Increasing distillable key rate from bound entangled states by using local filtration

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We show the enhancement of a distillable key rate for quantum key distribution (QKD) by local filtering for several bound entangled states. Through our paper, it becomes evident that the local filtration operations, whereas transforming one bound entangled state to another, have the potential to increase the utility of the new state for QKD. We demonstrate three examples of "one-way distillable key rate" enhancement by local filtering and, in this process, discover new bound entangled states which are key distillable.

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I. INTRODUCTION

A perfect cryptography protocol can be set up if one can distribute a private key between the trusted parties interested in secure communication [1]. It has been established that only quantum key distribution (QKD) protocols are fundamentally secure as they draw their security from the laws of quantum physics, as opposed to their classical counterparts where the security is based on the impossibility of solving certain mathematical problems in polynomial time [2–4]. The QKD protocols are either of the prepare and measure type, such as the BB84 protocol [5,6] or the entanglement assisted protocols, such as the E91 protocol [3]. These two classes of protocols are intimately connected, and entanglement is considered a fundamental resource for QKD [7].

Bipartite quantum entanglement involves nonclassical correlations between two parties and can occur in pure as well mixed states [8–10]. A powerful tool to identify entanglement is the transpose operation where entangled quantum states can transform to nonstates when we apply the transpose operation on one of the subsystems [11]. The states whose density operators become negative under such partial transposition are entangled and are called negative under partial transpose (NPT) and those which remain positive and valid are called positive under partial transpose (PPT). From the work of Peres and Horodeckis', it became clear that partial transpose is necessary and sufficient only for $2 \otimes 2$ or $2 \otimes 3$ bipartite systems, whereas for higher-dimensional systems, there can be entangled states which are PPT [12]. Due to the existence of PPT entangled states, the geometry of states in composite dimensions other than 4 and 6 has not been fully understood. The examples and methods to construct classes of PPT entangled states are sparse [13–15], although a plethora of literature exists on the subject [10,16,17].

Ideally, quantum information tasks require pure entangled states. Since mixedness caused by the environment is unavoidable, for operational purposes, it is necessary to recreate pure copies of the required entangled state via distillation protocols [18,19]. Entangled states from which we cannot distill any pure entangled states are called bound entangled, and their entanglement is called bound entanglement. All pure entangled states are NPT, and distillable entangled states from which maximally entangled states can be distilled, are also NPT. It can be shown that pure entangled states cannot be distilled from PPT entangled states and, hence, PPT entangled states are bound entangled [20].

Contrary to expectation, bound entanglement has been found to be a useful resource for performing a number of quantum information processing tasks. It was shown by Wild and Hsieh [21] that bound entanglement can be used for establishing superactivation of quantum channels by using the Smith-Yard protocol [22]. Using tensor products of bound entangled states to obtain distillable states and aspects related to superactivation and superadditivity are discussed in Refs. [23,24]. Usefulness of bound entanglement in metrology has been observed by Czekaj *et al.* [25] where they have given an example of a family of bound entangled states which can be used in quantum enhanced metrology with the precision advantage approaching the Heisenberg limit. Furthermore, Tóth and Vértesi [26] demonstrated that multipartite quantum states that have a positive partial transpose with respect to all bipartitions can outperform separable states in linear interferometers. Bound entanglement has also been shown to be a useful resource for quantum heat engines [27].

For QKD protocols, for a long time, it was believed that distillable entanglement from which one can obtain maximally entangled states is an essential resource [3] and bound entanglement is not such a resource. Thus, it came as a surprise when it was shown that PPT-bound entangled states may also be useful for QKD [28–30]. A key role is played by a new class of states called private states [28,29,31] which can

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be used to carry out QKD. These states need not be maximally entangled but still can be used to obtain a perfectly correlated key that is completely uncorrelated to any eavesdropper [28,29]. Furthermore, it was demonstrated that one can find PPT-bound entangled states that are arbitrarily close in trace norm to the private states [30]. Thus, using the idea of private states, it was shown that the PPT-bound entangled states can also be used for QKD [28,29]. The initial class of key distillable bound entangled states presented in Refs. [28,31] were of large dimensions and, hence, of limited use for experimental applications. A new class of low-dimensional-bound entangled states which can be used for QKD key was introduced in Ref. [32]. These have been further studied in Refs. [30,33,34].

Evaluating PPT entangled states for their utility for QKD is not always straightforward, and there are more than one criteria being used in the literature. The two most important parameters in this context are "distillable key rate" (K_D) defined in Ref. [28] and the "one-way distillable key rate" ($K_D^{\rm DW}$) given in the Devetak-Winter protocol [35]. As per the definition given in Ref. [28], a state is declared useful for QKD if $K_D > 0$. However, for the states defined in Ref. [32], whereas it has been shown that $K_D > 0$, we have $K_D^{\rm DW} < 0$ [35]. The situation, therefore, is delicate, and different definitions may not always agree [30]. However, for the cases that we consider in our paper, we have $K_D > K_D^{\rm DW}$, and, thus, we use the one way distillable key rate as a lower bound of the available key from a PPT state.

Local filtration is a process where starting with an ensemble of bipartite states, the members are selected or discarded based on the results of certain local measurements thereby obtaining a new smaller (filtered) ensemble of bipartite states [36]. Although local filtration can not change the nature of entanglement (PPT or NPT), the filtered ensemble can have different properties as compared to the original one. Particularly, such a filtration has been used to enhance the Bell violation [37] for detecting the entanglement of PPT entangled states [38] and for increasing the usefulness of quantum states for QKD [39]. In this paper, we use the local filtration on different PPT entangled states and obtain transformed states that have higher one-way distillable key rate. We demonstrate our method by considering three concrete examples of PPT entangled states, some of which are available in the literature. First, we apply local filtration to a family of states closely related to the states given in Ref. [32] and show that, under certain conditions, the one way distillable key rate of the Devetak-Winter protocol can be made positive. Using similar filtration methods, we show enhancement of one way distillable key rate for two more families of states considered by Chi *et al.* [32]. In all the cases, the enhancement is dramatic, and the one-way distillable key rate goes from a negative to a significant positive value making the transformed states explicitly usable for QKD. Since we quantitatively calculate the key rate, the added advantage is that the amount of key available can also be ascertained.

