Distillation of lossy hyperentangled states

Dan-Yang Chen,[1] Zhi Lin,[1,2,*] Ming Yang,[1,†] Qing Yang,[1] Xue-Ping Zang,[3,1] and Zhuo-Liang Cao[4,1,‡]
[1]*School of Physics & Materials Science, Anhui University, Hefei 230601, China*
[2]*Department of Physics and State Key Laboratory of Surface Physics, Fudan University, Shanghai 200433, China*
[3]*School of Mechanical and Electrical Engineering, Chizhou University, Chizhou 247000, China*
[4]*School of Physics & Materials Engineering, Hefei Normal University, Hefei 230601, China*

(Received 20 January 2020; revised 12 July 2020; accepted 3 August 2020; published 28 August 2020)

Hyperentanglement has a higher information density than conventional single-degree-of-freedom entanglement, which has attracted much attention due to its fascinating applications in quantum communication. However, since the inevitable interactions between quantum entangled systems and the environment will drive hyperentangled systems into less hyperentangled states or even mixed hyperentangled states, the efficiency and security of quantum communication will be greatly depressed. The currently existing distillation protocols are not universal, i.e., they only work for the lossy hyperentangled states in specific systems. In this paper, based on local positive-operator-valued measures (POVMs), we present a distillation protocol for lossy hyperentangled photonic Bell and Greenberger-Horne-Zeilinger (GHZ) states. The intrinsic property of our protocol is twofold, i.e., the POVM nature of the protocol guarantees that the protocol is a universal one, and the distillation operation on only one of the two degrees of freedom (DOFs) can enhance the fidelity of the system in both DOFs. Furthermore, in the implementation level, our hyperentanglement distillation protocol (HEDP) has other two merits: no auxiliary local entanglement resources and sophisticated single-photon detectors are required, and only one copy of the lossy state will be operated in each distillation round, which show that our HEDPs are relatively simple and feasible.

DOI: 10.1103/PhysRevA.102.022425

I. INTRODUCTION

Quantum entanglement is a crucial physical resource in quantum information processing, which has been widely applied in quantum teleportation [1], quantum key distribution [2], quantum computation [3], quantum cryptography [4,5], quantum repeaters [6], and so on. Maximally entangled states connecting two or more adjacent nodes in a quantum network are generally prepared locally, but they need to be shared nonlocally for accomplishing these quantum information processing tasks effectively. However, during the practical entanglement distribution, the inevitable interactions between quantum systems and the environment will deteriorate the maximally entangled states into partially entangled pure states or even mixed states [7].

To improve entanglement or fidelity of noisy entangled states, researchers have proposed several solutions, such as entanglement purification (EP) [8–17], entanglement distillation [18,19], entanglement concentration [20–22], etc. Entanglement purification is the process of extracting a subset of higher-fidelity entangled states from a set of mixed entangled states. In 1996, Bennett *et al.* [8] put forward the first entanglement purification protocol based on bilateral controlled-NOT (CNOT) gates on two copies of the Werner state. Subsequently, Deutsch *et al.* [9] improved this scheme based on special joint operations. To avoid the complicated CNOT and other joint operations, Pan *et al.* [10] proposed an efficient EP protocol based on linear optical elements by using parity-check measurement in polarization degree of freedom (DOF). Although all of the above-mentioned schemes have effectively improved the fidelity of noisy states, two new problems still exist, i.e., two copies of the initial entanglement resources are needed in each purification round and the limited fidelities of purifiable initial states are within the range $(1/2, 1]$. Aiming at these two issues, Gisin [18] and Huang *et al.* [19] put forward a single-copy-based entanglement distillation protocol for bipartite and multipartite entangled states, respectively, where the fidelities of purifiable initial states are extended to the whole range (0,1]. These schemes solve the above-mentioned two problems and greatly loosen the threshold condition of initial fidelity and the complexity of its realization. But, because these schemes only work for quantum states entangled in a single DOF, one may wonder whether there exist similar purification or distillation schemes which can work for quantum states entangled in multiple DOFs, i.e., hyperentangled states.

Hyperentangled states have attracted considerable attention, owing to their advantages of larger information capacity, higher security of quantum communication, stronger resistance against noise [23–28], etc. A hyperentangled state is defined as a quantum state entangled in multiple DOFs simultaneously, such as polarization-spatial-mode-time-energy DOF [29], polarization-momentum DOF [30],

*zhilin18@ahu.edu.cn
†mingyang@ahu.edu.cn
‡zlcao@ahu.edu.cn

polarization-time-bin DOF [31], polarization-orbital-angular-momentum DOF [32], hyperentanglement between two quantum memories [33], etc. However, hyperentangled states also suffer from the environment, and thus maximally hyperentangled states will decay into partially hyperentangled mixed states. Therefore, several hyperentanglement purification (HEP) schemes have also been proposed [34–37]. Similarly, the two weaknesses still exist in these schemes, i.e., the fidelity of the purifiable initial state has to be greater than $1/2$ in each DOF, and two copies of the noisy hyperentangled state are needed in each purification round. So it is of great importance to design a single-copy-based hyperentanglement distillation scheme working in the whole range of initial fidelities. Wang *et al.* proposed a linear-optical scheme for distilling single-photon two-qubit states and hyperentangled states undergoing inherent channel losses in a heralded distillation strategy [38], but the lossy mixed entangled state they distilled is only mixed in spatial DOF and the state in polarization DOF is still pure, and it requires W states as local auxiliary resources. Very recently a single-copy-based heralded amplification protocol for a two-photon spatial-mode-polarization hyperentangled mixed state was proposed, which can transform an input mixed hyperentangled state into an output mixed hyperentangled state with higher fidelity [39]. The teleportation property of these two heralded protocols indicates that they are not universal and only work for lossy photonic hyperentangled states. It is of great importance to design a universal scheme to distill hyperentangled mixed states (mixed in multiple DOFs) in a single-copy manner without auxiliary resources. In this paper, we will propose a single-copy-based hyperentanglement distillation protocol for two-photon lossy Bell state mixed entangled in both polarization and spatial-mode DOFs, in which the vacuum errors caused by the transmission losses can be corrected by performing local positive-operator-valued measurements (POVMs) only on the spatial-mode or the polarization DOF of one copy of the mixed hyperentangled state. Furthermore this protocol can be generalized to lossy N-photon hyperentangled Greenberger-Horne-Zeilinger (GHZ) states. The POVM nature of our protocol indicates that our hyperentanglement distillation protocol (HEDP) is a universal one, which applies to the lossy hyperentangled states in any quantum system. In addition, our HEDP has the following advantages in the implementation level: the distillable range of initial fidelity is broadened to the whole range (0,1], only one copy of the mixed hyperentangled photon sources (Bell state and GHZ state) is required in each distillation round, the key operation of our protocol is the POVM measurements rather than the complicated CNOT operations, and no auxiliary local entanglement resources and sophisticated single-photon detectors are involved. All of these merits indicate that our scheme is relatively simple and implementable with a wider application range.

This paper is organized as follows. In Sec. II, we introduce the single-copy-based distillation scheme for a lossy two-photon hyperentangled Bell state. As a generalization, a hyperentanglement distillation scheme for a noisy three-photon GHZ state is presented in Sec. III. The analysis and discussion are presented in Sec. IV, and the results are summarized in Sec. V.

