Enhanced phase estimation with coherently boosted two-mode squeezed beams and its application to optical gyroscopes

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Quantum techniques, developed in recent decades, provide new approaches to achieving high-precision measurements beyond the classical bounds. In this paper, we theoretically demonstrate a metrology method for improving the sensitivity of the interferometric optical gyroscope, robust against the loss, by using coherent-light stimulated two-mode squeezed beams as the light source. The detection protocol is based on a simple intensity measurement, and the quantum noise is far below the shot-noise limit. The enhancement factors for different coherent light fields are analyzed in detail. Additionally, the influence of loss during the propagation in the optical path is studied, and the conditions for achieving sub-shot-noise measurement sensitivity are obtained. We also find that the phase sensitivity of the proposed gyroscope scheme becomes closer to the quantum Cramér-Rao bound with increasing of the photon number of the coherent beams.

I. INTRODUCTION

The advent of the laser in 1960 evoked interest in the use of the Sagnac effect for sensing inertial rotation by optical means. Generally, in an optical gyroscope, there is a pair of counterpropagating light beams traveling around a ring cavity and the interference between them provides information about the rotation rate relative to an inertial frame. To be specific, the phase difference φ between the two counterpropagating beams after one round trip is given by $\varphi = 8\pi A\Omega/c\lambda$, where Ω is the angular velocity of the rotating cavity, λ is the wavelength, and A is the area enclosed by the light beams. The theoretical limit on the minimum detectable angular velocity $\Omega_{\min}$ is directly related to the uncertainty in measuring φ, given by [1]

$$\Omega_{\min} = \frac{c\lambda}{8\pi A}\Delta\varphi. \tag{1}$$

The sensitivity of the gyroscope is then determined by the phase resolution of the interferometers.

In general, the Sagnac effect has always used lasers, in spite of a few with atoms or single photons [2–5], and the minimum detectable phase shift is the shot-noise limit for $\Delta\varphi_{SN} = 1/\sqrt{\langle N \rangle}$, where $\langle N \rangle$ is the average number of photons. Recent theoretical and experimental work has proved that the advance in quantum techniques makes it possible to enhance the phase-sensitivity scale of an interferometer beyond the conventional bound; even approaching or beyond the Heisenberg limit, $\Delta\varphi_{HL} = 1/\langle N \rangle$ [5–14]. In particular, the squeezing light technique has gained much attention in improving measurement sensitivity of the interferometer [14–21]. The Mach-Zehnder interferometer with two independently-squeezed coherent states has been investigated with respect to quantum Fisher information (QIF) and the detection loss effect on phase sensitivity [19–21]. The two-mode squeezed state, which has a correlation feature between the two modes, has also been used as the input light source to enhance the phase sensitivity with parity measurements [14,15], coincidence measurements [16], and homodyne detection [17].

The improvement in accuracy implies a revolution in the field of interferometric sensing. In terms of the atomic gyroscope, in 1998 Dowling demonstrated a type of gyroscope based on an entangled two-mode Fock state, the sensitivity of which can achieve the Heisenberg limit [10]. Recently, Heisenberg-limited Sagnac interferometry with entangled-photon pairs from parametric down conversion has been reported by using high-order coincidence measurements [11]. In addition, a nonlinear Sagnac interferometer is proposed by replacing the beam splitter (BS) with a nonlinear optical process [22]. Although these works showed a great enhancement in phase sensitivity, the difficulty in producing the light source, or implementing the detection strategy, or even modifying the structure of the Sagnac interferometer, would undoubtedly put an obstacle in their way toward application with current technology.

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In this paper, we investigate the sensitivity of the optical gyroscope fed with a two-mode squeezed-light field generated from an optical parametric amplification (OPA) process with two-mode coherent light field inputs—the so-called coherent-boosted two-mode squeezed beams. The phase shift is estimated by measuring the intensity difference of the light beams directly, which is much simpler than the multiphoton coincidence measurement [11,16] or parity measurement [14,15] suggested in previous works. The sub-shot-noise sensitivity with robustness against the loss is the main focus of our discussion. The enhancement factor with respect to the shot-noise limit is studied in detail under the conditions that the two modes of the coherent field are balanced or unbalanced in modulus, as well as in phase. We note that the measurement accuracy has seldom been examined in the latter condition. In addition, since loss plays an important role in precision measurements, especially in the fiber gyroscope, we explore the influence of the loss arising from the light propagation. We analyze the requirements on the loss and reciprocity of the two paths for sub-shot-noise measurements. Moreover, we study the quantum Cramér-Rao bound for our gyroscope scheme to evaluate the efficiency of the intensity difference estimation strategy. It should be pointed out that the results obtained in this paper can be applied to other kinds of SU(2) interferometers such as the Mach-Zehnder interferometer [12].

Unlike the conventional optical gyroscope, the biased phase modulation is no longer necessary for the gyroscope discussed in the present paper, and thus the corresponding loss can be avoided. Meanwhile, the loss arising from the propagation in the gyroscope can be overcome by control of the OPA parameters. All of this means that our gyroscope scheme has advantages not only in measurement precision but also in robustness. On the other hand, compared with interferometric schemes with a two-mode squeezed vacuum state (TMSV) [17] or one squeezed input and one coherent input [18], our scheme contains more photons experiencing the phase shift. This is due to the coherent light field stimulation in the OPA, and thus a higher sensitivity can be reached. The measurement sensitivity of the proposed gyroscope is very close to that of the coherent boosted SU(1,1) interferometer [23] or the nonlinear interferometer [24]. At the same time, the tolerance to loss in the interferometer is better than the nonlinear interferometer in Ref. [24].

The paper is organized as follows. In Sec. II, our gyroscope scheme for phase measurement beyond the shot-noise limit is described. The influence of the loss due to propagation is investigated in Sec. III. In Sec. IV, the effect of the detection strategy is discussed. Then, in Sec. V, we compare the present gyroscope scheme with the traditional one and previous works. The conclusion is given in Sec. VI.

II. OPTICAL GYROSCOPE BEYOND THE SHOT-NOISE LIMIT

As described in Fig. 1, a two-mode coherent light field, corresponding to the annihilation operators $\hat{a}_0$ and $\hat{b}_0$, respectively, is seeded into an OPA pumped by a high-power laser. The operation of the OPA can be described by a two-mode squeeze operator $\hat{S} = e^{\xi^* \hat{a}\hat{b} - \xi \hat{a}^\dagger \hat{b}^\dagger}$ with $\xi = re^{i\theta}$, and the
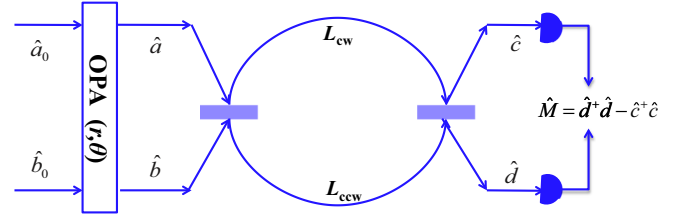

FIG. 1. The equivalent optical network diagram of the Sagnac interferometer for the coherent-boosted two-mode squeezed light field, where the first and second beam splitters are meant to be one and the same in the Sagnac interferometer. The two-mode squeezed light fields in $\hat{a}$ and $\hat{b}$, which are generated by the OPA (with the parameters r and θ), are injected into the gyroscope from the beam splitter. After traveling in the clockwise and counterclockwise directions, respectively, the two light fields combine at the beam splitter again, and then leave the gyroscope. Then, immediately, the intensity difference $\langle \hat{M} \rangle$ between the two output fields is detected to estimate the phase difference between the two arms.

field operators of the output modes $\hat{a}$ and $\hat{b}$ can be written as

$$\hat{a} = \hat{a}_0 \cosh r - e^{i\theta} \hat{b}_0^\dagger \sinh r, \tag{2}$$

$$\hat{b} = \hat{b}_0 \cosh r - e^{i\theta} \hat{a}_0^\dagger \sinh r, \tag{3}$$

where r is the parametric gain coefficient and θ is the phase of the pump laser [25].