This paper is organized as follows: In Sec. II, we recall the results that provide background for our paper, which includes the definition of private states and distillable key rate given in Sec. II A, privacy squeezing and one-way distillable key rate given in Sec. II B and a description of local filtration process given in Sec. II C. In Sec. III, we describe our main results where we show how the filtration process can enhance the one-way distillable key rate. Some concluding remarks are presented in Sec. IV.

II. BACKGROUND

In this section, we briefly recapitulate the results that our paper builds upon. In this context, private states and their roles in QKD protocols, classical-classical-quantum (ccq) states, the one-way distillable key rate and local filters that we will use in the next section for enhancement of the relevant key rates of certain bound entangled states are discussed.

A. Private states and distillable key

Consider a situation where Alice and Bob want to carry out QKD; Alice has systems A and A' of dimensions d_A and $d_{A'}$, respectively, whereas Bob has systems B and B' of dimensions d_B and $d_{B'}$, respectively. The key is contributed by the systems A and B, therefore, $d_A = d_B = d$. Let a state $\rho_{ABA'B'} \in B(\mathcal{H}^d \otimes \mathcal{H}^d \otimes \mathcal{H}^{d_{A'}} \otimes \mathcal{H}^{d_{B'}})$ be shared between Alice and Bob, and the eavesdropper has the standard purifying system E. The purification $|\psi_\rho\rangle_{ABA'B'E}$ of state $\rho_{ABA'B'}$ is called secure if, upon measurement on systems A and B in the basis $\{|ij\rangle_{AB}\}_{i,j=0}^{d-1}$, followed by tracing over the $A'B'$ subsystem, the joint state of systems A, B, and E takes the form of a ccq state:

$$\rho^{\rm ccq} = \sum_{i,j=0}^{d-1} p_{ij}|ij\rangle\langle ij|_{AB} \otimes \rho_E. \tag{1}$$

State $\rho_{ABA'B'}$ whose purification is secure is called a private state or a pdit, and $\rho^{\rm ccq}$ is called the ccq state corresponding to the secure state.

The private states are related to maximally entangled states of the system AB, and are obtained by applying twisting operations on the tensor product of a maximally entangled state of the AB system and some arbitrary state $\sigma_{A'B'}$ of the system $A'B'$ in the following way:

$$\gamma_{ABA'B'} = \frac{1}{d}\sum_{i,j=0}^{d-1} |ii\rangle\langle jj|_{AB} \otimes U^{ii}\sigma_{A'B'}U^{jj\dagger}. \tag{2}$$

The operators U^{ii} are arbitrary unitary transformations on the $A'B'$ system. Twisting does not change the security related properties of a state as the corresponding ccq state is invariant under the twisting operation. Therefore, the private states are as efficient for QKD as maximally entangled states and have the same amount of key. In fact, given a private state, we should always be able to find a twisting operation so that we recover the maximally entangled state of the AB system by twisting [28,31].

Since, private states are analogous to maximally entangled states in entanglement theory, similar to the definition of measure of distillable entanglement, one can define maximal distillable pdits. For any given state $\rho_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$, let us consider a sequence of local operations and classical communication (LOCC) operations $\{P_n\}$ such that $P_n(\rho_{AB}^{\otimes n}) = \phi_n$, where $\phi_n \in B(\mathcal{H}_A^{(n)} \otimes \mathcal{H}_B^{(n)})$. A set of operations $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ is called a pdit distillation protocol of state ρ_{AB} if there is a pdit γ_{d_n} whose key part is of dimension $d_n \times d_n$, satisfying

$$\lim_{n\to\infty} \|\phi_n - \gamma_{d_n}\| = 0. \quad (3)$$

For a given distillation protocol $\mathcal{P}$, the key rate is given by

$$\mathcal{R}(\mathcal{P}) = \limsup_{n\to\infty} \frac{\log_2 d_n}{n}. \quad (4)$$

The distillable key rate of state ρ_{AB} is given by maximizing over all possible protocols,

$$K_D(\rho_{AB}) = \sup_{\mathcal{P}} \mathcal{R}(\mathcal{P}). \quad (5)$$

If one can transform by LOCC, such as the recurrence protocol [19], sufficiently many copies of a state $\rho_{ABA'B'}$ into a state close enough to a private state in a trace norm, then $K_D(\rho_{ABA'B'}) > 0$.

It can be shown that there are PPT states for which the key rate K_D is positive [40,41]. In any case, since separable states, by construction, do not have any distillable key rate and PPT states are always bound entangled, we can conclude that these states are bound entangled states with a nonzero distillable key rate. We will consider a few of these states in the next section, and the detailed mathematical proofs are available in Refs. [31,32]. It is not always straightforward to calculate K_D for a given quantum state as an optimization over distillation protocols is involved.

B. One-way distillable key rate

In the special case when $d = 2$, the private state is called a pbit, and we will restrict to this case in the rest of this paper. In this scenario, consider a state $\rho_{ABA'B'} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^{d_{A'}} \otimes \mathcal{H}^{d_{B'}})$. Expanding this state in the computational basis of the qubits A and B, namely, $|ij\rangle$, with $i, j \in \{0, 1\}$,

$$\rho_{ABA'B'} = \begin{bmatrix} \sigma^{0000} & \sigma^{0001} & \sigma^{0010} & \sigma^{0011} \\ \sigma^{0100} & \sigma^{0101} & \sigma^{0110} & \sigma^{0111} \\ \sigma^{1000} & \sigma^{1001} & \sigma^{1010} & \sigma^{1011} \\ \sigma^{1100} & \sigma^{1101} & \sigma^{1110} & \sigma^{1111} \end{bmatrix}. \quad (6)$$

The term $|00\rangle\langle 11|_{AB} \otimes \sigma^{0011}$ (upper right-hand corner in the above equation), with σ^{0011} being a density operator for the $A'B'$ subsystem plays an important role. If state $\rho_{ABA'B'}$ was such that the AB subsystem is in a maximally entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then, this term and its Hermitian conjugate, namely, the term $|11\rangle\langle 00|_{AB} \otimes \sigma^{1100}_{A'B'}$ is the only nonzero of-diagonal term in the expansion and the trace norm $\|\sigma^{1100}\| = \|\sigma^{0011}\| = \frac{1}{2}$. For any private state, by an appropriate twisting operation, $\|\sigma^{1100}\|$ can always be brought to $\frac{1}{2}$. For an arbitrary state $\rho_{ABA'B'}$, in order to determine its ability to carry out QKD, we carry out an operation called privacy squeezing where we maximize $\|\sigma^{1100}\|$ by applying twisting operations. State ρ^{PS} that we obtain after such a maximization is called a privacy squeezed state corresponding to $\rho_{AA'BB'}$. It has been shown by a more detailed analysis [35] that if $\|\sigma^{0000}\| = \|\sigma^{0011}\| = \|\sigma^{1111}\|$ and $\|\sigma^{0101}\| < \|\sigma^{0011}\|$, $\|\sigma^{1010}\| < \|\sigma^{0011}\|$, then, $K_D(\rho_{ABA'B'}) > 0$.

