# Continuity equation for spin angular momentum in relation to optical chirality

An'an Wu,[1] Yoshito Y. Tanaka,[1,2,*] Ryoma Fukuhara,[1] and Tsutomu Shimura[1]
[1]*Institute of Industrial Science, The University of Tokyo, 4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan*
[2]*Japan Science and Technology Agency, PRESTO, 4-1-8 Honcho, Kawaguchi, Saitama 332-0012, Japan*

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Spin and orbital angular momenta are separately observable properties of light. Both of them can interact with matter, causing an exchange between each other and generating an optical torque on the matter. However, thus far, we have been unable to separately discuss the transfer of the spin and orbital angular momenta or the origin of the optical torques with regard to them. Here, we consider the optical chirality density and propose an approach to analyzing the transfer of spin and orbital angular momenta separately by considering this property. Our approach is based on a continuity equation for spin angular momentum density. This equation is derived from an extended proportional relation between the chirality flux density and spin angular momentum density in sources-added space. Our findings will greatly facilitate the study of angular momentum interaction between light and matter.

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## I. INTRODUCTION

Angular momentum, as a basic property to describe the rotation of light, can be separated into spin angular momentum (SAM) associated with the polarization of light and orbital angular momentum (OAM) determined by the spatial degrees of freedom of light [1–3]. The SAM was first explicitly considered by Poynting and experimentally demonstrated by Beth in 1936 [4,5]. However, it was a long time before the idea of the OAM of light was pioneered by Allen *et al.* in 1992 [6]. They discovered that a light beam with a helical phase front carries OAM. From then on, the study of optical angular momentum has developed into a flourishing field of research, leading to a rich variety of research and applications [7–11].

When light carrying angular momentum interacts with matter, an exchange between the spin and orbital parts of the angular momentum can occur. In recent years, this spin-orbit interaction of light has been attracting rapidly growing interest due to its promising potential in nano-optics, photonics, and plasmonics [12–18]. Therefore, it is of great significance to find a method that can analyze the angular momentum in the interaction between light and matter. The transfer of angular momentum between light and matter enables an optical torque that acts on the matter [19–22]. Optical torque has gained considerable attention owing to the crucial role it plays in optical manipulation, especially leading to a variety of applications in nanoelectromechanical systems [23–25], biological science [26–28], and chemistry [29]. In the analysis of the transfer of angular momenta between light and matter, the Maxwell stress tensor (MST) method has been extensively used to calculate the optical torque on the matter, and is based on the continuity equation for angular momentum [30]. Optical torque arises from the transfers of both the SAM and OAM of light [19–22]. The study of the respective contributions of SAM and OAM to the optical torque, that is, the separate characterizations of the SAM transfer and OAM transfer, will greatly facilitate the analysis of the spin-orbit interaction. However, the MST method cannot separate them because it is restricted to the optical toque resulting from the total angular momentum.

In this paper, we propose a method for analyzing the angular momentum in the interaction between light and matter by separately characterizing the transfer of SAM from that of OAM. In analogy with the MST method, we obtain the continuity equation for SAM density from the flux of optical chirality density. Our method can be applied to any size, shape, and constituent of matter in the presence of arbitrarily structured optical fields.

## II. THEORETICAL DESCRIPTION

The optical chirality density, which is a conserved quantity of the electromagnetic (EM) field, was introduced by Tang and Cohen as a measure of the local density of chirality of the EM field [31–33]. It has recently been widely studied and employed in the chirality characterization of molecules and EM fields [34–41]. The optical chirality density in a lossless medium with electric permittivity $\varepsilon$ and magnetic permeability $\mu$ can be expressed as [42]

$$C = \frac{\varepsilon}{2}\mathbf{E}\cdot\nabla\times\mathbf{E} + \frac{1}{2\mu}\mathbf{B}\cdot\nabla\times\mathbf{B}, \tag{1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the time-dependent electric and magnetic fields, respectively. In an EM field, the conserved quantities should satisfy their continuity equations, indicating their exchange between the field and sources. Poynting's theorem is the most elementary continuity equation, which explicates the conservation law of energy [43]. In related to an EM field,