The output light of the OPA is fed into the Sagnac interferometer, the equivalent optical path diagram of which is depicted in Fig. 1. We assume that the BS of the gyroscope is 50-50, and the phase shift on transmission at the BSs for both the input and the output light fields is 0, and that on reflection is $\pi/2$. After passing through the BS for the first time, the light has two possible paths, one is in the clockwise direction L_{cw} and the other is counterclockwise L_{ccw}. If the interferometer is not rotating, the optical paths for both directions are the same; otherwise, the optical path difference will be induced. When the two light beams recombine at the BS for the second time, the light emerges from the two output ports and is detected by two detectors. We describe the light field leaving the interferometer by operators $\hat{c}$ and $\hat{d}$ in the following form:

$$\hat{c} = \frac{1}{2}[(e^{i\mu} - e^{i\nu})\hat{a} + (ie^{i\mu} + ie^{i\nu})\hat{b}], \tag{4}$$

$$\hat{d} = \frac{1}{2}[(ie^{i\mu} + ie^{i\nu})\hat{a} - (e^{i\mu} - e^{i\nu})\hat{b}], \tag{5}$$

where $\mu \propto L_{\text{cw}}$ and $\nu \propto L_{\text{ccw}}$ are the phases proportional to the length of the two light paths [25].

Suppose the initial two-mode coherent light fields $\hat{a}_0$ and $\hat{b}_0$ are in the state $|\psi\rangle = |\alpha\rangle|\beta\rangle$. The average number of the total photons exiting the interferometer then is

$$\begin{aligned} \langle \hat{N} \rangle_{\text{tot}} &= \langle \hat{c}^\dagger \hat{c} + \hat{d}^\dagger \hat{d} \rangle = \langle \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} \rangle \\ &= (|\alpha|^2 + |\beta|^2)\cosh 2r - (\alpha^* \beta^* e^{i\theta} + \alpha\beta e^{-i\theta})\sinh 2r \\ &\quad + 2\sinh^2 r, \end{aligned} \tag{6}$$

where the loss in the optical paths is omitted. We notice that, generally, the average photon numbers of the coherent light fields $|\alpha|^2$, $|\beta|^2 \gg \sinh^2 r$ (this assumption is used for the rest of Secs. II and III), and then the above equation can be

approximated as

$$\langle \hat{N} \rangle_{\text{tot}} \approx (|\alpha|^2 + |\beta|^2) \cosh 2r - (\alpha^* \beta^* e^{i\theta} + \alpha \beta e^{-i\theta}) \sinh 2r. \tag{7}$$

To find out the phase signal produced by the difference of the two light paths, we measure the intensity difference operator between the two output beams, that is,

$$\begin{aligned} \hat{M} &= \hat{d}^\dagger \hat{d} - \hat{c}^\dagger \hat{c} \\ &= (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) \cos \varphi + (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) \sin \varphi, \end{aligned} \tag{8}$$

where $\varphi = \mu - \nu$ is the phase difference between the two beams. Hence, the output signal is given by

$$\begin{aligned} \langle \hat{M} \rangle &= (|\alpha|^2 - |\beta|^2) \cos \varphi + \big\{ (\alpha^* \beta + \alpha \beta^*) \cosh 2r \\ &\quad - \tfrac{1}{2}[e^{i\theta}(\alpha^{*2} + \beta^{*2}) + e^{-i\theta}(\alpha^2 + \beta^2)] \sinh 2r \big\} \sin \varphi. \end{aligned} \tag{9}$$

The uncertainty of the phase signal is given by the linear error propagation method [25,26],

$$\Delta \varphi^2 = \frac{\Delta M^2}{|\partial \langle \hat{M} \rangle / \partial \varphi|^2}, \tag{10}$$

which is dependent on the phase shift φ. The linear error propagation method is conventionally used for the local phase estimation, in which a small change of signal is monitored at a fixed reference point [27]. As for the gyroscope, we are concerned with the sensitivity at the point where the phase shift is close to zero, under the assumption that the rotation we desire to detect is usually very small. Thus, we analyze the uncertainty of the phase in the case of $\varphi = 0$. We have

$$\Delta \varphi^2 = \frac{|\alpha|^2 + |\beta|^2}{\left| (\alpha^* \beta + \alpha \beta^*) \cosh 2r - \tfrac{1}{2}[e^{i\theta}(\alpha^{*2} + \beta^{*2}) + e^{-i\theta}(\alpha^2 + \beta^2)] \sinh 2r \right|^2}. \tag{11}$$

The uncertainty of the phase signal is not only dependent on the OPA parameters r and θ, but is also determined by the complex amplitudes of the initial two-mode coherent light fields α and β.

On the other hand, according to Eq. (7), the shot-noise limit for the phase measurement, i.e., the variance of the phase shift for the classical gyroscope with the coherent light source containing the same average photon number, is

$$\Delta \varphi_{\text{SN}}^2 = \frac{1}{\langle \hat{N} \rangle_{\text{tot}}} = \frac{1}{(|\alpha|^2 + |\beta|^2) \cosh 2r - (\alpha^* \beta^* e^{i\theta} + \alpha^* \beta^* e^{-i\theta}) \sinh 2r}. \tag{12}$$

Next, we will discuss the behavior of the phase sensitivity in the following three cases classified according to the complex amplitudes of the initial coherent light field. This will make it easy to compare the relationship between the variance of the phase shift $\Delta \varphi$ and the shot-noise limit $\Delta \varphi_{\text{SN}}$. To make the comparison convenient, we define a factor $R = \Delta \varphi^2 / \Delta \varphi_{\text{SN}}^2$ to indicate the enhancement of the phase sensitivity. The uncertainty of phase $\Delta \varphi$ will be below the shot-noise limit if $R < 1$.

(i) Case I: $\alpha = \beta$

We first focus on the case that the complex amplitude of the input coherent light field for each mode is the same, that is, $\alpha = \beta = \eta = \sqrt{n} e^{i\gamma}$ (assuming n and γ to be real), where n is the average photon number of each coherent beam. Therefore, the average total number of the photons of the initial coherent light field is $\langle \hat{N} \rangle_{\text{coh}} = 2n$. In this condition, the average number of the photons [based on Eq. (7)] exiting from the Sagnac ring is

$$\begin{aligned} \langle \hat{N} \rangle_{\text{tot}} &= 2n \cosh 2r - 2n \sinh 2r \cos(\theta - 2\gamma) \\ &= 2nG = \langle \hat{N} \rangle_{\text{coh}} G, \end{aligned} \tag{13}$$

where $G = \cosh 2r - \sinh 2r \cos(\theta - 2\gamma)$, which is the parametric power gain of the OPA. It varies with the parametric gain coefficient r, the phase of the pumping laser θ, and the phase of the initial coherent light field γ. The shot-noise limit is $\Delta \varphi_{\text{SN}}^2 = 1/(2nG)$. Meanwhile, according to Eq. (11), we

can calculate the uncertainty of the phase shift as

$$\Delta \varphi^2 = \frac{1}{2nG^2} = \frac{1}{\langle \hat{N} \rangle_{\text{coh}} G^2} = \frac{1}{G} \Delta \varphi_{\text{SN}}^2, \tag{14}$$

For $G > 1$, the minimum detectable phase shift $\Delta \varphi$ is less than the shot noise while, for $G < 1$, it is greater than the shot noise. The maximum value of G, given by $G_{\max} = e^{2r}$, is obtained when $\cos(\theta - 2\gamma) = -1$, i.e., $\theta = \pi + 2\gamma$. Hence, the variance of the phase shift will be

$$\Delta \varphi^2 = \frac{1}{2n(e^{2r})^2} = e^{-2r} \Delta \varphi_{\text{SN}}^2. \tag{15}$$

Here, we see that our gyroscope is shown to be doubly enhanced over the conventional (classical) counterpart: First, the shot noise itself is reduced by the increase of input photon number due to the parametric amplification [see Eq. (13)]. Then, there is the signal-to-noise improvement via squeezing as shown in Eq. (15). At this stage, the enhancement factor R of the phase uncertainty can be written as

$$R = \Delta \varphi^2 / \Delta \varphi_{\text{SN}}^2 = e^{-2r}. \tag{16}$$

The above equation shows that the uncertainty of the phase shift is decreased with the parametric gain coefficient r relative to the shot-noise limit, which means the uncertainty in measuring the angular velocity Ω will be reduced accordingly.

(ii) Case II: $|\alpha| \neq |\beta|$ but $\gamma_\alpha = \gamma_\beta = \gamma$, where γ_α and γ_β are the phases of the complex amplitudes α and β, respectively.