II. SINGLE-COPY-BASED DISTILLATION SCHEME FOR LOSSY HYPERENTANGLED BELL STATE

Since the inevitable interaction between a quantum entangled system and its environment will cause the degradation of entanglement and coherence of the quantum system in the practical entanglement distributed process, the quantum entangled states actually available are almost mixed states. Usually, the ideal hyperentangled state can be generated locally before entanglement distribution. Here, we consider a pure hyperentangled two-photon source generated at Alice's side whose state is expressed as $|\varphi_0\rangle_{AB} = \frac{1}{2}(|HH\rangle + |VV\rangle)_{A_pB_p}(|a_1b_1\rangle + |a_2b_2\rangle)_{A_sB_s}$, where the subscripts A and B indicate the hyperentangled photons, and the first and second terms indicate hyperentangled two-photon states in polarization and spatial-mode DOFs, respectively. a_1, a_2 (b_1, b_2) are the spatial modes of the photon A (B). Suppose that the photon B is sent to Bob through two lossy spatial channels described by two independent transmission coefficients $T_1[\in (0, 1)]$ and $T_2[\in (0, 1)]$. As we only consider the vacuum errors of photons caused by transmission losses, corresponding to the energy dissipation process, we can describe this case by using a typical amplitude-damping noise model [40]. As a result, the spatial-mode and polarization DOFs of the transmitted photons will be affected by channel noises. After the transmission, the channel noises would deteriorate the pure hyperentangled state into the following mixed hyperentangled state:

$$\rho_{AB} = \left(\frac{1-T_1}{4}|a_1\rangle\langle a_1| + \frac{1-T_2}{4}|a_2\rangle\langle a_2|\right) \otimes I_p^A + \frac{T_1+T_2}{2}|\tau\rangle_{AB}\langle\tau| \otimes |\phi^+\rangle_{AB}\langle\phi^+|, \quad (1)$$

where $|\tau\rangle_{AB} = (\alpha_0|a_1b_1\rangle + \beta_0|a_2b_2\rangle)_{AB}$ with two coefficients $\alpha_0 = \sqrt{T_1/(T_1+T_2)}$ and $\beta_0 = \sqrt{T_2/(T_1+T_2)}$, $I_p^A = (|H\rangle\langle H| + |V\rangle\langle V|)$, and $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)_{AB}$ is one of the four Bell states in the polarization DOF. It can be easily seen that the photon B was lost with probability $P = (2 - T_1 - T_2)/2$ after the transmission. The fidelity of the damped two-photon hyperentangled state ρ_{AB} with respect to the initial ideal hyperentangled state $|\varphi_0\rangle_{AB}$ is $F_0^{(2)} = (\alpha_0 + \beta_0)^2(T_1 + T_2)/4$.

From the noisy hyperentangled photonic state in Eq. (1), one can see that the spatial channel loss will lead to the decoherence not only in the spatial DOF but also in the polarization DOF of the photon pair. To enhance the fidelity of the state in Eq. (1), Bob needs to implement a POVM measurement on the polarization or spatial-mode DOF of photon B. The POVM measurements performed in the spatial-mode or polarization DOF can be described by the POVM elements $E_{s_1}^B = |b_1\rangle\langle b_1| + (1-\lambda)|b_2\rangle\langle b_2|$, $E_{s_2}^B = \lambda|b_2\rangle\langle b_2|$ or $E_{p_1}^B = |H\rangle\langle H| + (1-\lambda)|V\rangle\langle V|$, $E_{p_2}^B = \lambda|V\rangle\langle V|$, respectively. Here λ is a parameter indicating the strength of the POVM in disturbing the state under consideration, and when $\lambda = 1$ it becomes a strong (projective) measurement, which destroys the superposition between the two eigenvectors of the measurement operator. When $\lambda = 0$, it becomes an identity operator, which does not disturb the state. Evidently, in order to make the expression a positive operator, the value of λ should be within the scope $0 < \lambda < 1$. The following analyses of our scheme show that the hyperentanglement distillation effect in the spatial-mode DOF is equal to that in the polarization DOF when the transmission coefficients T_1 and T_2 are equal, so we will mainly discuss the HEDP with POVMs on only one DOF, such as the spatial-mode DOF. In addition, notice that in our HEDP POVM measurements are only required at Bob's side, so classical communication is not required during the whole hyperentanglement distillation process, which greatly simplifies the realization complexity of it. In order to enhance the fidelity of the noisy hyperentangled state ρ_{AB}, we will select the measurement result E_{s_1} for the spatial-mode DOF of photon B, and the corresponding output state is

$$\begin{aligned}\widetilde{\rho}_{AB} &= I_s^A I_p^A \sqrt{E_{s_1}^B} I_p^B \rho_{AB} I_p^B \sqrt{E_{s_1}^B} I_p^A I_s^A \\ &\quad \Big/ \operatorname{tr}\left(I_s^A I_p^A \sqrt{E_{s_1}^B} I_p^B \rho_{AB} I_p^B \sqrt{E_{s_1}^B} I_p^A I_s^A\right) \\ &= |\widetilde{\varphi}\rangle_{AB}\langle\widetilde{\varphi}|, \end{aligned} \tag{2}$$

where $|\widetilde{\varphi}\rangle_{AB} = (m_1|a_1b_1\rangle + n_1|a_2b_2\rangle)_{AB} \otimes |\phi^+\rangle_{AB}$, whose coefficients are

$$m_1 = \frac{\alpha_0}{\sqrt{\alpha_0^2 + \beta_0^2(1-\lambda)}}, \quad n_1 = \frac{\beta_0\sqrt{1-\lambda}}{\sqrt{\alpha_0^2 + \beta_0^2(1-\lambda)}}. \tag{3}$$

From Eq. (2), we can see that the output state $\widetilde{\rho}_{AB}$ is actually a less-hyperentangled pure state relative to the initial maximally hyperentangled source $|\varphi_0\rangle_{AB}$, which indicates that the vacuum part of the noisy state ρ_{AB} is eliminated successfully. Furthermore, this unknown state $\widetilde{\rho}_{AB}$ can be concentrated into a maximally hyperentangled state using the Schmidt-projection method [41].

The new fidelity F_1 of the output state $\widetilde{\rho}_{AB}$ can be expressed as

$$F_1^{(2)} = \frac{(\alpha_0 + \sqrt{1-\lambda}\beta_0)^2}{2[\alpha_0^2 + \beta_0^2(1-\lambda)]}. \tag{4}$$

To clearly show the effect of the hyperentanglement distillation scheme, we plotted (in Fig. 1) the fidelity difference $F_1^{(2)} - F_0^{(2)}$ as a function of the transmission coefficient T and the POVM parameter λ in the case of $T_1 = T_2 = T$. The figure shows that the scheme succeeds, i.e., $F_1^{(2)} - F_0^{(2)} > 0$, for most of the values of T and λ. Furthermore, as long as $T_1 = T_2 = T$ and $\lambda = 1 - T$, our hyperentanglement distillation scheme works for the noisy states with arbitrary values for T and λ in (0,1). This means that all of the noisy states ρ_{AB} within the fidelity range $F_0^{(2)} \in (0, 1)$ can be distilled under the condition of $T_1 = T_2 = T$ and $\lambda = 1 - T$.