For a specific form of $\rho_{ABA'B'} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2)$,

$$\begin{aligned}\rho_{ABA'B'} &= |\phi^+\rangle\langle\phi^+| \otimes \sigma_0 + |\phi^-\rangle\langle\phi^-| \otimes \sigma_1 \\ &\quad + |\psi^+\rangle\langle\psi^+| \otimes \sigma_2 + |\psi^-\rangle\langle\psi^-| \otimes \sigma_3, \end{aligned} \quad (7)$$

where $|\phi^\pm\rangle$ and $|\psi^\pm\rangle$ are Bell states in $\mathcal{H}^2 \otimes \mathcal{H}^2$. Then, if $\|\sigma_0 - \sigma_1\| > \frac{1}{2}$ and $\text{tr}(\sigma_0\sigma_1) = 0$, then, $K_D(\rho_{ABA'B'}) > 0$.

The one-way key distillation protocol expresses the key distillation rate for the ccq state in terms of the Devetak-Winter function K_D^{DW} [35]. Devetak-Winter function or the one-way distillable key rate, K_D^{DW} is the difference between the mutual information of the Alice-Bob subsystem and the Alice-Eve subsystem,

$$K_D^{\text{DW}} = I(A{:}B) - I(A{:}E), \quad (8)$$

where the mutual information between A and B is given as $I(A{:}B) = S(A) + S(B) - S(AB)$ and between A and E is given as $I(A{:}E) = S(A) + S(E) - S(AE)$ with $S(X)$ being the Von-Neumann entropy of the subsystem X.

For a state $\rho_{ABA'B'}$ of the form given in Eq. (6), let us define the following parameters:

$$\begin{aligned} x &= (\|\sigma^{0000}\| + \|\sigma^{1111}\|)/2 + \|\sigma^{0011}\|, \\ y &= (\|\sigma^{0000}\| + \|\sigma^{1111}\|)/2 - \|\sigma^{0011}\|, \\ z &= (\|\sigma^{0101}\| + \|\sigma^{1010}\|)/2 + \|\sigma^{0110}\|, \\ w &= (\|\sigma^{0101}\| + \|\sigma^{1010}\|)/2 - \|\sigma^{0110}\|. \end{aligned} \quad (9)$$

In terms of these parameters, define

$$S(E) = -x\log_2 x - y\log_2 y - z\log_2 z - w\log_2 w, \quad (10)$$

The analysis given in Ref. [35] proves that

$$K_D^{\text{DW}}\big([\rho^{PS}]^{\text{ccq}}_{ABE'}\big) = 1 - S(E). \quad (11)$$

This is the key formula that we will use to compute the one-way distillable key rates for various states in the next section.

For any bipartite state $\rho_{ABA'B'} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^d \otimes \mathcal{H}^{d'})$, it can be shown that

$$K_D(\rho_{ABA'B'}) \geqslant K_D^{\text{DW}}\big([\rho^{PS}]^{\text{ccq}}_{ABE'}\big), \quad (12)$$

where $[\rho^{PS}]^{\text{ccq}}_{ABE'}$ is the ccq state corresponding to the privacy squeezed state of $\rho_{ABA'B'}$. Combining Eq. (10) with Eq. (12) will allow us to prove the positive distillable key rates of several bound entangled states.

C. Local filters

In a local filtering process that we consider, Alice and Bob perform local positive operator-valued measures on state $\rho_{AA'BB'} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2)$ defined by the operators L_A and L_B. They retain the cases when both of them get positive outcome for the operators L_A and L_B and discard the other cases. The state after such an operation changes as follows:

$$\rho \to (L\rho L^\dagger)/(\text{Tr}\, L\rho L^\dagger), \quad L = L_A \otimes L_B. \quad (13)$$

In our case, L_A and L_B are 4×4 diagonal matrices leading to

$$L = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}_A \otimes \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & u \end{bmatrix}_B. \quad (14)$$

The 16×16 matrix L can also be written as

$$L = \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & L_4 \end{bmatrix}, \tag{15}$$

where L_1, L_2, L_3, and L_4 all are 4×4 diagonal matrices determined in terms of L_A and L_B and we have

$$L_1 = \begin{bmatrix} ar & 0 & 0 & 0 \\ 0 & as & 0 & 0 \\ 0 & 0 & at & 0 \\ 0 & 0 & 0 & au \end{bmatrix}, \quad L_2 = \begin{bmatrix} br & 0 & 0 & 0 \\ 0 & bs & 0 & 0 \\ 0 & 0 & bt & 0 \\ 0 & 0 & 0 & bu \end{bmatrix},$$
$$L_3 = \begin{bmatrix} cr & 0 & 0 & 0 \\ 0 & cs & 0 & 0 \\ 0 & 0 & ct & 0 \\ 0 & 0 & 0 & cu \end{bmatrix}, \quad L_4 = \begin{bmatrix} dr & 0 & 0 & 0 \\ 0 & ds & 0 & 0 \\ 0 & 0 & dt & 0 \\ 0 & 0 & 0 & du \end{bmatrix}. \tag{16}$$

The parameters $\{a, b, c, d, r, s, t, u\} \in (0, 1]$ define the local filter. The filtration is typically considered as an operator on a large ensemble of identically prepared states $\rho_{AA'BB'}$ and lead to a smaller ensemble of filtered states. The success probability P of a filter is defined as the probability with which both the measurements give the positive result together and the size of the ensemble after filtration is P times the size of the original ensemble.