Here we consider the case that there is a difference in the intensity but not in the phase between the two amplitudes of the initial coherent light. Suppose that $\alpha = \sqrt{n}e^{i\gamma}$, $\beta = \epsilon\sqrt{n}e^{i\gamma}$, where $0 < \epsilon \leqslant 1$ indicates the ratio between the intensity of the two modes $\hat{a}_0$ and $\hat{b}_0$, then we have $|\beta|/|\alpha| = \epsilon$. According to Eq. (11), the variance of the phase takes the following form:

$$\Delta\varphi^2 = \frac{1 + \epsilon^2}{n|2\epsilon\cosh 2r - (1 + \epsilon^2)\sinh 2r\cos(\theta - 2\gamma)|^2}.$$

$$(17)$$

Based on Eq. (7), the shot-noise limit is

$$\Delta\varphi_{\text{SN}}^2 = \frac{1}{n[(1 + \epsilon^2)\cosh 2r - 2\epsilon\sinh 2r\cos(\theta - 2\gamma)]}. \quad (18)$$

Both $\Delta\varphi^2$ and $\Delta\varphi_{\text{SN}}^2$ will each have their own minimum when the condition $\theta = \pi + 2\gamma$ is satisfied, and the ratio between

them is

$$R = \frac{(1 + \epsilon^2)[(1 + \epsilon^2)\cosh 2r + 2\epsilon\sinh 2r]}{|2\epsilon\cosh 2r + (1 + \epsilon^2)\sinh 2r|^2}. \quad (19)$$

Figure 2 describes the dependence of the enhancement factor R on ϵ and the parametric gain coefficient r, respectively. When $\epsilon = 1$, we have $R = e^{-2r}$, which corresponds to case I. As shown in Fig. 2(a), with the decrease of the parameter ϵ, the factor R increases, which means the uncertainty of phase $\Delta\varphi$ increases and is generally close to or larger than the shot-noise limit $\Delta\varphi_{\text{SN}}$. However, we notice that even in the case of $\epsilon \to 0$, i.e., $|\alpha| \gg |\beta|$ ($R \to \frac{\cosh 2r}{|\sinh 2r|^2}$ in this condition), the phase variance below the shot-noise limit is also obtained if the parametric gain coefficient r satisfies the condition $\cosh 2r/\sinh^2 2r < 1$ [see Fig. 2(b)].

(iii) Case III: $|\alpha| = |\beta| = \sqrt{n}$, but $\gamma_\alpha \neq \gamma_\beta$.

Lastly, we deal with the case that the two coherent states in modes $\hat{a}_0$ and $\hat{b}_0$ have the same intensity but are different in the phase of the complex amplitude. We now assume that the phase difference between the complex amplitude of the two states $|\alpha\rangle$ and $|\beta\rangle$ is δ, and $\alpha = \sqrt{n}e^{i\gamma}$ and $\beta = \sqrt{n}e^{i(\gamma+\delta)}$. Following Eq. (11), it is straightforward to get that the variance of the phase

$$\Delta\varphi^2 = \frac{2}{n|2\cosh 2r\cos\delta - \sinh 2r[\cos(\theta - 2\gamma) + \cos(\theta - 2\gamma - 2\delta)]|^2} \quad (20)$$

and the corresponding shot-noise limit is

$$\Delta\varphi_{\text{SN}}^2 = \frac{1}{2n[\cosh 2r - \sinh 2r\cos(\theta - 2\gamma - \delta)]}. \quad (21)$$

The optimal OPA occurs when $\theta = \pi + 2\gamma + \delta$, and in this case the factor R between $\Delta\varphi^2$ and $\Delta\varphi_{\text{SN}}^2$ takes the following form:

$$R = \frac{1}{\cos^2\delta(\cosh 2r + \sinh 2r)} = \frac{e^{-2r}}{\cos^2\delta}. \quad (22)$$

The factor R varies periodically with parameter δ. As shown in Fig. 3(a), for one period $[0, \pi]$, R increases as the parameter δ increases in the interval $[0, \pi/2)$ and decreases in the interval $(\pi/2, \pi]$. And when $\delta \to \pi/2$, the ratio $R \to \infty$. This periodic variation is due to the interference between the two coherent beams with different phases. The changes of the R caused by the parametric gain coefficient r and parameter δ are displayed in Fig. 3(b). The curves show that the sub-shot-noise limit phase sensitivity is not always achieved if parameter δ does not equal 0 or π. But, for a fixed value of δ the ratio R will decrease, even below one, with the increase of the parametric gain coefficient r.

By comparing these three cases, it is not difficult to find that the optimal performance of the phase sensitivity takes place in the case that the input coherent light fields for the OPA are balanced (i.e., case I), in spite of the fact that the unbalanced input can also result in supersensitivity in phase measurements. The enhancement in phase measurements can be attributed, in part, to the suppression of quantum noise of the intensity difference between the two squeezed light

beams $\hat{a}$ and $\hat{b}$ at the input ports. Here we use a quantity, noise reduction factor (NRF) to indicate this suppression (see Appendix B).

In case I with $\theta = \pi + 2\gamma$, there is NRF $= G^{-1} = e^{-2r}$ [see Eq. (B5) in Appendix B], which is exactly equal to the phase-sensitivity enhancement factor R. For case II, we have [see Eq. (B6) in Appendix B]

$$\text{NRF} = \frac{1 + \epsilon^2}{(1 + \epsilon^2)\cosh 2r + 2\epsilon\sinh 2r}, \quad (23)$$

which increases with the decreasing of the parameter ϵ. In other words, the increasing imbalance between the intensity of the two states $|\alpha\rangle$ and $|\beta\rangle$ will diminish the noise reduction, and at the same time, the phase sensitivity of the gyroscope becomes worse.

Note, however, the phase-sensitivity enhancement of the gyroscope is not only determined by the noise suppression of the intensity difference of the light source of the gyroscope. As for case III, when the condition $\theta = \pi + 2\gamma + \delta$ is satisfied, the NRF for the two-mode squeezed light beams will also be e^{-2r} [see Eq. (B7) in Appendix B]. Although the factor NRF is the same as that in case I, the variance of the phase is not the same [see Eqs. (16) and (22)]. Notice that the complex amplitude of the two squeezed coherent light beams $\hat{a}$ and $\hat{b}$ are given as follows:

$$\langle\hat{a}\rangle = \sqrt{n}(e^{i\gamma}\cosh r - e^{i(\theta-\gamma-\delta)}\sinh r) = Ae^{i\gamma_a}, \quad (24)$$

$$\langle\hat{b}\rangle = \sqrt{n}(e^{i(\gamma+\delta)}\cosh r - e^{i(\theta-\gamma)}\sinh r) = Be^{i\gamma_b}. \quad (25)$$
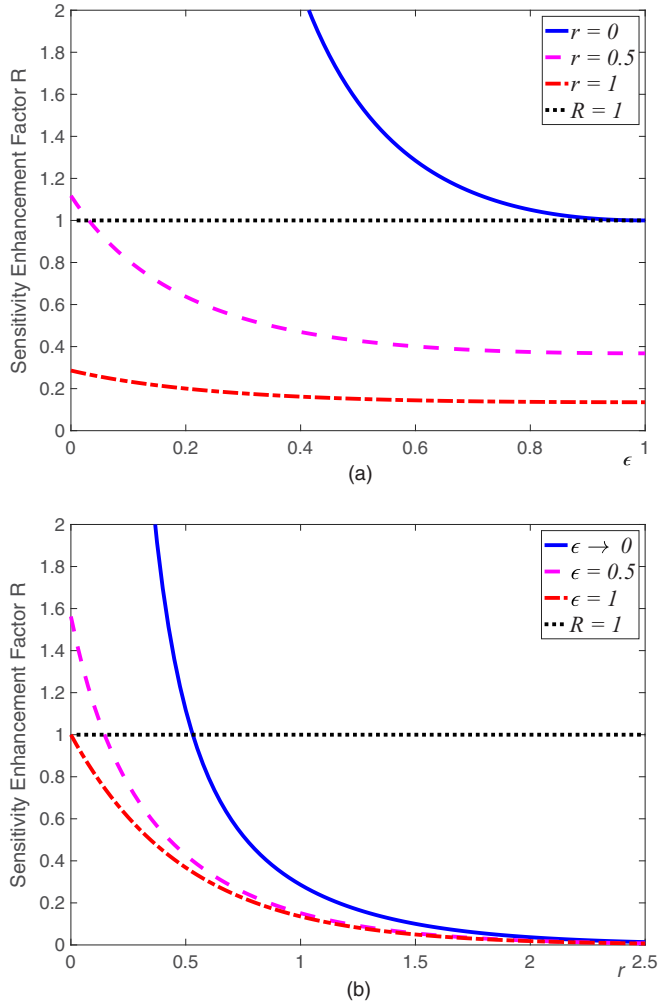
FIG. 2. For case II, the sensitivity enhancement factor R between the variance of the phase $\Delta\varphi^2$ and shot-noise limit $\Delta\varphi_{\mathrm{SN}}^2$ varies with (a) the parameter ϵ and (b) the parametric gain coefficient r. $R = 1$ (the black dot line) means the phase sensitivity reaches the shot-noise limit.