*yoshito@iis.u-tokyo.ac.jp

 

we regard a particle as a distribution of charges with density $\rho(\mathbf{r},t)$ and currents with density $\mathbf{j}(\mathbf{r},t)$. The continuity equation for optical chirality density is given in analogy with the Poynting's theorem. By taking the time derivative of Eq. (1) and using Maxwell's equations, we get [32,42]

$$\frac{\partial C}{\partial t}+\nabla\cdot\mathbf{F}=-\frac{1}{2}[\mathbf{j}\cdot(\nabla\times\mathbf{E})+\mathbf{E}\cdot(\nabla\times\mathbf{j})], \tag{2}$$

where

$$\mathbf{F}=\frac{1}{2\mu}[\mathbf{E}\times(\nabla\times\mathbf{B})-\mathbf{B}\times(\nabla\times\mathbf{E})] \tag{3}$$

is the chirality flux density. Equations (2) and (3) have been widely used for characterizing the optical chirality density [44–46]. Considering a monochromatic field in a source-free space ($\mathbf{j}=0$), $\mathbf{E}(\mathbf{r},t)=\mathrm{Re}[\widetilde{\mathbf{E}}(\mathbf{r})e^{-i\omega t}]$ and $\mathbf{B}(\mathbf{r},t)=\mathrm{Re}[\widetilde{\mathbf{B}}(\mathbf{r})e^{-i\omega t}]$, some studies have shown that $\mathbf{F}$ is proportional to the SAM density of light [45–49], one of the well-known important physical quantities, which is defined as [21,50]

$$\boldsymbol{\sigma}=\frac{1}{4\omega}\left\{\varepsilon\,\mathrm{Im}[\widetilde{\mathbf{E}}^{*}\times\widetilde{\mathbf{E}}]+\frac{1}{\mu}\,\mathrm{Im}[\widetilde{\mathbf{B}}^{*}\times\widetilde{\mathbf{B}}]\right\}, \tag{4}$$

where $*$ denotes the conjugation operation, and $\omega$ is the angular frequency.

By using Maxwell's equations, $\mathbf{F}$ can be written as

$$\mathbf{F}=\mathbf{R}+\frac{1}{2}\mathbf{E}\times\mathbf{j}, \tag{5}$$

$$\mathbf{R}=\frac{1}{2}\left(\varepsilon\mathbf{E}\times\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{\mu}\mathbf{B}\times\frac{\partial\mathbf{B}}{\partial t}\right). \tag{6}$$

$\mathbf{R}$, the first term on the right-hand side in Eq. (5), can be expressed as

$$\mathbf{R}=\frac{\omega}{4}\left\{\varepsilon\,\mathrm{Im}[\widetilde{\mathbf{E}}^{*}\times\widetilde{\mathbf{E}}]+\frac{1}{\mu}\,\mathrm{Im}[\widetilde{\mathbf{B}}^{*}\times\widetilde{\mathbf{B}}]\right\} \tag{7}$$

in a monochromatic field, which is proportional to Eq. (4). The second term on the right-hand side in Eq. (5) is related to the sources, which shows the same intrinsic physics as that of Eq. (6). Therefore, in a monochromatic field, the quantity

$$\frac{\mathbf{F}}{\omega^2}=\frac{1}{2\omega^2}\left(\varepsilon\mathbf{E}\times\frac{\partial\mathbf{E}}{\partial t}+\frac{1}{\mu}\mathbf{B}\times\frac{\partial\mathbf{B}}{\partial t}+\mathbf{E}\times\mathbf{j}\right) \tag{8}$$

can be regarded as the SAM density of the total field, including the contribution from the first two terms by the light field and the last term by the current.

The angular momentum transfer between light and matter can induce an optical torque acting on the matter [19–22]. Hence, the transfer of the quantity $\mathbf{F}/\omega^2$ between light and matter can be related to the optical torque. Before discussing this relation, we summarize the well-known MST method to calculate optical torque. In an EM field, we have the continuity equation for linear momentum [30]:

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{S}}{c^2}\right)-\nabla\cdot\mathbf{T}_M=-(\rho\mathbf{E}+\mathbf{j}\times\mathbf{B}), \tag{9}$$

$$\mathbf{T}_M=\varepsilon[\mathbf{E}\otimes\mathbf{E}+c^2\mathbf{B}\otimes\mathbf{B}]-U\mathbf{I}, \tag{10}$$

where $U=\varepsilon(\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B})/2$ and $c$ is the speed of light in the medium. $\mathbf{S}$ is the Poynting vector, which represents the energy flux density. $\mathbf{S}/c^2$ is the linear momentum density of the field. $\mathbf{T}_M$ is the MST and $-\mathbf{T}_M$ represents the flux density of linear momentum. $\otimes$ denotes the operation of the dyadic product. $\mathbf{I}$ is the unit dyadic. The term on the right-hand side of Eq. (9) represents the optical force density on the particle. Accordingly, one can define an angular momentum density with the cross product of the position vector and linear momentum. Thus, the continuity equation for angular momentum can be written as [30]

$$\frac{\partial}{\partial t}\left(\mathbf{r}\times\frac{\mathbf{S}}{c^2}\right)-\nabla\cdot(\mathbf{r}\times\mathbf{T}_M)=-\mathbf{r}\times(\rho\mathbf{E}+\mathbf{j}\times\mathbf{B}), \tag{11}$$

where $\mathbf{r}$ is the position vector and $-\mathbf{r}\times\mathbf{T}_M$ represents the flux density of angular momentum. Hence, we can get the time-averaged optical torque on the particle:

$$\boldsymbol{\tau}_{\mathrm{MST}}=\oint_S(\mathbf{r}\times\overline{\mathbf{T}}_M)\cdot\mathbf{n}\,dS, \tag{12}$$

where $\overline{\mathbf{T}}_M$ is the time-averaged MST, $\mathbf{n}$ is the outward normal unit vector to its surface, and $S$ is an arbitrary closed surface surrounding the particle.

It is worth mentioning that light possesses both SAM and OAM. Both can interact with matter to produce optical torques acting on the matter. Consequently, the optical torque calculated by the MST method, i.e., using Eq. (12), includes the contributions from both the SAM transfer and OAM transfer. In other words, the MST method has not been able to characterize the SAM transfer and OAM transfer separately thus far.

As we discussed in Eq. (8), $\mathbf{F}/\omega^2$ is the SAM density of the field, which is related to the chirality flux density. This relation is similar to that between the linear momentum density of the field and energy flux density. Accordingly, by taking the time derivative of $\mathbf{F}/\omega^2$ and applying Maxwell's equations, we obtain the exact continuity equation for SAM density:

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{F}}{\omega^2}\right)-\nabla\cdot\mathbf{T}_S=-\frac{c^2}{2\omega^2}[\rho(\nabla\times\mathbf{E})+\mathbf{j}\times(\nabla\times\mathbf{B})+(\nabla\times\mathbf{j})\times\mathbf{B}], \tag{13}$$

$$\mathbf{T}_S=\frac{\varepsilon c^2}{2\omega^2}\{[\mathbf{E}\otimes(\nabla\times\mathbf{E})+(\nabla\times\mathbf{E})\otimes\mathbf{E}]+c^2[\mathbf{B}\otimes(\nabla\times\mathbf{B})+(\nabla\times\mathbf{B})\otimes\mathbf{B}]\}-\frac{c^2}{\omega^2}C\mathbf{I}. \tag{14}$$

The symmetric tensor $\mathbf{T}_S$ resembles the MST $\mathbf{T}_M$. $-\mathbf{T}_S$ represents the flux density of SAM. The terms on the right-hand side of Eq. (13) account for the loss rate of the SAM density of the field. Equation (13) indicates the SAM transfer between the light and matter. Equation (13), as well as the continuity equations mentioned above, are exact consequences of Maxwell's equations, which hold instantaneously for each position without any averaging. Note that Eq. (11)