III. ENHANCEMENT OF THE DISTILLABLE KEY OF BOUND ENTANGLED STATES BY LOCAL FILTRATION

We are now ready to consider the possibility of enhancing the distillable key rate from bound entangled states by a local filtration process. The starting states that we consider are bound entangled because they are PPT and the local filters cannot change states from PPT to NPT and, hence, the filtered states are also bound entangled. We, then, analyze the distillable key rate from the new and old states by specifically computing the one-way distillable key rate defined in the Devetak-Winter protocol by a procedure described in Sec. II. In every case, the one distillable key rate goes from a negative to a positive value.

A. Example 1

Consider a one-parameter family of states $\rho_p^{(1)} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2)$ parametrized by a real parameter with $p \in [0, 1/2]$ defined as

$$\rho_p^{(1)} = \frac{1}{1+2p} \begin{bmatrix} 2p\tau_1 & 0 & 0 & 2p\tau_1 \\ 0 & \left(\frac{1}{2}-p\right)\tau_2 & 0 & 0 \\ 0 & 0 & \left(\frac{1}{2}-p\right)\tau_2 & 0 \\ 2p\tau_1 & 0 & 0 & 2p\tau_1 \end{bmatrix}, \tag{17}$$

with matrices τ_1 and τ_2 given as

$$\tau_1 = \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{bmatrix} \quad \text{and} \quad \tau_2 = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}. \tag{18}$$

This family of states is PPT, and, hence, any entanglement if present, has to be bound entanglement.

Since this family of states belongs to $B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2)$ as per Eq. (12), K_D is always greater than $K^{\rm DW}$, and, therefore, $K_D^{\rm DW}$ provides us with a lower bound for the distillable key rate. We calculate the value of $K_D^{\rm DW}$ for these states by exploiting the fact that the structure of the family of states is the same as that given in Eq. (6) for which we can calculate $K_D^{\rm DW}$. Comparison with Eq. (10) reveals that, for this family of states, we have $x = 4|\frac{p}{1+2p}||$, $y = 0$, $z = \frac{1}{2}|\frac{-1+2p}{1+2p}|$, and $w = \frac{1}{2}|\frac{-1+2p}{1+2p}|$. Therefore, using formula (10) and Eq. (11), we can calculate $K_D^{\rm DW}([\rho_p^{(1)\,PS}]^{\rm ccq}_{ABE'})$. The graph of $K_D^{\rm DW}([\rho_p^{(1)\,PS}]^{\rm ccq}_{ABE'})$ as a function of p is plotted in Fig. 1(a). The value of $K_D^{\rm DW}$ is negative up to a certain values of p and, then, crosses over to a positive value at approximately $p = 0.31$.

Now, we propose local filtration operation on the family of states $\rho_p^{(1)}$ with a view to raise the value of the corresponding $K_D^{\rm DW}([\rho_p^{(1)\,PS}]^{\rm ccq}_{ABE'})$. Consider the application of the local filter described in Eqs. (13) on state $\rho_p^{(1)}$. The state after filtration is given by

$$\rho_p^{(1)} \rightarrow \frac{L\rho_p^{(1)}L^\dagger}{{\rm Tr}\big(L\rho_p^{(1)}L^\dagger\big)} = \rho_p^{(1)'}. \tag{19}$$

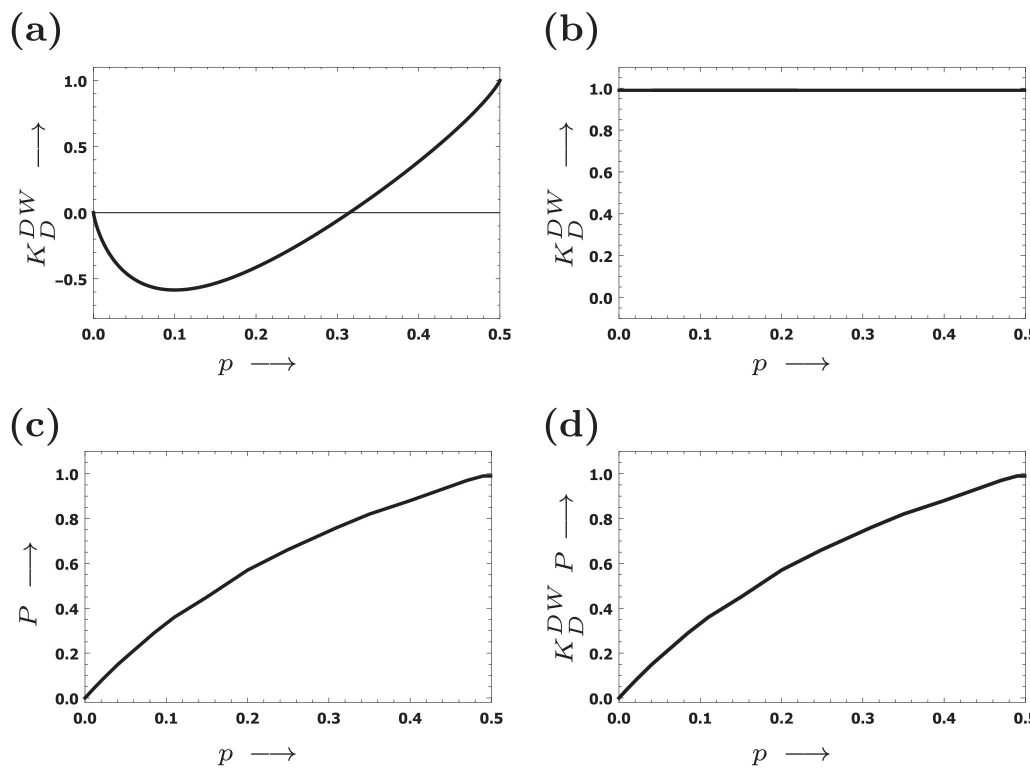

FIG. 1. Improvement in the value of $K_D^{\rm DW}$ after local filtration operation on the one-parameter family of quantum states $\rho_p^{(1)}$. (a) The value of the one-way distillable key rate $K_D^{\rm DW}$ for the family of states $\rho_p^{(1)}$ as a function of p. (b) The value of $K_D^{\rm DW}$ after optimal local filtration operation on $\rho_p^{(1)}$. (c) The probability of success P of the optimal local filter. (d) The effective one-way distillable key rate after filtration obtained by multiplying the value of $K_D^{\rm DW}$ with the success probability of the optimal filter. For all plots, the x axis is the parameter $p \in [0, 1/2]$. The enhancement of the one-way distillable key rate can be seen by comparing the graphs shown in (a) and (d).