The phase of these two light beams then takes form

$$\gamma_a = \arctan \frac{\cosh r \cos \gamma_\alpha - \sinh r \cos(\theta - \gamma_\beta)}{\cosh r \sin \gamma_\alpha - \sinh r \sin(\theta - \gamma_\beta)}, \quad (26)$$

$$\gamma_b = \arctan \frac{\cosh r \cos \gamma_\beta - \sinh r \cos(\theta - \gamma_\alpha)}{\cosh r \sin \gamma_\beta - \sinh r \sin(\theta - \gamma_\alpha)}. \quad (27)$$

It is clear that γ_a and γ_b are equal to each other in cases I and II, but are not the same in case III. In particular, when $\delta = \pi/2$, there is $\tan \gamma_a \tan \gamma_b = -1$, i.e., the phases of the two beams are separated by $\pi/2$ and the uncertainty of the phase approaches infinity (Fig. 3). Obviously, for case III, an additional interference happens between two light beams with different phases, which makes the phase sensitivity worse.

III. INFLUENCE OF LOSS IN THE OPTICAL PATH

In practice, there must be loss arising from the propagation of the light in the gyroscope. There are various origins of this loss, such as the changes in polarization, the widening of the
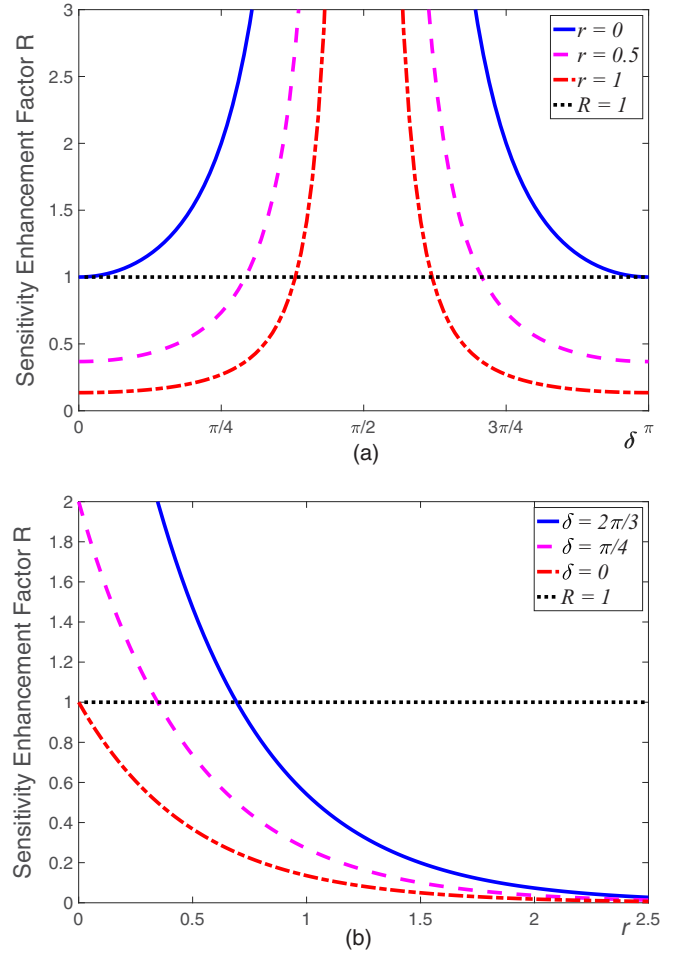

FIG. 3. For case III, the ratio R between the variance of the phase $\Delta\varphi^2$ and shot-noise limit $\Delta\varphi_{\mathrm{SN}}^2$ varies with (a) the parameter δ and (b) the parametric gain coefficient r. $R = 1$ (the black dot line) means the phase sensitivity reaches the shot-noise limit.

spectrum, the optical Kerr effect, and so on. The effect of the loss due to the propagation may be modeled by fictitious beam splitters BS_a and BS_b, placed in the two optical paths [28,29] as described in Fig. 4. $\hat{S}_i$ denotes the field operator of the beam in the gyroscope at the input port of the BS_i and $\hat{S}_i'$ denotes the field operator at the output port of the BS_i, $(i = a, b)$. The
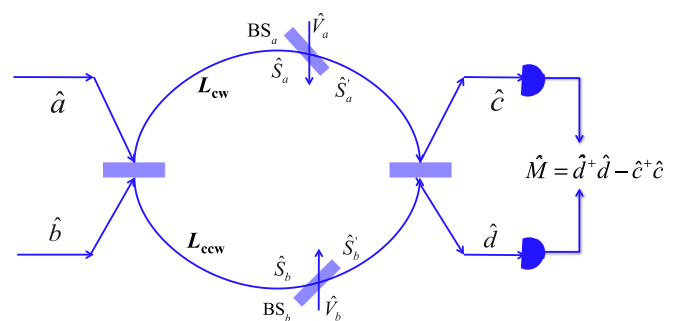

FIG. 4. The Sagnac interferometer in presence of loss in light propagation process, where the loss is modeled by the fictitious beam splitters.

relation between $\hat{S}_i$ and $\hat{S}_i'$ can be expressed as follows:

$$\hat{S}_i' = t_i\hat{S}_i + r_i\hat{V}_i, \tag{28}$$

where $\hat{V}_i$ is the field operator for the environment, which is assumed to be vacuum, and r_i and t_i are the reflection and the transmission coefficients for mode i, which are assumed to be real. Clearly, $r_i^2 = 1 - t_i^2$ stands for the loss in the mode i.

As mentioned in the preceding section, under the condition $\alpha = \beta$, the phase sensitivity is enhanced compared with the conventional gyroscope by a factor of $1/G$. When the loss is taken into account, following the argument in Sec. II, we find the variance of the phase shift φ is given by

$$\Delta\varphi^2 = \frac{1}{2n^2G^2} + \frac{t_a^2r_b^2 + t_b^2r_a^2}{4t_a^2t_b^2n^2G}$$

$$= \left(\frac{1}{G} + \frac{t_a^2r_b^2 + t_b^2r_a^2}{2t_a^2t_b^2}\right)\Delta\varphi_{\text{SN}}^2, \tag{29}$$

where $\Delta\varphi_{\text{SN}}$ is the shot-noise limit obtained for the ideal, no-loss condition. The phase sensitivity decreases due to the loss in the propagation. In the case $\theta = \pi + 2\gamma$, the enhancement factor of the phase variance R becomes

$$R = e^{-2r} + \frac{t_a^2r_b^2 + t_b^2r_a^2}{2t_a^2t_b^2}. \tag{30}$$

When the transmission coefficients t_a and t_b approximate to 0, the enhancement factor R is going to be infinity because, under this situation, no signal can be detected, and no information of the phase can be obtained either.

Figure 5 shows the factor R as a function of the transmission coefficients t_a^2 and t_b^2 for various values of the parametric gain coefficients r. It is clear that the area where $R < 1$ corresponds to the situation where the phase sensitivity is below the shot-noise limit. When the parametric gain coefficient r is fixed, the bigger the transmission coefficients t_a^2 and t_b^2 are, the smaller the ratio R is, and the better the precision of the phase measurement. Meanwhile, for given t_a^2 and t_b^2, the area where $R < 1$ increases with the increase in the coefficient r, and the phase sensitivity is improved. That is to say, the uncertainty of the phase measurement caused by the loss in light paths L_{cw} and L_{ccw} can be compensated by increasing the parametric gain coefficient r.

The reciprocity of the wave propagation in two directions plays an important role in the accuracy of the optical gyroscope. Let us consider the influence of the loss coming from the light propagation on the sub-shot-noise phase sensitivity under the conditions that the optical paths are reciprocal ($t_a^2 = t_b^2$) and nonreciprocal ($t_a^2 \neq t_b^2$), respectively.