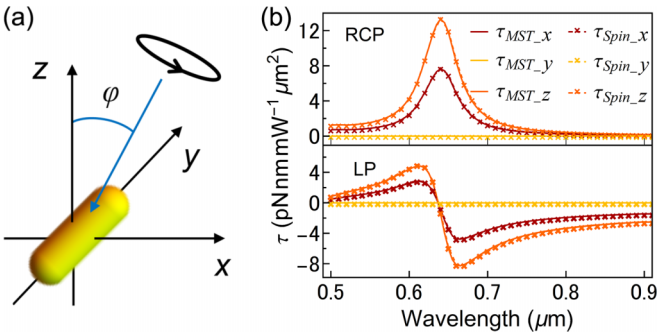


FIG. 1. (a) Geometric model of a gold nanorod (length 120 nm, diameter 40 nm) illuminated by circularly or linearly polarized light. (b) Wavelength dependence of optical torques $\boldsymbol{\tau}_{\mathrm{MST}}$ and $\boldsymbol{\tau}_{\mathrm{Spin}}$ calculated by Eqs. (12) and (15), respectively, with right circularly (top) and linearly (bottom) polarized light incident at an angle of $\varphi = 30^\circ$. The polarization direction of the linearly polarized light is at an angle of 45° with respect to the longitudinal axis of the nanorod. $\tau_{MST\_i}$ and $\tau_{Spin\_i}$ $(i = x, y, z)$ represent the different components of the optical torque.

describes the total angular momentum transfer between light and matter. Consequently, the OAM transfer can be obtained by combining Eqs. (11) and (13). This makes it possible to separately characterize the SAM transfer and OAM transfer. As a result of the SAM transfer and OAM transfer, optical torques originating from them can be produced on the matter; we call them the spin-transfer torque and orbit-transfer torque, respectively. Thus, the terms on the right-hand side of Eq. (13) can be regarded as the spin-transfer torque density on the particle. Based on the flux density of SAM, we can obtain the time-averaged spin-transfer torque:

$$\boldsymbol{\tau}_{\mathrm{Spin}} = \oint_S \overline{\mathbf{T}}_S \cdot \mathbf{n}\, dS, \tag{15}$$

where $\overline{\mathbf{T}}_S$ is the time-averaged $\mathbf{T}_S$. Equation (15) is a general equation and the integral surface $S$ is an arbitrary closed surface containing the particle, hence Eq. (15) can be applied to particles of any shape and composition in the presence of arbitrarily structured optical fields. The orbit-transfer torque can also be obtained by subtracting the spin-transfer torque from the total optical torque, i.e., Eq. (12). This may help to study the rotational states of nanoparticles and molecules in some optomechanical systems.

## III. VERIFICATION AND APPLICATION

Let us apply Eq. (15) to several examples. *A* and *B* are given as an applicability verification and *C* is an application of our method to analyze the angular momentum transfer between light and matter. The EM fields were calculated using the finite element method. All the following results were obtained with air as the medium, and the circularly and linearly polarized light used in *A* and *C* were plane waves. In *A* and *B*, since the transverse plasmon modes of the nanorods with the dimensions we used are negligible relative to their longitudinal plasmon modes, the nanorods can be treated as "needle" particles and the scattered light from them is linearly polarized without SAM or OAM. Therefore, there is no spin-orbit transformation in the interaction between the light and nanorods in the examples. This makes it much easier to analyze the physical mechanism of the spin- and orbit-transfer torque based on the incident light.

*A. Gold nanorod illuminated by circularly or linearly polarized light.* A circularly polarized light intrinsically has a SAM of $\pm\hbar$ per photon depending on the field rotation direction. On the other hand, a linearly polarized light has no angular momenta because it is given by the sum of the circularly polarized lights with different signs for SAM. Figure 1(a) schematically shows the geometric model. A nanorod lying along the $y$ axis is illuminated by a plane wave with an incident angle of $\varphi$ in the $xz$ plane. For the circularly polarized incident light, the optical torque arises from the absorption of the SAM carried by the incident light and the scattering of the incident light with the radiation of the linearly polarized light, i.e., the extinction of the incident SAM. For the linearly polarized incident light without any angular momenta, the optical torque is produced by the generation of SAM in the interference field between the scattered light and the incident light. In both cases, only SAM transfer exists. Therefore, the spin-transfer torque ($\boldsymbol{\tau}_{\mathrm{Spin}}$) calculated by Eq. (15) should be equal to the total optical torque ($\boldsymbol{\tau}_{\mathrm{MST}}$) calculated by Eq. (12), as shown in Fig. 1(b).