Although fidelity is the most important parameter of the output state in entanglement distillation, the success probability is crucial for the distillation scheme too. The success probability of the HEDP for the noisy two-photon Bell state is $P^{(2)} = T(2-\lambda)/2$. In order to clearly illustrate the relationship between the fidelity difference and the success probability, we also numerically simulated $P^{(2)}$ and $F_1^{(2)} - F_0^{(2)}$ in the cases of the POVM parameters $\lambda = 0.2$, $\lambda = 0.5$, and $\lambda = 0.8$, respectively, in Fig. 1(b). The figure shows that the fidelity gain of our HEDP is acquired at the cost of success

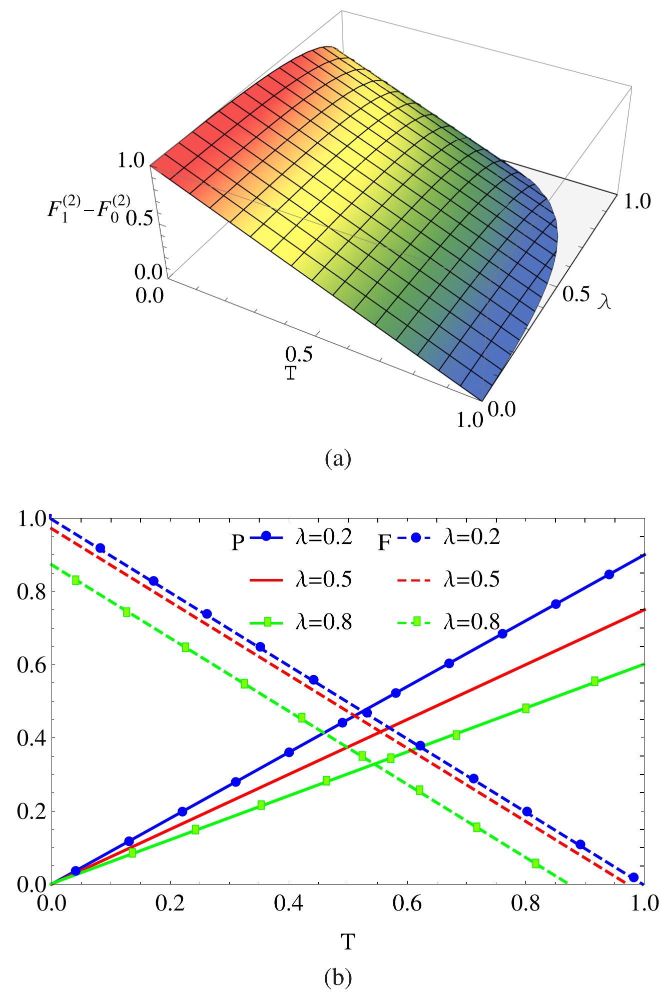

FIG. 1. For the noisy hyperentangled Bell state, (a) the fidelity difference $F_1^{(2)} - F_0^{(2)}$ in the POVM-based hyperentanglement distillation scheme as a function of the transmission coefficient T and the POVM parameter λ with $T_1 = T_2 = T$. (b) The success probability $P^{(2)}$ (P) versus the fidelity difference $F_1^{(2)} - F_0^{(2)}$ (F) in the cases of the POVM parameters $\lambda = 0.2$, $\lambda = 0.5$, and $\lambda = 0.8$.

probability, there does exist a tradeoff between the success probability and the fidelity gain, e.g., a high fidelity gain means a low success probability.

III. SINGLE-COPY-BASED DISTILLATION SCHEME FOR LOSSY HYPERENTANGLED GHZ STATE

As a generalization of our scheme in the preceding section, a single-copy-based scheme for distilling noisy hyperentangled GHZ states will be proposed in this section. Similarly, we first introduce the noisy state induced by the lossy channels. Suppose that the initial hyperentangled three-photon GHZ source is generated at Alice's side, $|\varphi_0\rangle_{ABC} = \frac{1}{2}(|HHH\rangle + |VVV\rangle)_{A_pB_pC_p}(|a_1b_1c_1\rangle + |a_2b_2c_2\rangle)_{A_sB_sC_s}$, and the photon B (C) is sent to Bob (Charlie) through two lossy spatial channels b_1, b_2 (c_1, c_2) described by two independent transmission coefficients T_1, T_2 (T_3, T_4). So, the noisy state for the hyperentangled three-photon system can be expressed as follows:

$$
\begin{aligned}
\rho_{ABC} = & \frac{T_1T_3+T_2T_4}{2}|\tau_1\rangle^s_{ABC}\langle\tau_1|\otimes|\mathrm{GHZ}\rangle_{ABC}\langle\,\mathrm{GHZ}| \\
& +\left[\frac{(1-T_1)T_3}{4}|a_1c_1\rangle\langle a_1c_1|+\frac{(1-T_2)T_4}{4}|a_2c_2\rangle\langle a_2c_2|\right]\otimes|(|HH\rangle\langle HH|+|VV\rangle\langle VV|)_{AC} \\
& +\left[\frac{(1-T_3)T_1}{4}|a_1b_1\rangle\langle a_1b_1|+\frac{(1-T_4)T_2}{4}|a_2b_2\rangle\langle a_2b_2|\right]\otimes(|HH\rangle\langle HH|+|VV\rangle\langle VV|)_{AB} \\
& +\left[\frac{(1-T_2)(1-T_4)}{4}|a_2\rangle\langle a_2|+\frac{(1-T_1)(1-T_3)}{4}|a_1\rangle\langle a_1|\right]\otimes(|H\rangle\langle H|+|V\rangle\langle V|)_A, \qquad (5)
\end{aligned}
$$

where, $|\tau_1\rangle_{ABC}=(\alpha_1|a_1b_1c_1\rangle+\beta_1|a_2b_2c_2\rangle)_{ABC}$ with two coefficients $\alpha_1=\sqrt{T_1T_3/(T_1T_3+T_2T_4)}$ and $\beta_1=\sqrt{T_2T_4/(T_1T_3+T_2T_4)}$, and $|\mathrm{GHZ}\rangle_{ABC}=1/\sqrt{2}(|HHH\rangle+|VVV\rangle)_{ABC}$. The fidelity of the noisy hyperentangled GHZ state relative to the initial maximally hyperentangled source $|\varphi_0\rangle_{ABC}$ can be written as $F_0^{(3)}=(T_1T_3+T_2T_4)(\alpha_1+\beta_1)^2/4$.

