topics in quantum resource theories too.

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