*B. Gold nanorod illuminated by a Laguerre Gaussian (LG) beam.* The properties of an LG beam can be quantified by its azimuthal mode index $l = 0, \pm1, \pm2, \ldots$, radial mode index $p = 0, 1, 2, \ldots$, and the handedness of circular polarization denoted by $s = \pm1$ [51,52]. $l$ and $s$ give rise to an OAM of $l\hbar$ and a SAM of $s\hbar$ per photon, respectively. Here, we confine our discussion to $l = 1$, $p = 0$, that is, a single-ringed beam

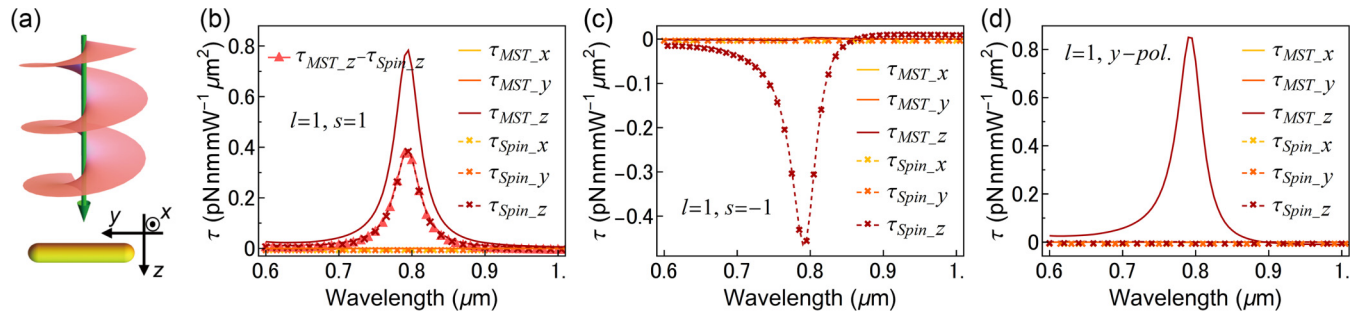


FIG. 2. (a) Geometric model of a gold nanorod (length 400 nm, diameter 40 nm) illuminated by a LG beam. (b)–(d) Wavelength dependence of optical torques $\boldsymbol{\tau}_{\mathrm{MST}}$ and $\boldsymbol{\tau}_{\mathrm{Spin}}$ with the incident light of LG beam with different modes: (b) $l = 1, s = 1$, (c) $l = 1, s = -1$, and (d) $l = 1$, $y$ polarized.

with an OAM of $\hbar$ per photon. For an LG beam with $l = 1$, the spin-transfer torque can be added to or subtracted from the orbit-transfer torque to obtain the total optical torque [53].

As shown in Fig. 2(a), a nanorod lies along the $y$ axis at the center of the incident LG beam propagating along the $z$ axis from the negative to the positive direction. (i) $l = 1$, $s = 1$. The scattered light from nanorod is linearly polarized without SAM or OAM. Therefore, optical torque is produced by the extinction of the incident angular momentum. As each photon has SAM and OAM with the same sign, the spin-transfer torque ($\boldsymbol{\tau}_{\mathrm{Spin}}$) and the orbit-transfer torque ($\boldsymbol{\tau}_{\mathrm{MST}} - \boldsymbol{\tau}_{\mathrm{Spin}}$) should be equal. (ii) $l = 1$, $s = -1$. The optical torque is produced by the same mechanism as that in (i). The incident light has a negative SAM, producing a negative spin-transfer torque ($\tau_{Spin\_z}$) which has a direction different from that in (i). Additionally, because of the different signs of SAM and OAM, the produced optical torques have the same magnitude but opposite directions, resulting in a total optical torque ($\boldsymbol{\tau}_{\mathrm{MST}}$) of 0. This indicates that, although the properties of SAM and OAM are very different, the spin-transfer torque and orbit-transfer torque can cancel each other out. (iii) $l = 1$, $y$ polarized. In this case, only OAM transfer occurs between the nanorod and the light field, inducing an orbit-transfer torque on the nanorod. Thus, the spin-transfer torque ($\boldsymbol{\tau}_{\mathrm{Spin}}$) should be 0, while the total optical torque should be nonzero. The calculation results of each case are shown in Figs. 2(b), 2(c), and 2(d).

All the above calculation results are consistent with those of previous studies and the analysis of physical mechanisms, indicating that Eq. (15) is effective for calculating the spin-transfer torque separately from the orbit-transfer torque. Furthermore, the combination with Eq. (12) provides the potential to analyze the angular momentum transfer in interactions between light and matter. In order to obtain a physical insight into such analysis, we consider the following example.