To distill the noisy hyperentangled state in Eq. (5), both Bob and Charlie would implement a POVM measurement on the polarization or spatial-mode DOF of the photons B and C, respectively. Similarly, we only consider the case where the local POVM measurements, i.e., the distillation operations, are performed on the spatial-mode DOF. Moreover, classical communication is allowed in this scheme so that the two users Bob and Charlie can coordinate their POVM measurements and post-select their measurement results. Assume that the POVM measurements carried out by the two users are the same form of the spatial-mode DOF, and can be expressed by the POVM elements $E^i_{s_1}=|k_1\rangle_i\langle k_1|+(1-\lambda)|k_2\rangle_i\langle k_2|$ and $E^i_{s_2}=\lambda|k_2\rangle_i\langle k_2|$, $k=b,c$, $i=B,C$. We will select the measurement events where both of two users get the same result E_{s_1}, and the corresponding output state is

$$
\begin{aligned}
\widetilde{\rho}_{ABC} &= I_s^AI_p^A\sqrt{E^C_{s_1}}\sqrt{E^B_{s_1}}I_p^CI_p^B\rho_{ABC}I_p^BI_p^C \\
&\quad \otimes\sqrt{E^B_{s_1}}\sqrt{E^C_{s_1}}I_p^AI_s^A\Big/\operatorname{tr}\Big(I_s^AI_p^A\sqrt{E^C_{s_1}}\sqrt{E^B_{s_1}}I_p^C \\
&\quad \otimes I_p^B\rho_{ABC}I_p^BI_p^C\sqrt{E^B_{s_1}}\sqrt{E^C_{s_1}}I_p^AI_s^A\Big) \\
&= |\widetilde{\varphi}\rangle_{ABC}\langle\widetilde{\varphi}|, \qquad (6)
\end{aligned}
$$

where $|\widetilde{\varphi}\rangle_{ABC}=(m_2|a_1b_1c_1\rangle+n_2|a_2b_2c_2\rangle)_{ABC}\otimes|\mathrm{GHZ}\rangle_{ABC}$ with coefficients

$$
m_2=\frac{\alpha_1}{\sqrt{\alpha_1^2+\beta_1^2(1-\lambda)^2}},\ n_2=\frac{\beta_1(1-\lambda)}{\sqrt{\alpha_1^2+\beta_1^2(1-\lambda)^2}}.
$$

Obviously, the state $|\widetilde{\varphi}\rangle_{ABC}$ is still a less-hyperentangled pure state, which can be concentrated into maximally hyperentangled state $|\varphi_0\rangle_{ABC}$ via the schemes in Ref. [42,43]. The fidelity of this post-selected state $\widetilde{\rho}_{ABC}$ is

$$
F_1^{(3)}=\frac{[\alpha_1+(1-\lambda)\beta_1]^2}{2[\alpha_1^2+\beta_1^2(1-\lambda)^2]}. \qquad (7)
$$

The fidelity difference $F_1^{(3)}-F_0^{(3)}$ is plotted in Fig. 2 as a function of the transmission coefficient T and the POVM parameter λ with $T_1=T_2=T_3=T_4=T$. From Fig. 2, we can see that, if we also choose the POVM parameter λ such that $\lambda=1-T$, our hyperentanglement distillation scheme can work for the noisy states with arbitrary $T\in(0,1)$. That is to say, all of the noisy states ρ_{ABC} within the fidelity range $F_0^{(3)}\in(0,1)$ can be distilled if $T_1=T_2=T_3=T_4=$

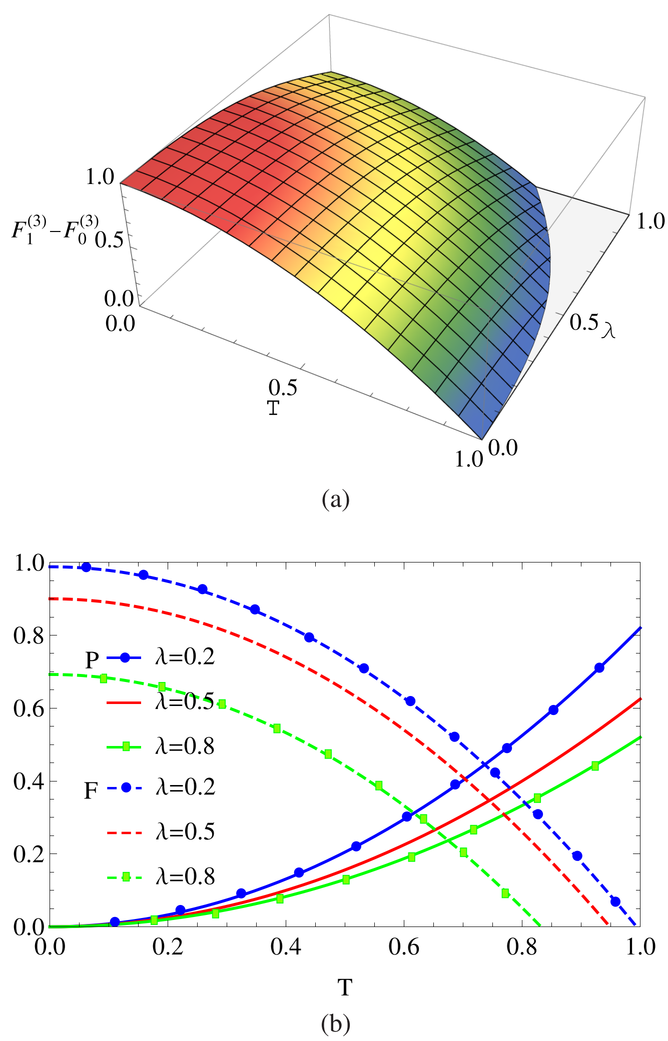

FIG. 2. For the noisy three-photon hyperentangled GHZ state, (a) the fidelity difference $F_1^{(3)}-F_0^{(3)}$ in the POVM-based hyperentanglement distillation scheme as a function of the transmission coefficient T and the POVM parameter λ with $T_1=T_2=T_3=T_4=T$. (b) The success probability $P^{(3)}$ (P) versus the fidelity difference $F_1^{(3)}-F_0^{(3)}$ (F) in the cases of the POVM parameters $\lambda=0.2$, $\lambda=0.5$, and $\lambda=0.8$.

T. In this case, the success probability can be expressed as $P^{(3)} = T^2[1 + (1 - \lambda)^2]/2$, which is plotted in Fig. 2(b) as a function of the transmission coefficient T in the cases of the POVM parameters $\lambda = 0.2$, $\lambda = 0.5$, and $\lambda = 0.8$, with $T_1 = T_2 = T_3 = T_4 = T$. Similarly, the figure shows that there does exist a tradeoff between the success probability and the fidelity gain for the distillation of noisy hyperentangled GHZ states too.

In principle, the above-proposed POVM-based hyperentanglement distillation scheme can be generalized to the noisy N-photon GHZ state as well. When the transmission coefficients T_i $(i = 1, 2, \cdots, 2N - 2)$ of spatial modes are all equal, i.e., $T_1 = T_2 = \cdots = T_{2(N-1)} = T$, the fidelities of the input noisy N-photon mixed GHZ states and the output state of our HEDP can be expressed, respectively, as

$$F_0^{(N)} = T^{(N-1)}, \tag{8}$$

$$F_1^{(N)} = \frac{\left[1 + (1 - \lambda)^{\frac{N-1}{2}}\right]^2}{2[1 + (1 - \lambda)^{(N-1)}]}. \tag{9}$$

The success probability of this HEDP is:

$$P^{(N)} = \tfrac{1}{2}T^{(N-1)}[1 + (1 - \lambda)^{(N-1)}]. \tag{10}$$

Figure 3 shows that our scheme still has a good hyperentanglement distillation effect for noisy ten-photon mixed GHZ states, and there exists a tradeoff between the success probability and the fidelity gain too. As long as the transmission coefficient T is not very small, one can distill the noisy states into pure high-fidelity output states with a relatively considerable probability.