Exploiting the structure of the filtration operation detailed in Eqs. (14)–(16), we can write $\rho_p^{(1)'}$ in explicit matrix form as

$$\rho_p^{(1)'} = \frac{1}{(1+2p)M}\begin{bmatrix} L_1(2p\tau_1)L_1^\dagger & 0 & 0 & L_1(2p\tau_1)L_4^\dagger \\ 0 & L_2\left(\frac{1}{2}-p\right)\tau_2 L_2^\dagger & 0 & 0 \\ 0 & 0 & L_3\left(\frac{1}{2}-p\right)\tau_2 L_3^\dagger & 0 \\ L_4(2p\tau_1)L_1^\dagger & 0 & 0 & L_4(2p\tau_1)L_4^\dagger \end{bmatrix}, \tag{20}$$

where $M = \mathrm{Tr}(L\rho_p^{(1)}L^\dagger)$. For calculating the one-way distillable key rate, we use the following Eq. (10) to ascertain the values of the parameters $x,\ y,\ z,$ and w for this state as

$$\begin{aligned} x &= [\|L_1 2p(\tau_1)L_1^\dagger/(1+2p)M\| + \|L_4 2p(\tau_1)L_4^\dagger/(1+2p)M\|]/2 + \|L_1 2p(\tau_1)L_4^\dagger/(1+2p)M\|, \\ y &= [\|L_1 2p(\tau_1)L_1^\dagger/(1+2p)M\| + \|L_4 2p(\tau_1)L_4^\dagger/(1+2p)M\|]/2 - \|L_1 2p(\tau_1)L_4^\dagger/(1+2p)M\|, \\ z &= \left[\left\|L_2\left(\tfrac{1}{2}-p\right)\tau_2 L_2^\dagger/(1+2p)M\right\| + \left\|L_3\left(\tfrac{1}{2}-p\right)\tau_2 L_3^\dagger/(1+2p)M\right\|\right]/2, \\ w &= \left[\left\|L_2\left(\tfrac{1}{2}-p\right)\tau_2 L_2^\dagger/(1+2p)M\right\| + \left\|L_3\left(\tfrac{1}{2}-p\right)\tau_2 L_3^\dagger/(1+2p)M\right\|\right]/2. \end{aligned} \tag{21}$$

The value of $K_D^{\rm DW}([\rho_p^{(1)\,PS}]_{ABE'}^{\rm ccq})$ can be again calculated following Eqs. (10) and (11) as was performed before filtering.

A given filter has a success probability P which is defined as the fraction of cases the filter gives a positive answer. In real terms, for the purposes of using an ensemble of states for QKD, we must multiply the key rate with the success probability to get the effective key rate. This is particularly important if we want to make any comparison with the key rate before filtration. A filter may give us a very high key rate, however, if its probability of success is small the effective key rate may, in fact, be small. For a given value of p, we numerically optimize the product $K_D^{\rm DW}P$ to obtain the optimal filter. It turns out that, for this family of states, the parameters of the optimal filters are

$$\begin{aligned} &a = d = r = s = t = u = 1, \\ &b = c = (\text{small } p\text{-dependent value}). \end{aligned} \tag{22}$$

The results are shown in Fig. 1. In Fig. 1(a), the value of one-way distillable key rate $K_D^{\rm DW}$ for the family of states $\rho_p^{(1)}$ is plotted as a function of $p \in [0, 1/2]$ before filtration. This value is clearly negative for p values up to a certain value and becomes positive only after p approximately crosses 0.31. In Fig. 1(b), the value of $K_D^{\rm DW}$ is plotted as a function p after the optimal local filtration operation. It is noteworthy that we are always able to find a filter such that the value of the one-way distillable key rate is equal to its maximum possible value of 1. In Fig. 1(c), we plot the success probability P of the optimal local filter as a function of p which clearly is small for small values of p but quickly becomes significant as the value of p increases and approaches 1 as p approaches 0.5. In Fig. 1(d), we plot the effective one-way distillable key rate obtained by multiplying $K_D^{\rm DW}$ and P for the optimal filter as a function p. It is clear form the graphs that the filtration process is able to make the one-way distillable key rate positive for all values p. We must compare the values in Fig. 1(a) with values in Fig. 1(d) to access the enhancement achieved through filtration. The comparison clearly shows that every member of the family of states $\rho_p^{(1)}$ after filtration has a positive value of $K_D^{\rm DW}$. This implies that this family of PPT states has a nonzero one-way distillable key rate after filtration. In other words, $\rho_p^{(1)}$ is a family of bound entangled states with a nonzero value of $K_D^{\rm DW}$. Since for this class of states $K_D > K_D^{\rm DW}$, they, thus, have a nonzero distillable key rate. Any PPT entangled state remains PPT entangled under a local filtration operation, this implies that all states in the original family(before filtration) are PPT entangled states which can be employed for QKD after filtration.

As it turns out, this family of states is closely related to the states considered and analyzed by Horodecki *et al.* [31]. They consider states $\rho_{(p,d,k)}$ which, for $d = 2,\ k = 1$, are defined as

$$\rho_{(p,2,1)}\begin{bmatrix} \frac{p}{2}(\tau_1+\tau_2) & 0 & 0 & \frac{p}{2}(\tau_1-\tau_2) \\ 0 & \left(\frac{1}{2}-p\right)\tau_2 & 0 & 0 \\ 0 & 0 & \left(\frac{1}{2}-p\right)\tau_2 & 0 \\ \frac{p}{2}(\tau_1-\tau_2) & 0 & 0 & \frac{p}{2}(\tau_1+\tau_2) \end{bmatrix}, \tag{23}$$

Consider a projection operator A such that

$$A = \frac{1}{2}\begin{bmatrix} I & 0 & 0 & I \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ I & 0 & 0 & I \end{bmatrix}. \tag{24}$$

where I is an identity operator on a $2 \otimes 2$ space. The families of states $\rho_p^{(1)}$ and $\rho_{(p,2,1)}$ are connected with each other via the operator A as follows:

$$\rho_p^{(1)} = \frac{A\rho_{(p,2,1)}A^\dagger}{\mathrm{Tr}(A\rho_{(p,2,1)}A^\dagger)}. \tag{25}$$