*C. Gold nanosphere illuminated by circularly polarized light.* The geometric model is shown in Fig. 3(a). A nanosphere is illuminated by a right circularly polarized light propagating along the $z$ axis from the positive to the negative direction. Interestingly, even for the incident light without OAM, the spin-transfer torque ($\tau_{Spin\_z}$) is not equal to the total optical torque ($\tau_{MST\_z}$), as shown in Fig. 3(b). Thus, the OAM along the $z$ axis should be generated by the interaction between the nanosphere and circularly polarized light. The generation of OAM produced a recoil orbit-transfer torque on the nanosphere. Considering this in Eq. (13), the terms on the right-hand side, that is, spin-transfer torque density, include the part that contributes to the generation of OAM due to the spin-orbit interaction, which leads to the difference between the spin-transfer torque and total optical torque in Fig. 3(b). To confirm the OAM generation, we calculated the phase distribution of the scattered light around the nanosphere as shown in Figs. 3(d) and 3(f). We can observe the phase gradient around the nanosphere in the plane parallel to the $xy$ plane, which shows the OAM along the $z$ axis in the scattering field. Additionally, some previous studies have shown the OAM in the scattering field around the nanosphere [54–58]. These findings may imply that all directions around the nanoparticle should be considered to analyze the OAM of a scattering field.

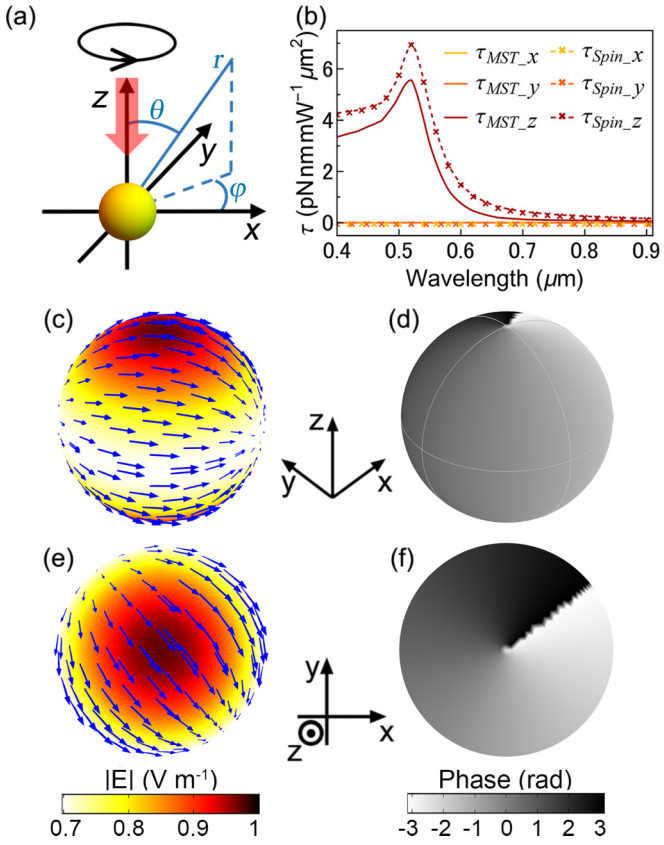


FIG. 3. (a) Geometric model of a gold nanosphere (diameter 100 nm) illuminated by right circularly polarized light. (b) Wavelength dependence of optical torques $\boldsymbol{\tau}_{\mathrm{MST}}$ and $\boldsymbol{\tau}_{\mathrm{Spin}}$. Electric field (c), (e) and phase distribution (d), (f) of the scattered light from the nanosphere illuminated by circularly polarized light in the far field at different viewpoints. The arrows in (c) and (e) show the instantaneous polarization states at different positions on the far-field surface. The colors in (c) and (e) show the normalized amplitude of the electric field in different directions of the scattered light. The grayscale in (d) and (f) show the phase distribution on the far-field surface. The phase is defined as the phase of the $\varphi$ component of the electric field in the spherical coordinate system $(r, \theta, \varphi)$ transformed from the Cartesian coordinate system $(x, y, z)$ in the figure.

Besides, as the spin-transfer torque and orbit-transfer torque are produced by the extinction of the incident SAM and the generation of OAM, respectively, our method to separately calculate them could be used to quantitatively characterize the transformation efficiency from SAM to OAM with the ratio of the orbit-transfer torque to spin-transfer torque. This quantitative analysis method for the spin-orbit transformation may provide a new route to understand and design the spin-orbit interaction systems.