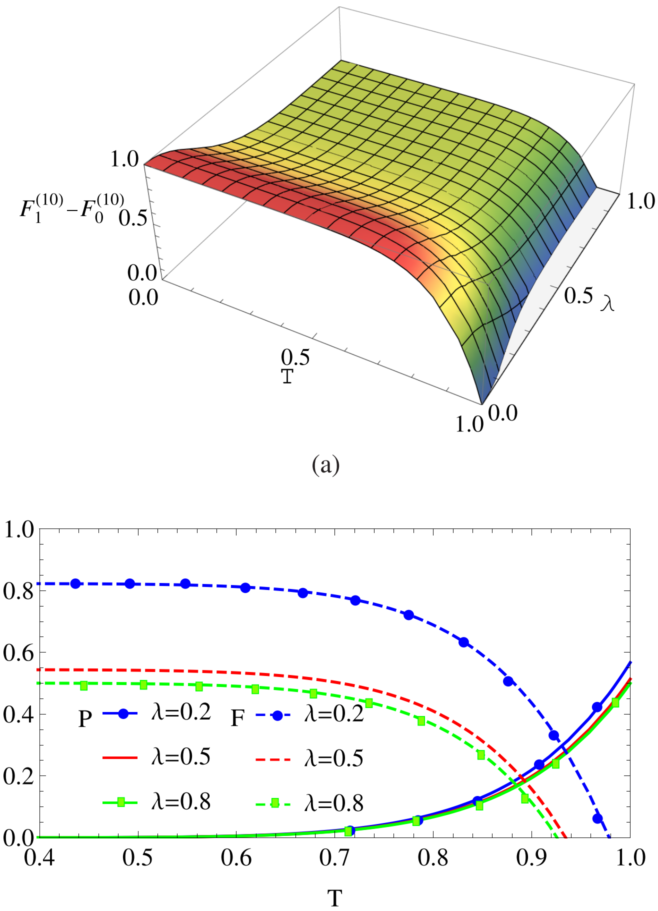

FIG. 3. For the noisy ten-photon hyperentangled GHZ state, (a) the fidelity difference $F_1^{(10)} - F_0^{(10)}$ in the POVM-based hyperentanglement distillation scheme as a function of the transmission coefficient T and the POVM parameter λ with $T_1 = T_2 = \cdots = T_{18} = T$. (b) The success probability $P^{(10)}$ (P) versus the fidelity difference $F_1^{(10)} - F_0^{(10)}$ (F) in the cases of the POVM parameters $\lambda = 0.2$, $\lambda = 0.5$, and $\lambda = 0.8$.

IV. ANALYSIS AND DISCUSSION

The key operation of our HEDP is the POVM measurement, whose optical realization only involves conventional single-photon detectors instead of complicated photon-number-resolution detectors [39,44,45]. More notably no auxiliary local entanglement resources are needed, such as the GHZ state or W state in [36,37,46].

For simplicity, we have only presented the special case where $T_i = T_j$ so far. However, the transmission coefficients T_i for different spatial modes are not equal in general, so it is necessary to consider the case where $T_1 \neq T_2$. To investigate the influence of $T_i \neq T_j$ on our HEDP we take the noisy hyperentangled Bell state as an example. To clearly show the effect of our HEDP for the $T_1 \neq T_2$ case, we plotted (in Fig. 4) the fidelity difference $F_1^{(2)} - F_0^{(2)}$ as a function of the transmission coefficient T_1 and T_2 in the cases of $\lambda = 0.2$ (a), $\lambda = 0.5$ (b), and $\lambda = 0.8$ (c), respectively. As in the case of $T_1 = T_2$, Fig. 4 shows that our HEDP succeeds, i.e., $F_1^{(2)} - F_0^{(2)} > 0$, for most of the values of T and λ in the case of $T_1 \neq T_2$.

Up to now we only considered the HEDP with the POVMs on the spatial-mode DOF of photon. If the POVMs are performed on the polarization DOF or on both the polarization and the spatial-mode DOFs simultaneously, can we realize a better distillation effect? A detailed calculation shows that our HEDP with POVM on the polarization DOF has a distillation effect similar to the HEDP with POVM on the spatial-mode DOF in both success probability and fidelity gain aspects. Meanwhile, the distillation effect of the HEDP with POVM on one DOF is better than that of the HEDP with POVMs on both DOFs. It is not difficult to understand these comparison results. The lossy hyperentangled state to be distilled is a mixture of a pure near-ideal hyperentangled state and the vacuum noise state, and in this state the two DOFs play the same role. So neither of these two DOFs is superior to the other, and thus the POVMs on the two DOFs have similar hyperentanglement distillation effects. Our HEDP is a single-copy-based protocol, and the POVM can filter out the noise component and keep the success case with some probability. That is to say, the hyperentanglement fidelity gain during our HEDP is acquired at the cost of success probability. So in most of the cases, the output fidelities and the success probabilities are smaller than unity when the POVM is performed on only one DOF, and thus the HEDP with POVMs on both the polarization and the spatial-mode DOFs will succeed with a relatively smaller probability and fidelity gain. To show this point more clearly, we calculated the fidelities