(i) $t_a^2 = t_b^2 = T$

In general, the two optical paths should be reciprocal when the optical gyroscope is stationary. This means the losses in the two paths should be equal, thus we can set $t_a^2 = t_b^2 = T$. At this stage, the variance of the phase signal will be

$$\Delta\varphi^2 = \left(\frac{1}{G} + \frac{1-T}{T}\right)\Delta\varphi_{\text{SN}}^2 \tag{31}$$

and Eq. (30) is rewritten as
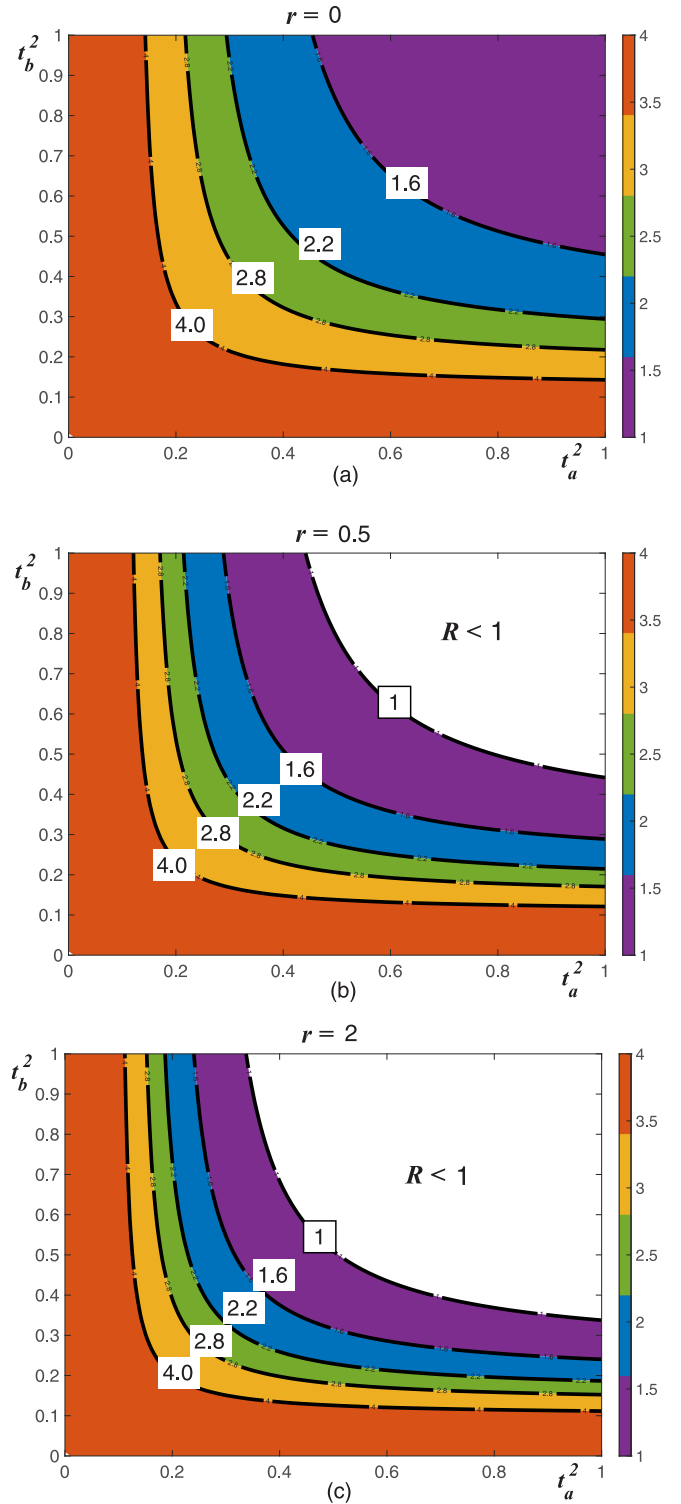
$$R = e^{-2r} + \frac{1-T}{T}. \tag{32}$$

FIG. 5. Contour plot of the enhancement factor R varying with the transmission coefficients t_a^2 and t_b^2. The area for $R < 1$ (white colored) corresponds to the situation in which the phase sensitivity is below the shot-noise limit.

It is easy to show that to obtain the sub-shot-noise limit, the transmissivity needs to be

$$T > \frac{1}{2 - e^{-2r}} > \frac{1}{2}. \tag{33}$$

The above equation implies that it is always possible to obtain the sub-shot-noise limit if the loss in the optical path is no more than 3dB.

(ii) $t_a^2 \neq t_b^2$

In many applications, the losses in the two optical paths are most likely unequal, i.e., $t_a^2 \neq t_b^2$. Suppose $t_a^2/t_b^2 = \kappa$ and $t_b^2 = T$, such that $0 < \kappa \leqslant 1$. The parameter κ reflects the degree of the reciprocity, the two paths tend to be reciprocal as the parameter κ approaches one, and vice versa. According to Eq. (29), we have

$$\Delta\varphi^2 = \left(\frac{1}{G} + \frac{1}{2T} + \frac{1}{2\kappa T} - 1\right)\Delta\varphi_{\mathrm{SN}}^2, \qquad (34)$$

and in the case of $\theta = \pi + 2\gamma$, the ratio between $\Delta\varphi^2$ and $\Delta\varphi_{\mathrm{SN}}^2$ takes the following form:

$$R = e^{-2r} + \frac{1}{2T} + \frac{1}{2\kappa T} - 1. \qquad (35)$$

The required value of κ for sub-shot-noise phase sensitivity ($R < 1$), of course, depends on T. From the above equation, it is always possible to achieve the sub-shot-noise limit measurement if $\kappa > 1/3$ for $T = 1$, or $\kappa > 1/2$ for $T = 3/4$.

If we were to have a single-port input, by adjusting the transmission coefficient of the BS, we would be able to compensate the unbalance of the loss. In our scheme, we have a two-port input before the first BS. And the coherent beams will experience the OPA and the BS, and then go through the two paths of the interferometer. That means the amplitudes of the initial two coherent beams are not equal to those of the beams passing through the two arms. To overcome the unbalance between two paths by using the initial coherent states of different amplitudes, it is necessary to know the degree of the nonreciprocity between the two paths, which we define as $t_a^2/t_b^2 = \kappa$ here, and then adjust the parameters of OPA and BS to ensure the amplitudes of the lights going through the two arms are balanced. On the other hand, there are many factors that can induce the nonreciprocity, such as the thermal fluctuation, light scattering, or other disturbing from outside. The degree of the nonreciprocity usually is unknown in practice. Therefore, optimizing the two amplitudes of the initial coherent states could be possible via an adaptive control with feedback.

In addition to the loss in propagation, the photon loss can occur in the detection, and alters the phase sensitivity of an interferometer sensor as well [19,21]. The effect of the photon loss in the detection is described in Appendix A. Comparing Eq. (31) with Eq. (A11), we notice that the loss in the detection and the loss in the propagation have the same effect on the phase sensitivity. After all, the final result is to shrink the number of photons to be detected no matter where the loss occurs (in the propagation or in the detection). From that point of view, the loss we discussed in this section can be regarded as the one that occurs not only in the propagation but also in the detection. If the two kinds of loss are considered at the same time, there will be a multiplication of the two factors; T is to be replaced by $T\tau$, where τ is the detection efficiency, shown in (A11).

IV. QUANTUM CRAMÉR-RAO BOUND

For a probing system, the limit of the accuracy of the parameter estimation is given by its quantum Cramér-Rao bound (QCRB), which is independent of the measurement strategy, and the limit may not always be achieved by any kind of measurement. In this section, we investigate the phase sensitivity for the gyroscope system to show whether the quantum Fisher information is saturated by intensity difference measurement. Unlike the numerical result reported in Ref. [15], here we will provide an analytical expression of the QCRB of the two-mode squeezed coherent state.

According to the discussion in Sec. II, for the present measurement strategy the optimal phase sensitivity, which is $\Delta\varphi^2 = 1/[2n(e^{2r})^2]$, is achieved under the condition that the complex amplitudes of the two initial coherent beams are the same, i.e., $\alpha = \beta = \eta = \sqrt{n}e^{i\gamma}$, and the phase match $\theta = \pi + 2\gamma$ is satisfied (see case I in Sec. II). Therefore, the state of the light field injected into the gyroscope can be expressed as $|\Psi\rangle = \hat{S}|\eta, \eta\rangle$. And the gyroscope can be represented by a unitary transformation $\hat{U}_{\mathrm{gyro}}$ in terms $\hat{a}$ and $\hat{b}$ (see Appendix C),

$$\hat{U}_{\mathrm{gyro}} = e^{\frac{1}{2}\varphi(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})} = e^{-i\varphi\hat{G}_{\mathrm{gyro}}}, \qquad (36)$$

where $\hat{G}_{\mathrm{gyro}} = \frac{i}{2}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})$. Therefore, the quantum Fisher information is $4\Delta G_{\mathrm{gyro}}^2$ and the QCRB of the system is given by [30]

$$\Delta\varphi_{\mathrm{QCRB}}^2 = \frac{1}{4\Delta\hat{G}_{\mathrm{gyro}}^2} = \frac{1}{2n(e^{2r})^2 + \sinh^2 2r}. \qquad (37)$$

The phase sensitivity obtained by intensity-difference measurement will be equal to the QCRB in the case that the average photon number of the initial coherent field $\langle\hat{N}\rangle_{\mathrm{coh}}$ is far greater than that of the TMSV field $\langle\hat{N}\rangle_{\mathrm{TMSV}}$, i.e., $2n \gg 2\sinh^2 r$. As shown in Fig. 6, the curves of the phase sensitivity and the QCRB tend to overlap with the increase in the average photon number of each initial coherent beam n and the average total photon number at the output port of the gyroscope $\langle\hat{N}\rangle_{\mathrm{tot}}$. This indicates that the intensity difference measurement is capable of saturating the QCRB under the condition $n \gg \sinh^2 r$.