B. Example 2

Next, we consider examples of families of states considered by Chi *et al.* [32]. The family of states $\rho_p^{(2)} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2 \otimes \mathcal{H}^2)$ is defined as follows:

$$\rho_p^{(2)} = \frac{1}{2}\begin{bmatrix} \sigma_0+\sigma_1 & 0 & 0 & \sigma_0-\sigma_1 \\ 0 & \sigma_2+\sigma_3 & \sigma_2-\sigma_3 & 0 \\ 0 & \sigma_2-\sigma_3 & \sigma_2+\sigma_3 & 0 \\ \sigma_0-\sigma_1 & 0 & 0 & \sigma_0+\sigma_1 \end{bmatrix}, \tag{26}$$

with

$$\sigma_0 = \frac{1}{2}\begin{bmatrix} p & 0 & 0 & p \\ 0 & 2p & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & p \end{bmatrix}, \quad \sigma_1 = \frac{1}{2}\begin{bmatrix} p & 0 & 0 & -p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2p & 0 \\ -p & 0 & 0 & p \end{bmatrix},$$

$$\sigma_2 = \frac{1}{2}\begin{bmatrix} 1-4p-2\sqrt{2}p & 0 & 0 & 0 \\ 0 & (\sqrt{2}+1)p & p & 0 \\ 0 & p & (\sqrt{2}-1)p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

$$\sigma_3 = \frac{1}{2}\begin{bmatrix} 1-4p-2\sqrt{2}p & 0 & 0 & 0 \\ 0 & (\sqrt{2}-1)p & -p & 0 \\ 0 & -p & (\sqrt{2}+1)p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

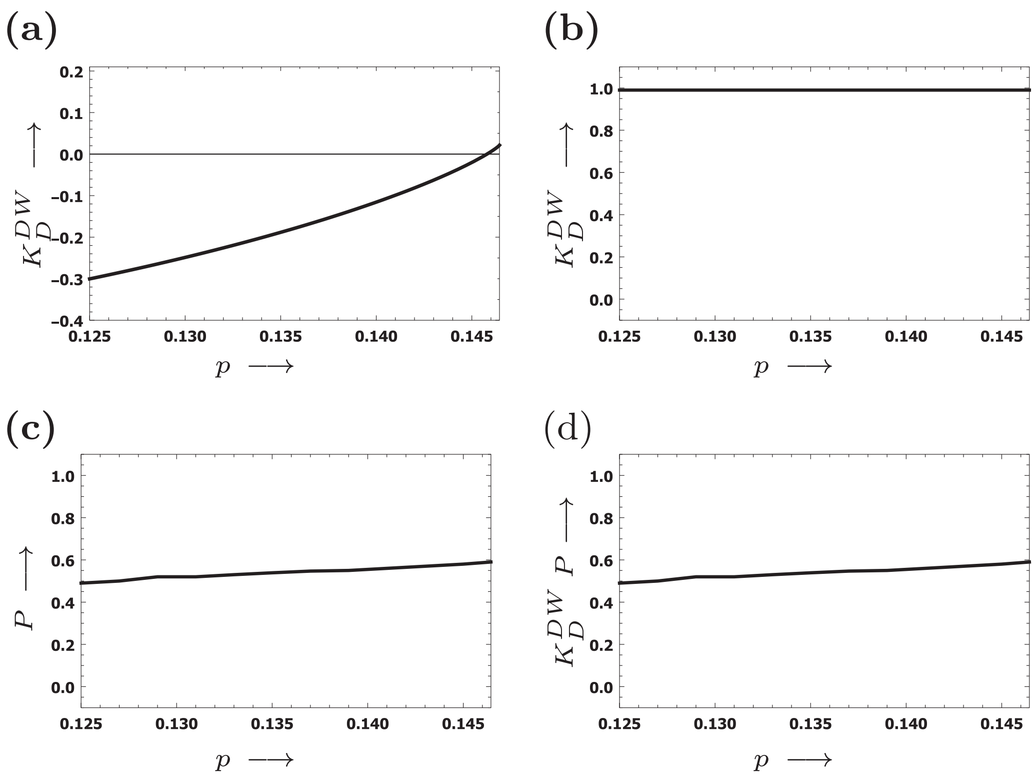

FIG. 2. Results for the family of states $\rho_p^{(2)}$. Various quantities are plotted as a function of the parameter p in the range of $(\frac{1}{8}, \frac{1}{4+2\sqrt{2}})$. (a) Displays $K_D^{\rm DW}$ for the states $\rho_p^{(2)}$ before filtration. (b) Displays $K_D^{\rm DW}$ for $\rho_p^{(2)'}$ which is the state after optimal filtration. (c) Displays the success probability P of the optimal filter. (d) Displays the product $K_D^{\rm DW}P$ which represents the effective key rate after filtration. Comparison of plots (a) and (d) clearly shows that an enhancement of the key rate has been achieved by filtration.

This family of states is determined by single real parameter p in the range of $(\frac{1}{8}, \frac{1}{4+2\sqrt{2}})$. It is straightforward to see that the states in the family are PPT as $(\rho_p^{(2)})^\Gamma = \rho_p^{(2)}$ and, therefore, if they have entanglement, it has to be bound entanglement [41]. Since the family of states have a form given in Eq. (7) and, here, we have $\|\sigma_0 + \sigma_1\| = 4p$ and $\|\sigma_0 - \sigma_1\| = 4p > 1/2$, we have $K_D(\rho_p^{(2)}) > 0$ [32].

In order to calculate the one-way distillable key rate for this family of states, we first identify the parameters $x,\ y,\ z$, and w as per Eq. (10) which turn out to be

$$x = 4p, \quad y = 0, \\ z = \frac{1-4p+2\sqrt{2}p}{2} \quad \text{and} \quad w = \frac{1-4p-2\sqrt{2}}{2}, \tag{28}$$

Using Eqs. (10) and (11), we obtain the value of

$$K_D^{\rm DW}\left(\left[\rho_p^{(1)\,PS}\right]_{ABE'}^{\rm ccq}\right) = 1 + 4p\log_2 4p + \left(\frac{1-4p-2\sqrt{2}}{2}\right)\log_2\left(\frac{1-4p-2\sqrt{2}}{2}\right) \\ + \left(\frac{1-4p+2\sqrt{2}p}{2}\right)\log_2\left(\frac{1-4p+2\sqrt{2}p}{2}\right). \tag{29}$$

Here, $[\rho_p^{(2)\,PS}]_{ABE'}^{\rm ccq}$ is the ccq state for the privacy squeezed state of $\rho_p^{(2)}$. For this class of states, the values of $K_D^{\rm DW}$ is negative for most values of p, therefore, although the $K_D > 0$ utility of the states for QKD is not clear. The exact plot of $K_D^{\rm DW}$ as function of p is shown in Fig. 2(a).