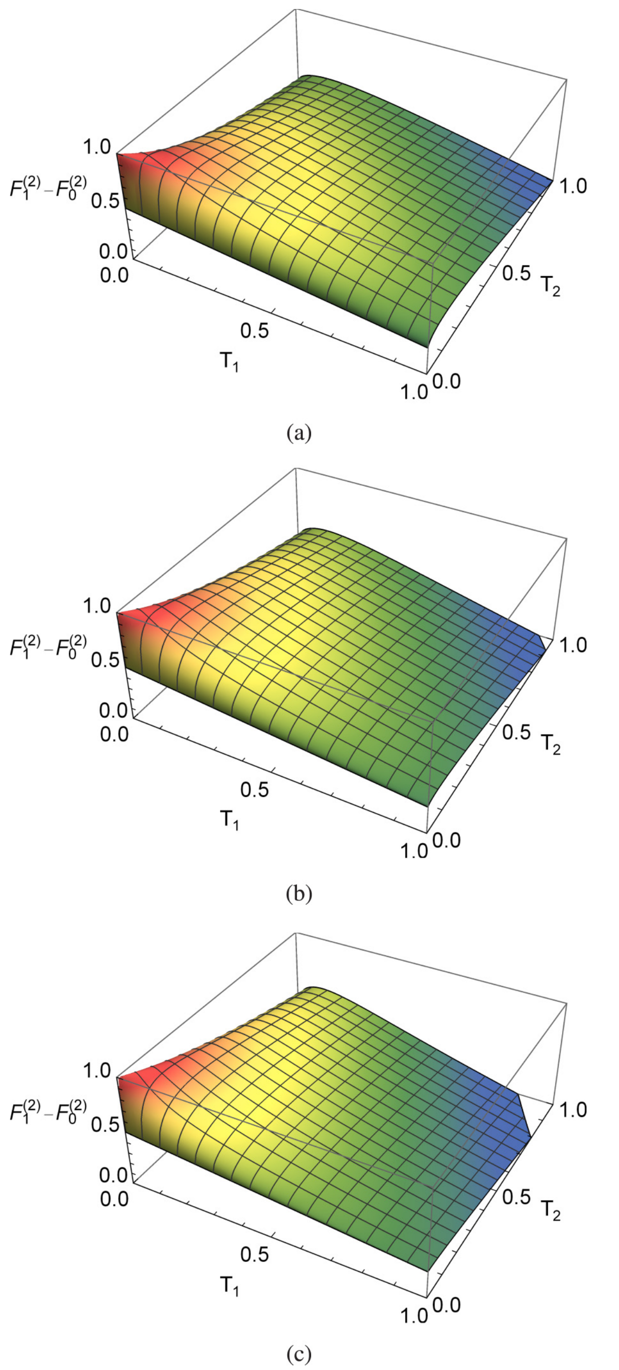

FIG. 4. For the noisy hyperentangled Bell state, the fidelity differences $F_1^{(2)} - F_0^{(2)}$ of our HEDP for the POVM parameters $\lambda = 0.2$ (a), $\lambda = 0.5$ (b), and $\lambda = 0.8$ (c).

$F_1'^{(N)}$ and success probabilities $P'^{(N)}$ of output states for N-photon hyperentangled GHZ cases after performing POVM measurements on two DOFs, and found that they are smaller than those [$F_1^{(N)}$ in Eq. (9), $P^{(N)}$ in Eq. (10)] of output states after performing POVM measurements on single DOF:

$$F_1'^{(N)} = \frac{[1+(1-\lambda)^{\frac{N-1}{2}}]^4}{4[1+(1-\lambda)^{(N-1)}]^2} = \left(F_1^{(N)}\right)^2, \tag{11}$$

$$P'^{(N)} = \frac{T^{(N-1)}}{4}[1+(1-\lambda)^{(N-1)}]^2. \tag{12}$$

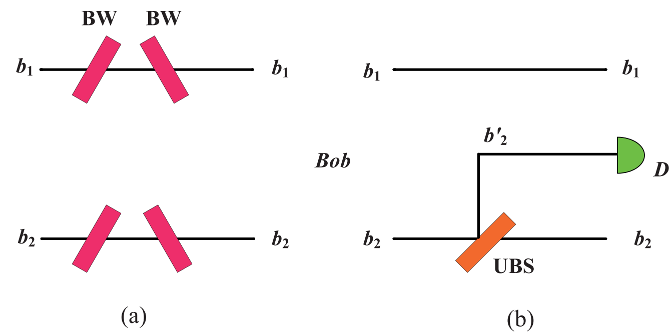

FIG. 5. The schematic diagram for realizing the POVMs on polarization DOF (a) and spatial-mode DOF (b) of photon B, respectively. BW is a Brewster window tilted near Brewster's angle θ such that $\sin 2\theta = \sqrt{1-\lambda}$. UBS represents an unbalanced beam splitter with the transmission coefficient $\sqrt{1-\lambda}$. D denotes the conventional single-photon detector.

Here all of the $2(N-1)$ transmission coefficients are supposed to be equal too. Since the POVM parameter λ and $F_1^{(N)}$ are smaller than 1, we have $F_1'^{(N)} < F_1^{(N)}$ and $P'^{(N)} < P^{(N)}$ for an arbitrary value N. In addition, the HEDP with POVM on one DOF is easier to implement than the HEDP with POVMs on both DOFs, which will reduce the realization complexity of the scheme. Therefore, in this paper, we only focus on the HEDP with POVM on only one DOF (say, the spatial-mode DOF).

To demonstrate the feasibility of the POVM-based hyperentanglement distillation protocols, we design the corresponding example schemes for realizing the local filtering operations, i.e., local POVMs, in the linear optical system. For the hyperentanglement distillation scheme of noisy two-photon Bell state, the distillation operation, i.e., the local POVM on the polarization DOF can be realized by two Brewster window (BW) pairs placed on two spatial modes (b_1, b_2) of the B photon [Fig. 5(a)], respectively, where the BWs are tilted near Brewster's angle θ such that $\sin 2\theta = \sqrt{1-\lambda}$ and are symmetrically arranged in pairs to compensate for transverse displacements. Usually, the refractive index of the BW material is larger than that of air, so Brewster's angle θ is larger than 45°, which means that λ is increasing with the increase of θ. Meanwhile, from the distillation effects of our HEDP, we can see that our HEDP has a better fidelity gain and a wider distillable range when λ is smaller. So we will choose the appropriate BW material so that the value of Brewster's angle θ is slightly larger than 45°, and thus the value of the realized POVM parameter λ is slightly larger than zero. Similarly, as shown in Fig. 5(b), the distillation operation, i.e., the local POVM, on the spatial-mode DOF of B photon can be realized by using an unbalanced BS (i.e., UBS) [47] on spatial mode b_2 with transmission coefficient $\sqrt{1-\lambda}$. When the photons B do not emit from the spatial mode b_2', this POVM on the spatial-mode DOF succeeds. In addition, because the POVMs performed by the operational users are considered to be the same in the corresponding DOF, in the optical realization scheme for three-photon hyperentanglement distillation Charlie uses the same optical filtering device as Bob (as depicted in Fig. 6).

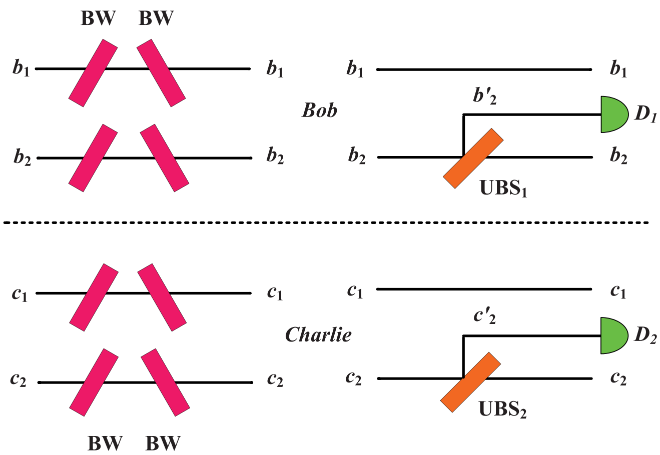

FIG. 6. The optical realization of a hyperentanglement distillation scheme for noisy three-photon GHZ states, where Charlie uses the same optical device as Bob. All required optical elements are the same as those of Fig. 5.

Notice that the success of the HEDP can be read from the event where there is no photon detected at the detectors. However, the efficiency of a single-photon detector is lower than 100% in practice, and thus a photon arriving at the detector may cause a no-photon-detection result because of the error induced by the quantum efficiency of the single-photon detector, so the success events maybe mixed with the failure events by error, which will lead to a higher success probability and a lower fidelity of the output state than those of the ideal case. Fortunately, the post-selection process is an alternative way of judging the success of the HEDP, i.e., the HEDP succeeds if Bob (and Charlie) detect the photons B (and C) emitting from the spatial modes b_1, b_2 (c_1, c_2), respectively, when Alice, Bob, and Charlie use the three photons ABC to complete their tasks in quantum communication. That is to say, the post-selection process of our HEDP and the next quantum communication tasks can be verified simultaneously. So, in this post-selection based judging scenario, the quantum efficiency of the single-photon detectors will lead to a lower success probability, but the fidelity of the output state is exactly the true one.