The average number of the total photons at the output port is $\langle\hat{N}\rangle_{\mathrm{tot}} = 2nG + 2\sinh^2 r$, based on Eq. (6). Thus the shot-noise limit and the Heisenberg limit are $\Delta\varphi_{\mathrm{SN}} = 1/\sqrt{2nG + 2\sinh^2 r}$ and $\Delta\varphi_{\mathrm{HL}} = 1/(2nG + 2\sinh^2 r)$, respectively, which are also shown in Fig. 6. When the average photon number of the initial coherent field $n = 0$, it means the light field used as the probe is the TMSV state. The total number of the photons exiting the gyroscope will then be $\langle\hat{N}\rangle_{\mathrm{tot}} = \langle\hat{N}\rangle_{\mathrm{TMSV}} = 2\sinh^2 r$, and the QCRB is

$$\Delta\varphi_{\mathrm{QCRB}}^2 = \frac{1}{\langle\hat{N}\rangle_{\mathrm{tot}}(\langle\hat{N}\rangle_{\mathrm{tot}} + 2)}, \qquad (38)$$

which is consistent with the result in Ref. [14]. Note that the QCRB for the TMSV state is below the Heisenberg limit, which indicates that the Heisenberg limit—given by the inverse of the total photon number—is not the ultimate limit of the phase variance [12,13]. As the phase sensitivity with intensity difference measurements becomes unbound,
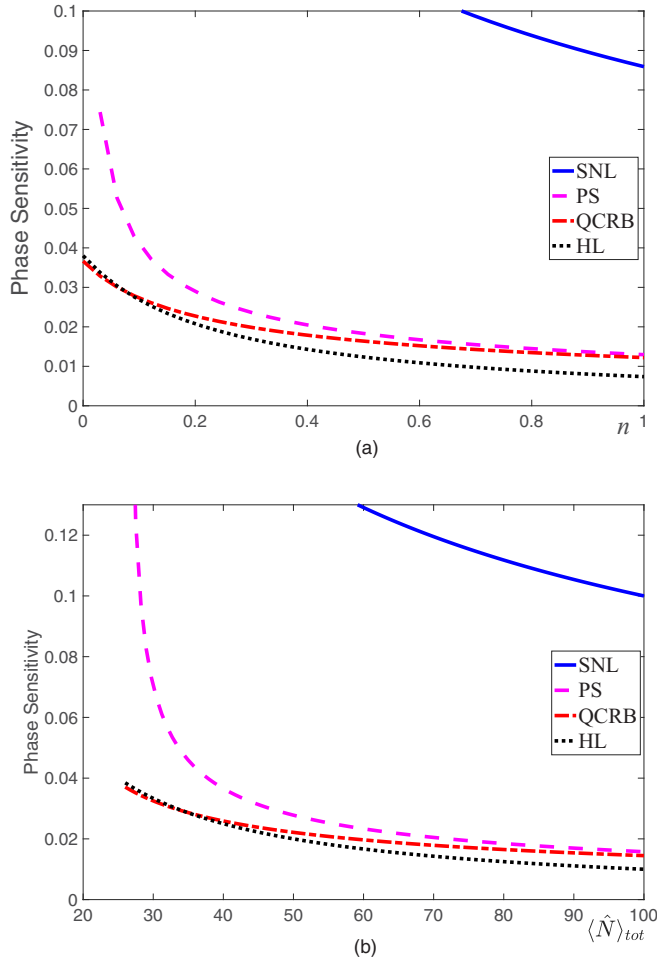
FIG. 6. The phase sensitivity $\Delta\varphi$ varying with (a) the average photon number of each one of the initial coherent beams n and (b) the average total photon number of the output light fields $\langle\hat{N}\rangle_{\text{tot}}$. The parametric gain coefficient r is set to be 2, that $\langle\hat{N}\rangle_{\text{tot}} \approx 26.3$ if the input coherent beams are in the vacuum state. SNL: shot-noise limit. PS: phase sensitivity of our interferometer scheme given by Eq. (15). QCRB: quantum Cramér-Rao bound. HL: Heisenberg limit.

the intensity difference measurement strategy is no longer a good measurement scheme when the probing light field is in the TMSV state [see Eq. (14)]. In this case, the signal obtained by intensity difference measurements does not contain any information of the phase shift [see Eq. (9)]. If losses were to be included, the QCRB can be calculated following, for example, Ref. [31]. Calculation of QCRB including losses, however, becomes significantly more cumbersome and complex.

V. DISCUSSION

Compared with the conventional gyroscope, a distinct advantage of our proposed scheme is the enhancement with a factor of e^{2r} in the phase sensitivity $\Delta\varphi^2$. As a consequence, the minimal detectable angular velocity will decrease with the factor e^{-r}, based on Eq. (1). In practice, the minimum detectable angular velocity for a conventional fiber gyroscope is given as [32,33]

$$\Omega_{\text{min}}^{\text{cg}} = \frac{\lambda c}{2\pi DL} \frac{\sqrt{2e(1+\cos\phi_b)\Delta f}}{\sqrt{R_D P}\sin\phi_b}$$

$$= \frac{\lambda c}{2\pi DL}\sqrt{\frac{2e\Delta f\lambda}{R_D hc}}\frac{1}{\sqrt{2}\sin(\phi_b/2)}\Delta\varphi_{\text{SN}}, \quad (39)$$

where L is the length of the fiber, coil diameter $D = 0.2$ m, P is the output power at the detectors, λ is the wavelength, R_D is the detector responsiveness which measures the input-output gain of a detector system (in the unit of A/W), ϕ_b is the bias phase, e is the electron charge, h is the Planck's constant, c is the velocity of light, and Δf is the count bandwidth of the output signal. If the two-mode squeezed coherent state is adopted, as discussed in Sec. II, according to Eq. (15), the minimum detectable angular velocity can be expressed as

$$\Omega_{\text{min}}^{\text{qg}} = \frac{\lambda c}{2\pi DL}\sqrt{\frac{2e\Delta f\lambda}{R_D hc}}\Delta\varphi. \quad (40)$$

It is obvious that the minimum detectable angular velocity decreases by a factor of $\sqrt{2}e^{-r}\sin(\phi_b/2)$. Note that for a fair comparison, we assume the total number of photons is the same for both the conventional gyroscope and the quantum gyroscope.

For example, for a conventional fiber gyroscope with the parameters $L = 5$ km, $D = 0.2$ m, $P = 250$ μW, $\lambda = 1550$ nm, $R_D = 1$A/W, $\phi_b = 3\pi/4$, and $\Delta f = 0.01$ Hz, the minimum detectable angular velocity $\Omega_{\text{min}}^{\text{cg}}$ is $4.2\times10^{-5}(°)$/h. Whereas, for the present gyroscope with the same parameters, the minimum detectable angular velocity $\Omega_{\text{min}}^{\text{qg}}$ can reach $5.5\times10^{-6}(°)$/h in the case of $r = 2.25$, which can be realized with the current technology [34]. Besides, if the interferometer is injected with light from the TMSV light field, as suggested in Ref. [17], to realize a phase sensitivity equivalent to $5.5\times10^{-6}(°)$/h, the demand on the parametric gain r would be no less than nine, which is hard to achieve in practice. That is the reason we adopt the output field of the OPA with the coherent light input as the light source of the gyroscope.