Next, we apply filtration operation on $\rho_p^{(2)}$ as described in Eq. (13),

$$\rho_p^{(2)} \rightarrow \frac{L\rho_p^{(2)}L^\dagger}{\mathrm{Tr}\,L\rho_p^{(2)}L^\dagger} = \rho_p^{(2)'}. \tag{30}$$

By using the form of filtration matrices given in Eqs. (14) and (15), the density operator after filtration operation can be written as

$$\rho_p^{(2)'} = \frac{1}{2M}\begin{bmatrix} L_1(\sigma_0+\sigma_1)L_1^\dagger & 0 & 0 & L_1(\sigma_0-\sigma_1)L_4^\dagger \\ 0 & L_2(\sigma_2+\sigma_3)L_2^\dagger & L_2(\sigma_2-\sigma_3)L_3^\dagger & 0 \\ 0 & L_3(\sigma_2-\sigma_3)L_2^\dagger & L_3(\sigma_2+\sigma_3)L_3^\dagger & 0 \\ L_4(\sigma_0-\sigma_1)L_1^\dagger & 0 & 0 & L_4(\sigma_0+\sigma_1)L_4^\dagger \end{bmatrix}. \tag{31}$$

Each element in the above expression is a 4×4 matrix and can be computed by using Eq. (16), here, $M{=}\mathrm{Tr}(L\rho_p^{(2)}L^\dagger)$. In order to calculate the value of $K_D^{\mathrm{DW}}([\rho_p^{(2)'\,PS}]^{\mathrm{ccq}}_{ABE'})$, we first map the parameters $x,\ y,\ z$, and w as per Eq. (10) which turn out to be

$$\begin{aligned} x &= [\|L_1(\sigma_0+\sigma_1)L_1^\dagger/2M\| + \|L_4(\sigma_0+\sigma_1)L_4^\dagger/2M\|]/2 + \|L_1(\sigma_0-\sigma_1)L_4^\dagger/2M\|, \\ y &= [\|L_1(\sigma_0+\sigma_1)L_1^\dagger/2M\| + \|L_4(\sigma_0+\sigma_1)L_4^\dagger/2M\|]/2 - \|L_1(\sigma_0-\sigma_1)L_4^\dagger/2M\|, \\ z &= [\|L_2(\sigma_2+\sigma_3)L_2^\dagger/2M\| + \|L_3(\sigma_2+\sigma_3)L_3^\dagger/2M\|]/2 + \|L_2(\sigma_2-\sigma_3)L_3^\dagger/2M\|, \\ w &= [\|L_2(\sigma_2+\sigma_3)L_2^\dagger/2M\| + \|L_3(\sigma_2+\sigma_3)L_3^\dagger/2M\|]/2 - \|L_2(\sigma_2-\sigma_3)L_3^\dagger/2M\|. \end{aligned} \tag{32}$$

We, now, calculate the value of $K_D^{\mathrm{DW}}([\rho_p^{(2)'\,PS}]^{\mathrm{ccq}}_{ABE'})$ by once again using Eqs (10) and (11).

The filter is numerically optimized to maximize the effective key rate which is the product of the one-way distillable key rate K_D^{DW} and the success probability of the filter P. The structure of the filter turns out to be similar to what was obtained for the previous example and is give in Eq. (23). The results are displayed in Fig. 2 where, in Fig. 2(a), we plot K_D^{DW} before filtration, in Fig. 2(b), we plot K_D^{DW} after filtration, in Fig. 2(c), we plot the success probability of the filter, and in Fig. 2(d), we plot the effective key rate which is the quantity $K_D^{\mathrm{DW}}P$ as functions of p. It is clear from a comparison of plots Figs. 2(a) and 2(d), that, for the entire family of states, the one-way distillable key rate turns positive from negative. By using the Devetak-Winter lower bound of distillable key rate for the ccq states, we can, therefore, state that distillable key rate $K_D(\rho_p^{(2)'}) \geqslant K_D^{\mathrm{DW}}([\rho_p^{(2)'\,PS}]^{\mathrm{ccq}}_{ABE'}) > 0$. This establishes the usefulness of this family of states for QKD and provides a quantitative estimate of the available key.

C. Example 3

We consider another example from the paper of Chi *et al.* [32], namely, the family of states $\rho_p^{(3)}$ given by the density operator,

$$\rho_p^{(3)} = \frac{1}{2}\begin{bmatrix} \sigma_0+\sigma_1 & 0 & 0 & \sigma_0-\sigma_1 \\ 0 & 2\sigma_2 & 0 & 0 \\ 0 & 0 & 2\sigma_2 & 0 \\ \sigma_0-\sigma_1 & 0 & 0 & \sigma_0+\sigma_1 \end{bmatrix}. \tag{33}$$

The operators σ_0 and σ_1 are defined in Eq. (27) and

$\sigma_2 = \frac{p}{\sqrt{2}}(|01\rangle\langle 01| + |10\rangle\langle 10|) + [\frac{1}{4} - (1+\frac{1}{\sqrt{2}})p](|00\rangle\langle 00| + |11\rangle\langle 11|)$. The states in this one-parameter family of states are well defined and are bound entangled states in the rangeof $\frac{1}{8} \leqslant p \leqslant \frac{1}{4+2\sqrt{2}}$. Again, as per Ref. [32], for this family of states $K_D(\rho_p^{(3)}) > 0$. However, by an explicit calculation, it was shown that $K_D^{\mathrm{DW}}([\rho_p^{(3)\,PS}]^{\mathrm{ccq}}_{ABE'}) < 0$ for $\frac{1}{8} \leqslant p \leqslant \frac{1}{4+2\sqrt{2}}$.