Although our single-copy-based HEDP is proposed for the two-photon lossy Bell hyperentangled state and three-photon lossy GHZ hyperentangled state in polarization and spatial-mode DOFs, it applies to any remote lossy hyperentangled state caused by the transmission loss. This point can be understood as follows. Any remote lossy hyperentangled state caused by the transmission loss can be regarded as a mixture of a pure partially hyperentangled state and the vacuum noise state, and the local POVM designed in our HEDP can be used to filter out the vacuum noise state caused by the transmission loss, so our HEDP is applicable to any remote lossy hyperentangled state caused by the transmission loss. In addition, the transmission loss will cause the decoherence of all the DOFs involved in the hyperentanglement, so our HEDP can be extended to other DOFs.

V. CONCLUSION

In conclusion, we proposed an efficient single-copy-based hyperentanglement distillation protocol for a two-photon lossy Bell state and a three-photon lossy GHZ state in polarization and spatial-mode DOFs, where the vacuum errors caused by the transmission loss can be filtered out via local POVM measurements performed on only one DOF. The POVM nature of our HEDP indicates that our HEDP can be applied to the lossy hyperentangled states in any quantum system, i.e., it is a universal HEDP, and the distillation operation on only one DOF will enhance the fidelity of the system in both DOFs. Besides these two intrinsic advantages, our HEDP also possesses the following advantages in the implementation level. Because only one copy of the lossy state will be needed in each round of our HEDP, our scheme has a wider range of distillable fidelities, and no auxiliary local entanglement resources are involved. All of these properties suggest that our HEDP is relatively simple and feasible in quantum communication.

One may want to know the detailed situations where our HEDP can be applied. These situations include quantum communications and quantum computation. Here, we will give two detailed situations where our HEDP can be applied.

The first one is that our HEDP can help us to enhance the information capacity of a practical quantum channel in quantum superdense coding. In a standard quantum superdense coding process, only 1.585 (rather than 2) classical bits will be sent by sending one qubit because of the impossibility of the complete and deterministic Bell state measurement with linear optics. With the assistance of hyperentanglement, complete and deterministic Bell state measurements become possible in an optical system [31,48,49], and thus the channel capacity of a hyperentangled quantum state is larger than that of an entangled state in single DOF in linear photonic superdense coding [50]. But the channel quality of the hyperentangled state will deteriorate during the transmission of the photon propagating between two users (the information decoder Alice and the encoder Bob), which will subsequently reduce the channel capacity. Alice will generate an ideal two-photon hyperentangled state and send one of the two photons to Bob. Due to the loss during the transmission process, Alice and Bob can only share a noisy two-photon hyperentangled state. Before information encoding, Alice and Bob can apply our HEDP to distill a high-quality hyperentangled state from the noisy one. If the distillation process succeeds, Bob will continue the information encoding process, otherwise they will discard the pair and start a new round of the superdense coding process.

The second one is hyperentanglement-assisted quantum error correction. Here, the unknown-state information photon will be protected via hyperentanglement-assisted quantum error correction code during the noisy transmission from the sender Alice to the receiver Bob [26,51]. To this end, Alice and Bob must share an ideal two-photon hyperentangled state. Usually the ideal two-photon hyperentangled state will be generated locally, and distributed among Alice and Bob. Due to the loss during the transmission process, Alice and Bob can only share a noisy two-photon hyperentangled state, which will inevitably reduce the protection effects of the hyperentanglement-assisted quantum error correction code. To partially or completely recover the error-correction function of the code, one needs our HEDP before the encoding process.

In addition, remote hyperentangled states also can be used as the quantum channels to teleport the states of multiple degrees of freedom of a single photon [52] and to realize the superdense teleportation of quantum state parameters [53], and the hyperentanglement quantum channel must be generated locally and distributed among remote users. However, during the practical entanglement distribution, the inevitable transmission loss will deteriorate maximally hyperentangled states into partially hyperentangled mixed states. So our HEDP provides an efficient way to overcome the transmission loss problem of quantum communication.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (NSFC) under Grant No. 11947102, the Natural Science Foundation of Anhui Province under Grants No. 2008085MA16, No. 2008085QA26, and No.1808085MA21, the Ph.D. Research Startup Foundation of Anhui University (Grant No. J01003310), and the Open Project of State Key Laboratory of Surface Physics in Fudan University (Grant No. KF2018_13).

[1] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an Unknown Quantum state via dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. **70**, 1895 (1993).

[2] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum cryptography, Rev. Mod. Phys. **74**, 145 (2002).

[3] R. Raussendorf, D. E. Browne, and H. J. Briegel, Measurement-based quantum computation on cluster states, Phys. Rev. A **68**, 022312 (2003).

[4] A. K. Ekert, Quantum Cryptography Based on Bell's Theorem, Phys. Rev. Lett. **67**, 661 (1991).

[5] C. H. Bennett, G. Brassard, and N. D. Mermin, Quantum Cryptography without Bell's Theorem, Phys. Rev. Lett. **68**, 557 (1992).

[6] H. J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Quantum Repeaters: The Role of Imperfect Local Operations in Quantum Communication, Phys. Rev. Lett. **81**, 5932 (1998).

[7] Y. B. Sheng, L. Zhou, S. M. Zhao, and B. Y. Zheng, Efficient single-photon-assisted entanglement concentration for partially entangled photon pairs, Phys. Rev. A **85**, 012307 (2012).

[8] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of Noisy Entanglement and Faithful Teleportation via Noisy Channels, Phys. Rev. Lett. **76**, 722 (1996).

[9] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Quantum Privacy Amplification and the Security of Quantum Cryptography over Noisy Channels, Phys. Rev. Lett. **77**, 2818 (1996).

[10] J. W. Pan, C. Simon, Č. Brukner, and A. Zellinger, Entanglement purification for quantum communication, Nature (London) **410**, 1067 (2001).

[11] C. Simon and J. W. Pan, Polarization Entanglement Purification using Spatial Entanglement, Phys. Rev. Lett. **89**, 257901 (2002).

[12] Y. B. Sheng, F. G. Deng, and H. Y. Zhou, Efficient polarization-entanglement purification based on parametric down-conversion sources with cross-Kerr nonlinearity, Phys. Rev. A **77**, 042308 (2008).

[13] Y. B. Sheng and F. G. Deng, Deterministic entanglement purification and complete nonlocal Bell-state analysis With hyperentanglement, Phys. Rev. A **81**, 032307 (2010).

[14] Y. B. Sheng and F. G. Deng, One-step deterministic polarization-entanglement purification using spatial entanglement, Phys. Rev. A **82**, 044305 (2010).

[15] X. H. Li, Deterministic polarization-entanglement purification using spatial entanglement, Phys. Rev. A **82**, 044304 (2010).

[16] F. G. Deng, One-step error correction for multipartite polarization entanglement, Phys. Rev. A **83**, 062316 (2011).