Typically, for the conventional optical gyroscope, the maximal sensitivity occurs at the point $\varphi = \pi/2$, and a bias phase ϕ_b is necessary to obtain the maximal sensitivity at the point $\varphi = 0$. For our gyroscope scheme, according to Eq. (9), when $|\alpha| = |\beta|$ the output signal is proportional to $\sin\varphi$, i.e., $\langle\hat{M}\rangle \propto \sin\varphi$. Thus $\partial\langle\hat{M}\rangle/\partial\varphi \propto \cos\varphi$, even if $r = 0$. That shows the maximal sensitivity can be achieved at the working point $\varphi = 0$ without the bias phase for our scheme. In turn, it indicates that the loss caused by the bias phase modulation in the conventional optical gyroscope can be avoided in our scheme. At the same time, our gyroscope scheme is robust against propagation loss for a wide range of the transmission coefficient T and nonreciprocal κ.

On the other hand, we can obtain the same phase sensitivity with a much smaller length of the light path, that is, $\Omega_{\text{min}}^{\text{cg}} = \Omega_{\text{min}}^{\text{qg}}$, when $L^{\text{qg}} = e^{-r}L^{\text{cg}}$. As a result, the loss due to the light propagation in the fiber will be reduced due to the smaller length. Hence, any kind of loss that is proportional to the length of the fiber, such as loss coming from backscattering, the Kerr effect, the Faraday magnetic-optical effect, etc., will decrease by a factor of e^{-r}. With a simple numeral calculation,

we find these kinds of losses in our gyroscope will decrease to 10% of those in the conventional one when we take the parametric gain of OPA $r = 2.25$.

In addition, several nonlinear interferometer schemes showing better performance than linear interferometers have been reported in recent years [23,24]. In the present paper, the uncertainty of the phase measurement can be expressed as $\Delta\varphi = e^{-2r}/\sqrt{\langle\hat{N}\rangle_{\rm coh}}$ by taking Eq. (14) when $\theta = \pi + 2\gamma$, which is comparable to the nonlinear interferometer in Ref. [23], with $\Delta\varphi = e^{-2r}/\sqrt{2\langle\hat{N}\rangle_{\rm coh}}$ in the high gain limit. The output detection adopted in Ref. [23], however, is the intensity-sum measurement. Our gyroscope scheme using the intensity-difference measurement might well have advantages in that various technical noises can be eliminated with differencing. Also, unlike the result in Ref. [24], the influence of the loss in propagation can be compensated by enhancing the parametric gain of the OPA in our scheme.

VI. CONCLUSION

In conclusion, we have analyzed a type of gyroscope with a coherent-boosted two-mode squeezed light field and showed its sub-shot-noise phase sensitivity. The phase sensitivity has been examined with different coherent light fields and the results show that the optimal sensitivity occurs when the complex amplitudes of the two modes of the coherent light are equal to each other. The corresponding minimum detectable angular velocity has an enhancement with a factor of e^{-r} compared with the conventional gyroscope with the same number of photons at the output—that factor would be e^{-2r} if the same number of photons at the input is assumed. The effect of the loss from the light propagating through the interferometer is another significant factor in high precision measurement.

The good news here is that it does not limit the enhancement of the measurement sensitivity, since it can also be offset by the increase of the parametric gain coefficient r. It is still possible to achieve sub-shot-noise sensitivity even in the noisy environment. In addition, we found that the intensity difference estimation strategy has the capability to reach Cramér-Rao bound. It should be pointed out that the technologies involved in our scheme, such as the OPA and the intensity difference measurement, are highly accessible and well-established techniques. We also notice that the use of the orbital angular momentum states of light has been demonstrated for measuring rotation angle or vibrations [35,36]. By contrast, our scheme is based on the current gyroscope technique and can be integrated with existing optical gyroscope systems. Therefore, we believe that our gyroscope scheme is positively suitable for practical applications of quantum technique and can be realized in the near future.

ACKNOWLEDGMENT

APPENDIX A: THE VARIANCE OF THE PHASE SHIFT

Assume the quantum efficiency of the two detectors are the same and equal to τ (where $\tau \leqslant 1$), the operators of the intensity detection are $\hat{m}_d = \tau\hat{d}^\dagger\hat{d}$ and $\hat{m}_c = \tau\hat{c}^\dagger\hat{c}$, respectively. In this condition, the operator for intensity difference is given as [37]

$$\hat{M}_\tau = \tau\hat{d}^\dagger\hat{d} - \tau\hat{c}^\dagger\hat{c} = \tau\hat{M}, \tag{A1}$$

$$\begin{aligned}\widehat{M_\tau^2} &= \tau^2(\hat{d}^\dagger\hat{d} - \hat{c}^\dagger\hat{c})^2 + \tau(1-\tau)(\hat{d}^\dagger\hat{d} + \hat{c}^\dagger\hat{c}) \\ &= \tau^2\hat{M}^2 + \tau(1-\tau)(\hat{d}^\dagger\hat{d} + \hat{c}^\dagger\hat{c}) \end{aligned} \tag{A2}$$

and the variance on the intensity difference measurement is

$$\begin{aligned}\Delta M_\tau^2 &= \langle\widehat{M_\tau^2}\rangle - \langle\hat{M}_\tau\rangle^2 \\ &= \tau^2\langle\hat{M}^2\rangle + \tau(1-\tau)\langle N\rangle_{\rm tot} - \tau^2\langle\hat{M}\rangle^2 \\ &= \tau^2\Delta M^2 + \tau(1-\tau)\langle\hat{N}\rangle_{\rm tot}, \end{aligned} \tag{A3}$$

where $\Delta M^2 = \langle\hat{M}^2\rangle - \langle\hat{M}\rangle^2$. Then, with the method of linear error propagation, the phase sensitivity is

$$\Delta\varphi^2 = \frac{\Delta M_\tau^2}{|\partial\langle M_\tau\rangle/\partial\varphi|^2} = \frac{\tau^2\Delta M^2 + \tau(1-\tau)\langle\hat{N}\rangle_{\rm tot}}{\tau^2|\partial\langle M\rangle/\partial\varphi|^2}. \tag{A4}$$

First, we are concerned with the case of idea detection, that is, $\tau = 1$, then the uncertainty of the phase shift φ takes the following form:

$$\Delta\varphi^2 = \frac{\Delta M^2}{|\partial\langle M\rangle/\partial\varphi|^2}, \tag{A5}$$

which is Eq. (10) in the Sec. II. According to Eq. (8), there is

$$\begin{aligned}\Delta M^2 &= \langle\hat{M}^2\rangle - \langle\hat{M}\rangle^2 \\ &= \langle[(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})\cos\varphi + (\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)\sin\varphi]^2\rangle \\ &\quad - \langle(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})\cos\varphi + (\hat{a}^\dagger\hat{b} + \hat{a}\hat{b}^\dagger)\sin\varphi\rangle^2. \end{aligned} \tag{A6}$$

Meanwhile, according to Eq. (9), we have

$$\begin{aligned}\frac{\partial\langle\hat{M}\rangle}{\partial\varphi} &= -(|\alpha|^2 - |\beta|^2)\sin\varphi + \Big\{(\alpha^*\beta + \alpha\beta^*)\cosh 2r \\ &\quad - \frac{1}{2}[e^{i\theta}(\alpha^{*2} + \beta^{*2}) + e^{-i\theta}(\alpha^2 + \beta^2)]\sinh 2r\Big\}\cos\varphi. \end{aligned} \tag{A7}$$

Since we focus on the working point at $\varphi = 0$, in the above two equations the terms containing $\sin\varphi$ can be neglected,

$$\begin{aligned}\Delta M^2 &= \langle(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2\rangle - \langle(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})\rangle^2 \\ &= |\alpha|^2 + |\beta|^2, \end{aligned} \tag{A8}$$

$$\begin{aligned}\frac{\partial\langle\hat{M}\rangle}{\partial\varphi} &= \Big\{(\alpha^*\beta + \alpha\beta^*)\cosh 2r - \frac{1}{2}[e^{i\theta}(\alpha^{*2} + \beta^{*2}) \\ &\quad + e^{-i\theta}(\alpha^2 + \beta^2)]\sinh 2r\Big\}, \end{aligned} \tag{A9}$$

where Eqs. (2) and (3) are used. Bringing the above two equations into Eq. (A5), we obtain the uncertainty of the phase shift with ideal detection,

$$\Delta\varphi^2 = \frac{|\alpha|^2 + |\beta|^2}{\left|(\alpha^*\beta + \alpha\beta^*)\cosh 2r - \frac{1}{2}[e^{i\theta}(\alpha^{*2} + \beta^{*2}) + e^{-i\theta}(\alpha^2 + \beta^2)]\sinh 2r\right|^2}, \tag{A10}$$

which is Eq. (11) in Sec. II.