## IV. CONCLUSION

In conclusion, we have presented an approach to analyzing the angular momentum transfer between light and matter based on the separate calculation of spin-transfer torque and orbit-transfer torque. By extending the proportional relation between the chirality flux density and SAM density to

sources-added space, we have derived the continuity equation for SAM density in analogy with the continuity equation for linear momentum density. The derived equation provides a method to calculate the spin-transfer torque separately from the orbit-transfer torque. This method makes it possible to realize the separate characterization of SAM transfer and OAM transfer between light and matter. In fact, we revealed that the scattered light of a nanosphere illuminated by circularly polarized light has OAM as well as SAM. The transformation capability from SAM to OAM can also be evaluated based on our method. The separation of spin-transfer torque and orbit-transfer torque may play a significant role in analyzing and designing a spin-orbit interaction system, which can be applied to many fundamental processes such as optical manipulation of nanoparticles and molecules, subwavelength optical probing, and generation of vortex beams. Our approach to separating the SAM transfer and OAM transfer will provide a better understanding of the fundamentals of the physics of the interaction between light and matter.

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[1] K. Y. Bliokh and F. Nori, Phys. Rep. **592**, 1 (2015).
[2] I. Bialynicki-Birula and Z. Bialynicka-Birula, J. Opt. **13**, 064014 (2011).
[3] S. M. Barnett, L. Allen, R. P. Cameron, C. R. Gilson, M. J. Padgett, F. C. Speirits, and A. M. Yao, J. Opt. **18**, 064004 (2016).
[4] J. H. Poynting, Proc. R. Soc. London Ser. A **82**, 560 (1909).
[5] R. A. Beth, Phys. Rev. **50**, 115 (1936).
[6] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
[7] M. Antognozzi, C. Bermingham, R. Harniman, S. Simpson, J. Senior, R. Hayward, H. Hoerber, M. Dennis, A. Bekshaev, K. Bliokh *et al.*, Nat. Phys. **12**, 731 (2016).
[8] L. Liu, A. Di Donato, V. Ginis, S. Kheifets, A. Amirzhan, and F. Capasso, Phys. Rev. Lett. **120**, 223901 (2018).
[9] L. Allen, M. J. Padgett, and M. Babiker, Prog. Opt. **39**, 291 (1999).
[10] S. Franke-Arnold, L. Allen, and M. J. Padgett, Laser Photonics Rev. **2**, 299 (2008).
[11] J. Ng, Z. Lin, and C. T. Chan, Phys. Rev. Lett. **104**, 103601 (2010).
[12] K. Y. Bliokh, F. J. Rodríguez-Fortuño, F. Nori, and A. V. Zayats, Nat. Photonics **9**, 796 (2015).
[13] F. J. Rodríguez-Fortuño, N. Engheta, A. Martínez, and A. V. Zayats, Nat. Commun. **6**, 8799 (2015).
[14] S. Sukhov, V. Kajorndejnukul, R. R. Naraghi, and A. Dogariu, Nat. Photonics **9**, 809 (2015).
[15] K. Y. Bliokh, D. Smirnova, and F. Nori, Science **348**, 1448 (2015).
[16] V. S. Liberman and B. Y. Zel'dovich, Phys. Rev. A **46**, 5199 (1992).
[17] F. Cardano and L. Marrucci, Nat. Photonics **9**, 776 (2015).
[18] J. E. Vázquez-Lozano and A. Martínez, Phys. Rev. A **97**, 033804 (2018).
[19] A. Ashkin, IEEE J. Sel. Top. Quantum Electron. **6**, 841 (2000).
[20] M. J. Padgett and R. Bowman, Nat. Photonics **5**, 343 (2011).
[21] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, Nat. Commun. **5**, 3300 (2014).
[22] D. Gao, W. Ding, M. Nieto-Vesperinas, X. Ding, M. Rahman, T. Zhang, C. Lim, and C.-W. Qiu, Light Sci. Appl. **6**, e17039 (2017).
[23] D. G. Grier, Nature (London) **424**, 810 (2003).
[24] M. Liu, T. Zentgraf, Y. Liu, G. Bartal, and X. Zhang, Nat. Nanotechnol. **5**, 570 (2010).
[25] A. Lehmuskero, R. Ogier, T. Gschneidtner, P. Johansson, and M. Käll, Nano Lett. **13**, 3129 (2013).
[26] Z. Bryant, M. D. Stone, J. Gore, S. B. Smith, N. R. Cozzarelli, and C. Bustamante, Nature (London) **424**, 338 (2003).
[27] F. M. Fazal and S. M. Block, Nat. Photonics **5**, 318 (2011).
[28] R. W. Bowman and M. J. Padgett, Rep. Prog. Phys. **76**, 026401 (2013).
[29] X. Liu, J. Li, Q. Zhang, and M. G. Dirbeba, Phys. Chem. Chem. Phys. **21**, 15339 (2019).
[30] P. H. Jones, O. M. Maragò, and G. Volpe, *Optical Tweezers: Principles and Applications* (Cambridge University Press, Cambridge, 2015).
[31] D. M. Lipkin, J. Math. Phys. **5**, 696 (1964).
[32] Y. Tang and A. E. Cohen, Phys. Rev. Lett. **104**, 163901 (2010).
[33] Y. Tang and A. E. Cohen, Science **332**, 333 (2011).
[34] A. Vázquez-Guardado and D. Chanda, Phys. Rev. Lett. **120**, 137601 (2018).
[35] M. Schäferling, D. Dregely, M. Hentschel, and H. Giessen, Phys. Rev. X **2**, 031010 (2012).
[36] N. Meinzer, E. Hendry, and W. L. Barnes, Phys. Rev. B **88**, 041407(R) (2013).
[37] T. J. Davis and E. Hendry, Phys. Rev. B **87**, 085405 (2013).
[38] E. Hendry, T. Carpy, J. Johnston, M. Popland, R. V. Mikhaylovskiy, A. J. Lapthorn, S. M. Kelly, L. D. Barron, N. Gadegaard, and M. Kadodwala, Nat. Nanotechnol. **5**, 783 (2010).
[39] L. V. Poulikakos, P. Thureja, A. Stollmann, E. De Leo, and D. J. Norris, Nano Lett. **18**, 4633 (2018).
[40] L. Kang, Q. Ren, and D. H. Werner, ACS Photonics **4**, 1298 (2017).
[41] A. Pham, A. Zhao, C. Genet, and A. Drezet, Phys. Rev. A **98**, 013837 (2018).
[42] J. E. Vázquez-Lozano and A. Martínez, Phys. Rev. Lett. **121**, 043901 (2018).
[43] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999).
[44] L. V. Poulikakos, P. Gutsche, K. M. McPeak, S. Burger, J. Niegemann, C. Hafner, and D. J. Norris, ACS Photonics **3**, 1619 (2016).