As was performed in the earlier cases, we apply filtration operation on $\rho_p^{(3)}$ as described in Eq. (13),

$$\rho_p^{(3)} \to \frac{L\rho_p^{(3)}L^\dagger}{\mathrm{Tr}\, L\rho_p^{(3)}L^\dagger} = \rho_p^{(3)'}. \tag{34}$$

Again, the explicit form of the density operator can be written in terms of filter parameters explicitly as, written as

$$\rho_p^{(3)'} = \frac{1}{2M}\begin{bmatrix} L_1(\sigma_0+\sigma_1)L_1^\dagger & 0 & 0 & L_1(\sigma_0-\sigma_1)L_4^\dagger \\ 0 & L_2(2\sigma_2)L_2^\dagger & 0 & 0 \\ 0 & 0 & L_3(2\sigma_2)L_3^\dagger & 0 \\ L_4(\sigma_0-\sigma_1)L_1^\dagger & 0 & 0 & L_4(\sigma_0+\sigma_1)L_4^\dagger \end{bmatrix}. \tag{35}$$

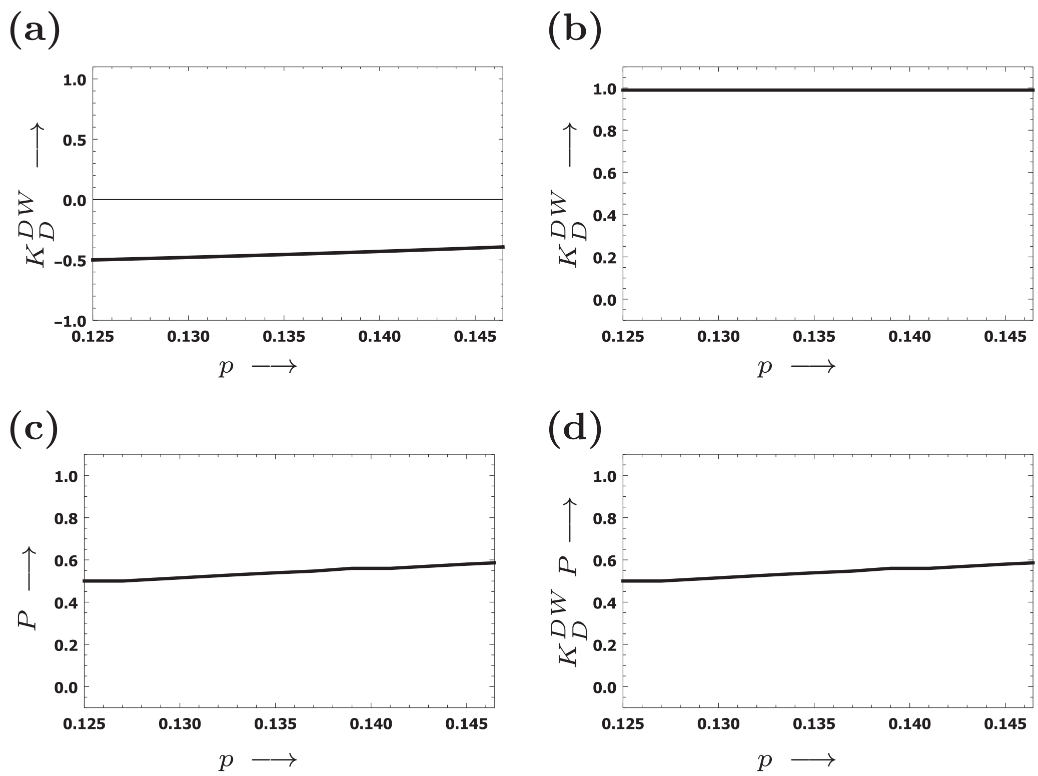

FIG. 3. Results for the family of states $\rho_p^{(3)}$. The plots of $K_D^{\rm DW}$ before and after filtration are shown in parts (a) and (b), the success probability P of the optimal filter is shown in (c) whereas in (d), the effective key rate is displayed in a way similar to Fig.s 1 and 2. The one-way key rate change from completely negative to positive is clearly evident.

Each element in the above expression is a 4×4 matrix and can be computed by using Eq. (16), here, $M = {\rm Tr}(L\rho_p^{(3)}L^\dagger)$. In order to calculate the value of $K_D^{\rm DW}([\rho_p^{(3)'\,PS}]_{ABE'}^{\rm ccq})$, we first map the parameters $x,\ y,\ z$, and w as per Eq. (10) which turn out to be

$$\begin{aligned}
x &= [\|L_1(\sigma_0+\sigma_1)L_1^\dagger/2M\| + \|L_4(\sigma_0+\sigma_1)L_4^\dagger/2M\|]/2 + \|L_1(\sigma_0-\sigma_1)L_4^\dagger/2M\|,\\
y &= [\|L_1(\sigma_0+\sigma_1)L_1^\dagger/2M\| + \|L_4(\sigma_0+\sigma_1)L_4^\dagger/2M\|]/2 - \|L_1(\sigma_0-\sigma_1)L_4^\dagger/2M\|,\\
z &= [\|L_2(2\sigma_2)L_2^\dagger/2M\| + \|L_3(2\sigma_2)L_3^\dagger/2M\|]/2,\\
w &= [\|L_2(2\sigma_2)L_2^\dagger/2M\| + \|L_3(2\sigma_2)L_3^\dagger/2M\|]/2.
\end{aligned} \tag{36}$$

The one-way distillable key rate is calculated for the numerically optimized filter which in this case again has the structure given in Eq. (23). The results are displayed in different parts of Fig. 3, where $K_D^{\rm DW}$ before and after filtration operation, the success probability of the filter and the effective key rate ($K_D^{\rm DW}P$) are plotted as functions of p in the relevant range of the parameter p. A comparison of Figs. 3(a) and 3(d) make it clear that the one-way distillable key rate has turned from negative to positive under filtration and we now have a filtered ensemble ready for QKD.

IV. CONCLUDING REMARKS

In this paper, we have explored the role of local filtration operations in enhancing the distillable key rate available from bound entangled states for QKD. The route we took was to calculate the Winter-Devetak function (also called the one-way distillable key rate) which provides a lower bound for the distillable key rate for the examples that we considered. Three examples of families of bound entangled states were analyzed from this point of view, and in each case a significant enhancement of the key rate was achieved by the filtration process. The filter was optimized so as to maximize the postfiltration one-way distillable key rate multiplied by the success probability of the filter. The one-way distillation key rate in each case turned from negative to positive and the effective key rate was significant. Our results provide quantitative estimates of available key rates for the family of states that we consider.

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