Next, we turn to a brief discussion about the effect of the quantum efficiency on the phase sensitivity. In the case of $\alpha = \beta = \sqrt{n}e^{i\gamma}$, based on Eq. (A4), we have

$$\Delta\varphi^2 = \frac{\tau^2(2n) + \tau(1-\tau)(2nG)}{\tau^2|2nG|^2}$$

$$= \left(\frac{1}{G} + \frac{1-\tau}{\tau}\right)\Delta\varphi_{\text{SN}}^2. \tag{A11}$$

Therefore, the the enhancement factor of the phase uncertainty becomes $R = \frac{1}{G} + \frac{1-\tau}{\tau}$. To make the phase uncertainty go below shot-noise limit, the condition $\tau > \frac{G}{2G-1}$ must be satisfied. For the optimal case of $\cos(\theta - 2\gamma) = -1$, the quantum efficiency of the detectors $\tau > \frac{1}{2}$ is required.

APPENDIX B: NOISE REDUCTION FACTOR

Much attention has been paid to the two-mode squeezed states due to the quantum correlation between its two modes. One of the characteristics of its quantum correlation is the suppression on the intensity difference noise between the two modes under certain conditions.

Assume the annihilation operators of two light beams are $\hat{i}_1$ and $\hat{i}_2$, then the intensity difference between the two beams can be expressed as $\hat{I}_{\text{diff}} = \hat{i}_1^\dagger\hat{i}_1 - \hat{i}_2^\dagger\hat{i}_2$. In general, for two independent coherent light beams, the state of which is $|\chi_1\rangle|\chi_2\rangle$, the variance of the intensity difference between the

two beams is

$$\Delta I_{\text{diff}} = \langle(\hat{i}_1^\dagger\hat{i}_1 - \hat{i}_2^\dagger\hat{i}_2)^2\rangle - \langle\hat{i}_1^\dagger\hat{i}_1 - \hat{i}_2^\dagger\hat{i}_2\rangle^2$$

$$= |\chi_1|^2 + |\chi_2|^2. \tag{B1}$$

This is the shot-noise limit of the intensity difference between any two light beams, which is equal to the total average photon number of the two light beams.

For the two-mode squeezed light beams, when we ignore the quantum correlation between the two modes, the uncertainty of the intensity difference should be equal to the total average photon number $\langle\hat{N}\rangle_{\text{tot}}$. That is to say the the shot-noise limit in detecting the intensity difference between the two modes $\hat{a}$ and $\hat{b}$, according to Eq. (7), is

$$\Delta k_{\text{SN}} = (|\alpha|^2 + |\beta|^2)\cosh 2r - (\alpha^*\beta^* e^{i\theta} + \alpha\beta e^{-i\theta})\sinh 2r$$

$$= (|\alpha|^2 + |\beta|^2)\cosh 2r$$

$$- 2|\alpha||\beta|\sinh 2r\cos(\theta - \gamma_\alpha - \gamma_\beta) \tag{B2}$$

in the case that $\alpha = |\alpha|e^{i\gamma_\alpha}$ and $\beta = |\beta|e^{i\gamma_\beta}$. However, for the two-mode squeezed state, i.e., $\hat{S}|\alpha\rangle|\beta\rangle$, the variance of the intensity difference between the two modes $\hat{a}$ and $\hat{b}$, in fact, is

$$\Delta k = \langle(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})^2\rangle - \langle\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\rangle^2$$

$$= |\alpha|^2 + |\beta|^2. \tag{B3}$$

The rate between the shot-noise limit Δk_{SN} and the variance Δk, NRF, is given as

$$\text{NRF} = \frac{\Delta\hat{k}^2}{\Delta\hat{k}_{\text{SN}}^2} = \frac{|\alpha|^2 + |\beta|^2}{[(|\alpha|^2 + |\beta|^2)\cosh 2r - 2|\alpha||\beta|\sinh 2r\cos(\theta - \gamma_\alpha - \gamma_\beta)]}. \tag{B4}$$

Obviously, when NRF < 1 the quantum noise is reduced, and vice versa.

When $\alpha = \beta = \sqrt{n}e^{i\gamma}$, there is

$$\text{NRF} = \frac{1}{\cosh 2r - \sinh 2r\cos(\theta - 2\gamma)} = \frac{1}{G}. \tag{B5}$$

It shows the quantum noise will decrease when the condition $G > 1$ is satisfied. The relation described in Eq. (B5) shows good agreement with the theoretic analysis and experimental results in Refs. [38–40], in which the reduction is attributed to the quantum correlation between the photon number of the two beams.

Based on Eq. (B4), the NRF for case II in Sec. II ($|\beta|/|\alpha| = \epsilon$, but $\gamma_\alpha = \gamma_\beta = \gamma$) is given as

$$\text{NRF} = \frac{1 + \epsilon^2}{(1 + \epsilon^2)\cosh 2r - 2\epsilon\sinh 2r\cos(\theta - 2\gamma)}. \tag{B6}$$

And there is

$$\text{NRF} = \frac{1}{\cosh 2r - \sinh 2r\cos(\theta - 2\gamma - \delta)} \tag{B7}$$

for case III in Sec. II ($|\alpha| = |\beta| = \sqrt{n}$, but $\gamma_\beta - \gamma_\alpha = \delta$).

APPENDIX C: CALCULATION FOR QUANTUM CRAMÉR-RAO BOUND

When a certain variable λ (for example) is estimated, we usually use the Cramér-Rao inequality to show the limit of the mean-square error of variable λ, that is, $\Delta\lambda^2 \geqslant 1/pF(\lambda)$ (where p is the is the number of measurements and $F(\lambda)$ is the so-called Fisher information). That means the limit of the variance, or the Cramér-Rao bound, is $\Delta\lambda_{\text{CRB}}^2 = 1/pF(\lambda)$.

Note that the probing light field is in the two-mode squeezed state, and the state to be detected is a pure state. According to Ref. [30], the quantum Fisher information has a

simple form $F(\lambda) = 4\langle\psi_0|\Delta\hat{G}_{\text{sys}}^2|\psi_0\rangle$, where $|\psi_0\rangle$ is the probing state and $\hat{G}_{\text{sys}}$ is the corresponding Hermitian generator of the system.

As described in Sec. IV, the probing state of our gyroscope is $|\Psi\rangle = \hat{S}|\eta,\eta\rangle$, then the light field to be detected will be in the state $|\psi(\varphi)\rangle = \hat{U}_{\text{gyro}}|\Psi\rangle$, where $\hat{U}_{\text{gyro}} = \hat{U}_{\text{BS}}^{-1}\hat{P}_\varphi\hat{U}_{\text{BS}}$. In terms of the operators of the modes $\hat{a}$ and $\hat{b}$, there are [41]

$$\hat{U}_{\text{BS}} = e^{i\frac{\pi}{4}(\hat{a}^\dagger b + \hat{b}^\dagger a)}, \tag{C1}$$

$$\hat{P}_\varphi = e^{i(-\frac{\varphi}{2}\hat{a}^\dagger\hat{a} + \frac{\varphi}{2}\hat{b}^\dagger\hat{b})} = e^{-i\frac{\varphi}{2}(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})}. \tag{C2}$$

Here we assume that the phase is imprinted symmetrically. That means the model with a symmetrically distributed phase shift is used [41]. We then get

$$\hat{U}_{\text{gyro}} = e^{\frac{1}{2}\varphi(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})} = e^{-i\varphi\hat{G}_{\text{gyro}}}, \tag{C3}$$

where $\hat{G}_{\text{gyro}} = \frac{i}{2}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})$.

Note that the probing light field is in the two-mode squeezed state, and the state $|\Psi(\varphi)\rangle = \hat{U}_{\text{gyro}}|\Psi\rangle$ we detected is a pure state. Therefore, according to Ref. [30], the quantum Fisher information can be written as

$$F(\varphi) = 4\langle\Psi|\Delta\hat{G}_{\text{gyro}}^2|\Psi\rangle$$

$$= 4(\langle\eta,\eta|\hat{S}^\dagger\hat{G}_{\text{gyro}}^2\hat{S}|\eta,\eta\rangle - \langle\eta,\eta|\hat{S}^\dagger\hat{G}_{\text{gyro}}\hat{S}|\eta,\eta\rangle^2)$$

$$= 2n(e^{2r})^2 + \sinh^2 2r, \tag{C4}$$

where Eqs. (2) and (3) are used. The QCRB of the system is then obtained as

$$\Delta\varphi_{\text{QCRB}}^2 = \frac{1}{4\Delta\hat{G}_{\text{gyro}}^2} = \frac{1}{2n(e^{2r})^2 + \sinh^2 2r}. \tag{C5}$$

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