[17] Y. B. Sheng and L. Zhou, Deterministic polarization entanglement purification using time-bin entanglement, Laser Phys. Lett. **11**, 085203 (2014).

[18] N. Gisin, Hidden quantum nonlocality revealed by local filters, Phys. Lett. A **210**, 151 (1996).

[19] Y. S. Huang, H. B. Xing, M. Yang, Q. Yang, W. Song, and Z. L. Cao, Distillation of multipartite entanglement by local filtering operations, Phys. Rev. A **89**, 062320 (2014).

[20] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, Concentrating partial entanglement by local operations, Phys. Rev. A **53**, 2046 (1996).

[21] H. K. Lo and S. Popescu, Concentrating entanglement by local actions: Beyond mean values, Phys. Rev. A **63**, 022301 (2001).

[22] T. J. Wang and G. L. Long, Entanglement concentration for arbitrary unknown less-entangled three-photon *W* states with linear optics, J. Opt. Soc. Am. B **30**, 1069 (2013).

[23] P. B. Dixon, G. A. Howland, J. Schneeloch, and J. C. Howell, Quantum Mutual Information Capacity for High-Dimensional entangled states, Phys. Rev. Lett. **108**, 143603 (2012).

[24] D. Bruss and C. Macchiavello, Optimal Eavesdropping in Cryptography with Three-Dimensional Quantum States, Phys. Rev. Lett. **88**, 127901 (2002).

[25] N. J. Cerf, M. Bourennane, A. Karlsson, and N. Gisin, Security of Quantum Key Distribution Using *d*-Level Systems, Phys. Rev. Lett. **88**, 127902 (2002).

[26] M. M. Wilde and D. B. Uskov, Linear-optical hyperentanglement-assisted quantum error-correcting code, Phys. Rev. A **79**, 022305 (2009).

[27] B. C. Ren and G. L. Long, General hyperentanglement concentration for photon systems assisted by quantum dot spins inside optical microcavities, Opt. Express **22**, 6547 (2014).

[28] Y. B. Sheng, J. Liu, S. Y. Zhao, and L. Zhou, Multipartite entanglement concentration for nitrogen-vacancy center and microtoroidal resonator system, Chin. Sci. Bull. **59**, 3507 (2013).

[29] J. T. Barreiro, N. K. Langford, N. A. Peters, and P. G. Kwiat, Generation of Hyperentangled Photon Pairs, Phys. Rev. Lett. **95**, 260501 (2005).

[30] M. Barbieri, C. Cinelli, P. Mataloni, and F. De Martini, Polarization-momentum hyperentangled states: Realization and characterization, Phys. Rev. A **72**, 052110 (2005).

[31] C. Schuck, G. Huber, C. Kurtsiefer, and H. Weinfurter, Complete Deterministic Linear Optics Bell State Analysis, Phys. Rev. Lett. **96**, 190501 (2006).

[32] D. Bhatti, J. von Zanthier, and G. S. Agarwal, Entanglement of polarization and orbital angular momentum, Phys. Rev. A **91**, 062303 (2015).

[33] W. Zhang, D. S. Ding, M. X. Dong, S. Shi, K. Wang, S. L. Liu, Y. Li, Z. Y. Zhou, B. S. Shi, and G. C. Guo, Experimental realization of entanglement in multiple degrees of freedom between two quantum memories, Nat. Commun. **7**, 13514 (2016).

[34] B.-C. Ren and F.-G. Deng, Hyperentanglement purification and concentration assisted by diamond NV centers inside photonic crystal cavities, Laser Phys. Lett. **10**, 115201 (2013).

[35] G. Y. Wang, Q. Liu, and F. G. Deng, Hyperentanglement purification for two-photon six-qubit quantum systems, Phys. Rev. A **94**, 032319 (2016).

[36] T. J. Wang, L. L. Liu, R. Zhang, C. Cao, and C. Wang, One-step hyperentanglement purification and hyperdistillation with linear optics, Opt. Express **23**, 009284 (2015).

[37] T. J. Wang, S. C. Mi, and C. Wang, Hyperentanglement purification using imperfect spatial entanglement, Opt. Exp. **25**, 2969 (2017).

[38] T. J. Wang, C. Cao, and C. Wang, Linear-optical implementation of hyperdistillation from photon loss, Phys. Rev. A **89**, 052303 (2014).

[39] G. Yang, Y. S. Zhang, Z. R. Yang, L. Zhou, Y. B. Sheng, Linear-optical heralded amplification protocol for two-photon spatial-mode-polarization hyperentangled state, Quantum Inf. Process. **18**, 317 (2019).

[40] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[41] B. C. Ren, F. F. Du, and F. G. Deng, Hyperentanglement concentration for two-photon four-qubit systems with linear optics, Phys. Rev. A **88**, 012302 (2013).

[42] X. H. Li and S. Ghose, Hyperconcentration for multipartite entanglement via linear optics, Laser Phys. Lett. **11**, 125201 (2014).

[43] H. J. Liu, Y. Xia, and J. Song, Efficient hyperentanglement concentration for N-particle Greenberger-Horne-Zeilinger state assisted by weak cross-Kerr nonlinearity, Quantum Inf. Process. **15**, 2033 (2016).

[44] A. J. Miller, S. W. Nam, J. M. Martinis, and A. V. Sergienko, Demonstration of a low-noise near-infrared photon counter with multiphoton discrimination, Appl. Phys. Lett. **83**, 791 (2003).

[45] C. F. Wildfeuer, A. J. Pearlman, J. Chen, J. Fan, A. Migdall, and J. P. Dowling, Resolution and sensitivity of a Fabry-Perot interferometer with a photon-number-resolving detector, Phys. Rev. A **80**, 043822 (2009).

[46] T. J. Wang and C. Wang, High-efficient entanglement distillation from photon loss and decoherence, Opt. Exp. **23**, 031550 (2015).

[47] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Experimental Realization of Any Discrete Unitary Operator, Phys. Rev. Lett. **73**, 58 (1994).

[48] P. G. Kwiat and H. Weinfurter, Embedded Bell-state analysis, Phys. Rev. A **58**, R2623(R) (1998).

[49] M. Barbieri, G. Vallone, P. Mataloni, and F. DeMartini, Complete and deterministic discrimination of polarization Bell states assisted by momentum entanglement, Phys. Rev. A **75**, 042317 (2007).

[50] J. T. Barreiro, T. C. Wei, and P. G. Kwiat, Beating the channel capacity limit for linear photonic superdense coding, Nat. Phys. **4**, 282 (2008).

[51] T. A. Brun, I. Devetak, and M. H. Hsieh, Correcting quantum errors with entanglement, Science **314**, 436 (2006).

[52] X. L. Wang, X. D. Cai, Z. E. Su, M. C. Chen, D. Wu, L. Li, N. L. Liu, C. Y. Lu, and J. W. Pan, Quantum teleportation of multiple degrees of freedom of a single photon, Nature (London) **518**, 516 (2015).

[53] T. M. Graham, H. J. Bernstein, T. C. Wei, M. Junge, and P. G. Kwiat, Superdense teleportation using hyperentangled photons, Nat. Commun. **6**, 7185 (2015).