[45] K. Y. Bliokh and F. Nori, Phys. Rev. A **83**, 021803(R) (2011).
[46] G. Nienhuis, Phys. Rev. A **93**, 023840 (2016).
[47] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, New J. Phys. **15**, 033026 (2013).
[48] R. P. Cameron, S. M. Barnett, and A. M. Yao, New J. Phys. **14**, 053050 (2012).
[49] T. G. Philbin, Phys. Rev. A **87**, 043843 (2013).
[50] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori, Phys. Rev. Lett. **119**, 073901 (2017).
[51] Q. Zhan, Adv. Opt. Photonics **1**, 1 (2009).
[52] L. Allen and M. J. Padgett, Opt. Commun. **184**, 67 (2000).
[53] N. B. Simpson, K. Dholakia, L. Allen, and M. J. Padgett, Opt. Lett. **22**, 52 (1997).
[54] C. Triolo, A. Cacciola, S. Patanè, R. Saija, S. Savasta, and F. Nori, ACS Photonics **4**, 2242 (2017).
[55] J. Olmos-Trigo, C. Sanz-Fernández, F. S. Bergeret, and J. J. Sáenz, Opt. Lett. **44**, 1762 (2019).
[56] C. Schwartz and A. Dogariu, Opt. Express **14**, 8425 (2006).
[57] K. Y. Bliokh, E. A. Ostrovskaya, M. A. Alonso, O. G. Rodríguez-Herrera, D. Lara, and C. Dainty, Opt. Express **19**, 26132 (2011).
[58] A. Y. Bekshaev, O. V. Angelsky, S. G. Hanson, and C. Y. Zenkova, Phys. Rev. A **86**, 023847 (2012).